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Torus knot and minimal model

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Abstract

We reveal an intimate connection between the quantum knot invariant for torus knot $T(s,t)$ and the character of the minimal model $\mathcal{M}(s,t)$, where s and t are relatively prime integers. We show that Kashaev's invariant, i.e., the N-colored Jones polynomial at the Nth root of unity, coincides with the Eichler integral of the character.

1. Introduction

After Jones polynomial was introduced [1], studies of quantum invariants have been extensively developed. These quantum knot invariants are physically interpreted as the Feynman path integral of the Wilson loop with the Chern–Simons action [2]. Though, geometrical interpretation of the quantum invariant is still not complete. Some time ago, Kashaev defined a quantum knot invariant based on the quantum dilogarithm function [3], and made a conjecture [4] that a limit of his invariant coincides with the hyperbolic volume of the knot complement [5]. This suggests an intimate connection between the quantum invariant and the geometry. Note that Kashaev's invariant was later identified with a specialization of the N-colored Jones polynomial at q being the Nth primitive root of unity [6].

In this Letter, we study Kashaev's invariant $\langle \mathcal{K} \rangle_N$ for the torus knot $\mathcal{K} = T(s,t)$, where s and t are coprime. See Fig. 1 for a projection of some torus knots. One may think that it is insignificant from a view point of the *volume conjecture* because the torus knot is not hyperbolic [5]. Although, the Chern–Simons invariant is considered as an imaginary part of the hyperbolic volume, and in fact the torus knot is supposed to have non-trivial Chern–Simons invariant. We shall show that the invariant exactly coincides with a limiting value of the Eichler integral of the character of the minimal model $\mathcal{M}(s,t)$ with q being the Nth root of unity.

This Letter is organized as follows. In Section 2 we recall a modular property of the character of the minimal model $\mathcal{M}(s,t)$. We define the Eichler integral, and give an explicit form of limiting value thereof when q is the Nth primitive root of unity. In Section 3 we study the colored Jones polynomial for the torus knot $T(s,t)$. We give a formula relating the quantum invariant with the Eichler integral. We further give some examples on q-series identities. Clarified

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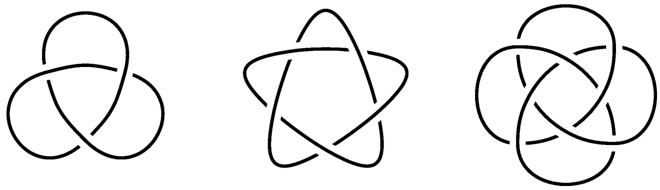

Fig. 1. Torus knot $T(s,t)$. From left to right, we depict trefoil $T(2,3)$, Solomon's seal knot $T(2,5)$, and $T(3,4)$, respectively.

is a relationship between the conformal weight and the Chern–Simons invariant of the minimal model. Section 4 is devoted to concluding remarks.

2. Eichler integral of the character

We study the character of the minimal model $\mathcal{M}(s,t)$, where s and t are coprime integers. The central charge of the minimal model $\mathcal{M}(s,t)$ is

$$c(s,t) = 1 - \frac{6(s-t)^2}{st}, \tag{1}$$

and the irreducible highest weight representation of the Virasoro algebra is given for the conformal weight

$$\Delta_{n,m}^{s,t} = \frac{(nt-ms)^2-(s-t)^2}{4st}, \tag{2}$$

where integers m and n are

$$1 \leqslant n \leqslant s-1, \qquad 1 \leqslant m \leqslant t-1.$$

The number of distinct fields in the theory is

$$D(s,t) = \frac{1}{2}(s-1)(t-1). \tag{3}$$

The character $\mathrm{ch}_{n,m}^{s,t}(\tau)$ for an irreducible highest weight representation of the Virasoro algebra with above central charge and weight, is computed as [7,8]

$$\begin{aligned}\mathrm{ch}_{n,m}^{s,t}(\tau) &= \mathrm{Tr}\, q^{L_0-\frac{1}{24}c(s,t)} \\ &= \frac{q^{\Delta_{n,m}^{s,t}-\frac{1}{24}c(s,t)}}{(q)_\infty}\sum_{k\in\mathbb{Z}} q^{stk^2} \\ &\quad \times \left(q^{k(nt-ms)} - q^{k(nt+ms)+mn}\right),\end{aligned} \tag{4}$$

where we set $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$. We see that

$$\mathrm{ch}_{n,m}^{s,t}(\tau) = \mathrm{ch}_{s-n,t-m}^{s,t}(\tau) = \mathrm{ch}_{m,n}^{t,s}(\tau) = \mathrm{ch}_{t-m,s-n}^{t,s}(\tau).$$

