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Fuzzy Topological Spaces and Fuzzy Compactness

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It is the purpose of this paper to go somewhat deeper into the structure of fuzzy topological spaces. In doing so we found we had to alter the definition of a fuzzy topology used up to now. We shall also introduce two functors $\tilde{\omega}$ and $\tilde{\iota}$ which will allow us to see more clearly the connection between fuzzy topological spaces and topological spaces. Finally we shall introduce the concept of fuzzy compactness as the generalization of compactness in topology. It will be shown in a following publication that contrary to the results obtained up to now, the Tychonoff-product theorem is safeguarded with fuzzy compactness.

1. Fuzzy Topological Spaces

Let E be a set and I the unit interval.

In [3] C. L. Chang defines a fuzzy topology on E as a subset $\delta \subset I^E$ such that

(i) $0, 1 \in \delta$,

(ii) $\forall \mu, \nu \in \delta \Rightarrow \mu \wedge \nu \in \delta$,

(iii) $\forall (\mu_j)_{j \in J} \subset \delta \Rightarrow \sup_{j \in J} \mu_j \in \delta$.

Several articles on the subject all involve this definition. Amongst these the most important ones are [3, 6, 11–14].

Although most results obtained in this work remain valid with Chang's definition (as can be seen from our former publications [8 and 9]) we would like to suggest an alternative and more natural definition. This involves the changing of condition (i) namely: $0, 1 \in \delta$ to (i)′ $\forall \alpha$ constant $\alpha \in \delta$. There are several reasons why the latter is to be preferred.

Intuitively it is clear that the general idea of the theory would involve this change. But then there are mathematically more sophisticated reasons available as well. Indeed

(1) It follows at once from the definition of ω in 2, that every topologically generated fuzzy topology fulfills condition (i)′.

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(2) It is easily seen that with Chang's definition constant functions between fuzzy topological spaces are not necessarily continuous. This can only be true in general if one uses the alternative definition. This is of course the most important argument in favor.

(3) The definition of fuzzy-compactness in Section 4, also suggests the alternative.

In view of all this we now put

DEFINITION 1.1. $\delta \subset I^E$ is a fuzzy topology on E iff

(i) $\forall \alpha$ constant, $\alpha \in \delta$,

(ii) $\forall \mu, \nu \in \delta \Rightarrow \mu \wedge \nu \in \delta$,

(iii) $\forall (\mu_j)_{j \in J} \subset \delta \Rightarrow \sup_{j \in J} \mu_j \in \delta$.

It is this concept of fuzzy topology that will be used throughout the sequel. Chang's definition we will refer to as quasi fuzzy topology.

The fuzzy sets in δ are called open fuzzy sets. A fuzzy set $\mu \in I^E$ is called closed iff μ^c is open. The closure and interior of a fuzzy set $\mu \in I^E$ are defined respectively

$$\bar{\mu} = \inf\{\nu: \nu \geqslant \mu \ \nu^c \in \delta\},$$

$$\mathring{\mu} = \sup\{\nu: \nu \leqslant \mu \ \nu \in \delta\}.$$

It is easily seen that $\bar{\mu}$ is the smallest closed fuzzy set larger than μ and that $\mathring{\mu}$ is the largest open fuzzy set smaller than μ.

DEFINITION 1.2. An operator $\psi: I^E \to I^E$ is a fuzzy closure operator iff

(i) $\psi(\alpha) = \alpha \qquad \forall \alpha$ constant,

(ii) $\psi(\mu) \geqslant \mu \qquad \forall \mu \in I^E$,

(iii) $\psi(\mu) \vee \psi(\nu) = \psi(\mu \vee \nu) \qquad \forall \mu, \nu \in I^E$,

(iv) $\psi(\psi(\mu)) = \psi(\mu) \qquad \forall \mu \in I^E$.

A fuzzy interior operator is defined dually, i.e.: An operator $\phi: I^E \to I^E$ is a fuzzy interior operator iff

(i) $\phi(\alpha) = \alpha \qquad \forall \alpha$ constant,

(ii) $\phi(\mu) \leqslant \mu \qquad \forall \mu \in I^E$,

(iii) $\phi(\mu) \wedge \phi(\mu) = \phi(\mu \wedge \nu) \qquad \forall \mu, \nu \in I^E$,

(iv) $\phi(\phi(\mu)) = \phi(\mu) \qquad \forall \mu \in I^E$.

