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The asymptotic behaviour of maxima of complete and incomplete samples from stationary sequences[☆,☆☆]

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Abstract

Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence of random variables and $M_n = \max\{X_1, X_2, \ldots, X_n\}$. Assume that some random variables $X_1, X_2, \ldots$ can be observed and the sequence of random variables $\boldsymbol{\varepsilon} = \{\varepsilon_n, n \geq 1\}$ indicate which $X_1, X_2, \ldots$ are observed, thus $M_n(\boldsymbol{\varepsilon}) = \max\{X_j : \varepsilon_j = 1, 1 \leq j \leq n\}$. In paper (Mladenovič and Piterbarg, 2006 [3]), the limiting behaviour $(M_n, M_n(\boldsymbol{\varepsilon}))$ is investigated under the condition

$$\frac{\sum_{j=1}^{n} \varepsilon_j}{n} \xrightarrow{P} p, \quad \text{as } n \to \infty,$$

for some real $p \in (0, 1)$. We generalize these results on the case, when for some random variable λ

$$\frac{\sum_{j=1}^{n} \varepsilon_j}{n} \xrightarrow{P} \lambda, \quad \text{as } n \to \infty.$$

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1. Introduction

Let $\{X_n, n \geq 1\}$ be a strictly stationary random sequence with the marginal distribution function $F(.)$ which belongs to the domain of attraction of a nondegenerate distribution function G (for short $F \in D(G)$), i.e. there exist sequences $a_n > 0$ and $b_n \in \Re,\ n \in N$, such that

$$\lim_{n\to\infty} F^n(a_n x + b_n) = G(x), \tag{1}$$

holds for every continuity point of G. The set of possible distribution functions $G()$ as well as the constants a_n, b_n are described, e.g. in [5]. For two fixed continuity points of $G, x < y$, we assume Condition $D(u_n, v_n)$ (Definition 2.3 in [3]):

Condition $D(u_n, v_n)$**.** *For all* $A_1, A_2, B_1, B_2 \subset \{1, 2, 3, \ldots, n\}$*, such that*

$$\max_{b\in B_1\cup B_2, a\in A_1\cup A_2} |b - a| \geq l, \qquad A_1 \cap A_2 = \emptyset, \qquad B_1 \cap B_2 = \emptyset,$$

the following inequality holds:

$$\left| P\left[\bigcap_{j\in A_1\cup B_1} \{X_j \leq u_n\} \cap \bigcap_{j\in A_2\cup B_2} \{X_j \leq v_n\}\right] - P\left[\bigcap_{j\in A_1} \{X_j \leq u_n\} \cap \bigcap_{j\in A_2} \{X_j \leq v_n\}\right] P\left[\bigcap_{j\in B_1} \{X_j \leq u_n\} \cap \bigcap_{j\in B_2} \{X_j \leq v_n\}\right]\right| \leq \alpha_{n,l},$$

and $\alpha_{n,l_n} \to 0$ *as* $n \to \infty$ *for some* $l_n = o(n)$.

It is easy to check that the sequence $\{X_n, n \geq 1\}$ of independent identically distributed random variables satisfy Condition $D(u_n, v_n)$ with $\alpha_{n,l} = 0$ for all $n \geq l$.

Condition $D'(u_n)$**.** *Let* $\{X_n, n \geq 1\}$ *be strictly stationary sequence of random variables and let* $\{u_n, n \geq 1\}$ *be a sequence of real numbers. We say that* $\{X_n, n \geq 1\}$ *satisfy the Condition*$D'(u_n)$ *iff*

$$\lim_{k\to\infty} \limsup_{n\to\infty} n \sum_{j=2}^{[n/k]} P[X_1 > u_n, X_j > u_n] = 0.$$

Obviously, if $\{X_n, n \geq 1\}$ is the sequence of independent identically distributed random variables with $\lim_{n\to\infty} nP[X_1 > u_n] = c$ then Condition $D'(u_n)$ holds.

Let among the sequence $X_1, X_2, \ldots$ some variables are observed. Let the random variable ε_k is the indicator of event that random variable X_k is observed. In paper [3] in Theorem 3.2 it was assumed that $\{\varepsilon_n, n \geq 1\}$ are dependent but independent of $\{X_n, n \geq 1\}$. This result is

Theorem 1. *Let:*

(a) $F \in D(G)$ *for some real constants* $a_n > 0, b_n$ *and every real* x *(i.e. (1) holds).*
(b) $\{X_n, n \geq 1\}$ *is strictly stationary random sequence, such that Conditions* $D(u_n, v_n)$ *and* $D'(u_n)$ *are satisfied for* $u_n = a_n x + b_n$ *and* $v_n = a_n y + b_n$*, where* $x < y$*.*
(c) $\boldsymbol{\varepsilon} = \{\varepsilon_n, n \geq 1\}$ *is the sequence of indicators such that*

$$\frac{S_n}{n} \xrightarrow{P} p, \quad \text{as } n \to \infty. \tag{2}$$

Then, the following equality holds for all real $x < y$*:*

$$\lim_{n\to\infty} P[M_n(\boldsymbol{\varepsilon}) \leq a_n x + b_n, M_n \leq a_n y + b_n] = G^p(x) G^{1-p}(y). \tag{3}$$

The general aim of this paper is to generalize Theorem 1 replacing condition (2) with

$$\frac{S_n}{n} \xrightarrow{P} \lambda, \quad \text{as } n \to \infty, \tag{4}$$

for some random variable λ. As a corollary we obtain Theorems 3.2 and 3.1 [3].