The character is modular covariant [9,10] as

$$\mathrm{ch}_{n,m}^{s,t}(\tau) = \sum_{n',m'} \mathbf{S}_{n,m}^{n',m'}\, \mathrm{ch}_{n',m'}^{s,t}(-1/\tau), \tag{5}$$

where sum runs over $D(s,t)$ distinct fields, and a matrix is explicitly written as

$$\begin{aligned}\mathbf{S}_{n,m}^{n',m'} &= \sqrt{\frac{8}{st}}\,(-1)^{nm'+mn'+1} \\ &\quad \times \sin\left(nn'\frac{t}{s}\pi\right)\sin\left(mm'\frac{s}{t}\pi\right).\end{aligned} \tag{6}$$

We rewrite the character of the minimal model as

$$\mathrm{ch}_{n,m}^{s,t}(\tau) = \frac{\Phi^{(n,m)}(\tau)}{\eta(\tau)}. \tag{7}$$

Here we have set the Dedekind η-function and $\Phi^{(n,m)}(\tau)$ as

$$\begin{aligned}&\eta(\tau) = q^{1/24}(q)_\infty, \\ &\Phi^{(n,m)}(\tau) = \sum_{k=0}^{\infty} \chi_{2st}^{(n,m)}(k)\, q^{\frac{1}{4st}k^2},\end{aligned} \tag{8}$$

where the function $\chi_{2st}^{(n,m)}(k)$ is periodic with modulus $2st$ as shown in Table 1.

From the modular property of the Dedekind η-function, we see that $\Phi^{(n,m)}(\tau)$ is modular with weight $1/2$, and spans $D(s,t)$ dimensional space; modular T- and S-transformations are respectively written as

$$\Phi^{(n,m)}(\tau+1) = \mathrm{e}^{\frac{(nt-ms)^2}{2st}\pi\mathrm{i}}\,\Phi^{(n,m)}(\tau), \tag{9}$$

$$\Phi^{(n,m)}(\tau) = \sqrt{\frac{\mathrm{i}}{\tau}}\sum_{n',m'} \mathbf{S}_{n,m}^{n',m'}\,\Phi^{(n',m')}(-1/\tau). \tag{10}$$

For the modular form with weight $w \in \mathbb{Z}_{>2}$, the period is defined by use of the classical Eichler integral, which is $w-1$ integrations of the modular form with respect to τ [11]. In a case of the half-integral weight modular form $\Phi^{(n,m)}(\tau)$, the Eichler

Table 1

$k \bmod 2st$	$nt-ms$	$nt+ms$	$2st-(nt+ms)$	$2st-(nt-ms)$	Others
$\chi_{2st}^{(n,m)}(k)$	1	−1	−1	1	0

integral is thus naively defined by the q-series as [12]

$$\widetilde{\Phi}^{(n,m)}(\tau)=-\frac{1}{2}\sum_{k=0}^{\infty}k\chi_{2st}^{(n,m)}(k)q^{\frac{1}{4st}k^2}. \tag{11}$$

A prefactor is for our convention. It can be seen that the former is regarded as a "half-derivative" ($\frac{1}{2}-1$ integration) of the modular form $\Phi^{(n,m)}(\tau)$ with respect to τ, as was originally studied in Ref. [12]. We consider a limiting value of the Eichler integral $\widetilde{\Phi}^{(n,m)}(\alpha)$ at $\alpha\in\mathbb{Q}$. Applying the Mellin transformation, we have

$$\widetilde{\Phi}^{(n,m)}\left(\frac{M}{N}+\mathrm{i}\frac{y}{2\pi}\right) \simeq -\frac{1}{2}\sum_{k=0}^{\infty}\frac{L_\omega(-2k-1,\chi_{2st}^{(n,m)})}{k!}\left(-\frac{y}{4st}\right)^k,$$

where $y\searrow 0$, and M, N are coprime integers. We mean that $L_\omega(k,\chi_{2st}^{(n,m)})$ is the twisted L-function defined by

