Pergamon

Computers Math. Applic. Vol. 35, No. 12, pp. 1–5, 1998

Printed in Great Britain
0898-1221/98 $19.00 + 0.00

PII: S0898-1221(98)00091-1

New Transformations of Cauchy Matrices and Trummer's Problem

V. Y. PAN
Graduate School and University Center, City University of New York
33 West 42nd Street, New York, NY 10036, U.S.A.
and
Department of Mathematics and Computer Science
Lehman College, City University of New York
Bronx, NY 10468, U.S.A.
vpan@alpha.lehman.cuny.edu

M. A. TABANJEH, Z. Q. CHEN, E. I. LANDOWNE AND A. SADIKOU
Graduate School and University Center, City University of New York
33 West 42nd Street, New York, NY 10036, U.S.A.

(Received December 1997; accepted January 1998)

Abstract—We show some new expressions for a Cauchy matrix, which enable us to simplify the solution of Trummer's problem, both in the general case and in the case where the input Cauchy matrix is fixed for the problem whereas the input vector varies.

Keywords—Cauchy matrix, Trummer's problem, Multipoint polynomial evaluation.

1. INTRODUCTION

The solution of Trummer's problem (that is, the problem of multiplication of an $n \times n$ Cauchy matrix C by a vector) is the basis for the solution of several important problems of scientific and engineering computing [1–10]. The straightforward algorithm solves Trummer's problem in $O(n^2)$ flops. The fast algorithm of [11] uses $O(n \log^2 n)$ flops but has poor numerical stability. Presently, the algorithm of choice in practical computations is the celebrated Multipoint Algorithm [10, 12–16, pp. 261–262], which belongs to the class of hierarchical methods [2,17,18]. The algorithm approximates the solution in $O(n \log n)$ flops in terms of n, but its cost estimate and even its ability to yield the desired approximation at all also depend on the bound on the approximation error and on the correlation between the entries of the pair of n-dimensional vectors defining the input matrix C.

The goal of the present paper is to enhance the power of the Multipoint Algorithm (as well as other solution algorithms for Trummer's problem) by showing some new expressions for a Cauchy matrix via other Cauchy matrices, which we may vary by changing one of their basis vectors. Under an appropriate choice of such a vector, the subsequent solution of Trummer's problem is simplified; in particular, the power of the Multipoint Algorithm can be enhanced.

Supported by NSF Grant CCR 9625344 and PSC CUNY Awards 667340 and 668365.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Technically, we achieve our goal by means of a simple transformation of the useful basic formulae of [19], and our resulting expressions for C give us further algorithmic opportunities. The underlying idea of the transformation of the basic vectors defining the problem is taken from [20], where this idea was used for multipoint polynomial evaluation and interpolation.

We use the following order of presentation. In the next section, we introduce the definitions, show how to avoid degeneration of Trummer's problem, and recall some basic formulae from [19]. In Section 3, we extend these formulae to yield the desired transformations of Cauchy matrices and Trummer's problem. In Section 4, we comment on the algorithmic aspects.

2. DEFINITIONS, BASIC EXPRESSIONS, AND TREATMENT OF DEGENERATION

DEFINITION 2.1. *For a pair of n-dimensional vectors* $\vec{a} = (a_i)_{i=0}^{n-1}$, $\vec{b} = (b_j)_{j=0}^{n-1}$, *let* $C(\vec{a}, \vec{b}) = (1/(a_i - b_j))_{i,j=0}^{n-1}$, $V(\vec{a}) = (a_i^j)_{i,j=0}^{n-1}$, $H(\vec{a}) = (h_{i,j})_{i,j=0}^{n-1}$, $h_{i,j} = a_{i+j}$ *for* $i + j \leq n - 1$, $h_{i,j} = 0$ *for* $i + j \geq n - 1$, *denote the associated* $n \times n$ *Cauchy, Vandermonde, and triangular Hankel matrices, respectively. For a vector* $\vec{a} = (a_i)_{i=0}^{n-1}$ *with* $a_i \neq a_j$ *for* $i \neq j$, *a Cauchy degenerate matrix* $C(\vec{a})$ *has the diagonal entries zeros and the* $(i,j)^{\text{th}}$ *entry* $1/(a_i - a_j)$ *for* $i \neq j$. W^{-1}, W^{T}, *and* $W^{-\mathsf{T}}$ *denote the inverse, the transpose, and the transpose of the inverse of a matrix* W, *respectively. Furthermore,* $p_{\vec{b}}(x)$ *and* $p'_{\vec{b}}(x)$ *denote the polynomial* $p_{\vec{b}}(x) = \prod_{j=0}^{n-1}(x - b_j)$ *and its derivative* $p'_{\vec{b}}(x) = \sum_{i=0}^{n-1} \prod_{j=0(j\neq i)}^{n-1}(x - b_j)$, *respectively. Finally,* $D(\vec{a}, \vec{b}) = \operatorname{diag}(p_{\vec{b}}(a_i))_{i=0}^{n-1} = \operatorname{diag}(\prod_{j=0}^{n-1}(a_i - b_j))_{i=0}^{n-1}$ *and* $D'(\vec{b}) = \operatorname{diag}(p'_{\vec{b}}(b_j))_{j=0}^{n-1} = \operatorname{diag}(\prod_{j=0(j\neq i)}^{n-1}(a_i - b_j))_{i=0}^{n-1}$ *denote a pair of* $n \times n$ *diagonal matrices, defined by the vectors* $\vec{a}$ *and* $\vec{b}$.

