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On the Factorization of Nuclear Operators on Hilbert Space

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1. Let $\mathscr{H}$ be a separable Hilbert space (which we take over the real scalar field for notational convenience) and T a nuclear (trace-class) operator on $\mathscr{H}$. It is well known that T can be factored as $T = T_2 \circ T_1$, where T_1 and T_2 are both Hilbert–Schmidt operators on $\mathscr{H}$ [8]. More generally, T can be factored as $T = T_2 \circ T_1$, where $T_1 \in C_p$ and $T_2 \in C_q$ for $1 < p < +\infty$ and $p^{-1} + q^{-1} = 1$ (see [3, 6] for a discussion of the operators in the classes C_p). Since it is known (e.g., [4]) that $C_p{}^*$ is isometrically isomorphic to $C_q(p^{-1} + q^{-1} = 1)$, the following problem is a very natural one.

Let α be a uniform crossnorm [8] and $J(\alpha)$ the completion of the finite dimensional operators on $\mathscr{H}$ under the norm α. Then $J(\alpha)$ can be identified with the separable norm ideal $\mathscr{H} \otimes_\alpha \mathscr{H}$ of compact operators on $\mathscr{H}$ [7]. Moreover, as Schatten has shown [7], $J(\alpha)^* = (\mathscr{H} \otimes_\alpha \mathscr{H})^*$ may be identified with a norm ideal of compact operators on $\mathscr{H}$, and if $T_1 \in J(\alpha)$ and $T_2 \in J(\alpha)^*$, then $T_2 \circ T_1$ is a nuclear operator on $\mathscr{H}$.

PROBLEM. For which separable norm ideals $J(\alpha)$ is it true that if T is a nuclear operator on $\mathscr{H}$, then T can be factored as $T = T_2 \circ T_1$ for $T_1 \in J(\alpha)$ and $T_2 \in J(\alpha)^*$?

In this note we show that this problem is equivalent to one concerning an analogous factorization of elements of the sequence space l^1.

2. If $\mathscr{H}$ is a separable Hilbert space, we denote by $B(\mathscr{H})$, $K(\mathscr{H})$, and $N(\mathscr{H})$, respectively, the spaces of bounded linear operators, compact operators, and nuclear operators on $\mathscr{H}$. If $T \in K(\mathscr{H})$, the sequence of s-numbers of T [4] is the sequence $\{s_n(T)\}_{n=1}^\infty$ of eigenvalues of $(T^*T)^{1/2}$ arranged in a sequence decreasing to zero and repeated according to their multiplicity. By a well known and fundamental result of Von Neumann, any operator $T \in K(\mathscr{H})$ has the representation $T = \sum_{n=1}^\infty s_n(T)\, \varphi_n \otimes \psi_n$ for

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(φ_n) and (ψ_n) orthonormal sequences in $\mathscr{H}$[8]. Moreover, it is a classical result of Calkin and Von Neumann [8] that there is a one-to-one correspondence between the collection of two-sided ideals in $B(\mathscr{H})$ and the collection of certain "ideal sets" of sequences in c_0. Extending the results of Schatten [8] and of Gohberg, and Krein [4], the author has recently shown that there is a one-to-one correspondence between the set of all separable norm ideals in $B(\mathscr{H})$ and the set of all separable symmetric Banach sequence spaces [5]. This correspondence is obtained, as in Calkin's result, by associating with any separable norm ideal $J(\alpha)$ the space μ of all sequences (a_n) in c_0 for which $\sum_n a_n \varphi_n \otimes \psi_n$ converges in $J(\alpha)$ for any orthonormal sequences (φ_n) and (ψ_n) in $\mathscr{H}$. It turns out that μ is then a symmetric solid Banach space with the property that $T \in J(\alpha)$ if and only if $(s_n(T)) \in \mu$. Moreover, if $\mu^x = \{(b_n) \in c_0 \mid \sum_{n=1}^{\infty} |a_n| \, |b_n| < +\infty$ for all $(a_n) \in \mu\}$, then $S \in J(\alpha)^*$ if and only if $(s_n(S)) \in \mu^x$. These results will enable us to transform the original problem into a fundamentally simpler one concerning sequence spaces.

