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Kernel Density Estimation on Random Fields

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Let Z^N, $N \geqslant 1$, denote the integer lattice points in the N-dimensional Euclidean space. Asymptotic normality of kernel estimators of the multivariate density of stationary random fields indexed by Z^N is established. Appropriate choices of the bandwiths are found. The random fields are assumed to satisfy some mixing conditions. The results apply to many spatial models. © 1990 Academic Press, Inc.

1. INTRODUCTION

The nonparametric estimation of a probability density $f(x)$ is an interesting problem in statistical inference and has an important role in communication and pattern recognition. The literature dealing with density estimation when the observations are independent is extensive. The reader is referred to Wegman [21] for a review. Our goal in this paper is to study density estimation for random variables which show spatial interaction.

Let Z^N, $N \geqslant 1$, denote the integer lattice points in the N-dimensional Euclidean space. Let $\{X_{\mathbf{n}}\}$ be a strictly stationary random field indexed by Z^N and defined on some probability space $(\Omega, \mathscr{F}, P)$. A point $\mathbf{n}$ in Z^N will be referred to as a site and written as $\mathbf{n} = \langle n_1, ..., n_N \rangle$. Let S and S' be two sets of sites. The Borel fields $\mathscr{B}(S) = \mathscr{B}(X_{\mathbf{n}}, \mathbf{n} \in S)$ and $\mathscr{B}(S') = \mathscr{B}(X_{\mathbf{n}}, \mathbf{n} \in S')$ are the σ-fields generated by the random variables $X_{\mathbf{n}}$ with $\mathbf{n}$ elements of S and S', respectively. Let $\hat{d}(S, S')$ be the Euclidean distance between S and S'. We will assume that $X_{\mathbf{n}}$ satisfies the following mixing condition: There exists a function $\varphi(t) \downarrow 0$ as $t \to \infty$, such that whenever S, $S' \subset Z^N$,

$$\alpha(\mathscr{B}(S), \mathscr{B}(S')) = \sup\{|P(AB) - P(A)\,P(B)|,\ A \in \mathscr{B}(S),\ B \in \mathscr{B}(S')\}$$
$$\leqslant \hat{f}(\text{Card}(S), \text{Card}(S'))\,\varphi(\hat{d}(S, S')), \qquad (1.1)$$

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where Card(S) denotes the cardinality of S. Here $\hat{f}$ is a symmetric positive function nondecreasing in each variable. Throughout the paper, assume that $\hat{f}$ satisfies either

$$\hat{f}(n, m) \leqslant \min\{m, n\} \tag{1.2}$$

or

$$\hat{f}(n, m) \leqslant C(n+m+1)^{\tilde{k}}, \tag{1.3}$$

for some $\tilde{k} > 1$ and some $C > 0$. If $\hat{f} \equiv 1$, then $X_{\mathbf{n}}$ is called strongly mixing. In the case $N = 1$, many stochastic processes and time series are known to be strongly mixing. Withers [22] has obtained various conditions for linear processes to be strongly mixing. Under certain weak assumptions autoregressive and more generally bilinear time series models are strongly mixing with exponential mixing rates. See Pham and Tran [15] and Pham [14]. Guyon [5] has shown that the results of Withers extend to random fields $X_{\mathbf{n}} = \Sigma_{Z^N} g_{\mathbf{j}} Z_{\mathbf{n}-\mathbf{j}}$ with the $g_{\mathbf{j}}$'s and $Z_{\mathbf{j}}$'s satisfying certain conditions. Here $Z_{\mathbf{j}}$'s are independent r.v.'s. Conditions (1.2) and (1.3) are the same as the mixing conditions used by Neaderhouser [12] and Takahata [20], respectively, and are weaker than the uniform strong mixing condition used by Nahapetian [10]. They are satisfied by many spatial models. Examples can be found in Neaderhouser [12], Rosenblatt [17], and Guyon [5]. For relevant works on random fields, see, e.g., Neaderhouser [12], Bolthausen [1], Guyon and Richardson [4], and Guyon [5].

Let $I_{\mathbf{n}}$ be a rectangular region defined by $I_{\mathbf{n}} = \{\mathbf{i}\colon \mathbf{i} \in Z^N,\ 1 \leqslant i_k \leqslant n_k,\ k = 1, ..., N\}$. Assume that we observe $\{X_{\mathbf{n}}\}$ on $I_{\mathbf{n}}$. Suppose $X_{\mathbf{n}}$ takes values in R^d and has density $f(x)$. We write $\mathbf{n} \to \infty$ if $\min\{n_k\} \to \infty$ and $|n_j/n_k| < C$ for some $0 < C < \infty$, $1 \leqslant j, k \leqslant N$. Let $\hat{\mathbf{n}} = n_1 \cdots n_N$. The kernel density estimator $f_{\mathbf{n}}(x)$ (see Rosenblatt [16]) of $f(x)$ is defined by

$$f_{\mathbf{n}}(x) = (\hat{\mathbf{n}} b_{\mathbf{n}}^d)^{-1} \sum_{\substack{j_k = 1 \\ k = 1, ..., N}}^{n_k} K((x - X_{\mathbf{j}})/b_{\mathbf{n}}), \tag{1.4}$$

where $b_{\mathbf{n}}$ is a sequence of bandwiths tending to zero as $\mathbf{n}$ tends to infinity. The letter C will be used to denote constants whose values are unimportant and may vary from line to line. All limits are taken as $\mathbf{n} \to \infty$. For a site $\mathbf{i}$, we denote $\|\mathbf{i}\| = (i_1^2 + \cdots + i_N^2)^{1/2}$.