THEOREM 2.1. *(See [19].) Let* $c_i \neq d_j$, $i, j = 0, 1, \ldots, n-1$. *Then*

$$C\left(\vec{c}, \vec{d}\right) = D\left(\vec{c}, \vec{d}\right)^{-1} V(\vec{c})\, H\left(\vec{d}\right) V\left(\vec{d}\right)^{\mathsf{T}}, \tag{2.1}$$

$$C\left(\vec{c}, \vec{d}\right) = D\left(\vec{c}, \vec{d}\right)^{-1} V(\vec{c})\, V\left(\vec{d}\right)^{-1} D'\left(\vec{d}\right). \tag{2.2}$$

DEFINITION 2.2. *Trummer's problem is the problem of computing the vector* $C(\vec{a}, \vec{b})\vec{v}$ *for three given vectors* $\vec{a} = (a_i)_{i=0}^{n-1}$, $\vec{b} = (b_j)_{j=0}^{n-1}$, *and* $\vec{v} = (v_j)_{j=0}^{n-1}$, *where* $a_i \neq b_j$ *for all pairs* i, j. *Trummer's degenerate problem is the problem of computing the vector* $C(\vec{a})\vec{v}$ *for two given vectors* $\vec{a} = (a_i)_{i=0}^{n-1}$ *and* $\vec{v} = (v_j)_{j=0}^{n-1}$, *where* $a_i \neq a_j$ *for* $i \neq j$.

DEFINITION 2.3. $\omega_k = \exp(2\pi\sqrt{-1}/k)$ *is a primitive* k^{th} *root of 1,* $\omega_k^k = 1, \omega_k^l \neq 1$, *for* $l = 1, \ldots, k-1$.

LEMMA 2.1. $\sum_{l=0}^{k-1} \omega_k^{gl} = 0$ *for* $g = 1, \ldots, k-1$.

Approximate solution of Trummer's degenerate problem can be reduced to Trummer's problem due to the next simple result.

LEMMA 2.2. $C(\vec{c}) = 1/h \sum_{g=0}^{h-1} C(\vec{c}, \vec{c} + \epsilon\omega_h^g \vec{e}) + O(\epsilon^h)$ *as* $\epsilon \to 0$, *where* $\vec{e} = (1)_{j=0}^{n-1}$ *is the vector filled with the values one and where* ϵ *is a scalar parameter.*

PROOF. $\sum_{g=0}^{h-1} 1/(c_i - c_j - \epsilon\omega_h^g) = 1/(c_i - c_j) \sum_{l=0}^{\infty} \sum_{g=0}^{h-1} (\epsilon\omega_h^g / c_i - c_j)^l = h/(c_i - c_j)(1 + O(\epsilon^h))$, due to Lemma 2.1. ∎

3. NEW TRANSFORMATIONS OF A CAUCHY MATRIX AND OF TRUMMER'S PROBLEM

THEOREM 3.1. *For a triple of n-dimensional vectors* $\vec{b} = (b_i)_{i=0}^{n-1}$, $\vec{c} = (c_j)_{j=0}^{n-1}$, $\vec{d} = (d_k)_{k=0}^{n-1}$, *where* $b_i \neq c_j$, $c_j \neq d_k$, $d_k \neq b_i$ *for* $i, j, k = 0, \ldots, n-1$, *we have the following matrix equations.*

$$C\left(\vec{c},\vec{d}\right) = D\left(\vec{c},\vec{d}\right)^{-1} V(\vec{c})\, V\left(\vec{b}\right)^{-1} D\left(\vec{b},\vec{d}\right) C\left(\vec{b},\vec{d}\right), \tag{3.1}$$

