Decompositions of complete graphs and complete bipartite graphs into isomorphic supersubdivision graphs

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Abstract

A graph $H$ is called a supersubdivision of a graph $G$ if $H$ is obtained from $G$ by replacing every edge $uv$ of $G$ by a complete bipartite graph $K_{2,m}$ ($m$ may vary for each edge) by identifying $u$ and $v$ with the two vertices in $K_{2,m}$ that form one of the two partite sets. We denote the set of all such supersubdivision graphs by $\text{SS}(G)$. Then, we prove the following results.

1. Each non-trivial connected graph $G$ and each supersubdivision graph $H \in \text{SS}(G)$ admits an $\alpha$-valuation. Consequently, due to the results of Rosa (in: Theory of Graphs, International Symposium, Rome, July 1966, Gordon and Breach, New York, Dunod, Paris, 1967, p. 349) and El-Zanati and Vanden Eynden (J. Combin. Designs 4 (1996) 51), it follows that complete graphs $K_{2q+1}$ and complete bipartite graphs $K_{mq, nq}$ can be decomposed into edge disjoined copies of $H \in \text{SS}(G)$, for all positive integers $m, n$ and $c$, where $q = |E(H)|$.

2. Each connected graph $G$ and each supersubdivision graph in $\text{SS}(G)$ is strongly $n$-elegant, where $n = |V(G)|$ and felicitous.

3. Each supersubdivision graph in $\text{EASS}(G)$, the set of all even arbitrary supersubdivision graphs of any graph $G$, is cordial.

Further, we discuss a related open problem. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

A function \( f \) is called a graceful labeling of a graph \( G \) with \( q \) edges if \( f \) is an injection from the vertices of \( G \) to the set \{0,1,2,\ldots,q\} such that when each edge \( uv \) is assigned the label \( |f(u) - f(v)| \), the resulting edge labels are distinct. A graceful labeling \( f \) is called an \( \alpha \)-valuation of \( G \) if there exists an integer \( \lambda \) such that \( f(u)\leq \lambda < f(v) \) for every \( uv \in E(G) \) with \( f(u) < f(v) \). A graph \( G \) admitting an \( \alpha \)-valuation is necessarily bipartite.

A graph \( G \) with \( q \) edges is called a strongly \( c \)-elegant if there is an injection \( f \) from the set of vertices of \( G \) to the set of integers \{0,1,2,3,\ldots,q\} such that when each edge \( xy \) is assigned the label \( f(x) + f(y) \) (mod \( q \)), the resulting edge labels are distinct.

A graph \( G \) with \( q \) edges is said to be felicitous if there is an injection from the set of vertices of a graph \( G \) to the set \{0,1,2,\ldots,q\} such that when each edge \( xy \) is assigned the label \( f(x) + f(y) \) (mod \( q \)), the resulting edge labels are distinct.

Let \( f \) be a function from the vertices of \( G \) to \{0,1\} and for each edge \( xy \) assign the label \( |f(x) - f(y)| \). Call \( f \) a cordial labeling of \( G \) if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1 and the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1.

Rosa [7] has proved the following significant theorem.

**Theorem.** If a graph \( G \) with \( q \) edges has an \( \alpha \)-valuation, then there exists a cyclic decomposition of the edges of the complete graph \( K_{2cq+1} \) into subgraphs isomorphic to \( G \), where \( c \) is an arbitrary natural number.

By a decomposition, we mean a set of subgraphs which partition the edges. For the definition and more details on cyclic decompositions of \( K_{2cq+1} \) into subgraphs refer to the classic paper by Rosa [7].

Rosa’s theorem provides motivation to construct families of bipartite graphs which admit an \( \alpha \)-valuation. In this direction some interesting results have been obtained in [1,5,7]. For an exhaustive survey on these topics, we refer to the excellent survey paper [4]. For a recent result in cordial labelling, we refer to [2].

We introduce a new operation called supersubdivision of graph to generate families of bipartite graphs. In the complete bipartite graph \( K_{2,m} \), we call the part consisting of two vertices, the 2-vertices part of \( K_{2,m} \) and the part consisting of \( m \) vertices, the \( m \)-vertices part of \( K_{2,m} \).