Density estimation for dependent observations ($N = 1$) has received increasing attention recently. See, e.g., Masry [8, 9], Ioannides and Roussas [7], Roussas [18, 19], and the references therein.

Our paper is organized as follows: In Section 2, some preliminaries are presented and the asymptotic variance of $f_{\mathbf{n}}(x)$ is computed. In Section 3, the asymptotic normality of $f_{\mathbf{n}}(x)$ is obtained for random fields satisfying (1.1) and (1.2). Section 4 considers $f_{\mathbf{n}}(x)$ under (1.1) and (1.3).

In the case of independent observations, the asymptotic normality of kernel density estimators was obtained by Parzen [13]. The key issue in this paper is to find appropriate conditions on $b_{\mathbf{n}}$ for $f_{\mathbf{n}}(x)$ to be asymptotically normal. These conditions are explicitly given. They are more involved than those in the independent case. The required rates at which $b_{\mathbf{n}}$ tends to infinity vary with the strengths of dependence.

2. Preliminaries

The following lemma can be found in Ibragimov and Linnik [6] or Deo [2].

LEMMA 2.1. (i) *Suppose* (1.1) *holds. Let* $\mathscr{L}_r(\mathscr{F})$ *denote the class of* $\mathscr{F}$*-measurable r.v.'s* X *satisfying* $\|X\|_r = (E|X|^r)^{1/r} < \infty$. *Let* $X \in \mathscr{L}_r(\mathscr{B}(S))$ *and* $Y \in \mathscr{L}_s(\mathscr{B}(S'))$. *Suppose* $1 \leqslant r, s, h < \infty$ *and* $r^{-1} + s^{-1} + h^{-1} = 1$, *then*

$$|EXY - EXEY| \leqslant C \|X\|_r \|Y\|_s \times \{\hat{f}(\mathrm{Card}(S), \mathrm{Card}(S'))\, \varphi(\hat{d}(S, S'))\}^{1/h}. \tag{2.1}$$

(ii) *For r.v.'s bounded with probability* 1, *the right-hand side of* (2.1) *can be replaced by* $C\hat{f}(\mathrm{Card}(S), \mathrm{Card}(S'))\, \varphi(\hat{d}(S, S'))$.

The kernel function K is assumed to satisfy the following conditions:

Assumption 1. (i) $|K(x)|$ is uniformly bounded by a constant $\tilde{K}$ and

$$\int_{R^d} |K(x)|\, dx < \infty. \tag{2.2}$$

(ii) Assume K has an integrable radial majorant, that is, $Q(x) \equiv \sup\{K(y) \colon \|y\| > \|x\|\}$ is integrable.

Let

$$K_{\mathbf{n}}(x) = (1/b_{\mathbf{n}}^d)\, K(x/b_{\mathbf{n}}), \tag{2.3}$$

$$\eta_{\mathbf{j}}(x) = K_{\mathbf{n}}(x - X_{\mathbf{j}}), \qquad \Delta_{\mathbf{j}}(x) = \eta_{\mathbf{j}}(x) - E\eta_{\mathbf{j}}(x). \tag{2.4}$$

Then

$$f_{\mathbf{n}}(x) = \hat{\mathbf{n}}^{-1} \sum_{\substack{j_k = 1 \\ k = 1, \ldots, N}}^{n_k} K_{\mathbf{n}}(x - X_{\mathbf{j}}). \tag{2.5}$$

Assumption 2. The joint probability density $f_{\mathbf{i},\mathbf{j}}(x, y)$ of $X_{\mathbf{i}}$ and $X_{\mathbf{j}}$ exists and satisfies $|f_{\mathbf{i},\mathbf{j}}(x, y) - f(x) f(y)| \leqslant C$ for some constant C and for all x, y and $\mathbf{i}$, $\mathbf{j}$.

In the case $N = 1$, Assumption 2 has been used by Masry [9]. The proof of the following result uses an argument similar to that of Theorem 3 of Masry.