$$C\left(\vec{c},\vec{d}\right) = D\left(\vec{c},\vec{d}\right)^{-1} D\left(\vec{c},\vec{b}\right) C\left(\vec{c},\vec{b}\right) D'\left(\vec{b}\right)^{-1} D\left(\vec{b},\vec{d}\right) C\left(\vec{b},\vec{d}\right), \tag{3.2}$$

$$C\left(\vec{c},\vec{d}\right) = C\left(\vec{c},\vec{b}\right) D\left(\vec{b},\vec{c}\right) V\left(\vec{b}\right)^{-\top} V\left(\vec{d}\right)^{\top} D\left(\vec{d},\vec{c}\right)^{-1}, \tag{3.3}$$

$$C\left(\vec{c},\vec{d}\right) = -C\left(\vec{c},\vec{b}\right) D\left(\vec{b},\vec{c}\right) D'\left(\vec{b}\right)^{-1} C\left(\vec{b},\vec{d}\right) D\left(\vec{d},\vec{b}\right) D\left(\vec{d},\vec{c}\right)^{-1}. \tag{3.4}$$

PROOF. From (2.1), we immediately deduce that $C(\vec{b},\vec{d})^{-1} = V(\vec{d})^{-\top}H(\vec{d})^{-1}V(\vec{b})^{-1}D(\vec{b},\vec{d})$. Substitute the latter matrix equation and the expression (2.1) for $C(\vec{c},\vec{d})$ into the trivial matrix identity $C(\vec{c},\vec{d}) = C(\vec{c},\vec{d})C(\vec{b},\vec{d})^{-1}C(\vec{b},\vec{d})$ and obtain (3.1). Extend (2.2) to a similar expression $C(\vec{c},\vec{b}) = D(\vec{c},\vec{b})^{-1}V(\vec{c})V(\vec{b})^{-1}D'(\vec{b})$ and deduce that $V(\vec{c})V(\vec{b})^{-1} = D(\vec{c},\vec{b})C(\vec{c},\vec{b})D'(\vec{b})^{-1}$. Substitute this expression into (3.1) and obtain (3.2). Observe that $C(\vec{c},\vec{d}) = -C(\vec{d},\vec{c})^{\top}$ and extend (2.2) to obtain that $-C(\vec{c},\vec{d}) = C(\vec{d},\vec{c})^{\top} = (D(\vec{d},\vec{c})^{-1}V(\vec{d})V(\vec{b})^{-1}D(\vec{b},\vec{c})C(\vec{b},\vec{c}))^{\top} = C(\vec{b},\vec{c})^{\top}D(\vec{b},\vec{c})V(\vec{b})^{-\top}V(\vec{d})^{\top}D(\vec{d},\vec{c})^{-1}$. Substitute $C(\vec{c},\vec{b}) = -C(\vec{b},\vec{c})^{\top}$ and obtain (3.3). Finally, extend (2.2) to obtain that $V(\vec{d})V(\vec{b})^{-1} = D(\vec{d},\vec{b})C(\vec{d},\vec{b})D'(\vec{b})^{-1}$ and consequently $V(\vec{b})^{-\top}V(\vec{d})^{\top} = D'(\vec{b})^{-1}C(\vec{d},\vec{b})^{\top}D(\vec{d},\vec{b})$. Substitute the latter matrix equation and the matrix equation $C(\vec{d},\vec{b})^{\top} = -C(\vec{b},\vec{d})$ into (3.3) and obtain (3.4). ∎

4. SOME ALGORITHMIC ASPECTS

The expressions (3.2) and (3.4) for $C(\vec{c},\vec{d})$ are Vandermonde-free and Hankel-free, but they enable us to transform the basis vectors $\vec{c}$ and $\vec{d}$ for the Cauchy matrix $C(\vec{c},\vec{d})$ into the two pairs of basis vectors $\vec{c}$, $\vec{b}$, and $\vec{b}$, $\vec{d}$ for any choice of the vector $\vec{b} = (b_j)$, $b_j \neq c_j$, $b_j \neq d_k$, $i,j,k = 0,\dots,n-1$. The associated Trummer's problem is reduced to

(a) the evaluation of the diagonal entries of the diagonal matrices $D'(\vec{b})^{-1}$, $D(\vec{f},\vec{g})$, and/or $D(\vec{f},\vec{g})^{-1}$, for $(\vec{f},\vec{g})$ denoting the pairs $(\vec{c},\vec{d})$, $(\vec{b},\vec{d})$, $(\vec{c},\vec{b})$, $(\vec{b},\vec{c})$, $(\vec{d},\vec{b})$, and/or $(\vec{d},\vec{c})$,
(b) recursive multiplication of these matrices and the Cauchy matrices $C(\vec{b},\vec{d})$ and $C(\vec{c},\vec{b})$ by vectors.