2. Main result

Let $\{X_n,\ n \geq 1\}$ be a strictly stationary sequence of random variables and F be a distribution function such that $F(x) = P[X_1 \leq x]$. Let $\boldsymbol{\varepsilon} = \{\varepsilon_n, n \geq 1\}$ be a sequence of indicator of events that random variable X_n is observed, respectively, and let

$$S_n = \sum_{i=1}^{n} \varepsilon_i.$$

Let $\boldsymbol{\alpha} = \{\alpha_n, n \geq 1\}$ be a sequence of 0 and 1 ($\boldsymbol{\alpha} \in \{0, 1\}^N$) and $\boldsymbol{\vartheta} = \{1\}^N$ be an infinite sequence of 1. For the arbitrary random or nonrandom sequence $\boldsymbol{\beta} = \{\beta_n, n \geq 1\}$ of 0 and 1 and subset $I \subset N$, let us put

$$M(I, \boldsymbol{\beta}) = \begin{cases} \max\{X_j : j \in I, \beta_j = 1\}, & \text{if } \max_{j\in I} \beta_j > 0, \\ \inf\{t : F(t) > 0\}, & \text{otherwise,} \end{cases}$$
$$M_n(\boldsymbol{\beta}) = M(\{1, 2, 3, \ldots, n\}, \boldsymbol{\beta}),$$
$$M(I) = M(I, \boldsymbol{\vartheta}) = \max\{X_j : j \in I\},$$
$$M_n = M(\{1, 2, 3, \ldots, n\}, \boldsymbol{\vartheta}) = M(\{1, 2, 3, \ldots, n\}) = \max_{1\leq j\leq n} X_j,$$
$$K_s = \{j : (s-1)m + 1 \leq j \leq sm\}, \quad 1 \leq s \leq k,$$
$$A_{sj} = \{X_{(s-1)m+j} > u_n\}, \quad 1 \leq j \leq k.$$

By $I_1, I_2, \ldots, I_k$ we will denote such subsets of $\{1, 2, 3, \ldots, n\}$ that $\min I_t - \max I_s \geq l$, for $k \geq t > s \geq 1$. For random variable λ such that $0 \leq \lambda \leq 1$ a.s., we put

$$B_{r,l} = \left\{ \omega : \lambda(\omega) \in \begin{cases} \left[0, \frac{1}{2^l}\right], & r = 0, \\ \left(\frac{r}{2^l}, \frac{r+1}{2^l}\right], & 0 < r \leq 2^l - 1 \end{cases} \right\},$$
$$B_{r,l,\boldsymbol{\alpha},n} = \{\omega : \varepsilon_j(\omega) = \alpha_j, 1 \leq j \leq n\} \cap B_{r,l}.$$

Theorem 2. *Let us suppose that the following conditions are satisfied:*

(a) $F \in D(G)$, *for some real constants* $a_n > 0$, b_n *and every real* x,
(b) $\{X_n, n \geq 1\}$ *is a strictly stationary random sequence satisfying* Conditions $D(u_n, v_n)$ *and* $D'(u_n)$ *for* $u_n = a_n x + b_n$ *and* $v_n = a_n y + b_n$, *where* $x < y$,
(c) $\boldsymbol{\varepsilon} = \{\varepsilon_n, n \geq 1\}$ *is a sequence of indicators that is independent of* $\{X_n, n \geq 1\}$ *and such that* (4) *holds for some random variable* λ.

Then the following equality holds for all real x *and* y, $x < y$:

$$\lim_{n\to\infty} P[M_n(\boldsymbol{\varepsilon}) \leq a_n x + b_n, M_n \leq a_n y + b_n] = E[G^{\lambda}(x) G^{1-\lambda}(y)].$$

Thus this result generalize that one in [3] where constant limit p of $\frac{S_n}{n}$ was considered instead of the random variable λ. Moreover as the corollary we may obtain:

Corollary 1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of i.i.d. random variables such that:*

(a) $F \in D(G)$, *(i.e.* (1) *holds),*
(b) $\boldsymbol{\varepsilon} = \{\varepsilon_n, n \geq 1\}$ *is a sequence of indicators that is independent of* $\{X_n, n \geq 1\}$ *and such that*

$$\frac{S_n}{n} \xrightarrow{P} \lambda, \quad \text{as } n \to \infty,$$

for some random variable λ.

Then, the following equality holds for all real $x < y$:

$$\lim_{n\to\infty} P[M_n(\boldsymbol{\varepsilon}) \leq a_n x + b_n, M_n \leq a_n y + b_n] = E[G^{\lambda}(x) G^{1-\lambda}(y)].$$

3. Proofs

Lemma 1. *For any sequence* $\boldsymbol{\alpha} = \{\alpha_j,\ j \in N\}$, *under conditions of* Theorem 2 *we have*

$$\left| P\left[\bigcap_{s=1}^{k} M(I_s, \boldsymbol{\alpha}) \leq u_n, M(I_s) \leq v_n\right] - \prod_{s=1}^{k} P[M(I_s, \boldsymbol{\alpha}) \leq u_n, M(I_s) \leq v_n] \right| \leq (k-1)\alpha_{n,l},$$

where $\alpha_{n,l_n} \to 0$ *as* $n \to \infty$ *for some* $l_n = o(n)$.

Proof of Lemma 1. For $k = 2$ we have Condition $D(u_n, v_n)$ where $A_1 = \{j \in I_1 : \alpha_j = 1\}$, $A_2 = I_1 \setminus A_1$, $B_1 = \{j \in I_2 : \alpha_j = 1\}$, $B_2 = I_2 \setminus A_2$. And the proof follows from induction similarly as the proof of Lemma 4.2 in [3]. □

Lemma 2. *With assumptions of* Lemma 1 *we have*

$$\left| P[M_n(\boldsymbol{\alpha}) \leq u_n, M_n \leq v_n] - \prod_{s=1}^{k} P[M(K_s, \boldsymbol{\alpha}) \leq u_n, M(K_s) \leq v_n] \right|$$
$$\leq \left[(k-1)\alpha_{n,l} + (4k+3)\frac{l}{n} \cdot n(1 - F(u_n))\right].$$

Proof of Lemma 2. The proof is similar to the proof of Lemma 4.3 in [3] but we use Lemma 1 instead of Lemma 4.2 [3]. □

Let $d(X, Y)$ stands for Ky Fan metric, $d(X, Y) = \inf\{\varepsilon : P[|X - Y| > \varepsilon] < \varepsilon\}$.

