An existence theorem for a singular third-order boundary value problem on \([0, +\infty)\)

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Received 20 August 2007; accepted 14 November 2007

Abstract

An existence result for a singular third-order boundary value problem is proved in this work. Here the nonlinearity is of the form \(f(y) = (1 - y)\lambda g(y)\), where \(\lambda > 0\) and \(g(y)\) is continuous and positive on \((0, 1]\), and the boundary conditions are \(y(0) = 0\), \(y(+\infty) = 1\), \(y'(+) = y''(+) = 0\). The problem arises in the study of draining and coating flows.

1. Introduction

In [1], Wang and Zhang have established, among other things, the existence of solutions to the singular third-order boundary value problem

\[
\begin{align*}
    y'''(x) &= f(y(x)), \quad 0 < x < +\infty \\
    y(0) &= 0, \quad \lim_{x \to +\infty} y(x) = 1, \quad \lim_{x \to +\infty} y'(x) = \lim_{x \to +\infty} y''(x) = 0.
\end{align*}
\]

(E)

They assumed that \(f(y) = (1 - y)^2 g(y)\), where \(\lambda > 0\) is a given constant and \(g(y)\) is continuous, positive, and nonincreasing in \((0, 1]\). They left the uniqueness of solutions open. The uniqueness and the existence of solutions have been recently resolved by Agarwal and Jiang [2], in the absence of monotonicity of \(g(y)\). Instead they assumed that there exists a function \(G(y) \in C(0, 1]\), which is nonincreasing in \(y\) such that

\[
0 < G(1) \leq g(y) \leq G(y), \quad \text{for all } y \in (0, 1].
\]

\((H_0)\)

When \(f(y)\) is singular at \(y = 0\), problems of this type arise in the study of draining and coating flows. In fact, several possible choices of \(f(y)\) have been listed in [3]. However (see [4,5]), one of the simplest and most important cases

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which has attracted considerable attention is
\[ f(y) = (1 - y) y^{-3}. \]
Motivated by the results of [1,2], the purpose of this work is to establish an existence result for the problem (E) in the absence of monotonicity of \( g(y) \) and/or the assumption (H₀).

2. Preliminaries

Agarwal and Jiang in [2] demonstrated that the BVP (E) is equivalent to the terminal value problem
\[
\begin{aligned}
  w''(t) &= \frac{2f(t)}{\sqrt{w(t)}}, \quad 0 < t < 1 \\
  \lim_{t \to 1^-} w(t) &= \lim_{t \to 1^-} w'(t) = 0.
\end{aligned}
\]  
(2.1)
They used the transformation
\[
w(t) = \left[y'(y^{-1}(x))\right]^2, \quad 0 \leq x < +\infty
\]
where \( y(x) \), \( 0 \leq x < +\infty \), is a strictly increasing solution of (E), and conversely
\[ y(x) = u^{-1}(t), \quad \text{where } u(t) = \int_0^t \frac{ds}{\sqrt{w(s)}}, \quad 0 \leq t < 1,
\]
and \( w(t) \), \( 0 \leq t < 1 \), is a solution of (2.1).

We will prove an existence result for the terminal value problem (2.1) by employing a classical method, namely, Kneser’s property (continuum) of the cross-sections of the solutions funnel. More precisely, the continuum property of the latter is taken as the cross-sections of the funnel and the boundary of a certain (possibly unbounded) set in the \((w, w')\) phase plane. The latter set is just a simple quadrant. This cross-section gives rise to the so called consequent mapping, the properties of which (see [6,7]) lead to our existence results.

For the convenience of the reader and to make the work self-contained we summarize here the basic notions used in the sequel. First we refer to the well-known Kneser’s theorem (see for example Coppel’s textbook [8]).

**Theorem 1.** Consider the system
\[ x' = f(t, x), \quad (t, x) \in [\alpha, \beta] \times \mathbb{R}^n \]
where \( f \) is continuous, and let \( K_0 \) be a continuum (i.e. compact and connected) subset of \( \{(t, x) : t = \alpha\} \) and \( \mathcal{X}(K_0) \) the family of all solutions emanating from \( K_0 \). If any solution \( x \in \mathcal{X}(K_0) \) is defined on the interval \([\alpha, \tau]\), then the cross-section
\[ \mathcal{X}(\tau ; K_0) = \{x(\tau) : x \in \mathcal{X}(K_0)\} \]
is a continuum in \( R^n \).

We recall that a set-valued mapping \( G \) which maps a topological space \( X \) into compact subsets of another space \( Y \) is called upper semicontinuous (usc) at the point \( x_0 \) iff for any open subset \( V \) in \( Y \) with \( G(x_0) \subseteq V \) there exists a neighborhood \( U \) of \( x_0 \) such that \( G(x) \subseteq V \) for every \( x \in U \).

