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Weighted-1-antimagic graphs of prime power order

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ABSTRACT

Suppose G is a graph, k is a non-negative integer. We say G is weighted-k-antimagic if for any vertex weight function $w : V \to \mathbb{N}$, there is an injection $f : E \to \{1, 2, \ldots, |E| + k\}$ such that for any two distinct vertices u and v, $\sum_{e\in E(v)} f(e) + w(v) \neq \sum_{e\in E(u)} f(e) + w(u)$. There are connected graphs $G \neq K_2$ which are not weighted-1-antimagic. It was asked in Wong and Zhu (in press) [13] whether every connected graph other than K_2 is weighted-2-antimagic, and whether every connected graph on an odd number of vertices is weighted-1-antimagic. It was proved in Wong and Zhu (in press) [13] that if a connected graph G has a universal vertex, then G is weighted-2-antimagic, and moreover if G has an odd number of vertices, then G is weighted-1-antimagic. In this paper, by restricting to graphs of odd prime power order, we improve this result in two directions: if G has odd prime power order p^z and has total domination number 2 with the degree of one vertex in the total dominating set not a multiple of p, then G is weighted-1-antimagic. If G has odd prime power order p^z, $p \neq 3$ and has maximum degree at least $|V(G)| - 3$, then G is weighted-1-antimagic.

1. Introduction

Assume G is a graph with vertex set $\{0, 1, \ldots, n-1\}$ and edge set $\{e_1, e_2, \ldots, e_m\}$. A labeling f of the edges of G with distinct integer labels is called *antimagic* if for any two distinct vertices i and j, $\sum_{e\in E(i)} f(e) \neq \sum_{e\in E(j)} f(e)$, where $E(i)$ is the set of edges incident to vertex i. If G has an antimagic labeling using labels $\{1, 2, \ldots, m+k\}$, then G is called *k-antimagic*. We call G *antimagic* if G is 0-antimagic. Hartsfield and Ringel [5] introduced the concept of antimagic labeling of graphs in 1990, and conjectured that every connected graph other than K_2 is antimagic. Alon et al. [2] proved that graphs G with minimum degree $\delta(G) \geq C \log |V(G)|$ (for some absolute constant C) or with maximum degree $\Delta(G) \geq |V(G)| - 2$ are antimagic. Kaplan et al. [8] proved that if a tree T has at most one vertex of degree 2, then T is antimagic (cf. [9]). The Cartesian products of various graphs are shown to be antimagic in [3,4,11,12].

In the study of antimagic labeling of graphs, Hefetz [6] introduced the concept of (w, k)-antimagic labeling of graphs. Suppose G is a graph and $w : V(G) \to \mathbb{N}$ is a vertex weight function, which assigns to each vertex v a weight $w(v)$. A labeling f of the edges of G with distinct integer labels is called a *w-antimagic* labeling of G if for any two distinct vertices i and j, $\sum_{e\in E(i)} f(e) + w(i) \neq \sum_{e\in E(j)} f(e) + w(j)$. The sum $\sum_{e\in E(i)} f(e) + w(i)$ is called the *vertex sum* at i (with respect to

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labeling f and w). Suppose k is a non-negative integer. A (w, k)-*antimagic labeling* of G is a w-antimagic labeling of G such that $f(e) \in \{1, 2, \dots, m+k\}$ for every edge e. We say G is *weighted-k-antimagic* if for any vertex weight function w, G has a (w, k)-antimagic labeling.

Observe that if G has a spanning subgraph H which is weighted-k-antimagic, then G itself is weighted-k-antimagic. It was proved in [6] that if H has a 2-factor consisting of circuits of length 3, and the total number of vertices is $n = 3^k$ for some positive integer k, then H is weighted-0-antimagic. As a consequence, if a graph G has $n = 3^k$ vertices and has a 2-factor consisting of circuits of length 3, then G is antimagic. This result is further improved in [7], where the number 3 is replaced with any prime number. I.e., if p is a prime, the number of vertices of G is a power of p, and G has a 2-factor consisting of circuits of length p, then G is weighted-0-antimagic. In particular, if G has a Hamilton cycle and its order is a prime, then G is weighted-0-antimagic, and hence antimagic.

The proof of Alon et al. [2] actually shows that graphs G with minimum degree $\delta(G) \geq C \log |V(G)|$ are weighted-0-antimagic.

Nevertheless, not every connected graph $G \neq K_2$ is weighted-0-antimagic. It is observed in [13] that any star is not weighted-0-antimagic, and any star on an even number of vertices is not weighted-1-antimagic. Then they asked the following questions.