Lemma 3. (a) *For arbitrary positive integers s, m, we have*

$$d\left(\frac{S_{ms} - S_{m(s-1)}}{m}, \lambda\right) \le (2s-1)\left[d\left(\frac{S_{ms}}{ms}, \lambda\right) + d\left(\frac{S_{m(s-1)}}{m(s-1)}, \lambda\right)\right].$$

(b) *If $\{X_n, n \ge 1\}$ and $\{Y_n, n \ge 1\}$ are such that $|X_n - Y_n| < 1$ a.s. then*

$$E|X_n - Y_n| \le 2d(X_n, Y_n).$$

Proof of Lemma 3.

(a) Because

$$\frac{S_{ms} - S_{m(s-1)}}{m} - \lambda = s\left(\frac{S_{ms}}{ms} - \lambda\right) - (s-1)\left(\frac{S_{m(s-1)}}{m(s-1)} - \lambda\right), \tag{5}$$

and for every random variables X and Y and arbitrary α we have

$$P[|X + Y| > \varepsilon] \le P[|X| + |Y| > \varepsilon] \le P[|X| > \alpha\varepsilon] + P[|Y| > (1-\alpha)\varepsilon], \tag{6}$$

thus using (5) and putting in (6) $X = s\left(\frac{S_{ms}}{ms} - \lambda\right), Y = -(s-1)\left(\frac{S_{m(s-1)}}{m(s-1)} - \lambda\right)$, and $\alpha = \frac{s}{2s-1}$, we get

$$P\left[\left|\frac{S_{ms} - S_{m(s-1)}}{m} - \lambda\right| > \varepsilon\right] \le P\left[\left|\frac{S_{ms}}{ms} - \lambda\right| > \frac{\varepsilon}{2s-1}\right] + P\left[\left|\frac{S_{m(s-1)}}{m(s-1)} - \lambda\right| > \frac{\varepsilon}{2s-1}\right],$$

what gives (a).

(b) We have

$$\begin{aligned} E|X_n - Y_n| &= E|X_n - Y_n| I[|X_n - Y_n| > d(X_n, Y_n)] \\ &\quad + E|X_n - Y_n| I[|X_n - Y_n| \le d(X_n, Y_n)] \\ &\le P[|X_n - Y_n| > d(X_n, Y_n)] + d(X_n, Y_n) \\ &\le 2d(X_n, Y_n), \end{aligned}$$

as $d(X_n - Y_n, 0) = d(X_n, Y_n)$. □

Proof of Theorem 2. Proceedings as in the proof of Theorem 3.2 [3] with the sequence of random variables $\{\varepsilon_n, n \ge 1\}$ replaced by the nonrandom sequence of $\{\alpha_n, n \ge 1\} \in \{0, 1\}^N$

for any $0 \le r \le 2^k - 1$, we have

$$\begin{aligned}
&\Big(\sum_{j\in K_s}\alpha_j\Big)\big(F(u_n)-F(v_n)\big)+\big(1-m(1-F(v_n))\big)\\
&= \left[1-\frac{mr}{2^k}(1-F(u_n))-m\left(1-\frac{r}{2^k}\right)(1-F(v_n))\right]\\
&\quad+\left[\frac{\sum_{j\in K_s}\alpha_j}{m}-\frac{r}{2^k}\right]m\big(F(u_n)-F(v_n)\big)\\
&\le P[M(K_s,\boldsymbol{\alpha})\le u_n, M(K_s)\le v_n]\\
&\le \left[1-\frac{mr}{2^k}(1-F(u_n))-m\left(1-\frac{r}{2^k}\right)(1-F(v_n))\right]\\
&\quad+m\sum_{j=2}^{m}P[A_{s1},A_{sj}]+\left[\frac{\sum_{j\in K_s}\alpha_j}{m}-\frac{r}{2^k}\right]m(F(u_n)-F(v_n)), \qquad (7)
\end{aligned}$$

where $A_{ij}=\{X_{(i-1)m+j}>u_n\}$, $j\in\{1,2,\ldots,m\}$ and $m=\left[\frac{n}{k}\right]$ for any fixed positive integer k. From the previous inequalities, Lemma 2, inequality

$$\left|\prod_{s=1}^{k}a_s-\prod_{s=1}^{k}b_s\right|\le\sum_{s=1}^{k}|a_s-b_s|, \qquad (8)$$

valid for all $a_s, b_s \in [0,1]$ as $0\le 1-m\left[\frac{r}{2^k}(1-F(u_n))+\left(1-\frac{r}{2^k}\right)(1-F(v_n))\right]\le 1$, and since $\{X_n, n\ge 1\}$ is strictly stationary, we have

$$\begin{aligned}
&\left|P\big[M_n(\boldsymbol{\alpha})\le u_n, M_n\le v_n\big]-\prod_{s=1}^{k}\left[1-\frac{\frac{r}{2^k}n(1-F(u_n))+\left(1-\frac{r}{2^k}\right)n(1-F(v_n))}{k}\right]\right|\\
&\le\left|P\big[M_n(\boldsymbol{\alpha})\le u_n, M_n\le v_n\big]-\prod_{s=1}^{k}P\big[M(K_s,\boldsymbol{\alpha})\le u_n, M(K_s)\le v_n\big]\right|\\
&\quad+\left|\prod_{s=1}^{k}P\big[M(K_s,\boldsymbol{\alpha})\le u_n, M(K_s)\le v_n\big]\right.\\
&\quad\left.-\prod_{s=1}^{k}\left[1-\frac{\frac{r}{2^k}n(1-F(u_n))+\left(1-\frac{r}{2^k}\right)n(1-F(v_n))}{k}\right]\right|\\
&=J_1+J_2,\quad\text{say.} \qquad (9)
\end{aligned}$$

