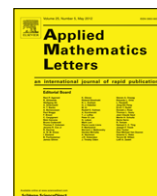


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A new generalized Ostrowski Grüss type inequality and applications

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ABSTRACT

Integration with weight functions is used in countless mathematical problems such as: approximation theory and spectral analysis, statistical analysis and the theory of distributions. The aim of this paper is to establish a new inequality using weight function which generalizes the inequalities of Barnett et al. in 2001 and Rafiq et al. in 2006. We also discuss some other interesting inequalities in special cases. Perturbed midpoint and trapezoid inequalities are also obtained. Some applications in numerical integration are also given.

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1. Introduction

In 1935, Grüss developed an integral inequality [1]. Ostrowski [2] established an interesting integral inequality associated with differentiable mappings in 1938. This Ostrowski type inequality has powerful applications in numerical integration, probability and optimization theory, stochastic, statistics, information and integral operator theory. During past few years, many researchers focused their great attentions on the study of the above two inequalities. In 1997, Dragomir and Wang [3] have investigated the Ostrowski type inequality in terms of the lower and upper bounds of the first derivative. In 2001, Barnett et al. [4] pointed out a similar result to the above for twice differentiable mappings in terms of the upper and lower bounds of the second derivative. Inspired and motivated by the work of Barnett et al. [4], we establish a new inequality, which is more generalized as compared to previous inequalities developed and discussed in [4,3,5]. Moreover, our results are in weighted form instead of previous results which are in non-weighted form. In addition, approach of Barnett et al. [4] for obtaining the bounds of a particular quadrature rule has depended on the peano kernel but we use weighted peano kernel in our findings. This approach not only generalizes the results of [4], but also gives some other interesting inequalities as special cases. Some closely related new results are also discussed. Perturbed midpoint and trapezoid inequalities are obtained in the form of remarks. In Section 4, we give some applications in numerical integration.

2. Preliminaries

In [2], Ostrowski proved the following interesting and useful integral inequality.

Theorem 1. Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) , whose derivative $f': (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e. $\|f'\|_\infty = \sup_{t \in [a, b]} |f'(t)| < \infty$ then

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$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty \quad (1)$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

For some applications of Ostrowski's inequality to special means and numerical quadrature rule, we refer to [6].

In [1], the Grüss inequality is defined as the integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals. The inequality is as follows.

Theorem 2. Let $f, g: [a, b] \rightarrow \mathbb{R}$ be integrable functions such that $\varphi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$, for all $x \in [a, b]$, where $\varphi, \Phi, \gamma, \Gamma$ are constants. Then

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma) \quad (2)$$

where the constant $\frac{1}{4}$ is sharp.

In [3], Dragomir and Wang improved the above inequality and proved the following Ostrowski type inequality in terms of the lower and upper bounds of the first derivative.

Theorem 3. Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) , whose derivative satisfies the condition: $\gamma \leq f'(x) \leq \Gamma$ for all $x \in [a, b]$, then we have the inequality,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(\frac{f(b) - f(a)}{b-a} \right) \left(x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma) \quad (3)$$

for all $x \in [a, b]$.

In [4], Barnett et al. pointed out a similar result to the above for twice differentiable mappings in terms of the upper and lower bounds of the second derivative.

Theorem 4. Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and twice differentiable on (a, b) , and assume that the second derivative $f'': (a, b) \rightarrow \mathbb{R}$ satisfies the condition: $\gamma \leq f''(x) \leq \Gamma$ for all $x \in [a, b]$. Then we have,

$$\begin{aligned} & \left| f(x) - \left(x - \frac{a+b}{2} \right) f'(x) + \left[\frac{(b-a)^2}{24} + \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 \right] \left(\frac{f'(b) - f'(a)}{b-a} \right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{8} (\Gamma - \gamma) \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^2 \end{aligned} \quad (4)$$

for all $x \in [a, b]$.

3. Some new results

We assume that the weight function (or density) $w: (a, b) \rightarrow [0, \infty)$ to be non-negative and integrable over its entire domain and consider $\int_a^b w(t) dt < \infty$.

The domain of w may be finite or infinite and may be vanish at the boundary point. We denote $m(a, b) = \int_a^b w(t) dt$. Now we give our main result.

Theorem 5. Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) , whose derivative satisfies the condition: $\varphi \leq f'(x) \leq \Phi$ for all $x \in (a, b)$. Then we have the weighted inequality:

$$\begin{aligned} & \left| \frac{1}{b-a} m(a, b) f(x) - w(x) \left(x - \frac{a+b}{2} \right) \left(\frac{f(b) - f(a)}{b-a} \right) - \frac{1}{b-a} \int_a^b f(t) w(t) dt \right| \\ & \leq \frac{1}{4} (\Phi - \varphi) \left(\frac{1}{2} m(a, b) + \frac{1}{2} \left| \int_a^b \operatorname{sgn}(t-x) w(t) dt \right| \right) \end{aligned} \quad (5)$$

for all $x \in [a, b]$.