Question 1. *Is it true that every connected graph $G \neq K_2$ is weighted-2-antimagic?*

Question 2. *Is it true that every connected graph G on an odd number of vertices is weighted-1-antimagic?*

In [13], it is proved that if G has an odd number of vertices and has domination number 1 (i.e., has a universal vertex), then G is weighted-1-antimagic; if $G \neq K_2$ has an even number of vertices and has domination number 1, then G is weighted 2-antimagic; if G has a prime number of vertices and having a Hamilton path, then G is weighted-1-antimagic.

A set X of $V(G)$ is called a *total dominating set* if every vertex of $V(G)$ (including vertices in X) is adjacent to some vertex in X. The *total domination number* of G is the cardinality of a smallest total dominating set. In this paper, by restricting to graphs of prime power order, we improve the result in [13] about graphs having a universal vertex in two directions: assume G has prime power number of vertices. If G has total domination number 2, then G is weighted-1-antimagic if the degree of one vertex in the total dominating set is not a multiple of p. If G is a graph on p^z vertices, where $p \geq 5$ is a prime and z is an integer, whose maximum degree is at least $|V(G)| - 3$, then G is weighted-1-antimagic.

2. Preliminaries

We associate to each edge e_j of G a variable x_j. For each vertex i of G, let $v_{G,\vec{x}}(i) = \sum_{e_j \in E(i)} x_j$. Let w be a vertex weight function of G, where w_i is the weight of i. Let $Q_{G,w}$ be the polynomial defined as

$$Q_{G,w}(x_1, x_2, \dots, x_m) = \prod_{1 \leq i < j \leq m} (x_i - x_j) \prod_{0 \leq i < j \leq n-1} (v_{G,\vec{x}}(i) + w_i - v_{G,\vec{x}}(j) - w_j).$$

It is obvious that a mapping $f : E(G) \to \mathbb{N}$ is a w-antimagic labeling of G if and only if $Q_{G,w}(f(e_1), f(e_2), \dots, f(e_m)) \neq 0$. So to find a w-antimagic labeling of G is equivalent to finding a non-zero assignment for the polynomial $Q_{G,w}$. For the purpose of proving the existence of such an assignment, we use Combinatorial Nullstellensatz.

Theorem 3 (*[1]*)**.** *Let F be a field and let $P(x_1, x_2, \dots, x_m)$ be a polynomial in $F[x_1, x_2, \dots, x_m]$. Suppose the degree of P is equal to $\sum_{j=1}^{m} t_j$ and the coefficient of $\prod_{j=1}^{m} x_j^{t_j}$ in the expansion of P is nonzero. Then for any subsets $S_1, S_2, \dots, S_m$ of F with $|S_j| = t_j + 1$, there exist $s_1 \in S_1, s_2 \in S_2, \dots, s_m \in S_m$ so that*

$$P(s_1, s_2, \dots, s_m) \neq 0.$$

The polynomial $Q_{G,w}$ has degree $\binom{n}{2} + \binom{m}{2}$. By Theorem 3, if $\binom{n}{2} + \binom{m}{2} = \sum_{i=1}^{m} t_i$ and the monomial $\prod_{i=1}^{m} x_i^{t_i}$ in the expansion of $Q_{G,w}$ has nonzero coefficient, then for any list assignment L which assigns to e_i a set $L(e_i)$ of $t_i + 1$ permissible labels, there is a w-antimagic labeling f of G with $f(e) \in L(e)$ for every edge e. Let

$$Q_G(x_1, x_2, \dots, x_m) = \prod_{1 \leq i < j \leq m} (x_i - x_j) \prod_{0 \leq i < j \leq n-1} (v_{G,\vec{x}}(i) - v_{G,\vec{x}}(j)).$$

If $\binom{n}{2} + \binom{m}{2} = \sum_{i=1}^{m} t_i$, then the monomial $\prod_{i=1}^{m} x_i^{t_i}$ has the same coefficient in $Q_{G,w}$ and Q_G. Thus we have the following lemma.

Lemma 4. *Let Q_G be the polynomial defined as above. If there is a monomial $\prod_{i=1}^{m} x_i^{t_i}$ with $\sum_{i=1}^{m} t_i = \binom{n}{2} + \binom{m}{2}$ and whose coefficient in the expansion of Q_G is nonzero, then for any vertex weight function w and for any list assignment L such that $|L(e_i)| \geq t_i + 1$, there is a w-antimagic labeling f of G with $f(e) \in L(e)$ for every edge e.*

Assume that G is a tree. Hence the number of edges is $n-1$.

Let a be the coefficient of the monomial $\prod_{j=1}^{n-1} x_j^{n-1}$ in $Q_G(x_1, x_2, \ldots, x_{n-1})$. By Lemma 4, if $a \neq 0$, then G is weighted-1-antimagic.

To calculate the coefficient a, we use the following lemma proved in [10].