From Lemma 2 and (7), (8) we have

$$J_1\le(k-1)\alpha_{n,l}+(4k+3)\frac{l}{n}\cdot n(1-F(u_n)), \qquad (10)$$

$$
\begin{aligned}
J_2 \leq & \sum_{s=1}^{k} \Bigg| P\big[M(K_s,\boldsymbol{\alpha}) \leq u_n, M(K_s) \leq v_n\big] \\
& - \left[1 - \frac{\frac{r}{2^k} n(1-F(u_n)) + \left(1-\frac{r}{2^k}\right) n(1-F(v_n))}{k}\right]\Bigg| \\
\leq & \; n \sum_{j=2}^{m} P[A_{1j}, A_{11}] + \sum_{s=1}^{k} \frac{\left|\sum_{i\in I_s} \frac{\alpha_i}{m} - \frac{r}{2^k}\right|}{k} n(F(u_n) - F(v_n)). \qquad (11)
\end{aligned}
$$

Furthermore, again from (8),

$$
\begin{aligned}
& E \sum_{\boldsymbol{\alpha}\in\{0,1\}^n} \Bigg| \prod_{s=1}^{k} \left[1 - \frac{\frac{r}{2^k} n(1-F(u_n)) + \left(1-\frac{r}{2^k}\right) n(1-F(v_n))}{k}\right] \\
& \quad - \prod_{s=1}^{k} \left[1 - \frac{\lambda n(1-F(u_n)) + (1-\lambda) n(1-F(v_n))}{k}\right]\Bigg| I\big[B_{r,k,\boldsymbol{\alpha},n}\big] \\
& \leq \sum_{s=1}^{k} E \left|\frac{r}{2^k} - \lambda\right| I\big[B_{r,k}\big] \frac{n(2-F(u_n)-F(v_n))}{k} \\
& \leq \frac{n(2-F(u_n)-F(v_n))}{2^k} P[B_{r,k}]. \qquad (12)
\end{aligned}
$$

From independency $\{X_n, n \geq 1\}$ and $\{\varepsilon_n, n \geq 1\}$, λ, we get

$$
\sum_{\boldsymbol{\alpha}\in\{0,1\}^n} E P[M_n(\boldsymbol{\alpha}) \leq u_n, M_n \leq v_n] I[B_{r,k,\boldsymbol{\alpha},n}] = P[M_n(\boldsymbol{\varepsilon}) \leq u_n, M_n \leq v_n, B_{r,k}]. \qquad (13)
$$

Now, taking into account (9)–(13), we get

$$
\begin{aligned}
J_{r,k} = & \sum_{\boldsymbol{\alpha}\in\{0,1\}^n} E \Bigg| P\big[M_n(\boldsymbol{\varepsilon}) \leq u_n, M_n \leq v_n\big] \\
& - \prod_{s=1}^{k} \left[1 - \frac{\lambda n(1-F(u_n)) + (1-\lambda) n(1-F(v_n))}{k}\right]\Bigg| I\big[B_{r,k,\boldsymbol{\alpha},n}\big] \\
\leq & \left((k-1)\alpha_{n,l} + (4k+3)\frac{l}{n} \cdot n(1-F(u_n))\right) P[B_{r,k}] + n \sum_{j=2}^{m} P[A_{1j}, A_{11}, B_{r,k}] \\
& + E \sum_{s=1}^{k} \frac{\left|\sum_{i\in I_s} \frac{\varepsilon_i}{m} - \frac{r}{2^k}\right|}{k} n(F(u_n) - F(v_n)) I[B_{r,k}] \\
& + \frac{n(2-F(u_n)-F(v_n))}{2^k} P[B_{r,k}]. \qquad (14)
\end{aligned}
$$

Now, we evaluate the third term on the right-hand side of (14). From triangle inequality and Lemma 3 we have

$$\sum_{r=0}^{2^k-1} E\left|\sum_{i\in I_s}\frac{\varepsilon_i}{m}-\frac{r}{2^k}\right| I[B_{r,k}] \le E\left|\sum_{i\in I_s}\frac{\varepsilon_i}{m}-\lambda\right| + \sum_{r=0}^{2^k-1} E\left|\lambda-\frac{r}{2^k}\right| I[B_{r,k}]$$
$$\le E\left|\frac{S_{ms}-S_{m(s-1)}}{m}-\lambda\right| + \frac{1}{2^k}$$
$$\le 2d\left(\frac{S_{ms}-S_{m(s-1)}}{m},\lambda\right)+\frac{1}{2^k}$$
$$\le 2\big(2s-1\big)\left(d\left(\frac{S_{ms}}{ms},\lambda\right)+d\left(\frac{S_{m(s-1)}}{m(s-1)},\lambda\right)\right)+\frac{1}{2^k} \quad (15)$$

thus taking a sum $\sum_{r=0}^{2^k-1}$ of the left- and right-hand side of (14) we get

$$\sum_{r=0}^{2^k-1} J_{r,k} \le \left[(k-1)\alpha_{n,l}+(4k+3)\frac{l}{n}\cdot n(1-F(u_n))\right]+n\sum_{j=2}^{m}P[A_{1j},A_{11}]$$
$$+\left[2\big(2s-1\big)\left(d\left(\frac{S_{ms}}{ms},\lambda\right)+d\left(\frac{S_{m(s-1)}}{m(s-1)},\lambda\right)\right)+\frac{1}{2^k}\right]\frac{n(F(u_n)-F(v_n))}{k}$$
$$+\frac{n(2-F(u_n)-F(v_n))}{2^k}. \quad (16)$$

Taking a limit $n \to \infty$ and then $m \to \infty$ from Lemma 4.1 in [3] by similar computations as those in the proof of Theorem 3.2 in [3] and because $\lim_{m\to\infty} d\left(\frac{S_{ms}}{ms},\lambda\right)=0$ we have

$$\left|\lim_{n\to\infty} P\big[M_n(\boldsymbol{\varepsilon})\le u_n, M_n\le v_n\big]-E\left[1-\frac{-\ln G^{\lambda}(x)-\ln G^{1-\lambda}(y)}{k}\right]^k\right|$$
$$\le ko\left(\frac{1}{k}\right)+\frac{-\ln G(y)}{2^{k-1}}.$$