Let us next specify parts (a) and (b).

(a) The evaluation of the entries of the matrices $D(\vec{f},\vec{g})$ and $D(\vec{f},\vec{g})^{-1}$ for a given pair of vectors $(\vec{f},\vec{g})$ and of the matrix $D'(\vec{g})$ for a given vector $\vec{g}$ can be reduced to the computation of the coefficients of the polynomial $p_{\vec{g}}(x) = \prod_{j=0}^{n-1}(x-g_j)$ and the subsequent evaluation of $p_{\vec{g}}(x)$ at the points f_i, $i = 0,\dots,n-1$ (for $D(\vec{f},\vec{g})$) and of its derivative $p'_{\vec{g}}(x)$ at the points g_i, $i = 0,\dots,n-1$ (for $D'(\vec{g})$).

The coefficients of the polynomial $p_{\vec{g}}(x)$ can be obtained by the fan-in method, consisting of the pairwise multiplication of the linear factors $x - g_j$ followed by recursive pairwise multiplication of the computed products (cf. [16, p. 25]). The computation is numerically stable and uses $O(n\log^2 n)$ flops.

Multipoint polynomial evaluation can be also done in $O(n\log^2 n)$ flops [16, p. 26], but due to the potential numerical stability problems, it seems more attractive to apply the more recent techniques of *fast multipoint polynomial approximation* [20–23]. We may very much simplify the evaluation of the matrices $D(\vec{f},\vec{g})$, $D(\vec{f},\vec{g})^{-1}$, and $D'(\vec{b})$, where $\vec{f} = \vec{b}$ or $\vec{g} = \vec{b}$ provided that we may choose a vector $\vec{b} = (b_i)_{i=0}^{n-1}$ at our convenience. For instance, let us fill this vector with the scaled n^{th} roots of 1, so that

$$b_i = a\omega_n^i, \qquad i = 0,1,\dots,n-1, \tag{4.1}$$

for a scalar a and for ω_n of Definition 2.3. Then $p_{\vec{b}}(x) = \prod_{i=0}^{n-1}(x - a\omega_n^i) = x^n - a^n$, $p'_{\vec{b}}(x) = nx^{n-1}$, and the matrices $D(\vec{f},\vec{b})$ and $D(\vec{b})$ can be immediately evaluated in

$O(n \log n)$ flops. Furthermore, the evaluation of any given polynomial $p(x)$ of degree n at the scaled n^{th} roots of 1 is immediately reduced to discrete Fourier transform and thus, can be performed in $O(n \log n)$ flops by means of FFT.

Finally, all the diagonal matrices involved in (3.1)–(3.4) can be precomputed once and for all if, in Trummer's problem of the computation of the vector $C(\vec{c}, \vec{d})\vec{v}$, the Cauchy matrix $C(\vec{c}, \vec{d})$ is fixed (e.g., $C(\vec{v}, \vec{d})$ is the Hilbert matrix $(1/(i+j+1))_{i,j=0}^{n-1}$), and only the vector $\vec{v}$ varies.

(b) The multiplication of the diagonal matrices by vectors is a trivial task. The multiplication of the Cauchy matrix $C(\vec{b}, \vec{d})$ or $C(\vec{c}, \vec{b})$ by a vector is Trummer's problem, whose solution can be simplified under an appropriate choice of the vector $\vec{b}$. In particular, even if we restrict $\vec{b}$ to be filled with scaled roots of 1 (cf. (4.1)), we still may choose the scaling parameter a to guarantee fast convergence of the power series of the Multipole Algorithm.

The above study can be extended to the expressions (3.1) and (3.3) for $C(\vec{c}, \vec{d})$. Each of them involves two Vandermonde matrices, but one of these matrices in each expression is defined by a vector $\vec{b}$ of our choice, and this enables us to yield simplification. In particular, for two given vectors $\vec{u} = (u_i)_{i=0}^{n-1}$ and $\vec{b} = (b_i)_{i=0}^{n-1}$ the vector $\vec{v} = V(\vec{b})^{-1}\vec{u}$ is the coefficient vector of the polynomial $v(x)$ that takes on the values u_k at the points b_k, $k = 0, \dots, n-1$. For b_k being a scaled n^{th} roots of 1, as in (4.1), the computation of $\vec{v}$ takes $O(n \log n)$ ops due to the inverse FFT. Similar comments apply to the multiplication of the matrix $V(\vec{b})^{-\top}$ by a vector.