LEMMA 2.2. *Assume Assumptions* 1 *and* 2 *hold and* $X_{\mathbf{n}}$ *satisfies* (1.1) *and* (1.2) *or* (1.3) *with* $\sum_{i=1}^{\infty} i^{N-1}(\varphi(i))^a < \infty$ *for some* $0 < a < \frac{1}{2}$. *Then*

$$\lim \hat{\mathbf{n}} b_{\mathbf{n}}^d \operatorname{var}[f_{\mathbf{n}}(x)] = f(x) \int_{R^d} K^2(u)\, du. \tag{2.6}$$

Proof. Let

$$\tilde{I}_{\mathbf{n}}(x) = \hat{\mathbf{n}}^{-2} \sum_{\substack{j_k = 1 \\ k = 1, \dots, N}}^{n_k} E\, \Delta_{\mathbf{j}}^2(x), \tag{2.7}$$

$$\tilde{R}_{\mathbf{n}}(x) = \hat{\mathbf{n}}^{-2} \sum_{\substack{j_k = 1 \\ k = 1, \dots, N}}^{n_k} \sum_{\substack{i_k = 1 \\ k = 1, \dots, N \\ i_k \neq j_k \text{ for some } k}}^{n_k} |E\, \Delta_{\mathbf{j}}(x)\, \Delta_{\mathbf{i}}(x)|. \tag{2.8}$$

Then $\operatorname{Var} f_{\mathbf{n}}(x) \leqslant \tilde{I}_{\mathbf{n}}(x) + \tilde{R}_{\mathbf{n}}(x)$. Observe that

$$\begin{aligned} \hat{\mathbf{n}} b_{\mathbf{n}}^d \tilde{I}_{\mathbf{n}}(x) &= b_{\mathbf{n}}^d E\, \Delta_{\mathbf{j}}^2(x) = b_{\mathbf{n}}^d [E\eta_{\mathbf{j}}^2 - (E\eta_{\mathbf{j}})^2] \\ &= (1/b_{\mathbf{n}}^d)[EK^2((x - X_{\mathbf{j}})/b_{\mathbf{n}}) - \{EK((x - X_{\mathbf{j}})/b_{\mathbf{n}})\}^2]. \end{aligned} \tag{2.9}$$

Under Assumption 1, by the Lebesgue density theorem (see Chapter 2 of Devroye and Gyorfi [3]),

$$\begin{aligned} &\lim \int_{R^d} (1/b_{\mathbf{n}}^d)\, K^2((x - u)/b_{\mathbf{n}})\, f(u)\, du = f(x) \int_{R^d} K^2(u)\, du, \\ &\lim \int_{R^d} (1/b_{\mathbf{n}}^d)\, K((x - u)/b_{\mathbf{n}})\, f(u)\, du = f(x) \int_{R^d} K(u)\, du. \end{aligned} \tag{2.10}$$

It is easily seen that $(1/b_{\mathbf{n}}^d)\{EK((x - X_{\mathbf{j}})/b_{\mathbf{n}})\}^2 = b_{\mathbf{n}}^d\{E[(1/b_{\mathbf{n}}^d) K((x - X_{\mathbf{j}})/b_{\mathbf{n}})]\}^2 \to 0$. From (2.9) and (2.10)

$$\lim \hat{\mathbf{n}} b_{\mathbf{n}}^d \tilde{I}_{\mathbf{n}}(x) = f(x) \int_{R^d} K^2(u)\, du. \tag{2.11}$$

Let $c_{\mathbf{n}} = b_{\mathbf{n}}^{-d(1-\gamma)/\nu}$, where $\nu = -N - \varepsilon + (1 - \gamma)\, Na^{-1}$ with γ and ε being small positive numbers such that $a^{-1}-(N+\varepsilon)(N(1-\gamma))^{-1}>1$. This can be done since $0<a<\frac{1}{2}$. Also note that $v>N(1-\gamma)$. Define

$$S_1=\{\mathbf{i},\mathbf{j}\in I_{\mathbf{n}} \mid 0<\hat{d}(\mathbf{i},\mathbf{j})\leqslant c_{\mathbf{n}}\},$$
$$S_2=\{\mathbf{i},\mathbf{j}\in I_{\mathbf{n}} \mid \hat{d}(\mathbf{i},\mathbf{j})>c_{\mathbf{n}}\}. \tag{2.12}$$

Split (2.8) into two separate summations over sites in S_1 and S_2. Let J_1 and J_2 be as defined in (2.14) and (2.16) below. Then

$$\tilde{R}_{\mathbf{n}}(x)\leqslant J_1+J_2. \tag{2.13}$$

Now

$$J_1=\hat{\mathbf{n}}^{-2}\sum_{\mathbf{i},\mathbf{j}\in S_1}\sum\iint_{R^d\times R^d}|K_{\mathbf{n}}(x-u)\,K_{\mathbf{n}}(x-v)|\;|f_{\mathbf{i},\mathbf{j}}(u,v)-f(u)\,f(v)|\,du\,dv. \tag{2.14}$$

Under Assumption 1,

$$J_1\leqslant C\hat{\mathbf{n}}^{-2}\left(\int_{R^d}K(v)\,dv\right)^2\sum_{\mathbf{i},\mathbf{j}\in S_1}\sum 1\leqslant C\hat{\mathbf{n}}^{-1}c_{\mathbf{n}}^N$$
$$=C\hat{\mathbf{n}}^{-1}b_{\mathbf{n}}^{-Nd(1-\gamma)/v}=o(\hat{\mathbf{n}}^{-1}b_{\mathbf{n}}^{-d}), \tag{2.15}$$

since $v>N(1-\gamma)$. Turning to J_2, we have

$$J_2=\hat{\mathbf{n}}^{-2}\sum_{\mathbf{i},\mathbf{j}\in S_2}\sum|\mathrm{cov}\{K_{\mathbf{n}}(x-X_{\mathbf{i}}),K_{\mathbf{n}}(x-X_{\mathbf{j}})\}|. \tag{2.16}$$