Now if we take $\lim_{k\to\infty}$ of the both sides we have

$$\left|\lim_{n\to\infty} P\big[M_n(\boldsymbol{\varepsilon})\le u_n, M_n\le v_n\big]-\lim_{k\to\infty} E\left(1+\frac{\ln G^{\lambda}(x)+\ln G^{1-\lambda}(y)}{k}\right)^k\right|=0,$$

so

$$\lim_{n\to\infty} P[M_n(\boldsymbol{\varepsilon})\le u_n, M_n\le v_n]=E[G^{\lambda}(x)G^{1-\lambda}(y)]. \quad \square$$

4. Examples and applications

Example 1. Let $\lambda \in [0,1]$, a.s., be an arbitrary random variable and let us define

$$\varepsilon_1 = 1 \quad \text{a.s.,}$$
$$\varepsilon_n = \begin{cases} 0, & \text{for } \lambda \in \bigcup_{r=1}^{n-1}\left(\frac{r-1}{n-1},\frac{r}{n}\right]\cup\{0\}, \\ 1, & \text{for } \lambda \in \bigcup_{r=1}^{n-1}\left(\frac{r}{n},\frac{r}{n-1}\right], \end{cases} \quad (17)$$

then

$$\frac{S_n}{n} \xrightarrow{P} \lambda, \quad \text{as } n \to \infty. \tag{18}$$

Let $\{X_n, n \geq 1\}$ be the family of stationary Farlie–Gumbel–Morgenstern sequence (cf. [4,2]), independent of λ with the law:

$$P\big[X_i < x, X_{i+j} < y\big] = F\big(x\big)F\big(y\big)\big(1 + \mu_j(1 - F(x))(1 - F(y))\big), \quad x, y \in \Re. \tag{19}$$

Then for arbitrary sequence $\boldsymbol{\mu} = \{\mu_n, n \geq 1\}$ and nondegenerate distribution function F such that $F \in D(G)$ the condition $D'(u)$ holds. If additionally $\{X_n, n \geq 1\}$ is the α-mixing sequence with $\alpha_n \to 0$, as $n \to \infty$ (it means, that $\mu_n \to 0$, as $n \to \infty$), then Condition $D(u_n, v_n)$ holds too. In this cases we have

$$\lim_{n\to\infty} P[M_n(\boldsymbol{\varepsilon}) \leq u_n, M_n \leq v_n] = E[G^{\lambda}(x)G^{1-\lambda}(y)].$$

For example, if λ is uniformly distributed on [0, 1] independent on α-mixing stationary family Farlie–Gumbel–Morgenstern laws with $F(x) = \frac{1}{2} + \frac{1}{\pi}\text{arctg}(x)$ (the Cauchy's law) then

$$\lim_{n\to\infty} P[M_n(\boldsymbol{\varepsilon}) \leq u_n, M_n \leq v_n] = \frac{xy}{x-y}\left(\mathrm{e}^{-\frac{1}{x}} - \mathrm{e}^{-\frac{1}{y}}\right).$$

Proof of Example 1. At first, we prove that

$$S_n = \sum_{k=1}^{n} \varepsilon_k = \begin{cases} 1, & \lambda \in \left[0, \frac{1}{n}\right] \\ 2, & \lambda \in \left(\frac{1}{n}, \frac{2}{n}\right], \\ \vdots & \\ n, & \lambda \in \left(\frac{n-1}{n}, 1\right], \end{cases} \qquad n \geq 1, \tag{20}$$

really, for $n = 1$ we have $S_1 = 1 = \varepsilon_1$. Assuming, (20) for some n we get

$$\begin{aligned}\big[S_{n+1} = k\big] &= \begin{cases} \big[S_n = 1, \varepsilon_{n+1} = 0\big], & \text{if } k = 1, \\ \big[S_n = k, \varepsilon_{n+1} = 0\big] \cup \big[S_n = k-1, \varepsilon_{n+1} = 1\big], & \text{if } 1 < k \leq n, \\ \big[S_n = n, \varepsilon_{n+1} = 1\big], & \text{if } k = n+1, \end{cases} \\ &= \begin{cases} \left[\lambda \in \left[0, \frac{1}{n}\right] \cap \left[0, \frac{1}{n+1}\right]\right], & \text{if } k = 1 \\ \left[\lambda \in \left(\frac{k-1}{n}, \frac{k}{n}\right] \cap \left(\frac{k-1}{n}, \frac{k}{n+1}\right] \right. & \\ \quad \left. \cup \left(\frac{k-2}{n}, \frac{k-1}{n}\right] \cap \left(\frac{k-1}{n+1}, \frac{k-1}{n}\right]\right], & \text{if } 1 < k \leq n, \\ \left[\lambda \in \left(\frac{n-1}{n}, 1\right] \cap \left(\frac{n}{n+1}, 1\right]\right], & \text{if } k = n+1, \end{cases}\end{aligned}$$

$$= \begin{cases} \left[\lambda \in \left[0, \dfrac{1}{n+1}\right]\right], & \text{if } k = 1, \\ \left[\lambda \in \left(\dfrac{k-1}{n+1}, \dfrac{k}{n+1}\right]\right], & \text{if } 1 < k \le n, \\ \left[\lambda \in \left(\dfrac{n}{n+1}, 1\right]\right], & \text{if } k = n+1, \end{cases}$$

what leads to

$$\left|\frac{S_n}{n} - \lambda\right| \le \frac{1}{n}, \quad \text{a.s.}, n \ge 1,$$

thus (18) holds.