Now let $\mu \cdot \mu^x = \{(c_n) \in c_0 \mid (c_n) = (a_n b_n)$ for $(a_n) \in \mu$ and $(b_n) \in \mu^x\}$ [2]. Then certainly $\mu \cdot \mu^x \subset l^1$. If $T \in N(\mathscr{H})$, then $(s_n(T)) \in l^1$ [8], so if $\mu \cdot \mu^x = l^1$ we must have $(s_n(T)) = (a_n b_n)$ for some $(a_n) \in \mu$ and $(b_n) \in \mu^x$. Consequently, since $T = \sum_n s_n(T) \varphi_n \otimes \psi_n$ for some orthonormal sets (φ_n) and (ψ_n) in $\mathscr{H}$, we have the factorization $T = T_2 \circ T_1$, where

$$T_1 = \sum_n a_n \varphi_n \otimes \varphi_n \text{ is in } J(\alpha),$$

and

$$T_2 = \sum_n b_n \varphi_n \otimes \psi_n \text{ is in } J(\alpha)^*.$$

That is, T factors through the pair of ideals $(J(\alpha), J(\alpha)^*)$.

We now show that the converse is also true. That is, the existence of such a factorization implies the indicated factorization of elements of l^1. Thus the problem mentioned earlier is equivalent to the problem of whether $l^1 = \mu \cdot \mu^x$ for the sequence space μ associated with $J(\alpha)$.

We begin with a technical result.

LEMMA. *Let A and B be compact operators on $\mathscr{H}$. Then there is a sequence (λ_n) convergent to zero for which $s_n(AB) \leqslant \lambda_n s_n(B)$ for all n.*

Proof. It is well known that if $T \in K(\mathscr{H})$, then $s_n(T) \leqslant \| T - F \|$ for any operator F on $\mathscr{H}$ of dimension $\leqslant (n-1)$[4].

Let $B = \sum_{n=1}^{\infty} s_n(B) \varphi_n \otimes \psi_n$ and denote by P_k the orthogonal projection of $\mathscr{H}$ onto $[\varphi_i]_{i=k}^{\infty}$. Then $P_k = I - Q_k$, where Q_k is the orthogonal projection onto $[\varphi_i]_{i=1}^{k-1}$, and $\| ABP_k \| = \| AB - ABQ_k \|$, where $\dim(ABQ_k) \leqslant K - 1$. Thus

$$s_k(AB) \leqslant \| ABP_k \| = \left\| A\left(\sum_{n=k}^{\infty} s_n(B)\, \varphi_n \otimes \psi_n\right)\right\|$$

$$= \left\| AR_k\left(\sum_{n=k}^{\infty} s_n(B)\, \varphi_n \otimes \psi_n\right)\right\|,$$

(where R_k is the orthogonal projection of $\mathscr{H}$ onto $[\psi_i]_{i=k}^{\infty}$)

$$\leqslant \| AR_k \| \left\| \sum_{n=k}^{\infty} s_n(B)\, \varphi_n \otimes \psi_n \right\| = \| AR_k \|\, s_k(B).$$

Now, $\| AR_k \| = \| R_k A^* \|$, implying

$$\| AR_k \| = \sup_{\|x\|=1} \| R_k A^*(x) \|$$

$$= \sup_{\|x\| \leqslant 1} \left\| \sum_{n=k}^{\infty} \langle A^*(x), \psi_n \rangle \psi_n \right\|.$$

But since A^* is compact, the set $\{A^*x \mid \| x \| \leqslant 1\}$ is compact in $\mathscr{H}$, and this last tends to zero as $k \to \infty$. Thus setting $\lambda_n = \| AR_n \|$, we have $s_n(AB) \leqslant \lambda_n s_n(B)$, where $\lambda_n \to 0$, and the lemma is proved.

We have seen that if $J(\alpha)$ is a separable norm ideal whose associated Banach sequence space is μ, then every nuclear operator on $\mathscr{H}$ factors through the pair $(J(\alpha), J(\alpha)^*)$ if $\mu \cdot \mu^x = l^1$. We can now prove the converse, thereby characterizing those norm ideals with this factorization property.

THEOREM. *Suppose every $T \in N(\mathscr{H})$ can be factored as $T = AB$ for $B \in J(\alpha)$ and $A \in J(\alpha)^*$. Then $\mu \cdot \mu^x = l^1$.*

Proof. By definition $\mu \cdot \mu^x \subset l^1$. To show the reverse inclusion, let $(c_i) \in l^1$. If (c_i) is finitely nonzero, then certainly $(c_i) \in \mu \cdot \mu^x$. Thus we may assume $c_i > 0$ for all i, and show $(c_i) = (a_i b_i)$ for $(a_i) \in \mu$ and $(b_i) \in \mu^x$.