$$L_\omega\big(k,\chi_{2st}^{(n,m)}\big) =\sum_{j=1}^{\infty}\chi_{2st}^{(n,m)}(j)\mathrm{e}^{\frac{M}{N}\frac{j^2}{2st}\pi\mathrm{i}}j^{-k} =\frac{1}{(2stN)^k}\sum_{j=1}^{2stN}\chi_{2st}^{(n,m)}(j)\mathrm{e}^{\frac{M}{N}\frac{j^2}{2st}\pi\mathrm{i}}\zeta\left(k,\frac{j}{2stN}\right),$$

where $\zeta(k,x)$ is the Hurwitz ζ function. By the analytic continuation, limiting value at $\tau\to M/N$ is then computed as

$$\widetilde{\Phi}^{(n,m)}(M/N) =\frac{stN}{2}\sum_{k=1}^{2stN}\chi_{2st}^{(n,m)}(k)\mathrm{e}^{\frac{k^2M}{2stN}\pi\mathrm{i}}B_2\left(\frac{k}{2stN}\right), \tag{12}$$

where $B_k(x)$ is the kth Bernoulli polynomial,

$$\frac{t\mathrm{e}^{xt}}{\mathrm{e}^t-1}=\sum_{k=0}^{\infty}\frac{t^k}{k!}B_k(x),$$

and especially $B_2(x)=x^2-x+\frac{1}{6}$.

This function fulfills a *nearly* modular property; for $N\in\mathbb{Z}$ we have an asymptotic expansion in $N\to\infty$,

$$\widetilde{\Phi}^{(n,m)}(1/N) +(-\mathrm{i}N)^{3/2}\sum_{n',m'}\mathbf{S}_{n,m}^{n',m'}\phi(n',m')\mathrm{e}^{-\frac{(n't-m's)^2}{2st}\pi\mathrm{i}N} \simeq\sum_{k=0}^{\infty}\frac{T^{(n,m)}(k)}{k!}\left(\frac{\pi}{2st\mathrm{i}N}\right)^k. \tag{13}$$

Here we have set

$$\phi(n,m)=\begin{cases}(s-n)m, & \text{if } nt>ms,\\ n(t-m), & \text{if } nt<ms,\end{cases} \tag{14}$$

and T-series is written in terms of the L-function associated with $\chi_{2st}^{(n,m)}$ as

$$T^{(n,m)}(k) =\frac{1}{2}(-1)^{k+1}L(-2k-1,\chi_{2st}^{(n,m)}) =\frac{1}{2}(-1)^k\frac{(2st)^{2k+1}}{2k+2}\sum_{j=1}^{2st}\chi_{2st}^{(n,m)}(j)B_{2k+2}\left(\frac{j}{2st}\right). \tag{15}$$

This can be shown as follows (see Refs. [12–15]). We define a variant of the Eichler integral

$$\widehat{\Phi}^{(n,m)}(z)=\sqrt{\frac{st\mathrm{i}}{8\pi^2}}\int_{z^*}^{\infty}\frac{\Phi^{(n,m)}(\tau)}{(\tau-z)^{3/2}}\,\mathrm{d}\tau. \tag{16}$$

This function is defined for z in the lower half plane, $z\in\mathbb{H}^-$, while the Eichler integral $\widetilde{\Phi}^{(n,m)}(z)$ is for the upper half plane, $z\in\mathbb{H}$. Using S-transformation (10), we have

$$\widehat{\Phi}^{(n,m)}(z)+\left(\frac{1}{\mathrm{i}z}\right)^{3/2}\sum_{n',m'}\mathbf{S}_{n,m}^{n',m'}\widehat{\Phi}^{(n',m')}(-1/z) =r^{(n,m)}(z;0), \tag{17}$$

where we have defined the period function

$$r^{(n,m)}(z;\alpha)=\sqrt{\frac{st\mathrm{i}}{8\pi^2}}\int_{\alpha}^{\infty}\frac{\Phi^{(n,m)}(\tau)}{(\tau-z)^{3/2}}\,\mathrm{d}\tau, \tag{18}$$

for $z \in \mathbb{H}^-$ and $\alpha \in \mathbb{Q}$. More generally, for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z}),$$