If $F \in D(G)$ then we show that Condition $D'(u_n)$ holds. Really, since F is nondegenerate, then there exists the real x_o, such that $0 < F(x_o) < 1$, then from (19)

$$0 \le P\big[X_i < x_o, X_{i+j} < x_o\big] = F^2\big(x_o\big)\big(1 + \mu_j (1 - F(x_o))^2\big) \le 1, \tag{21}$$

leads to

$$-\frac{1}{(1 - F(x_o))^2} \le \mu_j \le \frac{1 + F(x_o)}{F^2(x_o)(1 - F(x_o))},$$

which implies that $\sup |\mu_j| \le C$ for some absolute constants C. From

$$\begin{aligned} P[X_1 > x, X_j > y] &= 1 - P[X_1 < x, X_j > y] - P[X_1 > x, X_j < y] \\ &\quad - P[X_1 < x, X_j < y] \\ &= 1 - P[X_1 < x] - P[X_j < y] + P[X_1 < x, X_j < y], \end{aligned}$$

we get

$$P\big[X_1 > x, X_j > y\big] = \big(1 - F(x)\big)\big(1 - F(y)\big)\big(1 + \mu_j F(x)F(y)\big), \tag{22}$$

such that

$$\left| n \sum_{j=2}^{[n/k]} P\big[X_1 > u_n, X_j > u_n\big]\right| \le \frac{1 + C}{k}\big[n(1 - F(u_n))\big]^2,$$

and because $n(1 - F(u_n)) \to G(x)$ thus $D'(u_n)$ holds.

For the last fact of example, by Theorem 2 we have from l'Hospital theorem

$$\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = \lim_{t\to\infty} \frac{(1 + t^2)x}{1 + (tx)^2} = x^{-1},$$

thus from Theorem 2.1.1 [1] we have

$$\lim_{n\to\infty} P\left[\max\{X_1, X_2, \ldots, X_n\} < tg\left(\frac{\pi}{2} - \frac{\pi}{n}\right)x\right] = \exp\left(-\frac{1}{x}\right), \quad x > 0$$

therefore for $y > x > 0$ we have

$$\lim_{n\to\infty} P[M_n(\boldsymbol{\varepsilon}) \le u_n, M_n \le v_n] = E\mathrm{e}^{-\frac{\lambda}{x}-\frac{1-\lambda}{y}}$$
$$= \mathrm{e}^{-\frac{1}{y}} \int_0^1 \mathrm{e}^{t\left(\frac{1}{y}-\frac{1}{x}\right)} \mathrm{d}t$$
$$= \frac{xy}{x-y}\left(\mathrm{e}^{-\frac{1}{x}} - \mathrm{e}^{-\frac{1}{y}}\right). \quad \square$$

Example 2. Let $\{\xi_n, n \ge 1\}$ be a sequence of independent identically distributed random variables with the distribution function H and let $g(x_1, x_2, \ldots, x_m)$ be some measurable function. We put $X_n = g(\xi_n, \xi_{n+1}, \ldots, \xi_{n+m-1}), n \ge 1$, thus $\{X_n, n \ge 1\}$ is the sequence of m-dependent random variables, thus Condition $D(u_n, v_n)$ is satisfied. For some choice of $\{\xi_n, n \ge 1\}$ and function $g()$ the Condition $D'(u_n)$ may be fulfilled too. For example, if $g(x_1, x_2, \ldots, x_m) = \min\{x_1, x_2, \ldots, x_m\}$ and ξ_n are uniformly distributed on [0, 1] then

$$H(x) = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } x \in [0,1], \\ 1, & \text{if } x > 1, \end{cases}$$

$$F(x) = 1 - (1-H(x))^m = \begin{cases} 0, & \text{if } x < 0, \\ 1-(1-x)^m, & \text{if } x \in [0,1], \\ 1, & \text{if } x > 1. \end{cases}$$

Furthermore, for such defined sequence, we have that $F \in D(H_{2,m})$ where

$$H_{2,m} = \begin{cases} 1, & \text{if } x \ge 0, \\ \exp\bigl(-(-x)^m\bigr), & \text{if } x < 0, \end{cases} \tag{23}$$

with the centring and normalizing constants $a_n = 1, b_n = \frac{1}{\sqrt[m]{n}}$, i.e. for every $x \in R$,

$$P\left[\max\{X_1, X_2, \ldots, X_n\} < 1 + \frac{x}{\sqrt[m]{n}}\right] \longrightarrow H_{2,m}(x), \quad \text{as } n \to \infty.$$

Therefore, the appropriate sequence of $\{u_n, n \ge 1\}$ should be defined by $\left\{u_n = 1 + \frac{x}{\sqrt[m]{n}}, n \ge 1\right\}$. Furthermore Conditions $D'(u_n)$ and $D(u_n, v_n)$ hold. Hence if λ has the law with the density function $\alpha x^{\alpha-1}$ ($\{\varepsilon_n, n \ge 1\}$ is constructed as in Example 1) independent of $\{\xi_n, n \ge 1\}$ then

$$\lim_{n\to\infty} P[M_n(\boldsymbol{\varepsilon}) \le u_n, M_n \le v_n] = \alpha \mathrm{e}^{-(-y)^m} \int_0^1 t^{\alpha-1} \mathrm{e}^{t((-y)^m-(-x)^m)} \mathrm{d}t. \tag{24}$$

Proof of Example 2. We put $\omega(F) = 1$, $F^\star(x) = F\left(1 - \frac{1}{x}\right) = 1 - \frac{1}{x^m}$, (cf. [1], Section 2.1) and remark

$$\lim_{t\to\infty} \frac{1-F^\star(tx)}{1-F^\star(t)} = x^{-m},$$

such that from Theorem 2.1.2 [1] we have that $F \in D(H_{2,m})$. Now we check Condition $D'(u_n)$. For $1 < j \le m$ we have

$$
\begin{aligned}
P[X_1 > u_n, X_j > u_n] &= P[\min\{\xi_1, \xi_2, \ldots, \xi_{m+j-1}\} > u_n] \\
&= (1-u_n)^{m+j-1} \\
&= \left(\frac{x}{\sqrt[m]{n}}\right)^{m+j-1},
\end{aligned}
$$

thus

$$
\begin{aligned}
\lim_{k\to\infty} \limsup_{n\to\infty} n \sum_{j=2}^{[n/k]} P[X_1 > u_n, X_j > u_n] &= \limsup_{n\to\infty} n \sum_{j=2}^{m} \frac{x^{m+j-1}}{n^{(m+j-1)/m}} \\
&= \lim_{n\to\infty} \frac{x^{m+1}}{\sqrt[m]{n}} \cdot \frac{1-(\frac{x}{\sqrt[m]{n}})^{m-1}}{1-\frac{x}{\sqrt[m]{n}}} = 0.
\end{aligned}
$$

Equality (24) follows from our Theorem 2. □

We remark that in [3] there is an error in the proof of Lemma 4.2 [3].