Proof. The weighted integral inequality [7] is

$$f(x) = \frac{1}{m(a, b)} \int_a^b P(x, t) f'(t) dt + \frac{1}{m(a, b)} \int_a^b f(t) w(t) dt, \quad (6)$$

for all $x \in [a, b]$.

In [4], the approach of obtaining the bounds for a particular quadrature rule has depended on peano kernel $P : [a, b]^2 \longrightarrow \mathbb{R}$ defined by

$$P(x, t) = \begin{cases} t - a & \text{if } t \in [a, x] \\ t - b & \text{if } t \in (x, b]. \end{cases} \quad \text{where } t \in [a, b].$$

Here we use the weighted peano Kernel, $P(., .) : [a, b]^2 \longrightarrow \mathbb{R}$ defined by

$$P(x, t) = \begin{cases} \int_a^t w(u)du & \text{if } t \in [a, x] \\ \int_b^x w(u)du & \text{if } t \in (x, b] \end{cases} \tag{7}$$

that has been extensively used to obtain Ostrowski type results [7–9].

We observe that the mapping $P(., .) : [a, b] \longrightarrow \mathbb{R}$ satisfies the estimation:

$$0 \leq P(x, t) \leq \begin{cases} \int_x^b w(u)du, & \text{if } x \in \left[a, \frac{a+b}{2} \right) \\ \int_a^x w(u)du, & \text{if } x \in \left[\frac{a+b}{2}, b \right] \end{cases}. \tag{8}$$

Consider, $f(x) = P(x, t)$, $g(x) = f'(t)$.

Applying Grüss integral inequality for $P(x, t)$ and $f'(t)$, we get:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b P(x, t) f'(t) dt - \frac{1}{b-a} \int_a^b P(x, t) dt \cdot \frac{1}{b-a} \int_a^b f'(t) dt \right| \\ & \leq \frac{1}{4} (\Phi - \varphi) \begin{cases} \int_x^b w(u)du, & \text{if } x \in \left[a, \frac{a+b}{2} \right) \\ \int_a^x w(u)du, & \text{if } x \in \left[\frac{a+b}{2}, b \right]. \end{cases} \end{aligned} \tag{9}$$

Using (7), inequality (9) gives:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b P(x, t) f'(t) dt - w(x) \left(x - \frac{a+b}{2} \right) \left(\frac{f(b) - f(a)}{b-a} \right) \right| \\ & \leq \frac{1}{4} (\Phi - \varphi) \begin{cases} \int_x^b w(u)du, & \text{if } x \in \left[a, \frac{a+b}{2} \right) \\ \int_a^x w(u)du, & \text{if } x \in \left[\frac{a+b}{2}, b \right] \end{cases}. \end{aligned} \tag{10}$$

Using (6), inequality (10) gives:

$$\begin{aligned} & \left| \frac{1}{b-a} m(a, b) f(x) - \frac{1}{b-a} \int_a^b f(t) w(t) dt - w(x) \left(x - \frac{a+b}{2} \right) \left(\frac{f(b) - f(a)}{b-a} \right) \right| \\ & \leq \frac{1}{4} (\Phi - \varphi) \begin{cases} \int_x^b w(u)du, & \text{if } x \in \left[a, \frac{a+b}{2} \right) \\ \int_a^x w(u)du, & \text{if } x \in \left[\frac{a+b}{2}, b \right] \end{cases}. \end{aligned} \tag{11}$$

Now, let us observe that

$$\begin{aligned} \max \left(\int_x^b w(u)du, \int_a^x w(u)du \right) &= \begin{cases} \int_x^b w(u)du, & \text{if } x \in \left[a, \frac{a+b}{2} \right) \\ \int_a^x w(u)du, & \text{if } x \in \left[\frac{a+b}{2}, b \right] \end{cases} \\ &= \frac{1}{2} m(a, b) + \frac{1}{2} \left| \int_a^b \operatorname{sgn}(t-x) w(t) dt \right|. \end{aligned} \tag{12}$$

Using (12) in (11), we get our main result (5). \square

Remark 1. If we put $w(t) = 1$, in (5), we get a new inequality which is similar to inequality (4). Inequality (5) is a source of numerous particular inequalities that can be obtained by using different weights.