we have

$$\widehat{\Phi}^{(n,m)}(z) - \frac{1}{v^{(n,m)}(\gamma)}(cz+d)^{-3/2} \times \sum_{n',m'} (\mathbf{M}_\gamma)_{n,m}^{n',m'} \widehat{\Phi}^{(n',m')}\big(\gamma(z)\big) = r^{(n,m)}\big(z;\gamma^{-1}(\infty)\big), \tag{19}$$

where a matrix $\mathbf{M}_\gamma$ and $v^{(n,m)}(\gamma)$ are given from the modular transformation,

$$\sum_{n',m'} (\mathbf{M}_\gamma)_{n,m}^{n',m'} \Phi^{(n',m')}\big(\gamma(z)\big) = v^{(n,m)}(\gamma)\sqrt{cz+d}\,\Phi^{(n,m)}(z).$$

When we substitute Eq. (8) into Eq. (16) and perform an integration term by term in a limit $z \to \alpha \in \mathbb{Q}$, we see that

$$\widetilde{\Phi}^{(n,m)}(\alpha) = \widehat{\Phi}^{(n,m)}(\alpha).$$

Note that the left-hand side is given by Eq. (12) as a limit value from $\mathbb{H}$ while the right-hand side is a limit from $\mathbb{H}^-$. We can check for $N \in \mathbb{Z}$ that an asymptotic expansion of $r^{(n,m)}(1/N;0)$ gives a right-hand side of Eq. (13), and that from Eq. (12) we have

$$\widetilde{\Phi}^{(n,m)}(N+1) = \mathrm{e}^{\frac{(nt-ms)^2}{2st}\pi\mathrm{i}}\widetilde{\Phi}^{(n,m)}(N),$$
$$\widetilde{\Phi}^{(n,m)}(0) = \phi(n,m),$$

which shows

$$\widetilde{\Phi}^{(n,m)}(N) = \phi(n,m)\mathrm{e}^{\frac{(nt-ms)^2}{2st}\pi\mathrm{i}N}.$$

Combining these results we recover Eq. (13).

3. Quantum knot invariant for torus knot

We study the N-colored Jones polynomial $J_N(\mathcal{K})$ for the torus knot $\mathcal{K} = T(s,t)$. The torus knot $T(s,t)$ for coprime integers s, t is the knot which wraps around the solid torus in the longitudinal direction s times and in the meridinal direction t times. See Fig. 1. It is also represented as $(\sigma_1\sigma_2\cdots\sigma_{s-1})^t$ in terms of generators σ_j of the Artin braid group. An explicit form of the N-colored Jones polynomial is read as [16,17]

$$2\,\mathrm{sh}(N\hbar/2)\frac{J_N(\mathcal{K})}{J_N(\mathcal{O})} = \mathrm{e}^{-\frac{\hbar}{4}(\frac{t}{s}+\frac{s}{t})} \times \sum_{\varepsilon=\pm1}\sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \varepsilon \exp\left(\hbar st\left(k+\frac{s+\varepsilon t}{2st}\right)^2\right), \tag{20}$$

where we have set a parameter $q = \mathrm{e}^{\hbar}$, and $\mathcal{O}$ denotes unknot. As was shown in Ref. [6], Kashaev's invariant [3,4] coincides with a specialization $q \to \mathrm{e}^{2\pi\mathrm{i}/N}$ of the colored Jones polynomial. As the left-hand side of Eq. (20) vanishes in this substitution, Kashaev's invariant for the torus knot can be computed as a derivative of the right-hand side with respect to $\hbar$.

Here we recall the Eichler integral $\widetilde{\Phi}^{(n,m)}(1/N)$ which was computed in Eq. (12), and especially pay attention to a case of $(n,m) = (s-1,1)$. Using a property of the Gauss sum, we obtain from Eq. (12)

$$\widetilde{\Phi}^{(s-1,1)}(1/N) = \frac{st}{N}\mathrm{e}^{\frac{st}{2}N\pi\mathrm{i}+(s+t)\pi\mathrm{i}} \sum_{\varepsilon=\pm1}\sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \varepsilon\left(k+\frac{s+\varepsilon t}{2st}\right)^2 \times \mathrm{e}^{\frac{2\pi\mathrm{i}}{N}st\left(k+\frac{s+\varepsilon t}{2st}\right)^2}. \tag{21}$$