Lemma 5. *If $P(x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}[x_1, x_2, \ldots, x_{n-1}]$ is of degree $\leq s_1+s_2+\cdots+s_{n-1}$, where $s_1, s_2, \ldots, s_{n-1}$ are nonnegative integers, then*

$$\left(\frac{\partial}{\partial x_1}\right)^{s_1}\left(\frac{\partial}{\partial x_2}\right)^{s_2}\cdots\left(\frac{\partial}{\partial x_{n-1}}\right)^{s_{n-1}} P(x_1, x_2, \ldots, x_{n-1})$$
$$= \sum_{a_1=0}^{s_1}\cdots\sum_{a_{n-1}=0}^{s_{n-1}} (-1)^{s_1+a_1}\binom{s_1}{a_1}\cdots(-1)^{s_{n-1}+a_{n-1}}\binom{s_{n-1}}{a_{n-1}} P(a_1, \ldots, a_{n-1}).$$

Apply Lemma 5 to the polynomial Q_G with $s_i = n-1$ for every $1 \leq i \leq n-1$, we conclude that the coefficient a of the monomial $\prod_{j=1}^{n-1} x_j^{n-1}$ in Q_G satisfies the following equality.

$$\begin{aligned} a\cdot((n-1)!)^{n-1} &= \left(\frac{\partial}{\partial x_1}\right)^{n-1}\cdots\left(\frac{\partial}{\partial x_{n-1}}\right)^{n-1} Q_G(x_1, \ldots, x_{n-1}) \\ &= \sum_{a_1=0}^{n-1}\cdots\sum_{a_{n-1}=0}^{n-1}(-1)^{n-1+a_1+\cdots+a_{n-1}}\binom{n-1}{a_1}\cdots\binom{n-1}{a_{n-1}} Q_G(a_1, \ldots, a_{n-1}) \\ &= \sum_{\sigma}(-1)^{n-1+\sigma(1)+\cdots+\sigma(n-1)}\binom{n-1}{\sigma(1)}\cdots\binom{n-1}{\sigma(n-1)} Q_G(\sigma(1), \ldots, \sigma(n-1)), \end{aligned}$$

where the last sum runs over all the mappings $\sigma : \{1, 2, \ldots, n-1\} \to \{0, 1, 2, \ldots, n-1\}$. However, if σ is not injective, then $Q_G(\sigma(1), \sigma(2), \ldots, \sigma(n-1)) = 0$, so the sum can be taken to run over all injective mappings $\sigma : \{1, 2, \ldots, n-1\} \to \{0, 1, 2, \ldots, n-1\}$.

Let Γ be the set of injective mappings from $\{1, 2, \ldots, n-1\}$ to $\{0, 1, 2, \ldots, n-1\}$. For $\sigma \in \Gamma$, let

$$v_{G,\sigma}(i) = \sum_{e_j\in E(i)} \sigma(j),$$
$$a(\sigma) = \prod_{0\leq i<j\leq n-1} (v_{G,\sigma}(i) - v_{G,\sigma}(j)),$$
$$b(\sigma) = \binom{n-1}{\sigma(1)}\cdots\binom{n-1}{\sigma(n-1)} \prod_{1\leq i<j\leq n-1}(\sigma(i)-\sigma(j)).$$

The coefficient a of the monomial $\prod_{j=1}^{n-1} x_j^{n-1}$ is non-zero if and only if

$$\sum_{\sigma\in\Gamma}(-1)^{\sigma(1)+\sigma(2)+\cdots+\sigma(n-1)} b(\sigma)a(\sigma) \neq 0. \tag{A}$$

Let V' be the set of leaves of G. Let E' be the set of edges incident to V'. Assume $|V \setminus V'| = k$. Let the vertices in V' be labeled by $k, k+1, \ldots, n-1$ and let the edge of E' incident to $i \in V'$ be labeled by e_i. For $i \in \{k, k+1, \ldots, n-1\}$, vertex i is incident to e_i only, i.e., $E(i) = \{e_i\}$. Hence for $\sigma \in \Gamma$, for $i \in \{k, k+1, \ldots, n-1\}$, $\sigma(i) = v_{G,\sigma}(i)$.

Let Θ be the subgroup of the automorphism group $Aut(G)$ of G that fix every non-leaf vertex of G. Thus each automorphism in Θ is a permutation of $\{0, 1, \ldots, n-1\}$ which fixes $0, 1, \ldots, k-1$. Given a permutation τ of $\{1, 2, \ldots, n-1\}$ that fixes $\{1, 2, \ldots, k-1\}$, let v_τ be the permutation of $\{0, 1, \ldots, n-1\}$ that fixes $\{0, 1, \ldots, k-1\}$ and equals to τ on $\{k, k+1, \ldots, n-1\}$. It is obvious that if $v_\tau \in \Theta$ and $\sigma \in \Gamma$, then $v_{G,\sigma\circ\tau} = v_{G,\sigma} \circ v_\tau$. Moreover, it is easy to see that

$$a(\sigma\circ\tau) = \mathrm{sign}(\tau)a(\sigma),$$
$$b(\sigma\circ\tau) = \mathrm{sign}(\tau)b(\sigma).$$

