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Point regular groups of automorphisms of generalised quadrangles

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ABSTRACT

We study the point regular groups of automorphisms of some of the known generalised quadrangles. In particular we determine all point regular groups of automorphisms of the thick classical generalised quadrangles. We also construct point regular groups of automorphisms of the generalised quadrangle of order $(q-1, q+1)$ obtained by Payne derivation from the classical symplectic quadrangle $\mathsf{W}(3, q)$. For $q = p^f$ with $f \geqslant 2$ we obtain at least two nonisomorphic groups when $p \geqslant 5$ and at least three nonisomorphic groups when $p = 2$ or 3. Our groups include nonabelian 2-groups, groups of exponent 9 and nonspecial p-groups. We also enumerate all point regular groups of automorphisms of some small generalised quadrangles.

1. Introduction

In this paper we investigate the regular subgroups of some of the known generalised quadrangles. We demonstrate that the class of groups which can act as a point regular group of automorphisms of a generalised quadrangle is much wilder than previously thought.

A finite generalised quadrangle $\mathcal{Q}$ is a geometry consisting of a finite set of points and lines such that, if P is a point and ℓ is a line not on P, then there is a unique line through P which meets ℓ in a point. From this property, if there are at least three points of $\mathcal{Q}$ or there is a point on at least three lines, then one can see that there are constants s and t such that each line is incident with $s+1$ points, and each point is incident with $t+1$ lines. Such a generalised quadrangle is said to have *order* (s, t), and hence its point-line dual is a generalised quadrangle of order (t, s). The generalised quadrangle is said to be *thick* if $s, t \geqslant 2$.

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A permutation group G on a set Ω acts *regularly* on Ω if it acts transitively on Ω and only the identity of G fixes an element of Ω. Ghinelli proves in [10] that a Frobenius group or a group with a nontrivial centre cannot act regularly on the points of a generalised quadrangle of order (s, s), where s is even. S. De Winter and K. Thas [5] prove that if a finite thick generalised quadrangle admits an abelian group of automorphisms acting regularly on its points, then it is the Payne derivation of a translation generalised quadrangle of even order. Yoshiara [22] proved that there are no generalised quadrangles of order (s^2, s) admitting an automorphism group acting regularly on points.

Our first result is a complete classification of all regular subgroups of the thick classical generalised quadrangles.

Theorem 1.1. *Let $\mathcal{Q}$ be a finite thick classical generalised quadrangle and let G be a group of automorphisms that acts regularly on the points of $\mathcal{Q}$. Then one of the following holds*:

1. $\mathcal{Q} = \mathsf{Q}^-(5, 2)$ *and G is an extraspecial group of order* 27 *and exponent* 3.
2. $\mathcal{Q} = \mathsf{Q}^-(5, 2)$ *and G is an extraspecial group of order* 27 *and exponent* 9.
3. $\mathcal{Q} = \mathsf{Q}^-(5, 8)$ *and* $G \cong \mathrm{GU}(1, 2^9).9 \cong C_{513} \rtimes C_9$.

An alternative approach to the classification in Theorem 1.1 was independently undertaken in [7].

Most of the known generalised quadrangles are *elation generalised quadrangles*, and such a generalised quadrangle $\mathcal{Q}$ of order (s, t) has a group of automorphisms G which fixes a point x and each line on x, and acts regularly on the points not collinear with x. We call G an *elation group* and x a *base point* of $\mathcal{Q}$. Necessarily, G has order s^2t. The only known generalised quadrangles which are not elation generalised quadrangles are the *Payne derived* quadrangles and their duals.

Payne [18] gave a method for constructing a new generalised quadrangle from an old one. Take a generalised quadrangle $\mathcal{Q}$ of order (s, s) and suppose it has a point x such that for every point y not collinear with x, the set of points $\{x, y\}^{\perp\perp}$ has size $s + 1$, where we use the notation $S^\perp$ to denote the set of all points collinear with every element of the set S. A new generalised quadrangle $\mathcal{Q}^x$ can be constructed whose points are the points of $\mathcal{Q}$ not collinear with x and the lines of $\mathcal{Q}^x$ are: (i) the lines of $\mathcal{Q}$ not incident with x, and (ii) the *hyperbolic lines* $\{x, y\}^{\perp\perp}$ where y is not collinear with x. Thus $\mathcal{Q}^x$ is a generalised quadrangle of order $(s - 1, s + 1)$. If we take $\mathcal{Q}$ to be the elation generalised quadrangle $\mathsf{W}(3, q)$, then any point x will give rise to a Payne derived quadrangle $\mathcal{Q}^x$ of order $(q - 1, q + 1)$ and if G is the elation group of the classical symplectic quadrangle $\mathsf{W}(3, q)$ about the point x, then G is elementary abelian for q even and a Heisenberg group for q odd [13]. The stabiliser H of the point x in the full automorphism group of $\mathcal{Q}$ acts as a group of automorphisms of $\mathcal{Q}^x$ and contains G as a normal subgroup. In fact for $q \geqslant 5$, H is the full automorphism group of $\mathcal{Q}^x$ [11]. However, the full automorphism group of $\mathcal{Q}^x$ may contain point regular subgroups other than G. In Section 3 we exhibit several other infinite families of regular subgroups and the results are summarised in the following theorem.

Theorem 1.2. *Let $\mathcal{Q}^x$ be the generalised quadrangle of order $(q - 1, q + 1)$ obtained by Payne derivation from* $\mathsf{W}(3, q)$. *Then there exist distinct subgroups E and P of* $\mathrm{Aut}(\mathcal{Q}^x)$ *that act regularly on the points of $\mathcal{Q}^x$ and for q not a prime there also exists a further regular subgroup S such that E, P and S have the following properties*:

1. *E is an elation group of* $\mathsf{W}(3, q)$ *while P and S are not.*
2. $E \ncong S$ *and* $P \ncong S$.
3. $E \cong P$ *if and only if q is not a power of* 2 *or* 3.
4. *For q even,*
   (a) *E is elementary abelian while S and P have exponent* 4 *and are nonabelian except when* $q = 2$;
   (b) $P' < Z(P)$ *and* $S' < Z(S)$ (*in particular, P and S are not special*).
5. *For* $q = 3^f$, *E has exponent* 3 *while P and S have exponent* 9.
6. *For q odd,* $Z(P) = P' = Z(E) = E'$ *and* $Z(S) < S'$ (*in particular, P and E are special while S is not*).

More explicit details and constructions are given in Section 3. In particular, we construct more regular subgroups than those described in Theorem 1.2; see Remark 3.17. The generalised quadrangle of order $(2,4)$ obtained by Payne derivation from $\mathsf{W}(3,3)$ is isomorphic to $\mathsf{Q}^-(5,2)$ [17, §6.1]. The regular groups E and P occurring in Theorem 1.2 for this case are the two regular subgroups which appear in Theorem 1.1.

The reader may notice that the existence of a regular group of automorphisms implies that the point graph is a Cayley graph with the same automorphism group as the generalised quadrangle. Moreover, since E is normal in the full automorphism group of the Cayley graph, they are *normal* Cayley graphs for E. However P is not normal in the full automorphism group of the Cayley graph, and so when q is not a power of 2 or 3, the point graph is a normal and non-normal Cayley graph for two isomorphic groups. That this is possible answers a question posed to the authors by Yan-Quan Feng and Ted Dobson. The only previous instance of such a phenomenon in the literature known to the authors was a single example studied by Royle [20].