Let $\delta=2(1-\gamma)/\gamma$. Note that $\gamma=2/(2+\delta)$ and $\delta/(2+\delta)=1-\gamma$. Applying Lemma 2.1 with $r=s=2+\delta$, $h=(2+\delta)/\delta$,

$$|\mathrm{cov}\{K_{\mathbf{n}}(x-X_{\mathbf{i}}),K_{\mathbf{n}}(x-X_{\mathbf{j}})\}|$$
$$\leqslant C(E\,|K_{\mathbf{n}}(x-X_{\mathbf{i}})|^{2+\delta})^{\gamma}\,\{\hat{f}(1,1)\,\varphi(\hat{d}(\{\mathbf{i}\},\{\mathbf{j}\}))\,\mathrm{Card}(\{\mathbf{i}\})\}^{1-\gamma}$$
$$\leqslant C\left(\int_{R^d}|K_{\mathbf{n}}(x-u)|^{2+\delta}f(u)\,du\right)^{\gamma}\{\varphi(\|(\mathbf{i}-\mathbf{j}\|)\}^{1-\gamma}. \tag{2.17}$$

Employing (2.17)

$$J_2\leqslant C\hat{\mathbf{n}}^{-2}\sum_{\mathbf{i},\mathbf{j}\in S_2}\sum\left(\int_{R^d}|K_{\mathbf{n}}(x-u)|^{2+\delta}f(u)\,du\right)^{\gamma}\{\varphi(\|\mathbf{i}-\mathbf{j}\|)\}^{1-\gamma}$$
$$\leqslant C\hat{\mathbf{n}}^{-2}b_{\mathbf{n}}^{-\gamma d(1+\delta)}\left(\int_{R^d}(1/b_{\mathbf{n}}^d)\,|K((x-u)/b_{\mathbf{n}})|^{2+\delta}f(u)\,du\right)^{\gamma}$$
$$\times\sum_{\mathbf{i},\mathbf{j}\in S_2}\sum\{\varphi(\|\mathbf{i}-\mathbf{j}\|)\}^{1-\gamma}. \tag{2.18}$$

Clearly

$$\sum_{\mathbf{i},\mathbf{j}\in S_2}\sum \{\varphi(\|\mathbf{i}-\mathbf{j}\|)\}^{1-\gamma} \leqslant \hat{\mathbf{n}} \sum_{\|\mathbf{i}\|>c_{\mathbf{n}}} \{\varphi(\|\mathbf{i}\|)\}^{1-\gamma}. \tag{2.19}$$

Combining (2.18), (2.19),

$$\hat{\mathbf{n}} b_{\mathbf{n}}^d J_2 \leqslant b_{\mathbf{n}}^{-d(1-\gamma)} \left(\int_{R^d} (1/b_{\mathbf{n}}^d)\, |K((x-u)/b_{\mathbf{n}})|^{2+\delta} f(u)\, du \right)^{\gamma}$$

$$\times \sum_{\|\mathbf{i}\|>c_{\mathbf{n}}} \{\varphi(\|\mathbf{i}\|)\}^{1-\gamma}. \tag{2.20}$$

By assumption, $\sum_{i=1}^{\infty} i^{N-1}(\varphi(i))^a < \infty$. Thus $i^{N-1}(\varphi(i))^a = o(1/i)$ or $\varphi(i) = o(i^{-N/a})$ as $i \to \infty$. Since φ is a nonincreasing function, we have $\varphi(x) = o(x^{-N/a})$ as $x \to \infty$. Therefore

$$\|\mathbf{i}\|^{v} \{\varphi(\|\mathbf{i}\|)\}^{1-\gamma} = \|\mathbf{i}\|^{v}\, o(\|\mathbf{i}\|^{-N(1-\gamma)/a}) = o(\|\mathbf{i}\|^{-N-\varepsilon}),$$

since $v = -N - \varepsilon + (1-\gamma)\, Na^{-1}$. Thus

$$\sum_{\substack{i_k=1 \\ k=1,\ldots,N}}^{\infty} \|\mathbf{i}\|^{v} \{\varphi(\|\mathbf{i}\|)\}^{1-\gamma} < \infty. \tag{2.21}$$

Using (2.20), (2.21) and noting that $b_{\mathbf{n}}^{-d(1-\gamma)} c_{\mathbf{n}}^{-v} = 1$, we obtain

$$\begin{aligned} \limsup \hat{\mathbf{n}} b_{\mathbf{n}}^d J_2 &\leqslant C \limsup b_{\mathbf{n}}^{-d(1-\gamma)} \sum_{\|\mathbf{i}\|>c_{\mathbf{n}}} \{\varphi(\|\mathbf{i}\|)\}^{1-\gamma} \\ &\leqslant C \limsup b_{\mathbf{n}}^{-d(1-\gamma)} c_{\mathbf{n}}^{-v} \sum_{\|\mathbf{i}\|>c_{\mathbf{n}}} \|\mathbf{i}\|^{v} \{\varphi(\|\mathbf{i}\|)\}^{1-\gamma} \\ &\leqslant C \limsup \sum_{\|\mathbf{i}\|>c_{\mathbf{n}}} \|\mathbf{i}\|^{v} \{\varphi(\|\mathbf{i}\|)\}^{1-\gamma}, \end{aligned} \tag{2.22}$$

which tends to zero since $c_{\mathbf{n}} \to \infty$.