Since $(c_i) \in l^1$, $T = \sum_{i=1}^{\infty} c_i\, \varphi_i \otimes \varphi_i \in N(\mathscr{H})$ for (φ_i) some orthonormal basis for $\mathscr{H}$. By assumption, $T = AB$ for some $B \in J(\alpha)$ and $A \in J(\alpha)^*$.

Since $c_i > 0$, $c_i = s_i(AB)$ for all i.

CLAIM.

$$\frac{s_i(AB)}{s_i(A)} = \frac{c_i}{s_i(A)} \in \mu^x.$$

By the lemma, the sequence $s_i(AB)/s_i(A) \in c_0$ (where we note $s_i(A) > 0$ for all i since A is not finite dimensional). Thus we can arrange this sequence in decreasing order as $s_{\pi(i)}(AB)/s_{\pi(i)}(A)$ for some permutation π of the positive integers.

Now for any n, arrange the set $\{\pi(1), \pi(2),\ldots, \pi(n)\}$ as the increasing sequence of integers $1 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_n$. By the theorem of Amir-Moez [1] we have

$$\prod_{k=1}^{n} s_{j_k}(AB) \leqslant \left[\prod_{k=1}^{n} s_k(B)\right]\left[\prod_{k=1}^{n} s_{j_k}(A)\right].$$

Hence

$$\prod_{k=1}^{n} (s_{j_k}(AB)/s_{j_k}(A)) \leqslant \prod_{k=1}^{n} s_k(B),$$

or

$$\prod_{i=1}^{n} (s_{\pi(i)}(AB)/s_{\pi(i)}(A)) \leqslant \prod_{i=1}^{n} s_i(B).$$

The last inequality holds for all n, where $s_{\pi(i)}(AB)/s_{\pi(i)}(A)$ and $(s_i(B))$ are sequences decreasing to zero. Thus since the function $\ln x$ is increasing we have

$$\sum_{i=1}^{n} \ln(s_{\pi(i)}(AB)/s_{\pi(i)}(A)) \leqslant \sum_{i=1}^{n} \ln(s_i(B))$$

for all n, where $\ln(s_{\pi(i)}(AB)/s_{\pi(i)}(A))$ and $(\ln(s_i(B)))$ are decreasing sequences.

By the theorem of Hardy, Littlewood, and Polya (e.g., [4, p. 37]) (and by the fact that e^x is a convex function) we then have

$$\sum_{i=1}^{n} (s_{\pi(i)}(AB)/s_{\pi(i)}(A)) \leqslant \sum_{i=1}^{n} s_i(B) \qquad \text{for all } n.$$

Hence by a well-known result, if $(d_i) \in \mu$, $\|(d_i)\|\,\mu \leqslant 1$, and $d_i \downarrow 0$, then

$$\sum_{i=1}^{n} d_i(s_{\pi(i)}(AB)/s_{\pi(i)}(A)) \leqslant \sum_{i=1}^{n} d_i s_i(B) \qquad \text{for all } n.$$

Since the sequences $s_{\pi(i)}(AB)/s_{\pi(i)}(A)$ and $(s_i(B))$ are each decreasing, this implies

$$\sup_n \left\| \sum_{i=1}^{n} (s_{\pi(i)}(AB)/s_{\pi(i)}(A))\, e_i \right\|_{\mu^x} \leqslant \|(s_i(B))\|_{\mu^x}.$$

It follows that the sequence $s_{\pi(i)}(AB)/s_{\pi(i)}(A)$, and hence also $s_i(AB)/s_i(A)$ is in μ^x. But then since $(s_i(A)) \in \mu$ and $(c_i) = s_i(AB)$, we have

$$(c_i) = s_i(A) \cdot (s_i(AB)/s_i(A)) \in \mu \cdot \mu^x,$$

and the theorem is proved.

3. As we have remarked, the result proved in Section 2 shows that the original problem concerning factorization of nuclear operators on $\mathscr{H}$ is precisely equivalent to the problem of which symmetric solid Banach sequence spaces μ have the property that $\mu \cdot \mu^x = l^1$. Conceivably, this property may hold for every such sequence space, but it appears to be unknown. Thus we state the following.

PROBLEM. Characterize those separable symmetric solid Banach sequence spaces μ for which $\mu \cdot \mu^x = l^1$. In particular, is this the case for all such μ?

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