In Section 4 we list all regular subgroups of the small Payne derived generalised quadrangles. For q not a prime there are many more regular subgroups than just the groups E, P and S exhibited in Theorem 1.2. In Section 5 we give an account of how our results relate to previous results and conjectures in the literature.

Group theoretical terminology

Though our group theoretic notation is standard, we briefly review it for the sake of a reader whose interest lies more in geometry than in group theory. We denote a cyclic group of order n by C_n. If g and h are group elements, then we define their *commutator* as $[g,h]=g^{-1}h^{-1}gh$. The *centre* of a group G consists of those elements $z\in G$ that satisfy $[g,z]=1$ for all $g\in G$ and is usually denoted by $Z(G)$. For an element $g\in G$, the *centraliser* of g in G is the set of all elements $z\in G$ such that $[g,z]=1$ and is denoted by $C_G(g)$. If $H,\ K$ are subgroups of a group G, then the *commutator subgroup* $[H,K]$ is generated by all commutators $[h,k]$ where $h\in H$ and $k\in K$. The *derived subgroup* G' of G is defined as $[G,G]$ and is the smallest normal subgroup N such that G/N is abelian. The symbol $\gamma_i(G)$ denotes the i-th term of the *lower central series* of G; that is $\gamma_1(G)=G$, $\gamma_2(G)=G'$, and, for $i\geqslant 3$, $\gamma_{i+1}(G)=[\gamma_i(G),G]$. The *nilpotency class* of a p-group is the smallest c such that $\gamma_{c+1}(G)=1$. The *Frattini subgroup* $\Phi(G)$ of a finite group G is the intersection of all the maximal subgroups. If G is a finite p-group, then $\Phi(G)=G'G^p$ where G^p is the subgroup of G generated by the p-th powers of all elements of G. In particular, $\Phi(G)$ is the smallest normal subgroup N such that G/N is elementary abelian. The *exponent* of a finite group G is the smallest positive n such that $g^n=1$ for all $g\in G$.

A p-group P is called *special* if $Z(P)=P'=\Phi(P)$. It is called *extraspecial* if it is special and these three subgroups all have order p. Extraspecial groups have order p^{1+2n} for some positive integer n and there are two extraspecial groups of each order. When p is odd they are distinguished by their exponent: one has exponent p and the other has exponent p^2. One class of special p-groups are the (3-dimensional) *Heisenberg groups*. These are the p-groups which are isomorphic to a Sylow p-subgroup of $\mathrm{GL}(3,q)$ for $q=p^f$, that is, the group of lower triangular matrices with all entries on the diagonal equal to 1. When q is odd, the Heisenberg groups have exponent p and for $q=p$ are extraspecial.

2. The classical case

The *classical* generalised quadrangles are rank 2 polar spaces, whereby the points and lines are the singular one-dimensional and two-dimensional subspaces (resp.) of a vector space equipped with a quadratic or sesquilinear form. Below is a table listing the thick classical generalised quadrangles together with their orders and automorphism groups:

Generalised quadrangle	Order	Aut. Group
$\mathsf{H}(3,q^2)$	(q^2,q)	$\mathrm{P\Gamma U}(4,q)$
$\mathsf{H}(4,q^2)$	(q^2,q^3)	$\mathrm{P\Gamma U}(5,q)$
$\mathsf{W}(3,q)$	(q,q)	$\mathrm{P\Gamma Sp}(4,q)$
$\mathsf{Q}(4,q)$	(q,q)	$\mathrm{P\Gamma O}(5,q)$
$\mathsf{Q}^-(5,q)$	(q,q^2)	$\mathrm{P\Gamma O}^-(6,q)$

It is well known that the first and last examples above are dual pairs and the third and fourth examples are also dual. In a generalised quadrangle of order (s,t), a simple counting argument shows that the number of points is $(s+1)(st+1)$ and the number of lines is $(t+1)(st+1)$. Now we give a short proof of Theorem 1.1.

Proof of Theorem 1.1. The automorphism group of a classical generalised quadrangle acts primitively on the points of the generalised quadrangle (as it is the natural action of the classical group), and by hypothesis, it contains a regular subgroup G. The classification of all regular subgroups of almost simple primitive groups was established in the monograph [12] of Liebeck, Praeger and Saxl, from which the examples in the above table are precisely the examples that arise in our context. The groups of concern to us are dealt with in [12, Chapters 6 and 10]. The regular subgroups of $\mathrm{P\Gamma U}(n,q)$ in its action on totally isotropic i-spaces and some other actions of classical groups were independently determined by Baumeister [1,2]. The complete results of [12] require the Classification of Finite Simple Groups. However, for an individual family of groups it only requires precise information about the subgroup structure, and for low-dimensional classical groups this can be obtained without the Classification, for example [8,9,14–16,21,23]. □

Some other interesting consequences can be read off from the results of [12], namely: (i) only metacyclic groups can act regularly on the points of a Desarguesian projective plane (cf., [12, Chapter 5]); (ii) the classical generalised hexagons and octagons do not have a group of automorphisms which act regularly on points (cf., [12, Chapter 12] and [2, Theorem 3]).

The examples below can also be found in [2, §10.1].

First example: $\mathcal{Q} = \mathsf{Q}^-(5,2)$

The regular group G arising here is an extraspecial group of order 27 and exponent 3. Let $A \in O^-(2,2)$ be of order 3. Then

$$G = \left\langle \begin{pmatrix} A & 0 & 0 \\ 0 & A^{-1} & 0 \\ 0 & 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A^{-1} \end{pmatrix}, \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{pmatrix} \right\rangle$$

where G preserves an orthogonal decomposition of the 6-dimensional vector space into 3 anisotropic lines. If x is a nontrivial element of G with 1 as an eigenvalue then 1 has multiplicity 2 and the fixed-point space of x is an anisotropic line. Thus the only nontrivial element of G which fixes a singular vector is the identity. Since the order of G is equal to the number of points (i.e., 27), G is regular.

Second example: $\mathcal{Q} = \mathsf{Q}^-(5,2)$

Here G is an extraspecial group of order 27 and exponent 9. Let $A \in O^-(2,2)$ be of order 3. Then

$$G = \left\langle \begin{pmatrix} 0 & A & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & A \\ I & 0 & 0 \end{pmatrix} \right\rangle$$

where again G preserves an orthogonal decomposition of the 6-dimensional vector space into 3 anisotropic lines. The elements of order 9 act irreducibly on the vector space while the elements of order 3 are of the form

$$\begin{pmatrix} A^i & 0 & 0 \\ 0 & A^j & 0 \\ 0 & 0 & A^k \end{pmatrix}$$

where $A^{i+j+k} = I$ with $i, j, k \in \mathbb{Z}_3$ and at least two nonzero. The fixed-point spaces of such elements are anisotropic lines. Thus the only nontrivial element of G which fixes a singular vector is the identity. Since $|G| = 27$ it follows that G is regular.

Third example: $\mathcal{Q} = \mathsf{Q}^-(5,8)$

Here $G \cong \mathrm{GU}(1,2^9).9 \cong C_{513} \rtimes C_9$.