3. Asymptotic Normality of $f_{\mathbf{n}}(x)$ under (1.1) and (1.2)

We will need the following lemma from Nakhapeytyan [11]:

Lemma 3.1. *Let* $(\xi_1, \ldots, \xi_n)$ *be a random vector such that* $|E \prod_{s=i}^{n} \xi_i| < \infty$, $i = 1, \ldots, n-1$, $|C\xi_i| \leqslant 1$, $i = 1, \ldots, n$. *Then*

$$\left|E\prod_{s=1}^{n}\xi_s-\prod_{s=1}^{n}E\xi_s\right|$$
$$\leqslant\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left|E(\xi_i-1)(\xi_j-1)\right.$$
$$\left.\times\prod_{s=j+1}^{n}\xi_s-E(\xi_i-1)\,E(\xi_j-1)\prod_{s=j+1}^{n}\xi_s\right|.$$

LEMMA 3.2. *Suppose $X_{\mathbf{n}}$ satisfies (1.1) and (1.2) with $\sum_{i=1}^{\infty} i^{N-1}(\varphi(i))^a < \infty$ for some $0<a<\frac{1}{2}$. Suppose also that Assumptions 1 and 2 and the following conditions hold for some $0<\gamma<1$.*

(i) *The bandwith $b_{\mathbf{n}}$ tends to zero in a manner such that $\hat{\mathbf{n}} b_{\mathbf{n}}^{d(1+(1-\gamma)2N)} \to \infty$.*

(ii) *There exists a sequence of positive integers $q=q_{\mathbf{n}}\to\infty$ with $q=o((\hat{\mathbf{n}} b_{\mathbf{n}}^{d(1+(1-\gamma)2N)})^{1/(2N)})$ such that $\mathbf{n}\sum_{i=1}^{\infty} i^{N-1}\varphi(iq)\to 0$.*

(iii) *$b_{\mathbf{n}}$ tends to zero in such a manner that*

$$b_{\mathbf{n}}^{-d(1-\gamma)}\sum_{i=q}^{\infty} i^{N-1}(\varphi(i))^{1-\gamma}\to 0 \qquad as \quad \mathbf{n}\to\infty. \tag{3.1}$$

Let

$$\sigma^2=f(x)\int_{R^d}K^2(u)\,du. \tag{3.2}$$

Then $(\hat{\mathbf{n}} b_{\mathbf{n}}^d)^{1/2}([f_{\mathbf{n}}(x)-Ef_{\mathbf{n}}(x)]/\sigma)$ has a standard normal distribution as $\mathbf{n}\to\infty$.

Proof. By (i) and (ii), there exists a sequence of positive integers $\{s_{\mathbf{n}}\}$ tending to infinity such that

$$s_{\mathbf{n}}q_{\mathbf{n}}=o((\hat{\mathbf{n}} b_{\mathbf{n}}^{d(1+(1-\gamma)2N)})^{1/(2N)}). \tag{3.3}$$

Choose $p_{\mathbf{n}}=p=[(\hat{\mathbf{n}} b_{\mathbf{n}}^d)^{1/(2N)}/s_{\mathbf{n}}]$. By (3.3), $qp^{-1}\leqslant Cb_{\mathbf{n}}^{d(1-\gamma)}$ which tends to zero as $\mathbf{n}\to\infty$. Thus $q<Cp$. Multiplying $s_{\mathbf{n}}$ by a constant if necessary, it can be assumed without loss of generality that $q<p$. Assume for some integers $r_1, ..., r_N$, we have $n_1=r_1(p+q), ..., n_N=r_N(p+q)$. Define $Y_{\mathbf{j}}=b_{\mathbf{n}}^{d/2}\Delta_j$ and

$$S_{\mathbf{n}}=\sum_{\substack{j_k=1\\k=1,...,N}}^{n_k} Y_{\mathbf{j}}.$$

Then

$$S_{\mathbf{n}}=\hat{\mathbf{n}} b_{\mathbf{n}}^{d/2}[f_{\mathbf{n}}(x)-Ef_{\mathbf{n}}(x)].$$