Corollary 1. Under the assumptions of Theorem 5, we have the following perturbed midpoint inequality:

$$\begin{aligned} & \left| \frac{1}{b-a} m(a, b) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) w(t) dt \right| \\ & \leq \frac{1}{4} (\Phi - \varphi) \left(\frac{1}{2} m(a, b) + \frac{1}{2} \left| \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) w(t) dt \right| \right). \end{aligned} \quad (13)$$

Proof. Put $x = \frac{a+b}{2}$ in inequality (5), we get the above inequality (13). \square

Corollary 2. Under the assumptions of Theorem 5, we have the perturbed trapezoidal inequality:

$$\begin{aligned} & \left| \frac{1}{b-a} m(a, b) (f(a) + f(b)) - \frac{2}{b-a} \int_a^b f(t) w(t) dt + (w(a) - w(b)) \left(\frac{f(b) - f(a)}{2} \right) \right| \\ & \leq \frac{1}{4} (\Phi - \varphi) \left(m(a, b) + \frac{1}{2} \left| \int_a^b \operatorname{sgn}(t-a) w(t) dt \right| + \frac{1}{2} \left| \int_a^b \operatorname{sgn}(t-b) w(t) dt \right| \right). \end{aligned} \quad (14)$$

Proof. Put $x = a$ and $x = b$ in (5), then sum up these two inequalities. Using the triangle inequality and dividing by 2, we get (14). \square

4. Application in numerical integration

Let $I_n : a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$, $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, 1, \dots, n-1$) a sequence of intermediate points $h_i := x_{i+1} - x_i$ ($i = 0, 1, \dots, n-1$). We have the following quadrature formula:

Theorem 6. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) , and $f' : (a, b) \rightarrow \mathbb{R}$ satisfy the condition $\varphi \leq f'(x) \leq \Phi$, for all $x \in (a, b)$. Then, we have the following perturbed Riemann's type quadrature formula:

$$\int_a^b f(t) w(t) dt = A(f, \xi, I_n) + R(f, \xi, I_n)$$

where

$$A(f, \xi, I_n) \leq \sum_{i=0}^{n-1} m(x_i, x_{i+1}) f(\xi_i) - \sum_{i=0}^{n-1} h_i \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) w(\xi_i) f'(\xi_i) \quad (15)$$

and remainder satisfies the estimation:

$$R(f, \xi, I_n) \leq \frac{1}{8} (\Phi - \varphi) \sum_{i=0}^{n-1} h_i \left(m(x_i, x_{i+1}) + \left| \int_{x_i}^{x_{i+1}} \operatorname{sgn}(t - \xi_i) w(t) dt \right| \right) \quad (16)$$

for all $\xi_i \in [x_i, x_{i+1}]$, where $h_i := x_{i+1} - x_i$ ($i = 1, 2, 3, \dots, n-1$).

Proof. Apply Theorem 5 on the interval $[x_i, x_{i+1}]$, $\xi_i \in [x_i, x_{i+1}]$, where $h_i := x_{i+1} - x_i$, ($i = 1, 2, 3, \dots, n-1$) to get :

$$\begin{aligned} & \left| \int_{x_i}^{x_{i+1}} f(t) w(t) dt + h_i \left(\xi_i - \frac{x_{i+1} + x_i}{2} \right) w(\xi_i) f'(\xi_i) - m(x_i, x_{i+1}) f(\xi_i) \right| \\ & \leq \frac{1}{8} (\Phi - \varphi) h_i \left(m(x_i, x_{i+1}) + \left| \int_{x_i}^{x_{i+1}} \operatorname{sgn}(t - \xi_i) w(t) dt \right| \right). \end{aligned}$$

Summing over i from 0 to $n-1$

$$\begin{aligned} & \left| \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(t) w(t) dt + \sum_{i=0}^{n-1} h_i \left(\xi_i - \frac{x_{i+1} + x_i}{2} \right) w(\xi_i) f'(\xi_i) - \sum_{i=0}^{n-1} m(x_i, x_{i+1}) f(\xi_i) \right| \\ & \leq \frac{1}{8} (\Phi - \varphi) \sum_{i=0}^{n-1} h_i \left(m(x_i, x_{i+1}) + \left| \int_{x_i}^{x_{i+1}} \operatorname{sgn}(t - \xi_i) w(t) dt \right| \right) \end{aligned}$$

$$\int_a^b f(t)w(t)dt \leq \sum_{i=0}^{n-1} h_i \left(\xi_i - \frac{x_{i+1} + x_i}{2} \right) w(\xi_i)f'(\xi_i) - \sum_{i=0}^{n-1} m(x_i, x_{i+1})f(\xi_i) + \frac{1}{8} (\Phi - \varphi) \sum_{i=0}^{n-1} h_i \left(m(x_i, x_{i+1}) + \left| \int_{x_i}^{x_{i+1}} \operatorname{sgn}(t - \xi_i)w(t)dt \right| \right).$$