Let \( G \) be a graph with \( n \) vertices and \( t \) edges. A graph \( H \) is said to be a supersubdivision of \( G \), if \( H \) is obtained from \( G \) by replacing every edge \( e_i \) of \( G \) by a complete bipartite graph \( K_{2,m_i} \), for some \( m_i \), \( 1 \leq i \leq t \) (\( m_i \) may vary for each edge \( e_i \)) in such a way that the ends of \( e_i \) are merged with the two vertices of the 2-vertices part of \( K_{2,m_i} \) after removing the edge \( e_i \) from \( G \).

Let \( G \) be a graph with edge set \( E(G) = \{e_1,e_2,\ldots,e_q\} \). A supersubdivision graph of a graph \( G \) is called even arbitrary supersubdivision of \( G \) if each edge \( e_i \) is replaced by a complete bipartite graph \( K_{2,2m_i} \), where \( m_i \) is any positive integer, for \( 1 \leq i \leq q \). We denote the set of all even arbitrary supersubdivision graphs of a graph \( G \) by \( EASS(G) \).
In Section 2, we give an algorithm (Algorithm 2) which gives a method for constructing an infinite family of certain supersubdivision graphs from every non-trivial connected graph \( G \). We denote the set of all such supersubdivision graphs of a non-trivial connected graph \( G \) by \( \mathcal{SS}(G) \).

In Section 3, we prove our main result that each non-trivial connected graph \( G \) and each supersubdivision graph in \( \mathcal{SS}(G) \) admits an \( x \)-valuation. Consequently, due to the results of Rosa [7] and El-Zanati and Vanden Eynden [3], it follows that complete graphs \( K_{2c+1} \) and complete bipartite graph \( K_{qm, qn} \) can be decomposed into edge disjoint copies of any \( H \in \mathcal{SS}(G) \), for all positive integers \( m, n, c \), where \( q = |E(H)| \).

In Section 4, also, we show that each supersubdivision graph in the family \( \mathcal{SS}(G) \) is strongly \( n \)-elegant, where \( n = |V(G)| \) and felicitous.

Finally, we discuss a related open problem.

2. Construction of supersubdivision graphs of connected graph

In this section, we give an algorithm for the construction of certain supersubdivision graphs of a non-trivial connected graph.

Algorithm 1: Basic labeling for the vertices of a connected graph

Let \( G \) be a non-trivial connected graph with \( n \) vertices and \( t \) edges.

Step 1. Assign 0 to any vertex \( v \) of \( G \).

Step 2. Find the least labeled vertex of \( G \) which has adjacent unassigned vertices. If \( w_1, w_2, \ldots, w_r \) are the unassigned adjacent vertices of that vertex, then assign the numbers \( s+1, s+2, \ldots, s+r \), where \( s+1 \) is the least positive integer available for the assignment.

Step 3. Repeat Step 2 until all the vertices are assigned the numbers from 0 to \( n-1 \). We denote the labeled non-trivial connected graph obtained by using Algorithm 1 by \( B(G) \).

Remark 1. Observe that in \( B(G) \) vertices are labeled with numbers 0, 1, 2, \ldots, \( n-1 \). Since \( G \) is connected, in \( B(G) \) each vertex labeled \( i \) has at least one adjacent vertex labeled \( r \), such that \( r < i \), except for \( i = 0 \).

Remark 2. It is clear from Algorithm 1 that every non-trivial connected graph \( G \) has finitely many different \( B(G) \)'s.

Now we shall construct certain supersubdivision graphs of \( G \) by using the following algorithm (Fig. 1).

Algorithm 2: Construction of a supersubdivision graph of a non-trivial connected graph \( G \)

Step 1. Using Algorithm 1 label the vertices of \( G \) and hence obtain \( B(G) \).

Step 2. Let \( e_1, e_2, e_3, \ldots, e_t \) be the edges of \( B(G) \). For each edge \( e_i = xy \) of \( B(G) \), \( 1 \leq i \leq t \), define \( m_i = j_i |x - y| \), where \( j_i \) is any positive integer. Now in \( B(G) \), replace each...
edge \( e_i, 1 \leq i \leq t \), by \( K_{2,m_i} \) by merging the end vertices of \( e_i \) with the two vertices of 2-vertices part of \( K_{2,m_i} \) after removing the edge \( e_i \) of \( B(G) \).