It is easily seen that the operators $^-$ and $^\circ$ are respectively a fuzzy closure operator and a fuzzy interior operator. Thus we see that with a fuzzy topology we can associate a fuzzy closure operator (resp., a fuzzy interior operator). Vice versa with a given fuzzy closure operator (resp., fuzzy interior operator) we can associate a fuzzy topology in the following way. Let ψ be the fuzzy closure operator (resp., θ the fuzzy interior operator), then the associated fuzzy topology is given by

$$\delta(\psi) = \{\mu^c \colon \psi(\mu) = \mu\} \qquad (\text{resp., } \delta(\theta) = \{\mu \colon \theta(\mu) = \mu\}).$$

It is easily seen that these associations are reflexive in the sense that the associated fuzzy topology of $^-$ (resp., $^\circ$) in some fuzzy topology δ is δ itself, and that the associated fuzzy closure (resp., fuzzy interior operator) associated with the fuzzy topology of some fuzzy closure operator ψ (resp., fuzzy interior operator θ) is ψ (resp., θ) itself.

DEFINITION 1.3. A subset $\sigma \subset \delta$, where δ is a fuzzy topology on a set E, is a base for δ iff

$$\forall \mu \in \delta, \qquad \exists (\mu_j)_{j \in J} \subset \sigma \qquad \text{s.t.} \qquad \mu = \sup_{j \in J} \mu_j .$$

DEFINITION 1.4. A subset $\sigma' \subset \delta$ is a subbase for δ iff the family of finite infima of members of σ' is a base for δ.

2. THE FUNCTIONS ω AND ι

There is a natural way to associate a fuzzy topology with a given topology and vice versa. Put $\mathscr{T}(E)$ the set of all topologies on E and $\mathscr{W}(E)$ the set of all fuzzy topologies on E. On $\mathbb{R}$ we consider the topology $\mathscr{T}_r = \{]\alpha, \infty[\colon \alpha \in \mathbb{R}\} \cup \{\phi\}$. The topological space one obtains giving I the induced topology we denote I_r.

We then define the next two mappings

$$\iota \colon \mathscr{W}(E) \to \mathscr{T}(E) \colon \delta \to \iota(\delta)$$

where $\iota(\delta)$ is the initial topology on E for the family of "functions" δ and the topological space I_r.

$$\omega \colon \mathscr{T}(E) \to \mathscr{W}(E) \colon \mathscr{T} \to \omega(\mathscr{T})$$

where $\omega(\mathscr{T}) = \mathscr{C}(E, I_r)$, the continuous functions from $(E, \mathscr{T})$ to I_r.

It is trivial to check that $\omega(\mathscr{T})$ is indeed a fuzzy topology since it is the set of all lower semicontinuous functions from $(E, \mathscr{T})$ to the unit interval equipped with the usual topology.

If $\delta \in \mathscr{W}(E)$ equals $\omega(\mathscr{T})$ for some $\mathscr{T} \in \mathscr{T}(E)$ we say that δ is topologically generated.

PROPOSITION 2.1.

(i) $\iota \circ \omega = id_{\mathscr{T}(E)}$,

(ii) *ι and ω are respectively an isotone surjection and an isotone injection,*

(iii) *$\omega \circ \iota(\delta)$ is the smallest topologically generated fuzzy topology which contains δ. We denote it by $\bar{\delta}$.*

(iv) *δ is topologically generated iff $\delta = \bar{\delta}$.*

Proof. This is straightforward.

THEOREM 2.2. *(E, δ) is topologically generated iff for each continuous function $f \in \mathscr{C}(I_r, I_r)$ and for each $\nu \in \delta$ also $f \circ \nu \in \delta$.*

Proof. The only if part is trivial. Suppose $\mu \in \bar{\delta}$. Since a base for $\iota(\delta)$ is provided by the finite intersections

$$\bigcap_{i=1}^{n} \nu_i^{-1}(]\epsilon_i , 1]) \qquad \nu_i \in \delta, \quad \epsilon_i \in I;$$

this is equivalent to saying $\forall \epsilon \in I$, $\forall x \in \mu^{-1}(]\epsilon, 1])$ $\exists$ a finite set $I_{\epsilon,x}$ such that

$$x \in \bigcap_{i \in I_{\epsilon,x}} \nu_i^{-1}(]\epsilon_i , 1]) \subset \mu^{-1}(]\epsilon, 1]).$$