Consequently,

$$b(\sigma\circ\tau)a(\sigma\circ\tau) = b(\sigma)a(\sigma).$$

For $\sigma \in \Gamma$, let $[\sigma] = \{\sigma\circ\tau : v_\tau \in \Theta\}$. Then $\{[\sigma] : \sigma \in \Gamma\}$ partitions Γ into parts of cardinality $|\Theta|$. Thus

$$\sum_{\sigma\in\Gamma}(-1)^{\sigma(1)+\sigma(2)+\cdots+\sigma(n-1)} b(\sigma)a(\sigma) = |\Theta|\sum_{[\sigma]}(-1)^{\sigma(1)+\sigma(2)+\cdots+\sigma(n-1)} b(\sigma)a(\sigma),$$

where the second summation runs over all the equivalence classes $\{[\sigma] : \sigma \in \Gamma\}$.

In the following, we assume that $n = p^z$ is an odd prime power. For $\sigma \in \Gamma$, we define the sign of σ as (-1) to the power of the number of pairs $i < j$ such that $\sigma(i) > \sigma(j)$.

Lemma 6. *For any $\sigma \in \Gamma$, $b(\sigma) = \mathrm{sign}(\sigma)c$ for some constant c.*

Proof. Assume the range of σ is $\{0, 1, \ldots, n-1\} \setminus \{\ell\}$. Then

$$\begin{aligned} b(\sigma) &= \mathrm{sign}(\sigma) \prod_{i=0}^{n-1} \binom{n-1}{i} \prod_{0 \le i < j \le n-1} (i-j) \left(\binom{n-1}{\ell} \ell!(n-1-\ell)! \right)^{-1} \\ &= \mathrm{sign}(\sigma) \prod_{i=0}^{n-1} \binom{n-1}{i} \prod_{0 \le i < j \le n-1} (i-j) \left((n-1)!\right)^{-1}. \quad \square \end{aligned}$$

To prove (A), it is equivalent to prove that

$$\begin{aligned} \sum_{[\sigma]} (-1)^{\sigma(1)+\sigma(2)+\cdots+\sigma(n-1)} \mathrm{sign}(\sigma) a(\sigma) &= \frac{1}{|\Theta|} \sum_{\sigma \in \Gamma} (-1)^{\sigma(1)+\sigma(2)+\cdots+\sigma(n-1)} \mathrm{sign}(\sigma) a(\sigma) \\ &\neq 0. \end{aligned}$$

For an integer q, the *order* of q with respect to p is

$$\mathrm{ord}(q) = \max\{j : p^j | q\}.$$

Let $s = \mathrm{ord}\left(\prod_{0 \le i < j \le n-1} (i-j)\right)$. Instead of proving the inequality above directly, we prove the following stronger statement.

$$\frac{1}{|\Theta|} \sum_{\sigma \in \Gamma} (-1)^{\sigma(1)+\sigma(2)+\cdots+\sigma(n-1)} \mathrm{sign}(\sigma) a(\sigma) \not\equiv 0 \pmod{p^{s+1}}. \tag{B}$$

Lemma 7. *For $\sigma \in \Gamma$,*

$$\mathrm{ord}\left(\prod_{0 \le i < j \le n-1} (v_{G,\sigma}(i) - v_{G,\sigma}(j)) \right) \ge s$$

and equality holds if and only if $v_{G,\sigma}(i) \not\equiv v_{G,\sigma}(j) \pmod n$ for all $i \neq j$.

Proof. Assume $\sigma \in \Gamma$. For $i = 1, 2, \ldots$ and $j = 0, 1, \ldots, p^i - 1$, let

$$\alpha_{i,j} = |\{t : v_{G,\sigma}(t) \equiv j \pmod{p^i}\}|.$$

Then

$$\mathrm{ord}\left(\prod_{0 \le i < j \le n-1} (v_{G,\sigma}(i) - v_{G,\sigma}(j)) \right) = \sum_{i=1}^{\infty} \sum_{j=0}^{p^i-1} \binom{\alpha_{i,j}}{2}.$$

For each $i \ge 1$, $\sum_{j=0}^{p^i-1} \alpha_{i,j} = n = p^z$. It is obvious that the order

$$\mathrm{ord}\left(\prod_{0 \le i < j \le n-1} (v_{G,\sigma}(i) - v_{G,\sigma}(j)) \right)$$

is minimum if and only if for any i, $\alpha_{i,j} = n/p^i = p^{z-i}$ is a constant for all j. This happens if and only if $v_{G,\sigma}(i) \not\equiv v_{G,\sigma}(j) \pmod n$ for every $i \neq j$. In this case,

$$\mathrm{ord}\left(\prod_{0 \le i < j \le n-1} (v_{G,\sigma}(i) - v_{G,\sigma}(j)) \right) = s. \quad \square$$