Let \( SS(G) \) denote the set of all supersubdivision graphs of \( G \) constructed by using Algorithm 2.

**Remark 3.** From the definition of supersubdivision graphs of \( G \), observe that each supersubdivision graph has two types of vertices: vertices which originally belong to the graph \( G \) and the vertices which do not originally belong to the graph \( G \). In the supersubdivision graphs, the vertices which originally belong to the graph \( G \) are called **base vertices** and the other remaining vertices are called **non-base vertices**.

**Remark 4.** Observe from the Step 2 of Algorithm 2 that each edge \( e_i = xy \) in \( B(G) \) is replaced by \( K_{2,m_i} \), where \( m_i = j_i|x - y| \), and \( j_i \) is any positive integer. Thus, for each edge \( e_i \), there are infinitely many choices of \( K_{2,m_i} \) for its replacement in the construction of the supersubdivision graph.

Hence, from each edge of a non-trivial connected graph \( G \), an infinite family of supersubdivision graphs can be constructed. We denote the family of supersubdivision graphs of a connected graph \( G \) constructed using the Algorithm 2 by \( SS(G) \).

An illustration of a supersubdivision graph of the graph \( G \) given in Fig. 2.
3. Each supersubdivision graph in $SS(G)$ admits an $x$-valuation for any non-trivial connected graph $G$

In this section, we prove that for any non-trivial connected graph $G$ each supersubdivision graph of $G$ in $SS(G)$ admits an $x$-valuation.

**Observation 1.** For a non-trivial connected graph $G$, let $k_i$ denote the cardinality of the set of all edges of $B(G)$ incident with the vertex labeled with $i$ whose other ends have the labels less than $i$. Let $r_{i1}, r_{i2}, \ldots, r_{ik_i}$ be the vertices of $B(G)$ adjacent to $i$ with $r_{i1} < r_{i2} < r_{i3} < \cdots < r_{ik_i} < i$. Then to obtain a supersubdivision graph $H$ of $G$, by Step 2 of Algorithm 2, the edges $ir_{i1}, ir_{i2}, \ldots, ir_{ik_i}$ of $B(G)$ are replaced, respectively, by

$$K_{2, j_{i1}(i-r_{i1})}, K_{2, j_{i2}(i-r_{i2})}, \ldots, K_{2, j_{ik_i}(i-r_{ik_i})},$$

where, $j_{i1}, j_{i2}, \ldots, j_{ik_i}$ are arbitrary positive integers.
Therefore, we have
\[ |E(H)| = \sum_{i=1}^{n-1} \left( \sum_{\ell=1}^{k_i} 2(j_{i\ell}(i - r_{i\ell})) \right). \]

**Theorem 1.** For any non-trivial connected graph \( G \), each supersubdivision graph in \( \text{SS}(G) \) admits an \( \alpha \)-valuation.

**Proof.** Let \( G \) be a non-trivial connected graph and let \( H \) be any supersubdivision graph of \( G \) in \( \text{SS}(G) \). Let \( M \) be the number of edges of \( H \). By Observation 1, we have
\[ M = \sum_{i=1}^{n-1} \left( \sum_{\ell=1}^{k_i} 2(j_{i\ell}(i - r_{i\ell})) \right). \]

For convenience, denote \( N_0 = M \), and for \( 1 \leq i \leq n - 1 \), let
\[ i = N_0 - \sum_{q=1}^{i} \left( \sum_{\ell=1}^{k_q} 2(j_{q\ell}(q - r_{q\ell})) \right). \]

Now we shall first give a graceful labeling to the vertices of \( H \) and then show that this labeling is indeed an \( \alpha \)-valuation for \( H \). For the base vertices of \( H \), consider the same basic labels in \( B(G) \) as their labeling for the graceful labels. Now, for the remaining non-base vertices of \( H \), we shall give labels in the following way.

For each \( i, 1 \leq i \leq n - 1 \), we consider the non-base vertices of the complete bipartite graphs \( K_{2,j_{1}(i-r_{1})}, K_{2,j_{2}(i-r_{2})}, \ldots, K_{2,j_{p}(i-r_{p})}, \ldots, K_{2,j_{k_i}(i-r_{k_i})} \) of \( H \) which are obtained in the construction of \( H \) from \( B(G) \) by replacing all the edges \( r_{1}, r_{2}, \ldots, r_{p}, \ldots, r_{k_i} \) of \( B(G) \) such that \( r_{1} < r_{2} < \cdots < r_{p} < \cdots < r_{k_i} < i \).