Now fix x and let $\mu(x) = k_x$ then $\forall \epsilon < k_x$, $\exists I_\epsilon$ finite such that

$$x \in \bigcap_{i \in I_\epsilon} \nu_i^{-1}(]\epsilon_i , 1]) \subset \mu^{-1}(]\epsilon, 1]).$$

Then $\forall \epsilon < k_x$ and $\forall i \in I_\epsilon$ put

$$\mu_{i,\epsilon} = \epsilon \chi_{]\epsilon_i,1]} \circ \nu_i;$$

then $\mu_{i,\epsilon} \in \delta$ and clearly

$$\begin{aligned} \mu_{i,\epsilon}(y) &= \epsilon \Leftrightarrow \nu_i(y) > \epsilon_i \\ &= 0 \Leftrightarrow \nu_i(y) \leqslant \epsilon_i . \end{aligned}$$

Put $\nu_\epsilon^x = \inf_{i \in I_\epsilon} \mu_{i,\epsilon} \in \delta$ then clearly too

$$\begin{aligned} \nu_\epsilon^x(y) &= \epsilon \Leftrightarrow \nu_i(y) > \epsilon_i \qquad \forall i \in I_\epsilon \\ &= 0 \Leftrightarrow \exists j \in I_\epsilon , \qquad \nu_j(y) \leqslant \epsilon_j . \end{aligned}$$

Thus $\nu_\epsilon^x(y) = \epsilon \Rightarrow \mu(y) > \epsilon$ and $\forall \epsilon < k_x$, $\nu_\epsilon^x \leqslant \mu$. Now it is easily seen that

$$\mu = \sup_{x \in E} \sup_{\epsilon < k_x} \nu_\epsilon^x \in \delta.$$

3. Fuzzy Continuity

Let (E, δ) and (F, γ) be fuzzy topological spaces and f a function from E to F. In [3] Chang defines f fuzzy continuous iff

$$\forall \nu \in \gamma, \qquad f^{-1}(\nu) \in \delta.$$

It is shown that this is equivalent to

$$\forall \nu \in I^F, \qquad \nu^c \in \gamma, \qquad f^{-1}(\nu)^c \in \delta.$$

The fact that we use an alternative definition of fuzzy topological space obviously alters nothing to this result. As a matter of fact all results of [3 and 11] on this concept remain valid.

Definition 3.1. We shall say that a function

$$f: E, \delta \to F, \gamma$$

is continuous iff $f: E, \iota(\delta) \to F, \iota(\gamma)$ is continuous.

Proposition 3.1. *Consider the next properties for a function* $g: E, \delta \to F, \gamma$

(i) *g is fuzzy continuous*
(ii) *g is continuous*
(iii) *$g: E, \bar{\delta} \to F, \bar{\gamma}$ is fuzzy continuous*
(iv) *$g: E, \bar{\delta} \to F, \gamma$ is fuzzy continuous*

then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof. This is straightforward.

Now let us denote $\mathscr{C}(E, F)$ the set of all continuous functions from (E, δ) to (F, γ) and $\mathscr{C}_\omega(E, F)$ the set of all fuzzy continuous functions from (E, δ) to (F, γ) then we have

Corollary 3.2. *If δ is topologically generated then*

$$\mathscr{C}(E, F) = \mathscr{C}_\omega(E, F).$$

Proof. Trivial from 3.1.

The converse of this corollary is not true as is shown by the next:

COUNTEREXAMPLE. Let $E = I$ and F arbitrarily. Let δ be the fuzzy topology on E with subbase

$$\{\alpha: \alpha \text{ constant}\} \cup \{\text{identity}\}$$

and let γ be the discrete fuzzy topology on F, i.e., $\gamma = I^F$.

Clearly $\iota(\gamma)$ is discrete and, as it is easily checked, $\iota(\delta)$ is connected. Thus we have that

$$\mathscr{C}(E, F) = \{\text{constant functions from } E \text{ to } F\}.$$

From Proposition 3.1 it follows that always $\mathscr{C}_\omega(E, F) \subset \mathscr{C}(E, F)$ and since constant functions are fuzzy continuous we have $\mathscr{C}_\omega(E, F) = \mathscr{C}(E, F)$. Yet δ is not topologically generated since $\iota(\delta) = \mathscr{T}_{r/I}$ and $\mathscr{C}(I_r, I_r)$ is a much larger class than merely δ.