Now for the initial value problem
\[
\begin{aligned}
w''(t) &= F(t, w(t)) = \begin{cases} 
2(1 - t)^\lambda g(t)/\sqrt{w(t)}, & 0 \leq t < 1 \\
0, & t \geq 1
\end{cases} \\
w(\tau) &= w_0 \geq 0, \\
w'(\tau) &= w'_0 \leq 0,
\end{aligned}
\]  
(E₀)
we set
\[ (w(\tau), w'(\tau)) = (w_0, w'_0) = P \in (0, +\infty) \times (-\infty, 0) = \omega, \]
and let $\mathcal{X}(P)$ be the set of all (noncontinuable) solutions of IVP $(E_0)$. Consider also an open set $\Omega \subseteq \mathbb{R}^2$ such that $\omega \subseteq \Omega$.

A point $P = (w_0, w'_0) \in \partial \omega$ is a point of egress of $\omega$ (with respect to the system $(E_0)$) iff for any solution $w \in \mathcal{X}(P)$ there exist $\tau \in [0, 1)$ and $\epsilon > 0$ such that the graph of the restriction $w|_{[\tau - \epsilon, \tau]}$ is in $\omega^\circ$, i.e.

$$G(w|[\tau - \epsilon, \tau]; P) = \{(w(t), w'(t)) : \tau - \epsilon \leq t < \tau\} \subseteq \omega^\circ.$$ 

If moreover for all solutions $w \in \mathcal{X}(P)$ there is $\epsilon > 0$ such that

$$G(w|[(\tau + \epsilon), \tau]; P) \subseteq \Omega - \overline{\omega}$$

then $P$ is called a strict egress point. The set of semi-egress (strict semi-egress) points of $\omega$ will be denoted by $\omega^s$ (respectively $\omega^{s\#}$).

A point $P_1 = (w_1, w'_1) \in \omega^s$, is a consequent of another one $P_0 = (w_0, w'_0) \in K_0 \subseteq \omega^s$ iff there exists a solution $w \in \mathcal{X}(P_0, P_1) = \mathcal{X}(P_0) \cap \mathcal{X}(P_1)$ and $\tau_0 \leq \tau$ such that

$$P_0 = \left(w(\tau_0), w'(\tau_0)\right), \quad P = \left(w(\tau), w'(\tau)\right) \quad \text{and} \quad G(w|((\tau_0, \tau); P_0, P) \subseteq \omega^\circ.$$ 

The set of consequent points of $P_0 \in K_0$ will be denoted by $\mathcal{K}(P_0)$ and the set-valued mapping defined by

$$\mathcal{K} : K_0 \subseteq \omega^s \to \omega^s$$

will be referred to as the consequent mapping.

The following lemmas (see for example [7]) give sufficient conditions for the upper semicontinuity of the consequent mapping and some useful properties for a class of usc mapping. We notice that the consequent mapping is included in this class.

If $P \in K_0$ and every solution $w \in \mathcal{X}(P)$ egresses strictly from $\omega$, then the consequent map $\mathcal{K}$ is usc at the point $P$ and moreover the image $\mathcal{K}(P)$ is a continuum in $\partial \omega$.

**Lemma 1.** Let $X$ and $Y$ be metric spaces and let $F : X \to 2^Y$ be an usc mapping. If $A$ is a continuum subset of $X$, such that for every $w \in A$ the image $F(w)$ is a continuum, then the image $F(A) = \bigcup\{F(w) : w \in A\}$ is also a continuum subset of $Y$.

Let $w(t)$ be any solution of the third-order nonlinear singular boundary value problem

$$\begin{align*}
  &w''(t) = F(t, w(t)) \\
  &\lim_{t \to 1} w(t) = \lim_{t \to 1} w'(t) = 0.
\end{align*}$$

We shall consider the $(w, w')$ phase plane. It is easy to verify that

$$w''(t) = F(t, w(t)) > 0, \quad 0 \leq t < 1$$

as long as $w(t) > 0$. Thus, any trajectory $(w(t), w'(t))$, $t \geq 0$, emanating from any point in the fourth quadrant:

$$\{(w, w') : w > 0, w' < 0\},$$

evolves in a natural way, when $0 < t < 1$:

- toward the negative $w'$-semi-axis, crossing it for some $\tau \in (0, 1)$, i.e. $w(\tau) = 0$ and $w'(\tau) < 0$, and then, when $w(t) \leq 0$, toward the negative $w$-semi-axis;
- toward the positive $w$-semi-axis, crossing it for some $\tau \in (0, 1)$, i.e. $w(\tau) > 0$ and $w'(\tau) = 0$, and then remaining asymptotically in the first quadrant.

We notice that, in the first case the graph of the solution $w(t)$ may finally (for $t \geq 1$) remain in the third as well as in the first quadrant. In the second case, the graph of the solution remains finally in the first quadrant. These properties will be referred to as “the nature of the vector field”.