Now for each \( p, 1 \leq p \leq k_i \), we shall give labels to the non-base vertices of \( K_{2,j_{p}(i-r_{p})} \) as shown in Fig. 3 and Fig. 4.

Observe from the labeling that the labels of the non-base vertices in the first set of \( K_{2,j_{p}(i-r_{p})} \) form a monotonically decreasing sequence, thus the labels of any two non-base vertices differ by at least one. Hence the labels of all the non-base vertices of the first set of the \( K_{2,j_{p}(i-r_{p})} \) are distinct. Further, observe that the least value of the labels,
\[ N_{i-1} - 2 \sum_{\ell=1}^{p-1} j_{i\ell}(i - r_{i\ell}) + (i - r_{ip}) + r_{ip} + 1, \]

of the non-base vertices of the first set of \( K_{2,j_{p}(i-r_{p})} \) and the largest value,
\[ N_{i-1} - 2 \sum_{\ell=1}^{p-1} j_{i\ell}(i - r_{i\ell}) - 2(i - r_{ip}) + r_{ip}, \]

of the labels of the non-base vertices of the second set of \( K_{2,j_{p}(i-r_{p})} \) differ by \( i - r_{ip} + 1 \geq 2 \) (note that \( i > r_{ip} \)). As in the first set, the labels of the non-base
vertices of the second set also form a monotonically decreasing sequence, the labels of any two non-base vertices of the second set of the $K_{2,j_ip}(i-r_ip)$ will differ by at least one. Hence the labels of all the non-base vertices in the second set are also distinct. Similarly, the labels of the non-base vertices of the other remaining $j_ip - 2$ sets of $K_{2,j_ip}(i-r_ip)$ are all distinct, provided that the label

$$N_i - 2 \sum_{\ell=1}^{p-1} j_\ell (i - r_\ell) + r_ip$$

has not been used anywhere before for the labeling. When $p = 1$, observe from the above labeling that the largest value of the labels, $N_i + i - 1$, of the non-base vertices
of $K_{2,j,i(i-r)}$ and the least value, $N_{i-1} + (i - 1) + 1$, that is, $N_{i-1} + i$, of the labels of the non-base vertices of $K_{2,k(i-1)(i-1)(i-1)}$ differ by $i - r_{1}$. As $i > r_{1}$, the label $N_{i-1} + r_{1}$ has not been used anywhere before for the labeling. When $2 \leq p \leq k_{i}$, observe from the above labeling that the largest value of the labels,

$$N_{i-1} - 2 \sum_{\ell=1}^{p-1} j_{\ell}(i - r_{\ell}) + r_{p},$$

of the non-base vertices of $K_{2,j,i(p-i-p)}$ and the least value of the labels,

$$N_{i-1} - 2 \sum_{\ell=1}^{p-1} j_{\ell}(i - r_{\ell}) + i + 1,$$
of the non-base vertices of $K_{2,j_0(i-r_ip)}$ differ by $i - r_ip + 1$. As $i > r_ip$, the label

$$N_i - 2 \sum_{\ell=1}^{p-1} j_{\ell} (i - r_{\ell}) + r_ip$$

has not been used anywhere before for the labeling. Hence, the labels of all the non-base vertices of $K_{2,j_0(i-r_ip)}$ are all distinct, for $1 \leq p \leq k_i$. Consequently, for each $i$, $1 \leq i \leq n - 1$, the labels of the non-base vertices in the $K_{2,j_1(i-r_{i1})}, K_{2,j_2(i-r_{i2})}, \ldots$, $K_{2,j_k(i-r_{ik})}$ of $H$ are all distinct. Observe from the above labeling that the labels of the non-base vertices of $H$ form a monotonically decreasing sequence, hence the labels of all the non-base vertices of $H$ are distinct.

Therefore, the labels of all the vertices of $SS(G)$ are distinct.