Now we can consider the category of fuzzy topological spaces and fuzzy continuous mappings in the same way as the category of topological spaces and continuous mappings. It is easily seen that the functions ω and ι defined in 2 induce two covariant functors between these categories. Indeed let $\mathscr{C}$ be the category of topological spaces and $\mathscr{F}$ the category of fuzzy topological spaces and put

$$\tilde{\omega}: \mathscr{C} \to \mathscr{F}$$

defined by $\tilde{\omega}(E, \mathscr{T}) = (E, \omega(\mathscr{T}))$ and $\tilde{\omega}(f) = f$ and

$$\tilde{\iota}: \mathscr{F} \to \mathscr{C}$$

defined by $\tilde{\iota}(E, \delta) = (E, \iota(\delta))$ and $\tilde{\iota}(f) = f$.

Furthermore it follows immediately from Corollary 3.2 that $\tilde{\omega}(\mathscr{C})$ is a full subcategory of $\mathscr{F}$.

4. Fuzzy Compactness

In [3] Chang gives a definition of compactness for quasi fuzzy topological spaces which formally is the same one as for topological spaces. This definition has also been used in [6] and [12]. With this definition though one does not have that if $(X, \mathscr{T})$ is a compact space $(X, \omega(\mathscr{T}))$ is compact. Indeed not, consider the next

COUNTEREXAMPLE. Let $(X, \mathscr{T})$ be the unit interval I with the usual topology. Then $\forall x \in I$, $x \neq 0$, $x \neq 1$, let ν_x be the fuzzy set defined by

$$\nu_x(x) = 1,$$

$$\nu_x(y) = 0, \qquad \forall y \in \left[0, \frac{x}{2}\right] \cup \left[\frac{1+x}{2}, 1\right],$$

$$\nu_x \text{ is linear on } \left[\frac{x}{2}, x\right] \text{ and on } \left[x, \frac{1+x}{2}\right].$$

For

$$\begin{array}{llll} x = 0 & \text{put} & \nu_0(y) = -y + 1 & \forall y \in I, \\ x = 1 & \text{put} & \nu_1(y) = y & \forall y \in I. \end{array}$$

Then clearly $\forall x \in I$, $\nu_x \in \omega(\mathscr{T})$, and

$$\sup_{x \in I} \nu_x = 1,$$

but no finite subfamily has this property.

Therefore we introduce here an other form of compactness which we believe to be the correct one.

Let (E, δ) be a fuzzy topological (or quasi fuzzy topological) space.

DEFINITION 4.1. A fuzzy set $\nu \in I^E$ is fuzzy compact iff for all family

$$\beta \subset \delta \text{ such that } \sup_{\mu \in \beta} \mu \geqslant \nu$$

and for all $\epsilon > 0$, there exists a finite subfamily $\beta_0 \subset \beta$ such that

$$\sup_{\mu \in \beta_0} \mu \geqslant \nu - \epsilon.$$

Using this concept of fuzzy compact fuzzy set we now have

DEFINITION 4.2. The fuzzy topological (or quasi fuzzy topological) space (E, δ) is fuzzy compact iff each constant fuzzy set in (E, δ) is fuzzy compact.

Before proceeding, let us compare this definition with the former one's in [3, 6, and 12]. We remark that Chang's definition of compactness which we shall refer to as quasi fuzzy compactness only makes sense in the class of quasi fuzzy topological spaces. Indeed, no fuzzy topological space can be quasi fuzzy compact. Therefore whenever we use the concept of quasi fuzzy compactness the underlying space will be assumed taken out of the class of quasi fuzzy topological spaces. Otherwise all results would be vacuous.

Now the only relation to be found is that if (E, δ) is a quasi fuzzy compact space then the constant fuzzy set 1_E is fuzzy compact. This last property is a concept we have already introduced in [8 and 9]. To distinguish between it and Definition 4.2 we now put

DEFINITION 4.3. The fuzzy topological (or quasi fuzzy topological) space (E, δ) is weakly fuzzy compact iff 1_E is fuzzy compact.

Then we have that if (E, δ) is quasi fuzzy compact it is weakly fuzzy compact.

From now on, unless otherwise stated, all spaces will be fuzzy topological according to Definition 1.1.