The number of points of $\mathcal{Q} = \mathsf{Q}^-(5,8)$ is $(8+1)(8^3+1)$, and this value is divisible by a primitive prime divisor 19 of 8^6-1. The normaliser of the Sylow 19-subgroup of $\mathrm{P\Gamma O}^-(6,q)$ contains (and is in fact equal to) $(\mathrm{GU}(1,2^9).9):C_2$, where the involution on top is a field automorphism. This is a typical example of a maximal subgroup of $\mathrm{P\Gamma O}^-(6,q)$ in the "extension field groups" Aschbacher class. Now the subgroup $\mathrm{GU}(1,2^9).9$ is irreducible and has order $(8+1)(8^3+1)$. The normal subgroup $\mathrm{GU}(1,2^9)$ is the centraliser of the Sylow 19-subgroup and hence acts semi-regularly, whereas the 9 on top is an automorphism of the field extension, which ensures that $\mathrm{GU}(1,2^9).9$ also acts semi-regularly. Therefore $\mathrm{GU}(1,2^9).9$ acts regularly on the points of $\mathsf{Q}^-(5,8)$.

3. Payne derived generalised quadrangles

Let $q = p^f$ for some prime p and consider the generalised quadrangle $\mathcal{Q} = \mathsf{W}(3,q)$. Let $\mathbf{x}$ be a point of $\mathcal{Q}$. As outlined in the introduction, we can construct a new generalised quadrangle $\mathcal{Q}^x$ whose points are the points of $\mathsf{W}(3,q)$ which are not incident with $\mathbf{x}$ and whose lines of $\mathsf{W}(3,q)$ not incident with $\mathbf{x}$ together with the hyperbolic lines containing $\mathbf{x}$ but not in $\mathbf{x}^\perp$. This generalised quadrangle is referred to as a *Payne derived quadrangle* and has order $(q-1,q+1)$. The automorphism group of $\mathcal{Q}^x$ contains the stabiliser of $\mathbf{x}$ in $\mathrm{P\Gamma Sp}(4,q)$. When $q \geqslant 5$ this is the full automorphism group of $\mathcal{Q}^x$ [11, (2.4) Corollary].

We will use the following setup. Let V be a 4-dimensional vector space over $\mathrm{GF}(q)$, and consider the following alternating form on V:

$$\beta(x,y) := x_1y_4 - y_1x_4 + x_2y_3 - y_2x_3.$$

The totally isotropic subspaces yield the points and lines of the generalised quadrangle $\mathsf{W}(3,q)$, with isometry group $\mathrm{Sp}(4,q)$. Let $\mathbf{x} := \langle(1,0,0,0)\rangle$. Then

$$\mathrm{Sp}(4,q)_{\mathbf{x}} = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ \mathbf{u}^T & A & 0 \\ z & \mathbf{v} & \lambda^{-1} \end{pmatrix} \;\middle|\; A \in \mathrm{GL}(2,q),\ \mathbf{u},\mathbf{v} \in \mathrm{GF}(q)^2,\ z \in \mathrm{GF}(q),\right.$$
$$\left. AJA^T = J,\ \mathbf{u} = \lambda \mathbf{v} J A^T,\ \lambda \in \mathrm{GF}(q) \right\}$$

where $J := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Of particular importance is the subgroup

$$E := \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -c & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix} \;\middle|\; a,b,c \in \mathrm{GF}(q) \right\} \lhd \mathrm{Sp}(4,q)_{\mathbf{x}} \tag{1}$$

which has order q^3. Let $t_{a,b,c}$ be the element of E defined by

$$t_{a,b,c} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ -c & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix}.$$

Then a simple calculation shows that

$$t_{a,b,c}t_{x,y,z} = t_{a+x-bz+cy,b+y,c+z}$$

for any (a,b,c) and (x,y,z). In particular $(t_{a,b,c})^{-1} = t_{-a,-b,-c}$ and

$$[t_{a,b,c}, t_{x,y,z}] = t_{-a-x-bz+cy,-b-y,-c-z}t_{a+x-bz+cy,b+y,c+z} = t_{-2bz+2cy,0,0} \tag{2}$$

We record some properties of E in the following lemma.

Lemma 3.1. *Let E be the group defined in* (1)*.*

1. *E has exponent p.*
2. *For q even, E is an elementary abelian 2-group.*
3. *For q odd, $Z(E) = E' = \Phi(E) = \{t_{a,0,0} \mid a \in \mathrm{GF}(q)\}$.*
4. *For q odd, E is a special group, and for $q = p$, E is extraspecial of exponent p.*

Proof. The first two parts follow as $(t_{a,b,c})^p = t_{0,0,0}$ for all $a, b, c \in \mathrm{GF}(q)$. For q odd, we have that $Z(E) = \{t_{a,0,0} \mid a \in \mathrm{GF}(q)\}$. Since $E/Z(E)$ is elementary abelian, $E' \leqslant Z(E)$ and equality holds since by (2), each element of $Z(E)$ is a commutator. For p-groups, the Frattini subgroup is the smallest normal subgroup such that the quotient is elementary abelian, and so $\Phi(E) = E' = Z(E)$. Thus the last two parts follow. $\square$

Remark 3.2. The group E acts regularly on the points of $\mathsf{W}(3, q)$ not collinear with $\mathbf{x}$ and fixes each line through $\mathbf{x}$, that is, $\mathsf{W}(3, q)$ is an elation generalised quadrangle with elation group E. Moreover, for p odd, E is isomorphic to the (3-dimensional) Heisenberg group.

For $\alpha \in \mathrm{GF}(q)$ define

$$\theta_\alpha := \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha & 1 & 0 & 0 \\ -\alpha^2 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \end{pmatrix} \in \mathrm{Sp}(4, q)_{\mathbf{x}}.$$

Lemma 3.3. *Let $n \geqslant 1$ be an integer and $\alpha \in \mathrm{GF}(q)$. Then*

$$\theta_\alpha^n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -n\alpha & 1 & 0 & 0 \\ \frac{-n(n+1)}{2}\alpha^2 & n\alpha & 1 & 0 \\ \frac{-n(n^2-1)}{6}\alpha^3 & \frac{n(n-1)}{2}\alpha^2 & n\alpha & 1 \end{pmatrix}.$$

Proof. The lemma is clearly true for $n = 1$ so assume it is true for some $n = k - 1 \geqslant 1$. Then

$$\theta_\alpha^k = \theta_\alpha \theta_\alpha^{k-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha & 1 & 0 & 0 \\ -\alpha^2 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -(k-1)\alpha & 1 & 0 & 0 \\ \frac{-(k-1)k}{2}\alpha^2 & (k-1)\alpha & 1 & 0 \\ \frac{-(k-1)k(k-2)}{6}\alpha^3 & \frac{(k-1)(k-2)}{2}\alpha^2 & (k-1)\alpha & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -k\alpha & 1 & 0 & 0 \\ \frac{-k(k+1)}{2}\alpha^2 & k\alpha & 1 & 0 \\ \frac{-k(k^2-1)}{6}\alpha^3 & \frac{k(k-1)}{2}\alpha^2 & k\alpha & 1 \end{pmatrix}.$$

Thus the result follows by induction. $\square$

Corollary 3.4. *For $p > 3$ and $\alpha \in \mathrm{GF}(q)\backslash\{0\}$ the element θ_α has order p while for $p = 2, 3$ the element θ_α has order p^2. In all cases*

$$\theta_\alpha^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & -\alpha & 1 & 0 \\ 0 & \alpha^2 & -\alpha & 1 \end{pmatrix}.$$

Let

$$R := \{t_{a,b,0} \mid a, b \in \mathrm{GF}(q)\}.$$

Then R is an elementary abelian subgroup of E of order q^2. By Remark 3.2, R acts semi-regularly on the set of points of $\mathsf{W}(3,q)$ not collinear with $\mathbf{x}$. Let $Z := \{t_{a,0,0} \mid a \in \mathrm{GF}(q)\}$ and note that $Z = Z(E)$ when q is odd.