As seen from Eq. (20), this expression is proportional to the colored Jones polynomial at $\hbar \to 2\pi\mathrm{i}/N$. To conclude Kashaev's invariant $\langle\mathcal{K}\rangle_N$ for torus knot $\mathcal{K} = T(s,t)$ is identified with

$$\mathrm{e}^{-\frac{(st-s-t)^2}{2stN}\pi\mathrm{i}}\widetilde{\Phi}^{(s-1,1)}(1/N) = \langle T(s,t)\rangle_N. \tag{22}$$

We expect that the Eichler integrals $\widetilde{\Phi}^{(n,m)}(1/N)$ for other cases (n,m) are related with the quantum invariants of 3-manifolds. As a result Eq. (13) denotes an asymptotic expansion of Kashaev's invariant in $N \to \infty$. Note that an asymptotic behavior was studied in Refs. [18,19] in a different manner.

In general, we can construct q-series for the Eichler integrals based on the R-matrix [3]. We give some examples below (see Fig. 1). Hereafter we use a standard notation,

$$(x)_k = (x;q)_k = \prod_{j=1}^{k}\big(1-xq^{j-1}\big),$$

$$\begin{bmatrix} k \\ j \end{bmatrix} = \frac{(q)_k}{(q)_j (q)_{k-j}}.$$

- Trefoil $T(2,3)$,

$$\widetilde{\Phi}^{(1,1)}(\tau) \equiv -\frac{1}{2}\sum_{k=0}^{\infty} k\chi_{12}^{(1,1)}(k) q^{k^2/24}$$
$$= q^{1/24}\sum_{k=0}^{\infty}(q)_k.$$

This equality is Zagier's "strange" identity [12]; though both expressions do not converge simultaneously, the limiting values in q being roots of unity coincide. It is the Eichler integral of the Dedekind η-function.

- Solomon's Seal knot $T(2,5)$,

$$\widetilde{\Phi}^{(1,1)}(\tau) \equiv -\frac{1}{2}\sum_{k=0}^{\infty} k\chi_{20}^{(1,1)}(k) q^{k^2/40}$$
$$= q^{9/40}\sum_{k=0}^{\infty}(q)_k \sum_{j=0}^{k} q^{j(j+1)} \begin{bmatrix} k \\ j \end{bmatrix},$$
$$\widetilde{\Phi}^{(1,2)}(\tau) \equiv -\frac{1}{2}\sum_{k=0}^{\infty} k\chi_{20}^{(1,2)}(k) q^{k^2/40}$$
$$= q^{1/40}\sum_{k=0}^{\infty}(q)_k \sum_{j=0}^{k+1} q^{j^2} \begin{bmatrix} k+1 \\ j \end{bmatrix}.$$

The equalities in above formulae have same meaning with a case of trefoil [14]. These are the Eichler integral of the Rogers–Ramanujan q-series, which is the character of the Lee–Yang theory $\mathcal{M}(2,5)$.

- Knot $T(3,4)$,

$$\widetilde{\Phi}^{(1,1)}(\tau)$$
$$\equiv -\frac{1}{2}\sum_{k=0}^{\infty} k\chi_{24}^{(1,1)}(k) q^{k^2/48}$$
$$= q^{1/48}\sum_{k=0}^{\infty}(q)_k \left(\sum_{j=0}^{\lfloor k/2 \rfloor} q^{2j^2} \begin{bmatrix} k \\ 2j \end{bmatrix} + \sum_{j=0}^{\lfloor (k+1)/2 \rfloor} q^{2j^2} \begin{bmatrix} k+1 \\ 2j \end{bmatrix} \right),$$

$$\widetilde{\Phi}^{(1,2)}(\tau)$$
$$\equiv -\frac{1}{2}\sum_{k=0}^{\infty} k\chi_{24}^{(1,2)}(k) q^{k^2/48}$$
$$= 2q^{1/12}\sum_{k=0}^{\infty}(q^2;q^2)_k,$$
$$\widetilde{\Phi}^{(1,3)}(\tau)$$
$$\equiv -\frac{1}{2}\sum_{k=0}^{\infty} k\chi_{24}^{(1,3)}(k) q^{k^2/48}$$
$$= q^{25/48}\sum_{k=0}^{\infty}(q)_k \left(\sum_{j=0}^{\lfloor (k-1)/2 \rfloor} q^{2j(j+1)} \begin{bmatrix} k \\ 2j+1 \end{bmatrix} + \sum_{j=0}^{\lfloor k/2 \rfloor} q^{2j(j+1)} \begin{bmatrix} k+1 \\ 2j+1 \end{bmatrix} \right).$$

These are the Eichler integral of the Slater's q-series [20], which is the character of the Ising model $\mathcal{M}(3,4)$.