A mapping $\sigma \in \Gamma$ is called *faithful* if $v_{G,\sigma}(i) \not\equiv v_{G,\sigma}(j) \pmod n$ for any two distinct vertices i and j of G. Let

$$\Omega = \{\sigma \in \Gamma : \sigma \text{ is faithful}\}.$$

By Lemma 7, the summation in (B) can be restricted to faithful σ's in Γ, i.e., (B) is equivalent to

$$\frac{1}{|\Theta|} \sum_{\sigma \in \Omega} (-1)^{\sigma(1)+\sigma(2)+\cdots+\sigma(n-1)} \mathrm{sign}(\sigma) a(\sigma) \not\equiv 0 \pmod{p^{s+1}}. \tag{C}$$

Claim 1. *If σ is faithful, then σ is a permutation of $\{1, 2, \ldots, n-1\}$.*

Proof. As $\sigma \in \Gamma$ is an injection from $\{1, 2, \ldots, n-1\}$ to $\{0, 1, \ldots, n-1\}$, there is exactly one element $i \in \{0, 1, \ldots, n-1\}$ which is not in the range of σ. As n is odd, we have

$$1 + 2 + \cdots + (n-1) \equiv 0 \pmod{n},$$

and hence $\sum_{j=1}^{n-1} \sigma(j) \equiv -i \pmod{n}$. Since σ is faithful,

$$\sum_{j=0}^{n-1} v_{G,\sigma}(j) \equiv 0 \pmod{n}.$$

On the other hand,

$$\sum_{j=0}^{n-1} v_{G,\sigma}(j) = 2\sum_{j=1}^{n-1} \sigma(j) \equiv -2i \pmod{n}.$$

As n is odd, we have $i = 0$. Hence σ is a permutation of $\{1, 2, \ldots, n-1\}$. □

For $\sigma \in \Omega$, let $v^*_{G,\sigma}(i) \equiv v_{G,\sigma}(i) \pmod{n}$. Then $v^*_{G,\sigma}$ is a permutation of $\{0, 1, \ldots, n-1\}$ and σ is a permutation of $\{1, 2, \ldots, n-1\}$. By Lemma 7, to prove (C), it suffices to show that

$$\frac{1}{|\Theta|}\sum_{\sigma\in\Omega} \mathrm{sign}(\sigma)\mathrm{sign}(v^*_{G,\sigma}) \not\equiv 0 \pmod{p}. \tag{D}$$

Extend σ to a permutation of $\{0, 1, \ldots, n-1\}$ by letting $\sigma(0) = 0$. Then

$$\mathrm{sign}(\sigma)\mathrm{sign}(v^*_{G,\sigma}) = \mathrm{sign}(\sigma^{-1} \circ v^*_{G,\sigma}).$$

By our labeling of the vertices and edges of G, we know that $\sigma(i) = v^*_{G,\sigma}(i)$ for $i \in \{k, k+1, \ldots, n-1\}$. Hence, the restriction of $\sigma^{-1} \circ v^*_{G,\sigma}$ to $\{k, k+1, \ldots, n-1\}$ is identity, and the restriction of $\sigma^{-1} \circ v^*_{G,\sigma}$ to $\{0, 1, \ldots, k-1\}$, which we denote by σ^*, is a permutation of $\{0, 1, \ldots, k-1\}$. Moreover,

$$\mathrm{sign}(\sigma^{-1} \circ v^*_{G,\sigma}) = \mathrm{sign}(\sigma^*).$$

To prove the coefficient a is nonzero, it suffices to show that

$$\frac{1}{|\Theta|}\sum_{\sigma\in\Omega} \mathrm{sign}(\sigma^*) \not\equiv 0 \pmod{p}. \tag{E}$$

3. Double star

A tree whose non-leaf vertices induces a K_2 is called a *double star*. This section proves the following theorem.

Theorem 8. *If G is a double star of prime power order $n = p^z$ and the degree of one non-leaf vertex (and hence of both non-leaf vertices) is relatively prime to n, then G is weighted-1-antimagic.*

Proof. Assume G is a double star with exactly two non-leaf vertices: 0 and 1.

For $\sigma \in \Omega$, let σ^* be the permutation over $\{0, 1\}$ defined as in the previous section.

Let $T_1 = \{\sigma \in \Omega : \sigma^*(i) = i \text{ for } i = 0, 1\}$, and $T_2 = \{\sigma \in \Omega : \sigma^*(i) = 1 - i \text{ for } i = 0, 1\}$. So $\mathrm{sign}(\sigma^*) = 1$ if $\sigma \in T_1$ and $\mathrm{sign}(\sigma^*) = -1$ if $\sigma \in T_2$. To prove (E), we need to show that

$$\frac{1}{|\Theta|}(|T_1| - |T_2|) \not\equiv 0 \pmod{p}.$$

Let $U_1 = \{2, 3, \ldots, k\}$ be the set of leaves adjacent to 0, and let $U_2 = \{k+1, k+2, \ldots, n-1\}$ be the set of leaves adjacent to 1. Then $|\Theta| = (k-1)!(n-k-1)!$.