For $p = 3$ and $\alpha \in \mathrm{GF}(q)\backslash\{0\}$ we have

$$\theta_\alpha^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\alpha^2 & 0 & 0 & 1 \end{pmatrix} \in Z(E)$$

while for $p = 2$ we have

$$\theta_\alpha^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha^2 & 0 & 1 & 0 \\ \alpha^3 & \alpha^2 & 0 & 1 \end{pmatrix} \in R.$$

We collect together the following relations between the θ_α and elements of E.

Lemma 3.5. *Let $a, b, c, \alpha, \beta \in \mathrm{GF}(q)$.*

1. $\theta_\alpha^{-1} t_{a,b,c} \theta_\alpha = t_{-2\alpha^2 c - 2\alpha b + a, \alpha c + b, c}$.
2. $[t_{a,b,c}, \theta_\alpha] = t_{-\alpha(c^2 + 2\alpha c + 2b), \alpha c, 0}$.
3. $\theta_\alpha \theta_\beta = t_{\alpha^2\beta, \alpha\beta, 0}\theta_{\alpha+\beta}$.
4. $[\theta_\alpha, \theta_\beta] = t_{\alpha\beta(\alpha-\beta),0,0}$.

Proof. The first part follows as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & -\alpha & 1 & 0 \\ 0 & \alpha^2 & -\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -c & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha & 1 & 0 & 0 \\ -\alpha^2 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha - c & 1 & 0 & 0 \\ \alpha c + b & -\alpha & 1 & 0 \\ -\alpha^2 c - \alpha b + a & \alpha^2 + b & -\alpha + c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha & 1 & 0 & 0 \\ -\alpha^2 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -c & 1 & 0 & 0 \\ \alpha c + b & 0 & 1 & 0 \\ -2\alpha^2 c - 2\alpha b + a & \alpha c + b & c & 1 \end{pmatrix}.$$

The second part follows as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ c & 1 & 0 & 0 \\ -b & 0 & 1 & 0 \\ -a & -b & -c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -c & 1 & 0 & 0 \\ \alpha c + b & 0 & 1 & 0 \\ -2\alpha^2 c - 2\alpha b + a & \alpha c + b & c & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha c & 0 & 1 & 0 \\ -\alpha(c^2 + 2\alpha c + 2b) & \alpha c & 0 & 1 \end{pmatrix}.$$

The third part follows from

$$\theta_\alpha\theta_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha & 1 & 0 & 0 \\ -\alpha^2 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ -\beta^2 & \beta & 1 & 0 \\ 0 & 0 & \beta & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha-\beta & 1 & 0 & 0 \\ -\alpha^2-\alpha\beta-\beta^2 & \alpha+\beta & 1 & 0 \\ -\alpha\beta^2 & \alpha\beta & \alpha+\beta & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha\beta & 0 & 1 & 0 \\ \alpha^2\beta & \alpha\beta & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -(\alpha+\beta) & 1 & 0 & 0 \\ -(\alpha+\beta)^2 & \alpha+\beta & 1 & 0 \\ 0 & 0 & \alpha+\beta & 1 \end{pmatrix}.$$

Finally,

$$\theta_\alpha^{-1}\theta_\beta^{-1}\theta_\alpha\theta_\beta \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & -\alpha & 1 & 0 \\ 0 & \alpha^2 & -\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & -\beta & 1 & 0 \\ 0 & \beta^2 & -\beta & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha-\beta & 1 & 0 & 0 \\ -\alpha^2-\alpha\beta-\beta^2 & \alpha+\beta & 1 & 0 \\ -\alpha\beta^2 & \alpha\beta & \alpha+\beta & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha+\beta & 1 & 0 & 0 \\ -\alpha\beta & -\alpha-\beta & 1 & 0 \\ \alpha^2\beta & \alpha^2+\alpha\beta+\beta^2 & -\alpha-\beta & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha-\beta & 1 & 0 & 0 \\ -\alpha^2-\alpha\beta-\beta^2 & \alpha+\beta & 1 & 0 \\ -\alpha\beta^2 & \alpha\beta & \alpha+\beta & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha\beta(\alpha-\beta) & 0 & 0 & 1 \end{pmatrix}. \qquad \square$$

We will need the following lemma.

Lemma 3.6. *Let S be the Sylow p-subgroup of* $\mathrm{GL}(4,q)$ *for $q=p^f$. If $p>3$ then S has exponent p, while if $p=2,3$ then S has exponent p^2.*

Proof. Let $g\in S$. Then the Jordan blocks of g have sizes $1,2,3$ or 4. If $\ell\leqslant p$, a Jordan block of size ℓ has order p as an element of $\mathrm{GL}(\ell,q)$. Hence if $p\geqslant 5$ then g has exponent p. When $p=3$, Jordan blocks of size 4 have order 9, while for $p=2$, Jordan blocks of size 3 and 4 have order 4. Hence for $p=2,3$, S has exponent p^2. $\square$

We will also need the following.

Lemma 3.7. *Let $q=2^f$ with $f\geqslant 2$. Then* $\{\alpha\beta(\alpha+\beta)\mid\alpha,\beta\in\mathrm{GF}(q)\}=\mathrm{GF}(q)$ *for $f\geqslant 3$ and* $\{\alpha\beta(\alpha+\beta)\mid\alpha,\beta\in\mathrm{GF}(4)\}=\{0,1\}$.

Proof. Let $S = \{\alpha\beta(\alpha+\beta) \mid \alpha, \beta \in \mathrm{GF}(q)\}$. We can easily check that $S = \{0, 1\}$ when $q = 4$ so we may assume that $f \geqslant 3$. Clearly $S \neq \{0\}$ and so there exists $x, y \in \mathrm{GF}(q)$ such that $xy(x+y) = a \neq 0$. Then for all $\omega \in \mathrm{GF}(q)$, $(\omega x)(\omega y)(\omega x + \omega y) = \omega^3 a$. Thus if $a \in S$ then S contains $\{\omega^3 a \mid \omega \in \mathrm{GF}(q)\}$. Let T be the set of nonzero cubes in $\mathrm{GF}(q)$. If $T = \mathrm{GF}(q)\backslash\{0\}$ then $S = \mathrm{GF}(q)$ so we may assume that f is even. Then the set of nonzero elements of $\mathrm{GF}(q)$ can be partitioned into the three sets T, ξT and $\xi^2 T$, where ξ is a primitive element of $\mathrm{GF}(q)$. It remains to show that S contains at least one element from each of these three sets, and then the result will follow. Now S contains the subset $X = \{\alpha^2 + \alpha \mid \alpha \in \mathrm{GF}(q)\}$. The map $\alpha \mapsto \alpha^2 + \alpha$ is GF(2)-linear with kernel GF(2). Hence $|X| = 2^{f-1}$. For $f \geqslant 3$, we have $|X| > |T|$ and so X meets at least two of the sets T, ξT and $\xi^2 T$. Thus $|S| \geqslant 2(2^f - 1)/3 > |X|$ and so there exists $\mu \in \mathrm{GF}(q)$ such that the image Y of the GF(2)-linear map $\alpha \mapsto \alpha^2\mu + \alpha\mu^2$ is not X. Then as Y is another GF(2)-subspace of $\mathrm{GF}(q)$ not equal to X it follows that $|X \cap Y| = 2^{f-2}$ and so $|X \cup Y| = 2^f - 2^{f-2} > 2(2^f - 1)/3$. Hence S meets each of T, ξT and $\xi^2 T$ and so $S = \mathrm{GF}(q)$. □