THEOREM 4.1. *The fuzzy topological space $(E, \omega(\mathscr{T}))$ is fuzzy compact iff the space $(E, \mathscr{T})$ is compact.*

Proof. Let $\beta \subset \omega(\mathscr{T})$ be such that $\sup_{\mu \in \beta} \mu \geqslant \alpha > 0$ and ϵ such that $\alpha > \epsilon > 0$.

$$\forall \mu \in \beta \quad \text{put} \quad \mu^\epsilon = \mu + \epsilon, \qquad \text{and} \qquad [0, \alpha] = I_\alpha .$$

Then $\forall \mu \in \beta$, $\mathscr{L}(\mu^\epsilon) = \{(x, r): \mu^\epsilon(x) > r\}$ is an open set in $E \times \mathbb{R}$ and $\bigcup_{\mu \in \beta} \mathscr{L}(\mu^\epsilon) \supset E \times I_\alpha$.
Since $E \times I_\alpha$ is compact there exists a finite subfamily $\beta_0 \subset \beta$ such that

$$\bigcup_{\mu \in \beta_0} \mathscr{L}(\mu^\epsilon) \supset E \times I_\alpha \qquad \text{and then clearly} \qquad \sup_{\mu \in \beta_0} \mu \geqslant \alpha - \epsilon$$

For the converse suppose $\mathscr{B} \subset \mathscr{T}$ is an open cover of E, then

$$\sup_{A \in \mathscr{B}} \chi_A = 1$$

where χ_A is the characteristic function of A.

Choose $\epsilon \in]0, 1[$ then it follows from the fuzzy compactness of $(E, \omega(\mathscr{T}))$ that there exists a finite subset $\mathscr{B}_0 \subset \mathscr{B}$ such that

$$\sup_{A \in \mathscr{B}_0} \chi_A \geqslant 1 - \epsilon.$$

It is clear that $\mathscr{B}_0$ is a finite subcover of $\mathscr{B}$.

From this proof (or see e.g. [8]) it is perfectly clear that the same result holds if we replace fuzzy compactness by weak fuzzy compactness.

In [3] Chang proves that if f is a surjective fuzzy continuous function from (E, δ) onto (F, γ) and if (E, δ) is quasi fuzzy compact then so is (F, γ). The same holds for fuzzy compactness and we prove the more general result.

PROPOSITION 4.2. *If $f: (E, \delta) \to (F, \gamma)$ is fuzzy continuous and ν is a fuzzy compact fuzzy set in (E, δ) then $f(\nu)$ is fuzzy compact in (F, γ).*

Proof. Let $\beta \subset \gamma$ be such that

$$\sup_{\mu \in \beta} \mu \geqslant f(\nu)$$

then it is easily shown that $(f^{-1}(\mu))_{\mu \in \beta}$ fullfills

$$\sup_{\mu \in \beta} f^{-1}(\mu) \geqslant \nu.$$

Since $(f^{-1}(\mu))_{\mu \in \beta} \subset \delta$ and ν is fuzzy compact this implies that for any $\epsilon > 0$ there exists a finite subfamily $\beta_0 \subset \beta$ such that

$$\sup_{\mu \in \beta_0} f^{-1}(\mu) \geqslant \nu - \epsilon.$$

Now it is immediately clear that

$$\sup_{\mu \in \beta_0} \mu \geqslant f(\nu) - \epsilon.$$

From the fact that, if f is onto, each constant fuzzy set on F is the image through f of the constant fuzzy set on E with the same value we have the following.

COROLLARY 4.3. *If* (E, δ) *is fuzzy compact and* f *a fuzzy continuous mapping from* (E, δ) *onto* (F, γ) *then* (F, γ) *is fuzzy compact.*

Proof. Trivial.

Before coming to another important result we would like to show an equivalent condition for fuzzy compactness which is worth noticing.

PROPOSITION 4.4. (E, δ) *is fuzzy compact iff* $\forall \beta \subset \delta$ *and* $\forall \alpha > 0$ *such that* $\sup_{\mu \in \beta} \mu \geqslant \alpha$ *and* $\forall k > 1$ *there exists a finite subfamily* $\beta_0 \subset \beta$ *such that*

$$\sup_{\mu \in \beta_0} k\mu \geqslant \alpha.$$

Proof. If (E, δ) fulfills the condition of the theorem and $\beta \subset \delta$ such that $\sup_{\mu \in \beta} \mu \geqslant \alpha$ and $\epsilon > 0$ then $k = \alpha/(\alpha - \epsilon)$ will do.