Observe from the above labeling that the $2(i - r_ip)$ edges of the first set of the $K_{2,j_0(i-r_ip)}$ get the distinct values from

$$N_i - 2 \sum_{\ell=1}^{p-1} j_{\ell} (i - r_{\ell})$$

to

$$N_i - 2 \sum_{\ell=1}^{p-1} j_{\ell} (i - r_{\ell}) - 2(i - r_ip) + 1,$$

the $2(i - r_ip)$ edges of the second set of the $K_{2,j_0(i-r_ip)}$ get the distinct values from

$$N_i - 2 \sum_{\ell=1}^{p-1} j_{\ell} (i - r_{\ell}) - 2(i - r_ip)$$

to

$$N_i - 2 \sum_{\ell=1}^{p-1} j_{\ell} (i - r_{\ell}) - 4(i - r_ip) + 1,$$

and finally, the $2(i - r_ip)$ edges of the $j_ip$th set of the $K_{2,j_0(i-r_ip)}$ get the distinct values from

$$N_i - 2 \sum_{\ell=1}^{p-1} j_{\ell} (i - r_{\ell}) - 2(j_ip - 1)(i - r_ip)$$

to

$$N_i - 2 \sum_{\ell=1}^{p-1} j_{\ell} (i - r_{\ell}) - 2j_ip(i - r_ip) + 1,$$

that is, $N_i - 2 \sum_{\ell=1}^{p-1} j_{\ell} (i - r_{\ell}) + 1$ ($= N_i + 1$, when $p = k_i$).

Therefore, the $2 \sum_{p=1}^{k_i} j_ip(i - r_ip)$ edges of the $K_{2,j_0(i-r_{i1})}, K_{2,j_2(i-r_{i2})}, \ldots$, $K_{2,j_ip(i-r_{ip})}$ of $H$ get the distinct values from $N_i - 1$ to $N_i + 1$. Thus, it follows from the above labeling that all the edges of $H$ get distinct values from $M$ to 1. Hence, $H$ is graceful.

Observe that in $H$, one of the ends of each edge $e$ of $H$ is a base vertex and the other end is a non-base vertex. In the gracefully labeled $H$, the base vertex of each edge of $H$ gets a distinct basic label and the maximum of the set of basic labels is $n - 1$. Therefore, in the gracefully labeled $H$, for each edge $e = uv$, if $u$ is a base vertex
and \(v\) is a non-base vertex with labels \(f(u)\) and \(f(v)\), then \(f(u) \leq n - 1 < f(v)\), where \(n = |V(G)|\). Hence, \(H\) admits a \(z\)-valuation.

The following corollary is an immediate consequence of Rosa’s Theorem [7].

**Corollary 1.** For any non-trivial connected graph \(G\) and for each supersubdivision graph \(H \in SS(G)\), there exists a cyclic decomposition of the edges of the complete graphs \(K_{2q+1}\) into subgraphs isomorphic to \(H\), where \(q = |E(H)|\) and \(c\) is any arbitrary positive integer.

Recently, El-Zanati and Vanden Eynden [3] have proved that if \(G\) has \(q\) edges and admits an \(z\)-valuation then \(K_{qm,qn}\) can be partitioned into subgraphs isomorphic to \(G\) for all positive integers \(m\) and \(n\). Thus we have the following corollary.

**Corollary 2.** For any non-trivial connected graph \(G\) and for every supersubdivision graph \(H \in SS(G)\), there exists a partition of the edges of the complete bipartite graphs \(K_{mq,nq}\) into subgraphs isomorphic to \(H\), where \(q = |E(H)|\), and \(m\) and \(n\) are arbitrary positive integers.

### 4. For any non-trivial connected graph \(G\), each supersubdivision graph in \(SS(G)\) is strongly \(n\)-elegant and felicitous

In this section, we prove that for any non-trivial connected graph \(G\) each supersubdivision graph of \(G\) in \(SS(G)\) is strongly \(n\)-elegant, where \(n = |V(G)|\) and felicitous.