REMARK 4.1. Lemma 2.2 enables us to extend the above analysis to approximate solution of Trummer's degenerate problem.

REMARK 4.2. By Tellegen's theorem [24], the exact multiplication of the transposed Vandermonde matrix $V^{\top}(\vec{d})$ by a vector (cf. (2.1), (3.3)) can be reduced to the exact multiplication of $V(\vec{d})$ by a vector, that is, to exact multipoint polynomial evaluation, though Tellegen's theorem does not generally preserve the error bounds of algorithms for multipoint polynomial approximation, such as ones of [20–23].

REFERENCES

1. S.J. Aarseth, Dynamical evaluation of clusters of galaxies-I, *Mon. Not. R. Astron Soc.* **126**, 223–255, (1963).
2. A.W. Appel, An efficient program for many-body simulation, *SIAM J. on Sci. Stat. Computing* **6**, 85–103, (1985).
3. C.R. Anderson, A method of local corrections for computing the velocity field due to a distribution of vortex blobs, *J. Comput. Phys.* **62**, 111–123, (1986).
4. J. Barnes, Encounters of disk/halo galaxies, *The Astrophisical J.* **331**, 699, (1988).
5. A.J. Chorin, Numerical study of slightly viscous flaw, *Fluid Mechanics* **57**, 785–796, (1973).
6. J.M. Dawson, Particle simulation of plasmas, *Rev. Mod. Physics* **55**, 403–447, (1983).
7. R.W. Hockney and J.W. Eastwood, *Computer Simulation Using Particles*, McGraw-Hill, New York, (1981).
8. S.T. O'Donnell and V. Rokhlin, A fast algorithm for numerical evaluation of conformal mappings, *SIAM J. Sci. Stat. Comput.* **10** (3), 475–487, (1989).
9. A.M. Odlyzko and A. Schönhage, Fast algorithms for multiple evaluations of the riemann zeta function, *Trans. Am. Math. Soc.* **309** (2), 797–809, (1988).
10. V. Rokhlin, Rapid solution of integral equations of classical potential theory, *J. Comput. Physics* **60**, 187–207, (1985).
11. A. Gerasoulis, A fast algorithm for the multiplication of generalized Hilbert matrices with vectors, *Math. Comp.* **50** (181), 179–188, (1987).
12. L. Greengard and V. Rokhlin, A fast algorithm for particle simulation, *J. Comput. Physics* **73** (2), 325–348, (1987).
13. J. Ambrosiano, L. Greengard and V. Rokhlin, The fast multipole method for gridless particle simulation, *Computer Physics Communications* **48**, 115–125, (1988).
14. J. Carier, L. Greengard and V. Rokhlin, A fast adaptive multipole algorithm for particle simulation, *SIAM J. Sci. Stat. Comput.* **9** (4), 669–686, (1988).
15. L. Greengard, *The Rapid Evaluation of Potential Fields in Particle Systems*, MIT Press, Cambridge, MA, (1988).
16. D. Bini and V.Y. Pan, *Polynomial and Matrix Computations, Volume 1: Fundamental Algorithms*, Birkhäuser, Boston, (1994).

17. J. Barnes and P. Hut, A hierarchical $O(N \log N)$ force-calculation algorithm, *Nature* **324**, (1986).
18. L. van Dommelen and E.A. Rundensteiner, Fast adaptive summation of point forces in the two-dimentional poisson equation, *J. Comp. Physics* **83**, 126–147, (1989).
19. T. Fink, G. Heinig and K. Rost, An inversion formula and fast algorithms for Cauchy-Vandermonde matrices, *Linear Algebra Appl.* **183**, 179–191, (1993).
20. V.Y. Pan, E. Landowne, A. Sadikou and O. Tiga, A new approach to fast polynomial interpolation and multipoint evaluation, *Computers Math. Applic.* **25** (9), 25–30, (1993).
21. V. Rokhlin, A fast algorithm for the discrete Laplace transformation, *J. of Complexity* **4**, 12–32, (1988).
22. V.Y. Pan, An algebraic approach to approximate evaluation of a polynomial on a set of real points, *Advances in Computational Mathematics* **3**, 41–58, (1995).
23. V.Y. Pan, A. Zheng, X. Huang and Y. Yu, Fast multipoint polynomial evaluation and interpolation via computation with structured matrices, *Annals of Numerical Math.* **4**, 483–510, (1997).
24. P. Penfield, Jr., R. Spencer and S. Duinker, *Tellegen's Theorem and Electrical Networks*, MIT Press, Cambridge, MA, (1970).