Using the generalized triangular inequality, we deduce the desired estimation. \square

Remark 2. Choosing $\xi_i = \frac{x_i+x_{i+1}}{2}$, we have perturbed midpoint quadrature formula:

$$\int_a^b f(t)w(t)dt = A_M(f, \xi_i, I_n) + R_M(f, \xi_i, I_n)$$

where

$$A_M(f, \xi, I_n) \leq \sum_{i=0}^{n-1} m(x_i, x_{i+1})f\left(\frac{x_i + x_{i+1}}{2}\right), \tag{17}$$

and where the remainder $R_M(f, I_n)$ satisfies the estimation:

$$|R_M(f, I_n)| \leq \frac{1}{8}(\Phi - \varphi) \sum_{i=0}^{n-1} h_i \left(m(x_i, x_{i+1}) + \left| \int_{x_i}^{x_{i+1}} \operatorname{sgn}\left(t - \frac{x_i + x_{i+1}}{2}\right) w(t)dt \right| \right). \tag{18}$$

Corollary 3. The following perturbed trapezoid rule holds:

$$\int_a^b f(t)w(t)dt = A_T(f, \xi, I_n) + R_T(f, \xi, I_n),$$

where

$$A_T(f, I_n) \leq m(x_i, x_{i+1})\frac{f(x_i) + f(x_{i+1})}{2} - \frac{h_i^2}{2} w(x_{i+1})(f'(x_{i+1}) - w(x_i)f'(x_i)), \tag{19}$$

and the remainder $R_T(f, I_n)$ satisfies the estimation:

$$|R_T(f, , I_n)| \leq \frac{1}{4}(\Phi - \varphi) \sum_{i=0}^{n-1} h_i \left(m(x_i, x_{i+1}) + \frac{1}{2} \left| \int_{x_i}^{x_{i+1}} \operatorname{sgn}(t - x_i)w(t)dt \right| + \frac{1}{2} \left| \int_{x_i}^{x_{i+1}} \operatorname{sgn}(t - x_{i+1})w(t)dt \right| \right). \tag{20}$$

Remark 3. The above mentioned perturbed midpoint formula (18) and perturbed trapezoid formula (20) give better approximation of the integral $\int_a^b f(x) dx$ for general class of mapping.

5. Conclusion

Inspired and motivated by the work of Barnett et al. [4], we establish a new inequality, which is more generalized as compared to previous inequalities developed and discussed in [4,3,5]. In addition, approach of Barnett et al. [4] for obtaining the bounds of a particular quadrature rule has depended on the peano kernel but we use weighted peano kernel in our findings. This approach not only generalizes the results of [4], but also gives some other interesting inequalities as special cases. The applications in numerical integration have also given.

References

[1] D.S. Mitrinovic, J.E. Pečarić, A.M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, Dordrecht, 1993.
 [2] A.M. Ostrowski, Über die Absolutabweichung einer differenzierbaren Funktion von ihrem Integralmittelwert, Commentarii Mathematici Helvetici 10 (1938) 226–227.
 [3] S.S. Dragomir, S. Wang, An inequality of Ostrowski–Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, Computer and Mathematics with Applications 33 (1997) 15–20.
 [4] N.S. Barnett, P. Cerone, S.S. Dragomir, J. Roumeliotis, A. Sofo, A survey on Ostrowski type inequalities for twice differentiable mappings and applications, Inequality Theory and Applications 1 (2001) 24–30.
 [5] A. Rafiq, N.A. Mir, F. Zafar, A generalized Ostrowski–Grüss type inequality for twice differentiable mappings and applications, JIPAM 7 (4) (2006) article 124.

- [6] S.S. Dragomir, S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Applied Mathematics Letters* 11 (1998) 105–109.
- [7] A. Qayyum, A weighted Ostrowski–Grüss type inequality for twice differentiable mappings and applications, *International Journal of Mathematics and Computation* 1 (8) (2008) 63–71.
- [8] A. Qayyum, A Weighted Ostrowski–Grüss type inequality and applications, in: *Proceedings of the World Congress on Engineering 2009*, vol. II, WCE 2009, July 1–3, London, UK, 2009.
- [9] A. Rafiq, N.A. Mir, F. Ahmad, Weighted Ostrowski type inequality for differentiable mappings whose first derivatives belong to $L_p(a, b)$, *General Mathematics* 14 (3) (2006) 91–102.