For $\sigma \in \Omega$, let $X_\sigma = \sum_{i\in U_1} \sigma(i)$ and $Y_\sigma = \sum_{i\in U_2} \sigma(i)$. Since $\sigma \in \Omega$, $\sigma(i) \neq 0$ for every $1 \leq i \leq n-1$. Therefore $X_\sigma + Y_\sigma + \sigma(1) = 1 + 2 + \cdots + n - 1 \equiv 0 \pmod{n}$.

Observe that $\sigma^*(0) = 0$ means that $v^*_{G,\sigma}(0) = \sigma(0) = 0$, i.e., $\sum_{e_j\in E(0)} \sigma(j) = X_\sigma + \sigma(1) \cong 0 \pmod{n}$. This is equivalent to $Y_\sigma \cong 0 \pmod{n}$, as $X_\sigma + Y_\sigma + \sigma(1) = 1 + 2 + \cdots + n - 1 \equiv 0 \pmod{n}$. So the following equalities are equivalent:

$$\sigma^*(0) = 0$$
$$X_\sigma + \sigma(1) \equiv 0 \pmod{n}$$
$$Y_\sigma \equiv 0 \pmod{n}$$
$$\sigma^*(1) = 1.$$

Hence $\sigma \in T_1$ if and only if $X_\sigma + \sigma(1) \equiv 0 \pmod{n}$. Similarly, $\sigma \in T_2$ if and only if $X_\sigma \equiv 0 \pmod{n}$.

For $j = 1, 2, \ldots, n-1$, let A_j be the set of solutions to the equation

$$y_1 + y_2 + \cdots + y_j \equiv 0 \pmod{n}$$

subject to the condition that $y_i \in \{1, 2, \ldots, n-1\}$ and y_i are pairwise distinct. Let $\alpha_j = |A_j|$.

If $\sigma(1), \sigma(2), \ldots, \sigma(k)$ are chosen so that $X_\sigma + \sigma(1) \equiv 0 \pmod{n}$, then arbitrary assigning $\{1, 2, \ldots, n-1\} \setminus \{\sigma(1), \sigma(2), \ldots, \sigma(k)\}$ to $\sigma(k+1), \sigma(k+2), \ldots, \sigma(n-1)$, we obtain an element of T_1. So

$$|T_1| = \alpha_k \cdot (n-1-k)!.$$

Similarly, we have

$$|T_2| = \alpha_{k-1} \cdot (n-k)!$$

and hence

$$|T_2| - |T_1| = (n-k-1)!((n-k)\alpha_{k-1} - \alpha_k).$$

Observe that α_{k-1} is a multiple of $(k-1)!$, because given a solution to the equation $y_1 + y_2 + \cdots + y_{k-1} \equiv 0 \pmod{n}$ with y_i pairwise distinct, any permutation of $y_1, y_2, \ldots, y_{k-1}$ is also a solution. So

$$\frac{n\alpha_{k-1}}{(k-1)!} \equiv 0 \pmod{p}.$$

Hence

$$\frac{1}{|\Theta|}(|T_2| - |T_1|) = \frac{1}{(k-1)!(n-k-1)!}(|T_2| - |T_1|) \not\equiv 0 \pmod{p}$$

if and only if

$$\frac{1}{(k-1)!}(k\alpha_{k-1} + \alpha_k) \not\equiv 0 \pmod{p}.$$

Instead of calculating α_j directly, we consider a slightly different parameter. Let B_j be the set of solutions to the equation

$$y_1 + y_2 + \cdots + y_j \equiv 0 \pmod{n}$$

subject to the condition that $y_i \in \{0, 1, \ldots, n-1\}$ and y_i are pairwise distinct. Let $\beta_j = |B_j|$. There is a simple formula for β_j. Let

$$\Psi(j) = \{(y_1, y_2, \ldots, y_j) : y_i \in \{0, 1, \ldots, n-1\},\ \text{and } y_i \text{ are pairwise distinct}\}.$$

Then $|\Psi(j)| = n(n-1)\ldots(n-j+1)$.