**Remark 1.** The above properties can be extended whenever $\tau = 1$. Indeed, for the first case, by the definition of the modification $F(t, w)$, we have $w''(1) = 0$, $w(\tau) = 0$ and $w'(1) < 0$. Hence $w(t) < 0$ in a right neighborhood of
By Lemma 2, if \( \tau = 0 \) and \( w(\tau) = 0 \) and \( w'(\tau) < 0 \), then we get \( w(t) < 0 \) in a right neighborhood of \( t = 0 \), i.e. again \((w(0), w'(0))\) is a strict egress point of \( \omega \).

Now consider any point \( P_0 := (w_0, w'_0) \in K_0 \subseteq \omega \) and set

\[
E = \{(w, w') \in \partial \omega : w \geq 0 \text{ and } w' = 0\}
\]

and

\[
E' = \{(w, w') \in \partial \omega : w = 0 \text{ and } w' \leq 0\}.
\]

We assume that for any \( w \in \mathcal{X}(P_0) \) there exists \( \tau \in (0, 1] \) such that

\[
w(t) > 0, \quad w'(t) < 0, \quad 0 \leq t < \tau \quad \text{and} \quad w'(\tau) = 0.
\]

Then we notice (see [7]) that the set \( \{(w(\tau), w'(\tau)) = (w(\tau), 0) \in \partial \omega : w \in \mathcal{X}(P_0)\} \) is a continuum (notice that \( \tau \) is not unique and depends on the particular solution \( w \in \mathcal{X}(P_0) \)). As a result, we can define the (multi-valued) consequent mapping

\[
\mathcal{K}(P_0) := \{(w(\tau), w'(\tau)) = (w(\tau), 0) \in \partial \omega : w \in \mathcal{X}(P_0)\}.
\]

Similarly, assuming that for any \( w \in \mathcal{X}(P_0) \) there exists \( \tau \in (0, 1) \) such that

\[
w(t) > 0, \quad w'(t) < 0, \quad 0 \leq t < \tau \quad \text{and} \quad w(\tau) = 0,
\]

we define here the consequent mapping

\[
\mathcal{K}(P_0) := \{(w(\tau), w'(\tau)) = (0, w'(\tau)) \in \partial \omega : w \in \mathcal{X}(P_0)\}.
\]

**Remark 2.** If in the above definitions \( \tau = 1 \), by Remark 1, the consequent mapping is still well defined.

**Definition 1.** In the case when \( \mathcal{K}(P_0) \neq \emptyset, \; P_0 \in \omega \), and there is a solution \( w \in \mathcal{X}(P_0) \) such that \( \text{dom}(w) = [0, 1) \), and

\[
\lim_{t \to 1} w(t) = \lim_{t \to 1} w'(t) = 0,
\]

we say that \( P_0 \) is a singular point of the consequent map \( \mathcal{K} \).

**Remark 3.** If \( P_0 := (w_0, w'_0) \in K_0 \) is a nonsingular point of the consequent map \( \mathcal{K} \), then both the sets

\[
\mathcal{K}(P_0) \cap E_\alpha \quad \text{and} \quad \mathcal{K}(P_0) \cap E_\beta
\]

are compact, whenever these are not empty.

**Remark 4.** The point \( P_0 \) in the above definitions may be replaced by any continuum \( K \subset K_0 \).

We also need the following lemma from the classical topology.

**Lemma 2** ([9, Ch. V, & 47, point III, Th. 2]). If \( A \) is an arbitrary proper subset of a continuum \( B \) and \( S \) a connected component of \( A \), then

\[
\bar{S} \cap (B \setminus \bar{A}) \neq \emptyset \quad \text{i.e.} \quad \bar{S} \cap \partial A \neq \emptyset.
\]

**Proposition 1.** Let \( P_0 = (w_0, w'_0) \in K_0 \) be a singular point of the consequent map \( \mathcal{K} \). Then every connected component \( S \) of the set \( E \cap \mathcal{K}(P_0) \neq \emptyset \) \((E' \cap \mathcal{K}(P_0) \neq \emptyset)\) approaches the point \( (0, 0) \) of \( \partial \omega \) in the sense that \( (0, 0) \in \bar{S} \).