In case (E, δ) is fuzzy compact and $\beta \subset \delta$ is such that $\sup_{\mu \in \beta} \mu \geqslant \alpha$ and $k > 1$ then $\epsilon = \alpha - \alpha/k$ will do.

An analogous criterion holds for weak fuzzy compactness if one replaces everywhere α by 1.

If (E, δ) is a fuzzy compact space it is not necessarily so that each closed fuzzy set is fuzzy compact. We shall give a counterexample of this and some other phenomena after Theorems 4.6 and 4.7 but first we show

PROPOSITION 4.5. *If* (E, δ) *is a topologically generated fuzzy compact space* *i.e. there exists a compact topology $\mathscr{T}$ such that $\delta = \omega(\mathscr{T})$ then every closed fuzzy set is fuzzy compact.*

Proof. Consider α, $\alpha^c \in \omega(\mathscr{T})$ and $\beta \subset \omega(\mathscr{T})$ such that $\sup_{\mu\in\beta} \mu \geqslant \alpha$. Since $1 - \alpha \in \omega(\mathscr{T})$ we have that

$$\mathscr{U}(\alpha) = \{(x, r): \alpha(x) < r\} \qquad \text{is open in} \qquad E \times I$$

thus $\mathscr{U}(\alpha)^c$ is compact. Now choose $\epsilon > 0$ and put again

$$\mu^\epsilon = \mu + \epsilon$$

Clearly $\bigcup_{\mu\in\beta} \mathscr{L}(\mu^\epsilon) \supset \mathscr{U}(\alpha)^c$, thus there exists a finite subset $\beta_0 \subset \beta$ such that

$$\bigcup_{\mu\in\beta_0} \mathscr{L}(\mu^\epsilon) \supset \mathscr{U}(\alpha)^c$$

and clearly too $\sup_{\mu\in\beta_0} \mu \geqslant \alpha - \epsilon$.

The next theorem is a fuzzy form of Alexander's subbase lemma. It proves usefull in the study of products of fuzzy compact spaces and in several counterexamples.

THEOREM 4.6. *(E, δ) is fuzzy compact iff for any subbase σ for δ, for any $\beta \subset \sigma$ and for any $\alpha > \epsilon > 0$ such that $\sup_{\mu\in\beta} \mu \geqslant \alpha$ there exists a finite subset $\beta_0 \subset \beta$ such that*

$$\sup_{\mu\in\beta_0} \mu \geqslant \alpha - \epsilon.$$

Proof. The only if part is trivial; let then σ be a subbase for δ. It is clear that (E, δ) is fuzzy compact iff $\forall\beta \subset \delta$ such that there exist $\alpha > \epsilon > 0$ such that

$$\forall\beta_0 \in 2^{(\beta)} \ \sup_{\mu\in\beta_0} \mu \ngeqslant \alpha - \epsilon$$

it follows that

$$\sup_{\mu\in\beta} \mu \ngeqslant \alpha.$$

Put

$$\mathscr{C} = \{\beta \subset \delta: \forall\beta_0 \in 2^{(\beta)} \sup_{\mu\in\beta_0} \mu \ngeqslant \alpha - \epsilon\}.$$

Clearly $\mathscr{C}$ is of finite character and thus it follows from the Tukey lemma that for each family $\beta \in \mathscr{C}$ there exists a maximal family $\tilde{\beta} \in \mathscr{C}$ containing β.

Now one can easily show that if $\mu \in \tilde{\beta}$ such that

$$\gamma_1, \ldots, \gamma_n \in \delta \qquad \text{and} \qquad \mu \geqslant \gamma_1 \wedge \cdots \wedge \gamma_n \Rightarrow \exists k, \qquad \gamma_k \in \tilde{\beta}.$$

Now let $\beta \in \mathscr{C}$ then there exists a family $\tilde{\beta}$ maximal in $\mathscr{C}$ and containing β.

Consider $\tilde{\beta} \cap \sigma$. It is clear from the assumptions of the theorem that

$$\sup \tilde{\beta} \cap \sigma \not\geqslant \alpha.$$

We shall show that $\sup \tilde{\beta} \leqslant \sup \tilde{\beta} \cap \sigma$.