Let $\sim$ be the equivalence relation on $\Psi(j)$ defined as $(y_1, y_2, \ldots, y_j) \sim (y'_1, y'_2, \ldots, y'_j)$ if there is a constant d such that $y_i \equiv y'_i + d \pmod{n}$ for $i = 1, 2, \ldots, j$. Observe that if $(y_1, y_2, \ldots, y_j)$ and $(y'_1, y'_2, \ldots, y'_j)$ are equivalent but distinct, then $\sum_{i=1}^{j} y_i \equiv \sum_{i=1}^{j} y'_i + jd \pmod{n}$ for some $0 < d \le n-1$. If j and n are coprime, then there is no such d. Hence each equivalence class of $\sim$ contains exactly one j-tuple of B_j. As each equivalence class contains n tuples, we have

$$\beta_j = |B_j| = |\Psi(j)|/n = (n-1)(n-2)\ldots(n-j+1).$$

If $(y_1, y_2, \ldots, y_j) \in B_j$, then either none of the y_i's is equal to 0, and hence $(y_1, y_2, \ldots, y_j) \in A_j$ or exactly one of y_i's is 0. If $y_i = 0$, then $(y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_j) \in A_{j-1}$. Therefore

$$\beta_j = \alpha_j + j\alpha_{j-1}.$$

Since, by assumption, $(k, n) = 1$, we have

$$\frac{1}{(k-1)!}(k\alpha_{k-1} + \alpha_k) = \frac{1}{(k-1)!}\beta_k = \frac{(n-1)(n-2)\ldots(n-k+1)}{(k-1)!} \not\equiv 0 \pmod{p}$$

The last inequality holds because for $1 \le i \le n-1$, we have $\mathrm{ord}(i) = \mathrm{ord}(n-i)$. This completes the proof of Theorem 8. □

Corollary 9. *If G is of prime power order p^z and has a spanning tree which is a double star such that the degree of one non-leaf vertex is relatively prime to n, then G is weighted-1-antimagic. In particular, if G is of prime order and has total domination number 2, then G is weighted-1-antimagic.*

4. Graphs with large maximum degree

It was proved in [2] that graphs G of order n and maximum degree at least $n-2$ are antimagic, i.e., 0-antimagic. It was proved in [6] that for $k \geq 3$, graphs G of order n and maximum degree at least $n-k$ are $(3k-7)$-antimagic. In this section, we assume that $p \geq 5$ is a prime and z is an integer. We prove that if G is a graph of order $n = p^z$ and whose maximum degree is at least $n-3$, then G is weighted-1-antimagic.

Lemma 10. *Assume G is a tree with vertices $\{0, 1, \ldots, n-1\}$ and edges $e_i = 0i$ for $i = 4, 5, \ldots, n-1$ and $e_1 = 01, e_2 = 12, e_3 = 23$. Then G is weighted-1-antimagic.*

Proof. Given $\sigma \in \Omega$, let $X_\sigma = \sum_{i=4}^{n-1} \sigma(i)$. Then

$$v^*_{G,\sigma}(0) \equiv X_\sigma + \sigma(1) \pmod n,$$
$$v^*_{G,\sigma}(1) \equiv \sigma(1) + \sigma(2) \pmod n,$$
$$v^*_{G,\sigma}(2) \equiv \sigma(2) + \sigma(3) \pmod n.$$

As σ is faithful and $v^*_{G,\sigma}(i) = \sigma(i)$ for $i = 3, 4, \ldots, n-1$, we know that

$$\{v^*_{G,\sigma}(0), v^*_{G,\sigma}(1), v^*_{G,\sigma}(2)\} = \{0, \sigma(1), \sigma(2)\}.$$

Since none of $\sigma(1), \sigma(2)$ is congruent to 0 modulo n, we conclude that $v^*_{G,\sigma}(1) \neq \sigma(1), \sigma(2)$, and hence $v^*_{G,\sigma}(1) = 0$. Thus $\sigma(1) = n - \sigma(2)$. As $\sigma(3)$ is not congruent to 0 modulo n, we know that $v^*_{G,\sigma}(2) \neq \sigma(2)$ and hence

$$v^*_{G,\sigma}(2) = \sigma(1), \qquad v^*_{G,\sigma}(0) = \sigma(2).$$

This implies that for any $\sigma \in \Omega$, $\sigma^* = (210)$. Moreover, we have $\sigma(3) \equiv 2\sigma(1) \pmod n$. As $p \neq 3$, for any $a \in \mathbb{Z}_n \setminus \{0\}$, $a, n-a, 2a \pmod n$ are distinct elements. By assigning a to $\sigma(1)$, $n-a$ to $\sigma(2)$ and $2a \pmod n$ to $\sigma(3)$, and arbitrarily assigning the $n-4$ remaining elements in $\{1, 2, \ldots, n-1\}$ to the remaining edges, we obtain an element σ of Ω. Therefore $|\Omega| = (n-1) \cdot (n-4)!$. As $|\Theta| = (n-4)!$, we conclude that $\sum_{\sigma \in \Omega} \frac{1}{|\Theta|} \mathrm{sign}(\sigma^*) \not\equiv 0 \pmod p$. Hence G is weighted-1-antimagic. $\square$