**Proof.** By Definition 1, the set \( A = E \cap \mathcal{K}(P_0) \) is a subset of the continuum \( B = E \cup \{(0, 0)\} \). Let \( S \) be a connected component of the set \( A \). Then \( \bar{S} \), in view of being a compact and connected set in \( B \), is a continuum. Similarly, we get a continuum \( S' \subseteq E' \cap \mathcal{K}(P_0) \). By Lemma 2 and since the dimension \( \text{dim} \,(E) = 1 = \text{dim} \,(E') \), we obtain

\[
S = E \cap \mathcal{K}(P_0) \quad \text{and} \quad S' = E' \cap \mathcal{K}(P_0).
\]
We assume that \((0, 0) \notin S \cup S'\). Consider the points \(A = (w_0, 0)\) and \(B = (0, w_0')\), where
\[
 w_0 = \min \{ w : (w, 0) \in S \} \quad \text{and} \quad w_0' = \max \{ w' : (0, w') \in S' \}.
\]
Noticing the Remark 4, we get
\[
 A \in E \cap \mathcal{K}([A, B]) \neq \emptyset \quad \text{and} \quad B \in E' \cap \mathcal{K}([A, B]) \neq \emptyset.
\]
The above process holds for the segment \([A, B]\) instead of the point \(P_0\). Hence, we obtain the continuum \(S_1 = E \cap \mathcal{K}(\{A, B\})\). Therefore, a clever use of Lemma 2 contradicts the minimality of the point \(w_0\) and this yields the result. \(\square\)

**Theorem 2.** If \(P_0 = (w_0, w_0') \in K_0\) is a singular point of the consequent mapping \(K\) with respect to the set \(\omega\), then the boundary value problem
\[
\begin{cases}
 w''(t) = F(t, w(t)), & 0 < t < 1 \\
 \lim_{t \to 1^-} w(t) = \lim_{t \to 1^-} w'(t) = 0
\end{cases}
\]
has (at least) one solution \(w(t), 0 \leq t < 1\).

**Proof.** The result is obvious. \(\square\)

3. Main result

Consider the terminal value problem
\[
\begin{cases}
 w''(t) = 2(1 - t)^{\lambda} g(t) [w(t)]^{-1/2}, & 0 < t < 1 \\
 \lim_{t \to 1^-} w(t) = \lim_{t \to 1^-} w'(t) = 0,
\end{cases}
\]
where \(\lambda > 0\) is a given constant and \(g(t), 0 < t \leq 1\), is a positive and continuous function.

A function \(w(t) \in C([0, 1], (0, +\infty)) \cap C^2(0, 1)\) is a positive solution of (3.1) if
\[
w(t) = \int_t^1 2(s - t)(1 - s)^{\lambda} g(s) [w(s)]^{-1/2} ds.
\]
Then
\[
w'(t) = -\int_t^1 2(1 - s)^{\lambda} g(s) [w(s)]^{-1/2} ds < 0
\]
and this means that the projection
\[
 G(w) = \{ (w(t), w'(t)) : 0 < t < 1 \}
\]
of its graph is a subset of the fourth quadrant \(\{(w, w') : w \geq 0 \text{ and } w' \leq 0\}\) of \((w, w')\) phase plane.

The main result of this work is as follows.

**Theorem 3.** The problem (3.1) admits at least one increasing positive and convex solution.

**Proof.** Consider any segment \([A, B]\) where \(A \in E^\circ = \{(w, w') \in \partial \omega : w > 0 \text{ and } w' = 0\}\) and \(B \in (E')^\circ = \{(w, w') \in \partial \omega : w = 0 \text{ and } w' < 0\}\). Considering any solution \(w \in \mathcal{X}(B)\), we notice that \(\mathcal{K}(B) = \{B\}\), that is \(B\) is a nonsingular point of the consequent mapping \(K\). Similarly if \(w \in \mathcal{X}(A)\), then via Remark 1, \(A\) is a (strict) egress point and this again yields \(\mathcal{K}(A) = \{A\}\). Consequently
\[
 \mathcal{K}([A, B]) \cap E \neq \emptyset \quad \text{and} \quad \mathcal{K}([A, B]) \cap E' \neq \emptyset.
\]
Since the segment \([A, B]\) is a continuum, by virtue of Lemma 1 and the upper semicontinuity of the map \(K\), we should conclude that the set \(\mathcal{K}([A, B])\) is also a continuum. This however is not the case because \((0, 0)\) is a singular point for the differential equation in (3.1). Consequently the segment \([A, B]\) contains a singular point of the consequent mapping \(K\). An application of Theorem 2 guarantees that the boundary value problem (2.2) accepts a solution \(w(t)\). Finally by the above analysis of the underlined vector field, this solution is positive, and thus it is actually a solution of our BVP (3.1). \(\square\)
Corollary 1. Under assumption (H₀), the boundary value problem (E) admits at least one positive, increasing and concave solution.

Proof. We set

\[ y(x) = u^{-1}(t), \quad \text{where} \quad u(t) = \int_0^t \frac{ds}{\sqrt{w(s)}}, \quad 0 \leq t < 1, \]

where \( w(t) \) denotes the above obtained solution of (3.1). The function \( u(t) \) is clearly strictly increasing and thus \( u(0) = 0 \). Also \( y(x) \) satisfies all the conditions in the BVP (E) (see [2], for details). □

References