Example 3. Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with the exponential law $F(x) = (1-\mathrm{e}^{-x})I[x > 0]$. Obviously, for every x we have $F^n(\ln(n)+x) \to \mathrm{e}^{-\mathrm{e}^{-x}}$, as $n \to \infty$. Furthermore let $\{\varepsilon_n, n \geq 1\}$ be copies of random variable ε_1 (i.e. for every $i, j, \varepsilon_i = \varepsilon_j$) with the law $P[\varepsilon_1 = 1] = P[\varepsilon_1 = 0] = \frac{1}{2}$, and independent of $\{X_n, n \geq 1\}$. If we put $u_n = \ln(n) + 1$, $v_n = \ln(n) + 2$, thus $D(u_n, v_n)$ holds with $\alpha_{n,l} = 0$ for every $n, l \in N$. But, if we put $k = 2$, $I_1 = \{1\}$, $I_2 = \{2\}$ then

$$
\begin{aligned}
P\left[\bigcap_{s=1}^{2}\{M(I_s, \boldsymbol{\varepsilon}) \leq u_n, M(I_s) \leq v_n\}\right] &= P[\max\{X_1, X_2\} \leq u_n < v_n]P[\varepsilon_1 = \varepsilon_2 = 1] \\
&\quad + P[\max\{X_1, X_2\} \leq v_n]P[\varepsilon_1 = \varepsilon_2 = 0] \\
&= \frac{1}{2}((1-\mathrm{e}^{-1}/n)^2 + (1-\mathrm{e}^{-2}/n)^2)
\end{aligned}
$$

and

$$
\begin{aligned}
&\prod_{s=1}^{2} P\big[\{M(I_s, \boldsymbol{\varepsilon}) \leq u_n, M(I_s) \leq v_n\}\big] \\
&\quad = (P[X_1 \leq u_n < v_n]P[\varepsilon_1 = 1] + P[X_1 \leq v_n]P[\varepsilon_1 = 0]) \\
&\qquad \times (P[X_2 \leq u_n < v_n]P[\varepsilon_2 = 1] + P[X_2 \leq v_n]P[\varepsilon_2 = 0]) \\
&\quad = \frac{1}{4}((1-\mathrm{e}^{-1}/n) + (1-\mathrm{e}^{-2}/n))^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left| P\left[\bigcap_{s=1}^{k}\{M(I_s, \boldsymbol{\varepsilon}) \leq u_n, M(I_s) \leq v_n\}\right] - \prod_{s=1}^{k} P[M(I_s, \boldsymbol{\varepsilon}) \leq u_n, M(I_s) \leq v_n] \right| \\
&\quad = \frac{1}{4}\big((1-\mathrm{e}^{-1}/n) - (1-\mathrm{e}^{-2}/n)\big)^2 = \frac{(\mathrm{e}-1)^2}{4\mathrm{e}^4 n^2} > \alpha_{n,1} = 0.
\end{aligned}
$$

However the random variables $\boldsymbol{\varepsilon} = \{\varepsilon_n, n \geq 1\}$ defined in Example 3 do not satisfy the Weak Law of Large Numbers (WLLN). In the next example we construct the random variables satisfying WLLN but such that for every choice of subsets $\{I_s, 1 \le s \le k\}$ the Lemma 4.2 [3] fails. We begin with

Lemma 4. *Let $\{A_k, k \ge 1\}$ and $\{B_k, k \ge 1\}$ be such sequences of positive numbers, that for every $k \ge 1$, $A_k < B_k$. Then for every integer $n > 1$,*

$$\frac{1}{2}\prod_{i=1}^{n} A_i + \frac{1}{2}\prod_{i=1}^{n} B_i > \prod_{i=1}^{n} \frac{A_i + B_i}{2}.$$

The easy inductive proof we omitted.

Example 4. Let $\{X_n, n \ge 1\}$ be a sequence of independent identically distributed random variables with the law F, and let $\{u_n, v_n, n \ge 1\}$ be two sequences of reals such that $u_n < v_n$ and $F(u_n) < F(v_n)$, for every $n > 1$ (this sequences and F may be defined as in Example 3). Let us define two sequences of random variables $\{\xi_n, n \ge 0\}$ and $\{\eta_n, n \ge 1\}$ interindependent and independent of $\{X_n, n \ge 1\}$, with the law

$$P[\xi_n = 1] = P[\xi_n = 0] = \frac{1}{2}, \quad n \ge 0, \quad \text{and} \quad P[\eta_n = 1] = 1 - \frac{1}{n},$$
$$P[\eta_n = 0] = \frac{1}{n}, \quad n \ge 1.$$

Put

$$\varepsilon_n(\omega) = \begin{cases} \xi_0(\omega), & \text{for } \omega \in [\eta_n = 0], \\ \xi_n(\omega), & \text{for } \omega \in [\eta_n = 1]. \end{cases}$$

Then

$$\frac{S_n}{n} = \frac{\sum_{i=1}^{n} \varepsilon_i}{n} \xrightarrow{P} \frac{1}{2}, \quad \text{as } n \to \infty, \tag{25}$$

and for every $I_1, I_2, \ldots, I_k$ pairwise disjoint subsets of $\{1, 2, \ldots, n\}$ we have

$$P\left[\bigcap_{s=1}^{k} M(I_s, \boldsymbol{\varepsilon}) \le u_n, M(I_s) \le v_n\right] - \prod_{s=1}^{k} P[M(I_s, \boldsymbol{\varepsilon}) \le u_n, M(I_s) \le v_n] > 0, \tag{26}$$

whereas for every $l \in N$, $\alpha_{n,l} = 0$.