Construction 3.8. *Let $\{\alpha_1, \alpha_2, \ldots, \alpha_f\}$ be a basis for* $\mathrm{GF}(q)$ *over* $\mathrm{GF}(p)$. *Let*

$$P := \langle R, \theta_{\alpha_1}, \ldots, \theta_{\alpha_f} \rangle.$$

Note that P is independent of the choice of basis $\{\alpha_1, \alpha_2, \ldots, \alpha_f\}$ of $\mathrm{GF}(q)$ *over* $\mathrm{GF}(p)$ *since, by Lemma* 3.5(3), *P contains θ_α for all $\alpha \in \mathrm{GF}(q)$.*

Lemma 3.9. *The group P has order q^3 and has the following properties:*

1. *P/R is elementary abelian of order q.*
2. *For $q > 2$, P is nonabelian.*
3. *For $q = 2$, $P \cong C_4 \times C_2$.*
4. *For q odd, $Z(P) = P' = \Phi(P) = Z(E)$.*
5. *For $q > 2$ even $Z(P) = R$. Moreover, $P' = Z$ for $q \geqslant 8$ and $P' = \{t_{0,0,0}, t_{1,0,0}\}$ for $q = 4$.*
6. *For $p > 3$, P has exponent p.*
7. *For $p = 2, 3$, P has exponent p^2.*

Proof. By Lemma 3.5(1) each θ_{α_i} normalises R, and by Lemma 3.5(3), $\theta_\alpha\theta_\beta \in R\theta_{\alpha+\beta}$. Thus $|P| = q^3$ and P/R is isomorphic to the additive group of $\mathrm{GF}(q)$ (and so (1) holds). Hence $P' \leqslant R$. It follows from Lemma 3.5(1) that for $q > 2$, P is nonabelian (and so (2) holds). Moreover, for q odd, $C_R(\theta_\alpha) = Z(E)$ and it follows that $Z(P) = Z(E)$. For q even we have $C_R(\theta_\alpha) = R$ and if $q > 2$ it follows that $Z(P) = R$. For $q = 2$, P is an abelian group of exponent 4 and so is isomorphic to $C_4 \times C_2$. Thus (3) holds.

When q is odd, Lemma 3.5(1) and (3) implies that $P/Z(E)$ is an elementary abelian group of order q^2 and so $P' \leqslant Z(E)$. Moreover, by Lemma 3.5(2) each element of $Z(E)$ is a commutator of elements of P. Thus $P' = \Phi(P) = Z(E)$ and so (4) holds.

For q even, Lemma 3.5(4) implies that for $q > 2$ we have $X = \langle t_{\alpha\beta(\alpha+\beta)0,0} \mid \alpha, \beta \in \mathrm{GF}(q)\rangle \leqslant P'$. Lemma 3.7 implies that $X = Z$ when $q \geqslant 8$ while $X = \{t_{0,0,0}, t_{1,0,0}\}$ for $q = 4$. Since P/X is abelian (Lemma 3.5(2) and (4)), it follows that $P' = X$ and hence (5) holds.

Since P is contained in a Sylow p-subgroup of $\mathrm{GL}(4, q)$, Lemma 3.6 implies that the exponent of P is p for $p \geqslant 5$ (and (6) holds) and at most p^2 for $p = 2, 3$. For $p = 2, 3$, each $\theta_\alpha \in P$ has order p^2 and so the exponent of P is indeed p^2. Therefore, (7) holds. □

Corollary 3.10. *For $p = 2, 3$, $P \not\cong E$.*

Proof. This follows by comparing the exponents of E and P. □

Lemma 3.11. *For $q = p^f$ with $p > 3$, $E \cong P$.*

Proof. Define the map

$$\begin{aligned}\phi:\quad & P \to E,\\ t_{a,b,0} &\mapsto t_{a,b,0}, && a,b\in \mathrm{GF}(q),\\ \theta_{\alpha_i} &\mapsto t_{0,-\alpha_i^2/2,\alpha_i}, && i=1,\dots,f,\end{aligned}$$

which maps a set of generators of P to a set of generators for E. The generators for each group have order p. Since $[\theta_\alpha,\theta_\beta]=t_{\alpha\beta(\alpha-\beta),0,0}=[t_{0,-\alpha^2/2,\alpha},t_{0,-\beta^2/2,\beta}]$ and $[t_{a,b,0},\theta_\alpha]=t_{-2\alpha b,0,0}=[t_{a,b,0},t_{0,-\alpha^2/2,\alpha}]$ this map extends to an isomorphism. □

Lemma 3.12. *The group P acts regularly on the set of points of* W$(3,q)$ *not collinear with* $\mathbf{x}$*. Moreover, P fixes the line $\langle(1,0,0,0),(0,1,0,0)\rangle$ but transitively permutes the remaining q lines through* $\mathbf{x}$*.*

Proof. Consider the image of $\mathbf{y}=\langle(0,0,0,1)\rangle$ under $g\in P$. Then $g=t_{a,b,0}\theta_{\alpha_1}^{n_1}\theta_{\alpha_2}^{n_2}\dots\theta_{\alpha_f}^{n_f}$ for some $t_{a,b,0}\in R$ and integers $n_1,\dots,n_f$. The third coordinate of $(0,0,0,1)^g$ is equal to $n_1\alpha_1+\dots+n_f\alpha_f$. Thus if $g\in P_{\mathbf{y}}$ then $n_1\alpha_1+\dots+n_f\alpha_f=0$. Since $\{\alpha_1,\dots,\alpha_f\}$ is a linearly independent set over GF(p) it follows that p divides n_i for all i. Hence for each i, $\theta_{\alpha_i}^{n_i}\in R$ and so $g\in R_{\mathbf{y}}$. However, R acts semiregularly on the set of points not collinear with $\mathbf{x}$. Thus $g=1$ and since $|P|=q^3$, we have that P acts regularly on the set of points of W$(3,q)$ not collinear with $\mathbf{x}$.

Since each θ_{α_i} induces $\left(\begin{smallmatrix}1&0\\ \alpha_i&1\end{smallmatrix}\right)$ on $\mathbf{x}^\perp/\mathbf{x}$ it follows that P fixes one line through $\mathbf{x}$ and transitively permutes the remaining q. □

For $q=3$ the groups E and P are the two regular subgroups of $Q^-(5,2)$ given in Theorem 1.1.