**Observation 2.** For any non-trivial connected graph \(G\) for each \(i, 1 \leq i \leq n - 1\), let \(k_{n-i}\) denote the cardinality of the set of all edges of \(B(G)\) incident with the vertex labeled with \(n - i\) whose other ends have the labels less than \(n - i\). Let \(r_{(n-i)}(n-i), r_{(n-i)}(n-i-2), \ldots, r_{(n-i)}(n-i)\) be all the vertices adjacent to \((n-i)\) of \(B(G)\) with \(n-i > r_{(n-i)}(n-i) > r_{(n-i)}(n-i+2) > \cdots > r_{(n-i)}(n-i)\). Then to obtain \(SS(G)\), by Step 2 of the algorithm, the edges \((n-i)r_{(n-i)1}, (n-i)r_{(n-i)2}, \ldots, (n-i)r_{(n-i)(n-i)}\) of \(B(G)\) are replaced, respectively, by

\[
K_{2j_{(n-i)1}(n-i-r_{(n-i)1})}, K_{2j_{(n-i)2}(n-i-r_{(n-i)2})}, \ldots, K_{2j_{(n-i)(n-i)}(n-i-r_{(n-i)(n-i)})},
\]

where, \(j_{(n-i)1}, j_{(n-i)2}, \ldots, j_{(n-i)(n-i)}\) are any positive integers.

Therefore, we have

\[
|E(SS(G))| = M = \sum_{i=1}^{n-1} \left( \sum_{j=1}^{k_{n-i}} 2(j_{(n-i)}(n-i-r_{(n-i)})) \right).
\]

**Theorem 2.** For any non-trivial connected graph \(G\), each supersubdivision graph in \(SS(G)\) is strongly \(n\)-elegant, where \(n = |V(G)|\).
Proof. Let $G$ be a non-trivial connected graph and let $H$ be any supersubdivision graph of $G$ in $SS(G)$. Let $M$ be the number of edges of $H$. By Observation 2, we have

$$M = \sum_{i=1}^{n-1} \left( \sum_{\ell=1}^{k_{n-i}} 2(j_{(n-i)\ell}((n-i) - r_{(n-i)\ell})) \right).$$

For convenience, for $0 \leq i \leq n - 1$, let

$$N_{n-i} = M - \sum_{q=1}^{i} \left( \sum_{\ell=1}^{k_{n-q}} 2(j_{(n-q)\ell}((n-q) - r_{(n-q)\ell})) \right) + i.$$

Now we shall give strongly $n$-elegant labels to the vertices of $H$. For the base vertices of $H$, consider their same basic labels in $B(G)$ as their labels for the strongly $n$-elegant labels and for the remaining non-base vertices $H$ we shall give labels in the following way.

For each $n - i$, $1 \leq i \leq n - 1$, we consider the non-base vertices of the complete bipartite graphs

$$K_{2,j_{(n-i)}(n-i - r_{(n-i)1})}, K_{2,j_{(n-i)}2(n-i - r_{(n-i)2})}, \ldots, K_{2,j_{(n-i)p}(n-i - r_{(n-i)p})}, \ldots,$$

$$K_{2,j_{(n-i)}(n-i - r_{(n-i)k_{n-i}})}$$

which are obtained in the construction of $H$ from the $B(G)$ by replacing all the edges

$$(n-i) r_{(n-i)1}, (n-i) r_{(n-i)2}, \ldots, (n-i) r_{(n-i)p}, \ldots,$$

of $B(G)$ such that $n-i > r_{(n-i)1} > r_{(n-i)2} > \ldots > r_{(n-i)p} > \ldots > r_{(n-i)k_{n-i}}$.

For each $p$, $1 \leq p \leq k_{(n-i)}$, we obtain the labels of the vertices of $K_{2,j_{(n-i)p}(n-i - r_{(n-i)p})}$ from labels in Fig. 3 of the Proof of Theorem 1 in the following way. We replace the labels $i$ and $r_{ip}$ of the base vertices in Fig. 3, respectively, by $n-i$ and $r_{(n-i)p}$. In the label expression of each non-base vertex given in Fig. 3, we replace the $N_{n-(i-1)}$ and $\sum_{f=1}^{p-1} j_{f}(i - r_{ip})$ by $\sum_{f=1}^{p-1} j_{(n-i)f}((n-i) - r_{(n-i)p})$. If the label expression of a non-base vertex given in Fig. 3 contains $(i - r_{ip})$ or $(j_{ip} - 1)$ then we replace them, respectively, by $((n-i) - r_{(n-i)p})$ and $(j_{(n-i)p} - 1)$.

Then as in Theorem 1, all the vertex labels of $H$ are distinct.