The r.v.'s $Y_{\mathbf{j}}$'s are now set into large blocks and small blocks. Let

$$U(1, \mathbf{n}, x, \mathbf{j}) = \sum_{\substack{i_k = j_k(p+q)+1 \\ k=1,\dots,N}}^{j_k(p+q)+p} Y_{\mathbf{i}}(x),$$

$$U(2, \mathbf{n}, x, \mathbf{j}) = \sum_{\substack{i_k = j_k(p+q)+1 \\ k=1,\dots,N-1}}^{j_k(p+q)+p} \sum_{i_N = j_N(p+q)+p+1}^{(j_N+1)(p+q)} Y_{\mathbf{i}}(x),$$

$$U(3, \mathbf{n}, x, \mathbf{j}) = \sum_{\substack{i_k = j_k(p+q)+1 \\ k=1,\dots,N-2}}^{j_k(p+q)+p} \sum_{i_{N-1} = j_{N-1}(p+q)+p+1}^{(j_{N-1}+1)(p+q)} \sum_{i_N = j_N(p+q)+1}^{j_N(p+q)+p} Y_{\mathbf{i}}(x),$$

$$U(4, \mathbf{n}, x, \mathbf{j}) = \sum_{\substack{i_k = j_k(p+q)+1 \\ k=1,\dots,N-2}}^{j_k(p+q)+p} \sum_{i_{N-1} = j_{N-1}(p+q)+p+1}^{(j_{N-1}+1)(p+q)} \sum_{i_N = j_N(p+q)+p+1}^{(j_N+1)(p+q)} Y_{\mathbf{i}}(x)$$

and so on. Note that

$$U(2^{N-1}, \mathbf{n}, x, \mathbf{j}) = \sum_{\substack{i_k = j_k(p+q)+p+1 \\ k=1,\dots,N-1}}^{(j_k+1)(p+q)} \sum_{i_N = j_N(p+q)+1}^{j_N(p+q)+p} Y_{\mathbf{i}}(x).$$

Finally

$$U(2^N, \mathbf{n}, x, \mathbf{j}) = \sum_{\substack{i_k = j_k(p+q)+p+1 \\ k=1,\dots,N}}^{(j_k+1)(p+q)} Y_{\mathbf{i}}(x).$$

For each integer $1 \leqslant i \leqslant 2^N$, define

$$T(\mathbf{n}, x, i) = \sum_{\substack{j_k = 0 \\ k=1,\dots,N}}^{r_k - 1} U(i, \mathbf{n}, x, \mathbf{j}). \tag{3.4}$$

Clearly $S_{\mathbf{n}} = \sum_{i=1}^{2^N} T(\mathbf{n}, x, i)$. Note that $T(\mathbf{n}, x, 1)$ is the sum of the random variables $Y_{\mathbf{i}}$ in large blocks. The $T(\mathbf{n}, x, i)$, $2 \leqslant i \leqslant 2^N$ are sums of random variables in small blocks. If it is not the case that $n_1 = r_1(p+q), \dots, n_N = r_N(p+q)$ for some integers $r_1, \dots, r_N$, then a term, say, $T(\mathbf{n}, x, 2^N+1)$, containing all the $Y_{\mathbf{j}}$'s at the ends not included in the big or small blocks can be added. This term will not change the proof much. The general approach is to show that as $\mathbf{n} \to \infty$,

$$Q_1 \equiv \left| E\exp[iuT(\mathbf{n}, x, 1)] - \prod_{\substack{j_k = 0 \\ k=1,\dots,N}}^{r_k - 1} E\exp[iuU(1, \mathbf{n}, x, \mathbf{j})] \right| \to 0. \tag{3.5}$$

$$Q_2 \equiv \hat{\mathbf{n}}^{-1} E\left(\sum_{i=2}^{2^N} T(\mathbf{n}, x, i) \right)^2 \to 0. \tag{3.6}$$

$$Q_3 \equiv \hat{\mathbf{n}}^{-1} \sum_{\substack{j_k=0 \\ k=1,\dots,N}}^{r_k-1} E[U(1, \mathbf{n}, x, \mathbf{j})]^2 \to \sigma^2. \tag{3.7}$$

$$Q_4 \equiv \hat{\mathbf{n}}^{-1} \sum_{\substack{j_k=0 \\ k=1,\dots,N}}^{r_k-1} E[(U(1, \mathbf{n}, x, \mathbf{j}))^2 I\{|U(1,\mathbf{n},x,\mathbf{j})| > \varepsilon\sigma\hat{\mathbf{n}}^{1/2}\}] \to 0 \tag{3.8}$$

for every $\varepsilon > 0$. Note that

$$(\hat{\mathbf{n}} b_{\mathbf{n}}^d)^{1/2} [f_{\mathbf{n}}(x) - Ef_{\mathbf{n}}(x)]/\sigma = S_{\mathbf{n}}/(\sigma \hat{\mathbf{n}}^{1/2})$$
$$= T(\mathbf{n}, x, 1)/(\sigma \hat{\mathbf{n}}^{1/2}) + \sum_{i=2}^{2^N} T(\mathbf{n}, x, i)/(\sigma \hat{\mathbf{n}}^{1/2}).$$

The term $\sum_{i=2}^{2^N} T(\mathbf{n}, x, i)/(\sigma \hat{\mathbf{n}}^{1/2})$ is asymptotically negligible by (3.6). The r.v.'s $U(1, \mathbf{n}, x, \mathbf{j})$ are asymptotically independent by (3.5). The asymptotic normality of $T(\mathbf{n}, x, 1)/(\sigma \hat{\mathbf{n}}^{1/2})$ follows from (3.7) and the Lindeberg–Feller condition (3.8). Lemma 3.2 thus follows from (3.5)–(3.8). The argument here is reminiscient of those of Masry [9] and Nakhapetyan [11]. It remains to show (3.5) to (3.8).