Lemma 3.13. *Let $\mathcal{Q}^x$ be the generalised quadrangle obtained by Payne derivation from* W$(3,p)$ *for $p\geqslant 5$ a prime and suppose that G acts regularly on the set of points of $\mathcal{Q}^x$. Then $G\cong E\cong P$.*

Proof. Since $|G|=p^3$ and is contained in a Sylow p-subgroup of GL$(4,p)$, by Lemma 3.6, G has exponent p. Thus by inspecting the five groups of order p^3 (namely C_{p^3}, $C_{p^2}\times C_p$, C_p^3 and the two extraspecial groups) we deduce that either $G\cong E$ or G is elementary abelian. By [5, Main Theorem 2.6] the latter is not possible. □

Construction 3.14. *For $q=p^f$ with $f\geqslant 2$, let $U\oplus W$ be a decomposition of* GF(q) *into* GF(p)*-subspaces and let $\{\alpha_1,\dots,\alpha_k\}$ be a basis for U. Define*

$$S_{U,W}=\langle R,\theta_{\alpha_1},\dots,\theta_{\alpha_k},t_{0,0,w}\mid w\in W\rangle.$$

Note that $S_{U,W}$ is independent of the choice of basis $\{\alpha_1,\alpha_2,\dots,\alpha_k\}$ of U over GF(p) *since, by Lemma* 3.5(3)*, $S_{U,W}$ contains θ_α for all $\alpha\in U$.*

Lemma 3.15. *The group $S_{U,W}$ given by Construction* 3.14 *has order q^3 and has the following properties*:

1. *$S_{U,W}$ is nonabelian*;
2. *for q odd, $Z(S_{U,W})=Z(E)$ while*
$$(S_{U,W})'=\left\langle Z(E),t_{0,\alpha_1w_1+\dots+\alpha_kw_k,0}\;\middle|\; w_i\in W\right\rangle$$
which has order qp^ℓ where $\ell=\dim_{\mathrm{GF}(p)}(\alpha_1W+\dots+\alpha_kW)$;
3. *for q even, $Z(S_{U,W})=R$ and*
$$(S_{U,W})'=\begin{cases}\langle Z,t_{\alpha_1w_1^2+\dots+\alpha_kw_k^2,\alpha_1w_1+\dots+\alpha_kw_k,0}\mid w_i\in W\rangle & \text{for } k\geqslant 3,\\ \langle t_{1,0,0},t_{\alpha_1w_1^2+\alpha_2w_2^2,\alpha_1w_1+\alpha_2w_2,0}\mid w_i\in W\rangle & \text{for } k=2,\\ \langle t_{\alpha_1w_1^2,\alpha_1w_1}\mid w_1\in W\rangle & \text{for } k=1;\end{cases}$$

4. *for $p > 3$, $S_{U,W}$ has exponent p;*
5. *for $p = 2, 3$, $S_{U,W}$ has exponent p^2.*

Proof. Now $(\theta_{\alpha_i})^p \in R$ and by Lemma 3.5(1), each θ_{α_i} normalises $\langle R, t_{0,0,w} \mid w \in W\rangle$. Hence $|S_{U,W}| = q^3$. Since θ_{α_i} does not centralise elements $t_{0,0,w}$ for $w \in W\backslash\{0\}$, it follows that $S_{U,W}$ is nonabelian. Hence (1) holds.

For q odd we have that $Z(S_{U,W} \cap E) = Z(E)$. Since $C_E(\theta_{\alpha_i}) = Z(E)$ it follows that $Z(S_{U,W}) = Z(E)$. For q even we have that $S_{U,W} \cap E$ is elementary abelian. Moreover, by Lemma 3.5(1), θ_{α_i} centralises R and so $R \leqslant Z(S_{U,W})$. Since θ_{α_i} does not centralise any element of $S_{U,W} \cap E$ outside R it follows that $Z(S_{U,W}) = R$.

By Lemma 3.5(1), (3) and (4), $S_{U,W}/R$ is elementary abelian of order q and so $(S_{U,W})' \leqslant R$. For q odd all elements of $Z(E)$ can be written as commutators of θ_{α_1} and elements of R (Lemma 3.5(2)). Hence $Z(E) \leqslant (S_{U,W})'$. Moreover, $[\theta_{\alpha_i}, t_{0,0,w}] = t_{-\alpha_i(w^2+2\alpha_i w),\alpha_i w,0}$ for all $w \in W$. Thus

$$X = \bigl\langle Z(E), t_{0,\alpha_1 w_1+\cdots+\alpha_k w_k,0} \bigm| w_i \in W \bigr\rangle \leqslant (S_{U,W})'.$$

By Lemma 3.5(1) and (3), $S_{U,W}/X$ is abelian and so $(S_{U,W})' = X$, which has order qp^ℓ. Thus (2) holds.

For q even, by Lemma 3.5(2) and (4), $[\theta_{\alpha_i}, t_{0,0,w}] = t_{\alpha_i w^2,\alpha_i w,0}$ and $[\theta_{\alpha_i}, \theta_{\alpha_j}] = t_{\alpha_i\alpha_j(\alpha_i-\alpha_j),0,0}$. Thus

$$X = \bigl\langle t_{\alpha\beta(\alpha+\beta),0,0}, t_{\alpha_1 w_1^2+\cdots+\alpha_k w_k^2,\alpha_1 w_1+\cdots+\alpha_k w_k,0} \bigm| w_i \in W,\ \alpha, \beta \in \mathrm{GF}(q)\bigr\rangle \leqslant (S_{U,W})'.$$

Moreover, by Lemma 3.5(1), $\theta_{\alpha_i}^{-1} t_{a,b,c}\theta_{\alpha_i} = t_{a,\alpha_i c+b,c} = t_{\alpha_i c^2,\alpha_i c,0} t_{a,b,c}$. Since E is abelian and $t_{a,b,c} \in S_{U,W}$ if and only if $c \in W$, it follows that $S_{U,W}/X$ is elementary abelian. Hence $(S_{U,W})' = X$. The expression for $(S_{U,W})'$ given in (3) then follows from Lemma 3.7.

Since $S_{U,W}$ is contained in a Sylow p-subgroup of $\mathrm{GL}(4, q)$, Lemma 3.6 implies that the exponent of $S_{U,W}$ is p for $p \geqslant 5$ and at most p^2 for $p = 2, 3$. For $p = 2, 3$, each $\theta_{\alpha_i} \in S_{U,W}$ has order p^2 and so the exponent of $S_{U,W}$ is indeed p^2. Hence (4) and (5) hold. □

Corollary 3.16. *Let U be a 1-dimensional subspace of $\mathrm{GF}(q)$ over $\mathrm{GF}(p)$. Then $P \ncong S_{U,W} \ncong E$.*

Proof. This follows for $q > 4$ by comparing the orders of the derived subgroups. Note that $\ell = \dim(W) = f - 1$. A Magma [4] calculation verifies the result for $q = 4$. □

Remark 3.17. Note that $|S_{U,W} \cap E| = q^2 p^{f-k}$. Since $E \lhd \Gamma\mathrm{Sp}(4, q)_{\mathbf{x}}$ it follows that if $\dim(U_1) \neq \dim(U_2)$ then S_{U_1,W_1} is not conjugate to S_{U_2,W_2} in $\Gamma\mathrm{Sp}(4, q)_{\mathbf{x}}$. However, if $\dim(U_1) = \dim(U_2)$ it is possible for S_{U_1,W_1} to still not be conjugate to S_{U_2,W_2}. For example, when $q = 8$, Magma [4] calculations show that there are two conjugacy classes of subgroups $S_{U,W}$ with U a 2-dimensional subspace.