For each edge $uv$ of $H$ if we define edge value $f(u) + f(v)$, where $f(u)$ and $f(v)$ are labels assigned as above, then it follows that the $M$ edges of $H$ get distinct values from $M + n$ to $n$. Hence $H$ is strongly $n$-elegant.

Corollary 3. For any non-trivial connected graph $G$, each supersubdivision graph in $SS(G)$ is felicitous.

Proof. Let $G$ be any non-trivial connected graph and let $H \in SS(G)$. Let the labels of the vertices of $H$ be as defined in the proof of the Theorem 2. Then, if we define the edge value of each edge $xy$ as $f(x) + f(y)$ (mod $M$), where $M = |E(H)|$ we get distinct edge values 0, 1, 2, …, $M - 1$. Thus $H$ is felicitous.
5. Even arbitrary supersubdivision graphs of any graph are cordial

In this section, we prove that even arbitrary supersubdivision graphs of any graph are cordial.

**Theorem 3.** For any graph $G$, there exists an infinite family of graphs $EASS(G)$ such that each supersubdivision graph in $EASS(G)$ is cordial.

**Proof.** Let $G$ be any graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$ and the edge set $E(G) = \{e_1, e_2, \ldots, e_q\}$.

Let $H \in EASS(G)$. Then, by definition of even arbitrary supersubdivision of $G$, we assume that $H$ is constructed by replacing each edge $e_i$ of $G$ by $K_{2,2m_i}$, $1 \leq i \leq q$, where $m_i$ is any positive integer. Thus, $H$ has

$$p + 2 \sum_{i=1}^{q} m_i \text{ vertices and } 4 \sum_{i=1}^{q} m_i \text{ edges.}$$

For the base vertices $v_i$, $1 \leq i \leq p$, define

$$f(v_i) = \begin{cases} 
0 & \text{if } i \text{ is odd,} \\
1 & \text{if } i \text{ is even.}
\end{cases}$$

Let $u_{i,1}, u_{i,2}, \ldots, u_{i,2m_i}$, $1 \leq i \leq q$ be the set of non-base vertices of $H$. Then define

$$f(u_{i,j}) = \begin{cases} 
0 & \text{if } j \text{ is odd, } 1 \leq j \leq 2m_i \\
1 & \text{if } j \text{ is even, } 1 \leq j \leq 2m_i.
\end{cases}$$

Therefore, observe that for each $i$, $1 \leq i \leq q$, in the $K_{2,2m_i}$ (corresponding to edge $e_i$ of $G$) of $H$, the number of edges having the label 0 is $2m$ and the number of edges having the label 1 is $2m$.

Hence in the graph $H$, $2qm$ edges have edge labels 1 and $2qm$ edges have edge labels 0.

Further, if $p$ is even, then

$$\frac{p}{2} + \sum_{i=1}^{q} m_i \text{ vertices will have the label 0 and}$$

$$\frac{p}{2} + \sum_{i=1}^{q} m_i \text{ vertices will have the label 1 and}$$

if $p$ is odd, then

$$\frac{p+1}{2} + \sum_{i=1}^{q} m_i \text{ vertices will have the label 0 and}$$
\[ \frac{p - 1}{2} + \sum_{i=1}^{q} m_i \] vertices will have the label 1.

Thus, in \( H \), the number of edges having the label 0 is equal to the number of edges having the label 1 and number of vertices having the label 0 and number of vertices having the label 1 differ by at most 1.

Hence, \( H \) is cordial. \( \square \)

6. Discussion

In this paper, for any non-trivial connected graph \( G \), we have obtained an \( x \)-valuation for supersubdivision graphs in \( SS(G) \). We guess that it may not be possible to extend our method to obtain such supersubdivision graphs for disconnected graphs with an \( x \)-valuation. So we pose the question:

Is there any such supersubdivision for every disconnected graph which also admits \( x \)-valuation.

On the other hand, one can observe that each supersubdivision graph in \( SS(G) \) admits an \( x \)-valuation for every non-trivial connected graph mainly due to the appropriate choice “the \( K_{2,m_i} \)”, for replacing an edge \( e_i \) of \( G \) in the construction of the supersubdivision graphs in \( SS(G) \). So it is tempting to ask the question:

Is there any graph different from \( K_{2,m} \) that can be used to replace an edge \( e \) of a connected graph \( G \) to obtain a supersubdivision which admits an \( x \)-valuation.

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