Proof of (3.5). Enumerate the r.v.'s $U(1, \mathbf{n}, x, \mathbf{j})$ in an arbitrary manner and refer to them as $\tilde{U}_1, \dots, \tilde{U}_M$. Note that $M = \prod_{k=1}^N r_k = \hat{\mathbf{n}}(p+q)^{-N} \leqslant \hat{\mathbf{n}} p^{-N}$. Let

$$I(1, \mathbf{n}, x, \mathbf{j}) = \{i : j_k(p+q) + 1 \leqslant i_k \leqslant j_k(p+q) + p\}.$$

Distinct sets of sites $I(1, \mathbf{n}, x, \mathbf{j})$ are far apart by a distance of at least q. Clearly $I(1, \mathbf{n}, x, \mathbf{j})$ contains p^N sites. $I(1, \mathbf{n}, x, \mathbf{j})$ is the set of sites involved with $U(1, \mathbf{n}, x, \mathbf{j})$. Lemma 3.1 shows

$$Q_1 \leqslant \sum_{k=1}^{M-1} \sum_{j=k+1}^{M} \Big| E(\exp[iu\tilde{U}_k] - 1) \times (\exp[iu\tilde{U}_j] - 1) \prod_{s=j+1}^{M} \exp[iu\tilde{U}_s] - E(\exp[iu\tilde{U}_k] -) E(\exp[iu\tilde{U}_j] - 1) \prod_{s=j+1}^{M} \exp[iu\tilde{U}_s] \Big|.$$

Let $\tilde{I}_j$ be the sets of sites involved with $\tilde{U}_j$. An application of Lemma 2.1(ii) gives

$$|E(\exp[iu\tilde{U}_k] - 1)(\exp[iu\tilde{U}_j] - 1) - E(\exp[iu\tilde{U}_k] - 1) E(\exp[iu\tilde{U}_j] - 1)|$$
$$\leqslant C\varphi(\hat{d}(\tilde{I}_j, \tilde{I}_k))\, p^N$$

Thus

$$\begin{aligned}
Q_1 &\leqslant Cp^N \sum_{k=1}^{M-1} \sum_{j=k+1}^{M} \varphi(\hat{d}(\tilde{I}_j, \tilde{I}_k)) \\
&\leqslant Cp^N M \sum_{k=2}^{M} \varphi(\hat{d}(\tilde{I}_1, \tilde{I}_k)) \\
&\leqslant Cp^N M \sum_{i=1}^{\infty} \sum_{k: iq \leqslant \hat{d}(\tilde{I}_1, \tilde{I}_k) < (i+1)q} \varphi(\hat{d}(\tilde{I}_1, \tilde{I}_k)). \\
&\leqslant Cp^N M \sum_{i=1}^{\infty} i^{N-1} \varphi(iq) \\
&\leqslant C\hat{\mathbf{n}} \sum_{i=1}^{\infty} i^{N-1} \varphi(iq),
\end{aligned}$$

which tends to zero by Condition (ii).

Proof of 3.6. To prove (3.6), it is enough to show that

$$\hat{\mathbf{n}}^{-1} E[T(\mathbf{n}, x, i)]^2 \to 0 \qquad \text{for each} \quad 2 \leqslant i \leqslant 2^N. \tag{3.9}$$

Without loss of generality, consider $E[T(\mathbf{n}, x, 2)]^2$. Enumerate the r.v.'s $U(2, \mathbf{n}, x, \mathbf{j})$ in an arbitrary manner and refer to them as $\hat{U}_1, ..., \hat{U}_M$. Now

$$\begin{aligned}
E[T(\mathbf{n}, x, 2)]^2 &= \sum_{j=0}^{M} \operatorname{var}(\hat{U}_i) + 2 \sum_{i=1}^{M} \sum_{\substack{j=1 \\ i>j}}^{M} \operatorname{cov}(\hat{U}_i, \hat{U}_j) \\
&\equiv A_1 + A_2.
\end{aligned} \tag{3.10}$$

Since $X_{\mathbf{n}}$ is stationary,

$$\begin{aligned}
\operatorname{var}(\hat{U}_i) &= \operatorname{var}\Bigg(\sum_{\substack{i_k=1 \\ k=1,...,N-1}}^{p} \sum_{i_N=1}^{q} Y_{\mathbf{i}}(x) \Bigg)^2 \\
&= p^{N-1} q \operatorname{var} Y_{\mathbf{i}}(x) \\
&\quad + \sum_{\substack{j_k=1 \\ k=1,...,N-1}}^{p} \sum_{j_N=1}^{q} \sum_{\substack{i_k=1 \\ k=1,...,N-1 \\ i_k \neq j_k \text{ for some } 1 \leqslant k \leqslant N}}^{p} \sum_{i_N=1}^{q} EY_{\mathbf{j}}(x)\, Y_{\mathbf{i}}(x).
\end{aligned} \tag{3.11}$$