As for isomorphism classes, sometimes it can be read off from the order of derived subgroups that two such groups are nonisomorphic. For example when $q = 8$ comparing orders of derived subgroups yields $S_{U,W} \ncong S_{W,U}$ when U is a 1-space. Moreover, if U is a 2-space then $S_{U,W} \ncong P$ even though they have derived subgroups of the same order. It is even possible for $S_{U_1,W_1} \ncong S_{U_2,W_2}$ when $\dim(U_1) = \dim(U_2)$. Indeed for $q = 16$, Magma [4] calculations show that there are two isomorphism classes of subgroups $S_{U,W}$ with U a 3-dimensional subspace: one has Frattini subgroup of order 2^7 and one has Frattini subgroup of order 2^8.

Lemma 3.18. *The group $S_{U,W}$ acts regularly on the set of points of $\mathrm{W}(3, q)$ not collinear with $\mathbf{x}$. Moreover, $S_{U,W}$ fixes the line $\langle(1, 0, 0, 0), (0, 1, 0, 0)\rangle$ but permutes the remaining q lines through $\mathbf{x}$ in orbits of length p^k.*

Proof. Consider the image of $\mathbf{y} = \langle(0, 0, 0, 1)\rangle$ under $g \in S_{U,W}$. Then $g = t_{a,b,c}\theta_{\alpha_1}^{n_1}\theta_{\alpha_2}^{n_2}\ldots\theta_{\alpha_k}^{n_k}$ for some $t_{a,b,c} \in E \cap S_{U,W}$ and some integers n_i. The third coordinate of $(0, 0, 0, 1)^g$ is equal to $n_1\alpha_1 + \cdots + n_k\alpha_k + c$ where $c \in W$. Thus if $g \in (S_{U,W})_{\mathbf{y}}$ then $n_1\alpha_1 + \cdots + n_k\alpha_k + c = 0$. Since $\{\alpha_1, \ldots, \alpha_k, c\}$ is

Table 1
The isomorphism types of regular subgroups of the generalised quadrangle of order $(3, 5)$.

Group	#	Comment	Group	#	Comment	Group	#	Comment
9	1		90	7		215	1	special
18	1		92	1		219	1	special
23	6		102	1		224	1	special
32	5		136	2		226	1	special, $D_8 \times D_8$
33	3		138	2		227	1	special
34	1		139	2		232	2	special
35	4		193	1	$S_{U,W}$	241	1	special
56	1	P	199	1		242	1	special, Sylow 2-subgroup of GL(3, 4)
60	4		202	2		264	1	
88	1		206	1		267	1	E

a linearly independent set over GF(p) it follows that p divides each n_i. Hence each $\theta_{\alpha_i}^{n_i} \in R$ and so $g \in R_{\mathbf{y}}$. However, R acts semi-regularly on the set of points not collinear with $\mathbf{x}$. Thus $(S_{U,W})_{\mathbf{y}} = 1$.

Since θ_α induces $\left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)$ on $\mathbf{x}^\perp/\mathbf{x}$ it follows that $S_{U,W}$ fixes one line through $\mathbf{x}$ and permutes the remaining q in orbits of size p^k. $\square$

Proof of Theorem 1.2. The group E is the group defined in (1), the group P is as provided by Construction 3.8 and the group S can be taken to be $S_{U,W}$ given by Construction 3.14 with U a 1-dimensional subspace of GF(q) over GF(p). The theorem then follows from Lemmas 3.1, 3.9, 3.12, 3.15 and 3.18, and Corollaries 3.10 and 3.16. $\square$

4. Small generalised quadrangles

In general, the generalised quadrangle $\mathcal{Q}^x$ of order $(q-1, q+1)$ obtained by Payne deriving $\mathsf{W}(3, q)$ has more regular subgroups than those exhibited in Section 3. In this section we catalogue the point regular groups of automorphisms of small generalised quadrangles. The results suggest that the problem is wild.

For $q = 3$ we have $\mathcal{Q}^x \cong \mathsf{Q}^-(5, 2)$ and so the point regular groups of automorphisms are given by Theorem 1.1. They are simply conjugates of the groups E and P from Section 3.

For $q = 4$ the full automorphism group of $\mathcal{Q}^x$ is $C_2^6 \rtimes (3.A_6.2)$, which acts transitively on the lines of $\mathcal{Q}^x$ (see [19, §V]). Hence the dual of $\mathcal{Q}^x$ is a generalised quadrangle of order $(5, 3)$ with a point-transitive automorphism group.

Example 4.1. Let $\mathcal{Q}^x$ be the generalised quadrangle of order $(3, 5)$ obtained by Payne derivation from W(3, 4). A Magma [4] calculation[1] reveals that Aut($\mathcal{Q}^x$) has 58 conjugacy classes of regular subgroups with 30 different isomorphism classes occuring. In Table 1 we document the number in the Small Group Database of Magma of each isomorphism class and the number of conjugacy classes (indicated by the symbol #) of regular subgroups of that isomorphism type. We also give information about various groups in the list and identify E, P and the $S_{U,W}$. In this case all the $S_{U,W}$ are conjugate in $\mathrm{Sp}(4, 4)_{\mathbf{x}}$. We note that E is normal in Aut($\mathcal{Q}^x$) and is the only abelian regular subgroup. The groups occuring have nilpotency class 1, 2, 3 or 4. Note that the list of regular groups includes a group isomorphic to a Sylow 2-subgroup of GL(3, 4) so it is possible for a Heisenberg group of even order to act regularly on the points of a generalised quadrangle. This was previously believed to not be possible [6, p. 241].

Example 4.2. Let $\mathcal{Q}$ be the generalised quadrangle of order $(5, 3)$, the dual of the generalised quadrangle of order $(3, 5)$ in Example 4.1. Then $\mathcal{Q}$ has 96 points and 64 lines and has automorphism group $C_2^6 \rtimes (3.A_6.2)$ (see [19]). It is known that Aut(Q) contains a regular subgroup on points [7, p. 46]. In

[1] Our use of the computer was not complicated. We simply used the command `Subgroups(G:IsRegular)`, for the most part.

Table 2
Regular subgroups of the generalised quadrangle of order $(5,3)$.