Proof of Example 4. At first we compute the common law of $\{\varepsilon_n, n \ge 1\}$.

Lemma 5. *For $\{\varepsilon_n, n \ge 1\}$ defined as in Example 4 and every disjoint subsets of positive integers A and B such that $A \cup B \ne \emptyset$ we have*

$$P[\varepsilon_i = 0, i \in A; \varepsilon_j = 1, j \in B]$$
$$= \frac{1}{2^{\overline{\overline{A}}+\overline{\overline{B}}+1}}\left[\prod_{i \in A}\left(1 + \frac{1}{i}\right)\prod_{i \in B}\left(1 - \frac{1}{i}\right) + \prod_{i \in A}\left(1 - \frac{1}{i}\right)\prod_{i \in B}\left(1 + \frac{1}{i}\right)\right],$$

(where $\prod_{i \in \emptyset} a_i = 1$, $\overline{\overline{A}} = \mathrm{card}(A)$).

Proof of Lemma 5. The proof follows from

$$\begin{aligned}
&P[\varepsilon_i = 0, i \in A; \varepsilon_j = 1, j \in B; \xi_0 = 0]\\
&\quad = \sum_{K \subset A} P[\eta_i = 0, i \in K; \eta_j = 1, \xi_j = 0, j \in A \setminus K; \eta_p = 1, \xi_p = 1; p \in B; \xi_0 = 0]\\
&\quad = \sum_{K \subset A} \prod_{i \in K} \frac{1}{i} \prod_{i \in A \setminus K} \left(\frac{1}{2}\left(1 - \frac{1}{i}\right)\right) \prod_{i \in B} \left(\frac{1}{2}\left(1 - \frac{1}{i}\right)\right) \cdot \frac{1}{2}\\
&\quad = \frac{1}{2^{\overline{\overline{A}} + \overline{\overline{B}} + 1}} \prod_{i \in A} \left(1 + \frac{1}{i}\right) \prod_{i \in B} \left(1 - \frac{1}{i}\right),
\end{aligned}$$

and similarly

$$P[\varepsilon_i = 0, i \in A; \varepsilon_j = 1, j \in B; \xi_0 = 1] = \frac{1}{2^{\overline{\overline{A}} + \overline{\overline{B}} + 1}} \prod_{i \in A} \left(1 - \frac{1}{i}\right) \prod_{i \in B} \left(1 + \frac{1}{i}\right). \quad \square$$

Because

$$\operatorname{Cov}(\varepsilon_i, \varepsilon_j) = \operatorname{Var}(\xi_0) P[\eta_i = \eta_j = 0] = \frac{1}{4ij}, \quad i \neq j \geq 1,$$

and

$$\operatorname{Var}(\varepsilon_i) = \frac{1}{4}, \quad i \geq 1,$$

thus from Chebyshev's inequality we have, for every $\varepsilon > 0$,

$$P\left[\left|\sum_{i=1}^{n} (\varepsilon_i - E\varepsilon_i)\right| > n\varepsilon\right] \leq \frac{\operatorname{Var}\left(\sum_{i=1}^{n} \varepsilon_i\right)}{n^2 \varepsilon^2} = \frac{\sum_{i=1}^{n} \frac{1}{4} + 2 \sum_{1 \leq i < j \leq n} \frac{1}{4ij}}{n^2 \varepsilon^2} = O\left(\frac{1}{n}\right),$$

such that (25) holds.

Since for arbitrary $A \subset \{1, 2, \ldots, n\}$, from Lemma 5

$$\begin{aligned}
&P[M(A, \boldsymbol{\varepsilon}) \leq u_n, M(A) \leq v_n]\\
&\quad = \sum_{K \subset A} P\left[\max_{i \in K} X_i \leq u_n; \max_{i \in A \setminus K} X_i \leq v_n; \varepsilon_i = 0, i \in K; \varepsilon_j = 1, j \in A \setminus K\right]\\
&\quad = \sum_{K \subset A} \frac{F^{\overline{\overline{K}}}(u_n) F^{\overline{\overline{A \setminus K}}}(v_n)}{2^{\overline{\overline{K}} + \overline{\overline{A \setminus K}} + 1}} \left[\prod_{i \in K} \left(1 + \frac{1}{i}\right) \prod_{i \in A \setminus K} \left(1 - \frac{1}{i}\right)\right.\\
&\quad\quad \left. + \prod_{i \in K} \left(1 - \frac{1}{i}\right) \prod_{i \in A \setminus K} \left(1 + \frac{1}{i}\right)\right]\\
&\quad = \frac{1}{2} \sum_{K \subset A} \left[\prod_{i \in K} \frac{\left(1 + \frac{1}{i}\right) F(u_n)}{2} \prod_{i \in A \setminus K} \frac{\left(1 - \frac{1}{i}\right) F(v_n)}{2}\right.\\
&\quad\quad \left. + \prod_{i \in K} \frac{\left(1 - \frac{1}{i}\right) F(u_n)}{2} \prod_{i \in A \setminus K} \frac{\left(1 + \frac{1}{i}\right) F(v_n)}{2}\right]
\end{aligned}$$

$$= \frac{1}{2}\left[\prod_{i\in A}\frac{F(u_n)+F(v_n)-\frac{1}{i}(F(v_n)-F(u_n))}{2} + \prod_{i\in A}\frac{F(u_n)+F(v_n)+\frac{1}{i}(F(v_n)-F(u_n))}{2}\right],$$

thus putting for every $1 \le j \le k$,

$$A_j = \prod_{i\in I_j}\frac{F(u_n)+F(v_n)+\frac{1}{i}(F(v_n)-F(u_n))}{2},$$

and

$$B_j = \prod_{i\in I_j}\frac{F(u_n)+F(v_n)-\frac{1}{i}(F(v_n)-F(u_n))}{2},$$

the Lemma 4 ends the proof of (26). □

Because Lemma 4.2 [3] fails, thus the proof of Theorem 3.2 [3] is not correct, but this theorem follows from our Theorem 2 and allows true.

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