$\forall \mu \in \tilde{\beta}$ and $\forall x \in E$ such that $\mu(x) > 0$ and $\forall a > 0$, $a < \mu(x)$ $\exists \nu_1^a, \ldots, \nu_n^a \in \sigma$ such that

$$\nu_1^a \wedge \cdots \wedge \nu_n^a \leqslant \mu \qquad \text{and} \qquad \nu_1^a(x) \wedge \cdots \wedge \nu_n^a(x) > \mu(x) - a.$$

Since $\mu \in \tilde{\beta}$ and $\tilde{\beta}$ is maximal some $\nu_k^a \in \tilde{\beta}$.

Thus $\forall a > 0$, $\exists \nu_k^a$, $\nu_k^a(x) > \mu(x) - a$ and $\nu_k^a \in \tilde{\beta} \cap \sigma$.

Now fix x. Then $\forall \mu \in \tilde{\beta}$, $\mu(x) > 0$ and $\forall a > 0$, $a < \mu(x)$ there exists $\nu^a \in \tilde{\beta} \cap \sigma$ such that $\nu^a(x) > \mu(x) - a \Rightarrow \forall \mu \in \tilde{\beta}$

$$\sup(\tilde{\beta} \cap \sigma)(x) \geqslant \mu(x) \Rightarrow \sup \tilde{\beta} \cap \sigma \geqslant \sup \tilde{\beta}.$$

This implies $\sup \tilde{\beta} \not\geqslant \alpha$, which in turn implies $\sup \beta \not\geqslant \alpha$. Thus (E, δ) is fuzzy compact.

A perfectly analogous result holds for weakly fuzzy compact spaces, and this makes it possible to show that weak fuzzy compactness is not maintained when taking arbitrary products, but is preserved for finite products, as we shall prove in a following publication.

THEOREM 4.7. *(E, δ) is weakly fuzzy compact iff for any subbase σ for δ and for any $\beta \subset \sigma$ such that* $\sup_{\mu \in \beta} \mu = 1$ *and for all $\epsilon > 0$ there exists a finite subset $\beta_0 \subset \beta$ such that*

$$\sup_{\mu \in \beta_0} \mu \geqslant 1 - \epsilon.$$

Proof. Analogous to that of Theorem 4.6.

Now let us look at some

COUNTEREXAMPLES

Obviously if $(E, \iota(\delta))$ is compact (E, δ) is fuzzy compact. The converse is not necessarily true. Indeed not, let $E = I$ and let δ be the fuzzy topology with subbase

$$\{\alpha\colon \alpha \text{ constant}\} \cup \{\nu \in I^E\colon \nu(x) = x \text{ or } 0, \forall x \in E\} \cup \{\chi_0\}$$

where χ_0 is Dirac function in 0.

It is easily seen that (E, δ) is fuzzy compact but $\iota(\delta)$ is discrete.

This shows at the same time that the continuous image of a fuzzy topological fuzzy compact space needn't be fuzzy compact. Consider hereto (E, δ) fuzzy compact such that $(E, \bar{\delta})$ is not fuzzy compact and

$$id: (E, \delta) \to (E, \bar{\delta}).$$

As mentioned before Proposition 4.5 we will now show that if (E, δ) is fuzzy compact and μ closed in E, μ needn't be fuzzy compact.

Let $E = I$ and let δ be the fuzzy topology with subbase

$$\{\alpha: \alpha \text{ constant}\} \cup \{\mu_n: n \in \mathbb{N}\} \cup \{\mu, \mu^c\}$$

where

$$\begin{aligned} \forall n \in \mathbb{N} \qquad \mu_n(x) &= \tfrac{1}{3}, && \forall x \in [0, \tfrac{1}{2} - 1/n + 1] \cup [\tfrac{1}{2} + 1/n + 1, 1], \\ &= 0 && \text{elsewhere;} \\ \mu(x) &= \tfrac{1}{3}, && \forall x \neq \tfrac{1}{2}; \\ &= 0, && x = \tfrac{1}{2}. \end{aligned}$$

Then using Theorem 4.6 one can easily check that (E, δ) is fuzzy compact. Yet μ is closed and

$$\sup_{n \in \mathbb{N}} \mu_n = \mu$$

but for any $\epsilon > 0$, $\epsilon < \frac{1}{3}$ no finite subfamily of the $(\mu_n)_{n \in \mathbb{N}}$ covers $(\mu - \epsilon) \vee 0$.

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