Lemma 11. *Assume G is a tree with vertices $\{0, 1, \ldots, n-1\}$ and edges $e_i = 0i$ for $i = 5, 6, \ldots, n-1$ and $e_1 = 01, e_2 = 02, e_3 = 13, e_4 = 24$. Then G is weighted-1-antimagic.*

Proof. Given $\sigma \in \Omega$, let $X_\sigma = \sum_{i=5}^{n-1} \sigma(i)$. Then

$$v^*_{G,\sigma}(0) \equiv X_\sigma + \sigma(1) + \sigma(2) \pmod n$$
$$v^*_{G,\sigma}(1) \equiv \sigma(1) + \sigma(3) \pmod n$$
$$v^*_{G,\sigma}(2) \equiv \sigma(2) + \sigma(4) \pmod n.$$

Again, we have

$$\{v^*_{G,\sigma}(0), v^*_{G,\sigma}(1), v^*_{G,\sigma}(2)\} = \{\sigma(1), \sigma(2), 0\}.$$

Similarly as in the previous lemma, we know that $v^*_{G,\sigma}(1) \neq \sigma(1)$ and $v^*_{G,\sigma}(2) \neq \sigma(2)$. In other words, the permutation σ^* over $\{0, 1, 2\}$ satisfies $\sigma^*(1) \neq 1$ and $\sigma^*(2) \neq 2$. Therefore, σ^* has three possibilities:
(1) $\sigma^* = (021)$; (2) $\sigma^* = (012)$; (3) $\sigma^* = (12)$.

It is straightforward to verify that σ^* is of type (1) if and only if for some $a, b \in \{1, 2, \ldots, n-1\}$ such that $b \not\equiv a, -a, -2a, -\frac{1}{2}a \pmod n$, the following hold

$$\sigma(1) \equiv a + b \pmod n,$$
$$\sigma(2) = a,$$
$$\sigma(3) \equiv -(a+b) \pmod n,$$
$$\sigma(4) = b.$$

Since $p \geq 5$ is a prime and $n = p^z$ for a positive integer z, for any $a \in \{1, 2, \ldots, n-1\}$, the four elements $a, -a \pmod n, -2a \pmod n, -\frac{1}{2}a \pmod n$ are distinct. So a has $n-1$ choices, and b has $n-5$ choices, implying that there are $(n-1)(n-5)(n-5)!$ permutations $\sigma \in \Omega$ of type (1). Type (1) and type (2) are symmetric. So there are $(n-1)(n-5)(n-5)!$ permutations $\sigma \in \Omega$ are of type (2). A permutation $\sigma \in \Omega$ is of type (3) if and only if for some $a, b \in \{1, 2, \ldots, n-1\}$ such that $b \not\equiv a, -a, -2a \pmod n$, the following hold

$$\sigma(1) \equiv a + b \pmod n,$$
$$\sigma(2) = b,$$
$$\sigma(3) \equiv -a \pmod n,$$
$$\sigma(4) = a.$$

As $p \geq 5$, for any $a \in \{1, 2, \ldots, n-1\}$, the three elements $a, -a \pmod n, -2a \pmod n$ are distinct. So a has $n-1$ choices, and b has $n-4$ choices, implying that there are $(n-1)(n-4)(n-5)!$ permutations $\sigma \in \Omega$ are of type (3).

For σ of type (1) and type (2), $\text{sign}(\sigma^*) = 1$. For σ of type (3), $\text{sign}(\sigma^*) = -1$. Therefore

$$\sum_{\sigma \in \Omega} \frac{1}{|\Theta|} \text{sign}(\sigma^*) = (n-1)\left(2 \cdot (n-5) - (n-4)\right) \equiv (n-1)(n-6) \not\equiv 0 \pmod p.$$

This completes the proof of Lemma 11. $\square$

Theorem 12. *If $p \geq 5$ is a prime, G is a connected graph of order $n = p^z$ for some integer z and has maximum degree at least $n-3$, then G is weighted-1-antimagic.*

Proof. If G is a connected graph of maximum degree at least $n-3$, then G has a spanning tree which is either a star or a double star with one vertex of degree 2 or 3, or a tree as described in Lemma 10 or in Lemma 11. The results above show that such a tree is weighted-1-antimagic. Therefore G itself is weighted-1-antimagic. $\square$

Remark. We may define a graph G to be *weighted-k-antimagic choosable* if the following hold: for any list assignment L which assigns to each edge e a set $L(e)$ of $|E| + k$ permissible weights (integers) and for any weight function w on the vertex set of G, there is a mapping f which assigns to each edge e a distinct weight $f(e) \in L(e)$ so that for any two vertices i, j, $\sum_{e \in E(i)} f(e) + w(i) \neq \sum_{e \in E(j)} f(e) + w(j)$. The graphs proved to be weighted-1-antimagic are actually weighted-1-antimagic choosable graphs.

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