Group	Shape	Notes
H_1	$C_2^4 \rtimes S_3$	$Z(H_1)=1$, $H_1'=C_2^4 \rtimes C_3$
H_2	$2^{2+3} \rtimes C_3$	$Z(H_2)=1$, $H_2'=C_2^4$
H_3	$2^{2+3} \rtimes C_3$	$Z(H_3)=1$, $H_3'=C_4^2$
H_4	$C_4^2 \rtimes S_3$	$Z(H_4)=1$, $H_4'=C_4^2 \rtimes C_3$
H_5	$C_2^4 \rtimes S_3$	$\lvert Z(H_5)\rvert=2$, $H_5'=C_2^3 \rtimes C_3$
H_6	$2^{2+2} \rtimes S_3$	$\lvert Z(H_6)\rvert=2$, $H_6'=Q_8 \rtimes C_3$

Table 3
Numbers of conjugacy classes of point regular subgroups of the Payne derived generalised quadrangle from $W(3,q)$.

q	# regular subgroups	Comments
2	4	2^3, $C_4 \times C_2$, 2 D_8's, conjugacy in $\mathrm{P\Gamma Sp}(4,2)_x$
3	2	this is $\mathsf{Q}^-(5,2)$
4	58	30 isomorphism classes
5	2	E and P
7	2	E and P
8	14	8 isomorphism types, nilpotency class 1 or 2
		2 conjugacy classes of subgroups isomorphic to P
		1 conjugacy class of $S_{U,W}$ with U a 1-space
		2 conjugacy classes (but 1 isomorphism class) of $S_{U,W}$ with U a 2-space
		2 further conjugacy classes of groups isomorphic to $S_{U,W}$ with U a 2-space
9	5	distinct isomorphism types, nilpotency class 2 or 3
		1 class of $S_{U,W}$
11	2	E and P
13	2	E and P
16	231	1 conjugacy class of $S_{U,W}$ for U a 1-space
		2 isomorphism (and conjugacy) classes of $S_{U,W}$ for U a 3-space
		10 conjugacy classes of $S_{U,W}$ for U a 2-space (all isomorphic)
		nilpotency classes 1, 2, 3, 4, 5, 6 and 7
17	2	E and P
19	2	E and P
23	2	E and P
25	7	nilpotency class 2 or 3
		1 conjugacy class of $S_{U,W}$

fact the automorphism group contains 6 different conjugacy classes of regular subgroups on points (by a Magma calculation). They have shape as given in Table 2. By 2^{a+b} we mean a 2-group P with center an elementary abelian group of order 2^a and $P/Z(P)$ is elementary abelian of order 2^b.

For $q \geqslant 5$, the full automorphism group of $\mathcal{Q}^x$ is $\mathrm{P\Gamma Sp}(4,q)_{\mathbf{x}}$ [11], which is not transitive on the lines of $\mathcal{Q}^x$. In Table 3 we list, for certain values of q, the number of conjugacy classes of point regular subgroups of $\mathrm{Aut}(\mathcal{Q})$ where $\mathcal{Q}$ is the generalised quadrangle of order $(q-1,q+1)$ obtained from $\mathsf{W}(3,q)$ obtained by Payne derivation.

Example 4.3. Let $\mathcal{Q}$ be the generalised quadrangle of order $(15,17)$ arising from the Lunelli–Sce hyperoval. Then $\mathcal{Q}$ has 4096 points and 4608 lines. It follows from [3] that its automorphism group is isomorphic to $G = 2^{12} \rtimes H$ where H is the stabiliser in $\Gamma\mathrm{L}(3,16)$ of the hyperoval. The group H has shape $(3_+^{1+2} \times C_5) \rtimes (C_8 \times C_2)$.

A Magma [4] calculation shows that the group G contains 54 conjugacy classes of groups regular on points. This includes the elementary abelian 2-group which is the socle of G. There are 16 isomorphism types of groups and one further conjugacy class of subgroups for which we are unable to determine whether they are isomorphic to any of the former 16 types. There are no special 2-groups on the list. The nilpotency classes of the groups are 1, 2, 3, 4 and 7. There are two conjugacy classes of groups of nilpotency class 7 and all such groups are isomorphic. Every member of this isomorphism class has centre of order 2, derived subgroup of order 2^7 and exponent 16.

It is proved in [7, p. 46] that the generalised quadrangle of order $(17, 15)$ arising as the dual of $\mathcal{Q}$ has no point regular groups of automorphisms. This was confirmed by computer calculations.

5. Conjectures in the literature

We note that the second and third examples of Theorem 1.1 appear to have been overlooked in the classification of subgroups of $\mathrm{P\Gamma U}(4, q)$ transitive on lines given in [14, Corollary 5.12].

In [6] the authors prove

Theorem 5.1. *Let $\mathcal{Q}$ be a generalised quadrangle of order (s, t) admitting a point regular group G, where G is a p-group and p is odd. Suppose $|Z(G)| \geqslant \sqrt[3]{|G|}$. Then the following properties hold.*

1. *We have $t = s + 2$, and there is a generalised quadrangle $\mathcal{Q}'$ of order $s + 1$ with a regular point x, such that $\mathcal{Q}$ is Payne derived from $\mathcal{Q}'$ with respect to x. The generalised quadrangle $\mathcal{Q}'$ is an elation generalised quadrangle with elation group K isomorphic to G.*
2. *We have $|Z(G)| = \sqrt[3]{|G|}$, that is, $|Z(G)| = s + 1$.*

For q odd, the groups P and S from Theorem 1.2 satisfy the hypotheses of Theorem 5.1 with $\mathcal{Q}$ being a generalised quadrangle of order $(q - 1, q + 1)$. Hence there should be generalised quadrangles of order q with elation groups K and K' isomorphic to P and S respectively. However, the only generalised quadrangle of order 3 is $\mathsf{W}(3, 3)$ [17, §6.2] and this does not have an elation group isomorphic to P. The theorem seems to also fail with respect to P for larger values of q.

Thus we also have a counterexample to the following conjecture of [6].

Conjecture 1. *If $\mathcal{Q}$ is a generalised quadrangle admitting a point regular group of automorphisms G, then there exists an elation generalised quadrangle $\mathcal{Q}'$ of order s with elation group G', such that $\mathcal{Q}$ can be obtained from $\mathcal{Q}'$ by Payne derivation with respect to x, and such that $G \cong G'$.*

In [6] the authors also make the following conjecture.

Conjecture 2. *If a finite generalised quadrangle admits a point regular group of automorphisms G, such that G is a p-group, p odd, with the property that $|\mathsf{Z}(G)| \geqslant \sqrt[3]{|G|}$, then G is isomorphic to a Heisenberg group of dimension* 3 *over* $\mathrm{GF}(q)$, *where q is a power of p.*

Moreover, in [7, Conjecture 4.4.1] the following more general conjecture is made:

Conjecture 3. *If a finite thick generalised quadrangle $\mathcal{Q}$ admits a group of automorphisms G which acts regularly on the set of points, then either $\mathcal{Q}$ is the generalised quadrangle of order* $(5, 3)$, *or G is* (1) *an elementary abelian* 2*-group, or* (2) *an odd order Heisenberg group, and in* (1)–(2) *$\mathcal{Q}$ is a Payne derived generalised quadrangle arising in the usual way from an elation generalised quadrangle with elation group isomorphic to G.*

The authors state in [7] that perhaps 'Heisenberg' could be replaced by 'special' in the above conjecture. The groups P for $q = 3^f$ and $S_{U,W}$ for q odd and not a prime are not Heisenberg groups and so are counterexamples to Conjectures 2 and 3. The groups $S_{U,W}$ are not special. Moreover, when q is even, P and $S_{U,W}$ are nonabelian 2-groups acting regularly on a generalised quadrangle and so are further counterexamples.

The example of $C_{513} \rtimes C_9$ acting regularly on the points of $Q^-(5, 8)$ is a particularly interesting counterexample to Conjecture 3 as $Q^-(5, 8)$ is not Payne derived and the group is not nilpotent. It is however meta-abelian, but we saw in Example 4.1 that there are 4 groups acting regularly on the generalised quadrangle of order $(3, 5)$ that are not meta-abelian.

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