See that infinite sums in all those expressions reduce to a finite sum in a case $q \to \mathrm{e}^{2\pi \mathrm{i}/N}$.

Asymptotic behavior of Kashaev's invariant,

$$\lim_{N\to\infty} \frac{2\pi}{N} \log\langle\mathcal{K}\rangle_N,$$

is conjectured [4,6] to give the hyperbolic volume of the knot complement $M = S^3 \setminus \mathcal{K}$. In our case, the torus knot is not hyperbolic. We can rather expect from Eqs. (13) and (22) that a value

$$-\frac{(nt-ms)^2}{st}\pi^2 = -4\pi^2\left(\Delta_{n,m}^{s,t} - \frac{c(s,t)-1}{24}\right), \tag{23}$$

is related to the SU(2) Chern–Simons invariant,

$$\mathrm{CS}(M) = \frac{1}{4}\int_M \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right).$$

To see this fact, we recall that the fundamental group of $M = S^3 \setminus T(s,t)$ has a presentation

$$\pi_1(M) = \langle x, y \,|\, x^s = y^t \rangle. \tag{24}$$

As was shown in Ref. [21], the Chern–Simons invariant from two SU(2) representation ρ_0 and ρ_1 of $\pi_1(M)$

satisfies

$$\mathrm{CS}(M;\rho_1)-\mathrm{CS}(M;\rho_0)=-4\pi^2\int_0^1\beta(z)\alpha'(z)\,\mathrm{d}z. \tag{25}$$

Here $\alpha(z)$ and $\beta(z)$ are from the representation ρ_z, $z\in[0,1]$, of the meridian μ and the longitude λ up to conjugation,

$$\rho_z(\mu)=\begin{pmatrix} e^{2\pi i\alpha(z)} & \\ & e^{-2\pi i\alpha(z)}\end{pmatrix},\qquad \rho_z(\lambda)=\begin{pmatrix} e^{2\pi i\beta(z)} & \\ & e^{-2\pi i\beta(z)}\end{pmatrix}.$$

In a case of complement (24) of the torus knot, the longitude λ and the meridian μ are respectively given by x^s and $x^a y^b$, where $a,b\in\mathbb{Z}$ satisfies $as+bt=1$. As the longitude $\lambda=x^s=y^t$ is a center of group, it is sent to ±1. From relations $(x^a)^s=(x^s)^a$ and $(y^b)^t=(x^s)^b$ we see that x^a and y^b is conjugate to

$$\rho(x^a)\to\begin{pmatrix} e^{\pi i n/s} & \\ & e^{-\pi i n/s}\end{pmatrix},\qquad \rho(y^b)\to\begin{pmatrix} e^{\pi i m/t} & \\ & e^{-\pi i m/t}\end{pmatrix},$$

where n,m are integers. Correspondingly we find that a path of representation from a trivial representation $z=0$ is given by

$$\alpha(z)=\frac{1}{2}\left(\frac{n}{s}+\frac{m}{t}\right)z,\qquad \beta(z)=\frac{st}{2}\left(\frac{n}{s}+\frac{m}{t}\right).$$

Here $\beta(z)$ is constant along this path representation since the longitude is fixed to be ±1. Substituting into Eq. (25), we get a quantity (23) as the Chern–Simons invariant of M modulo $2\pi^2$.

4. Concluding remarks

We have revealed intriguing properties of the character of the minimal model $\mathcal{M}(s,t)$. We have shown that Kashaev's invariant, i.e., a specific value of the N-colored Jones polynomial, for the torus knot $T(s,t)$ is regarded as the Eichler integral of the character for $(n,m)=(s-1,1)$ with q being the Nth root of unity. It is natural to expect that general (n,m) case is also related to the quantum invariant of the 3-manifold.

As was shown in Ref. [15], the Eichler integral of the affine $\widehat{\mathrm{su}}(2)_{m+2}$ character, which is modular covariant with weight $3/2$, gives Kashaev's invariant for torus link $T(2,2m)$ when q is the Nth primitive root of unity. As the torus knot and link are not hyperbolic, we may regard the hyperbolic manifold as a (massive) deformation of the conformal field theory.

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