From (2.4) and the Lebesgue density theorem

$$\begin{aligned}
\operatorname{Var} Y_{\mathbf{i}}(x) &\leqslant b_{\mathbf{n}}^d E K_{\mathbf{n}}^2(x - X_{\mathbf{j}}) \\
&= \int_{R^d} (1/b_{\mathbf{n}}^d)\, K^2((x-u)/b_{\mathbf{n}})\, f(u)\, du \leqslant C.
\end{aligned} \tag{3.12}$$

Let $\delta = 2(1-\gamma)/\gamma$. Again employing Lemma 2.1 and the Lebesgue density theorem as in (2.17),

$$
\begin{aligned}
E\,|Y_{\mathbf{j}}(x)\;Y_{\mathbf{i}}(x)|
&\leqslant C\left(\int_{R^d} |b_{\mathbf{n}}^{-d/2}K((x-u)/b_{\mathbf{n}})|^{2+\delta} f(u)\,du\right)^{\gamma} \{\varphi(\hat{d}(\{\mathbf{i}\},\{\mathbf{j}\}))\}^{1-\gamma}\\
&\leqslant C b_{\mathbf{n}}^{-d(1-\gamma)}\{\varphi(\|\mathbf{i}-\mathbf{j}\|)\}^{1-\gamma}. \qquad (3.13)
\end{aligned}
$$

Applying (3.11), (3.12), and (3.13),

$$
\begin{aligned}
\operatorname{var}(\hat{U}_i) &\leqslant Cp^{N-1}q\left(1+b_{\mathbf{n}}^{-d(1-\gamma)} \sum_{\substack{i_k=1\\ k=1,\ldots,N-1}}^{p} \sum_{i_N=1}^{q} \{\varphi(\|\mathbf{i}\|)\}^{1-\gamma}\right)\\
&\leqslant Cp^{N-1}qb_{\mathbf{n}}^{-d(1-\gamma)} \sum_{\substack{i_k=1\\ k=1,\ldots,N-1}}^{p} \sum_{i_N=1}^{q} \{\varphi(\|\mathbf{i}\|)\}^{1-\gamma}. \qquad (3.14)
\end{aligned}
$$

By (3.10) to (3.14)

$$
A_1 \leqslant CMp^{N-1}qb_{\mathbf{n}}^{-d(1-\gamma)} \sum_{i=1}^{\infty} i^{N-1}(\varphi(i))^{1-\gamma}. \qquad (3.15)
$$

Let

$$
\begin{aligned}
I(2,n,x,\mathbf{j}) = \{\mathbf{i}\colon & j_k(p+q+1) \leqslant i_k \leqslant j_k(p+q)+p,\ 1\leqslant k\leqslant N-1,\\
& j_N(p+q)+p+1 \leqslant i_N \leqslant (j_N+1)(p+q)\}.
\end{aligned}
$$

Then $U(2,\mathbf{n},x,\mathbf{j})$ is the sum of $Y_{\mathbf{i}}$ with sites in $I(2,n,x,\mathbf{j})$. Since $p>q$, if $\mathbf{j}$ and $\mathbf{j}'$ belong to two distinct sets $I(2,\mathbf{n},x,\mathbf{j})$ and $I(2,\mathbf{n},x,\mathbf{j}')$, then $j_k \neq j'_k$ for some $1\leqslant k\leqslant N$ and $\|\mathbf{j}-\mathbf{j}'\|>q$. With (3.13), we obtain

$$
\begin{aligned}
A_2 &\leqslant C \sum_{\substack{j_k=1\\ k=1,\ldots,N}}^{n_k} \sum_{\substack{i_k=1\\ k=1,\ldots,N\\ \|\mathbf{i}-\mathbf{j}\|>q}}^{n_k} EY_{\mathbf{i}}(x)\;Y_{\mathbf{j}}(x)\\
&\leqslant Cb_{\mathbf{n}}^{-d(1-\gamma)}\hat{\mathbf{n}} \sum_{\substack{i_k=1\\ k=1,\ldots,N\\ \|i\|>q}}^{n_k} \{\varphi(\|\mathbf{i}\|)\}^{1-\gamma}\\
&\leqslant Cb_{\mathbf{n}}^{-d(1-\gamma)}\hat{\mathbf{n}} \sum_{i=q}^{\infty} i^{N-1}(\varphi(i))^{1-\gamma}. \qquad (3.16)
\end{aligned}
$$

From (3.10), (3.15), and (3.16),

$$\hat{\mathbf{n}}^{-1}E[T(\mathbf{n}, x, 2)]^2 \leqslant CMp^{N-1}q\hat{\mathbf{n}}^{-1}b_{\mathbf{n}}^{-d(1-\gamma)}\sum_{i=1}^{\infty} i^{N-1}(\varphi(i))^{1-\gamma}$$
$$+ Cb_{\mathbf{n}}^{-d(1-\gamma)}\sum_{i=q}^{\infty} i^{N-1}(\varphi(i))^{1-\gamma}. \tag{3.17}$$

Next,

$$\begin{aligned} Mp^{N-1}q\hat{\mathbf{n}}^{-1}b_{\mathbf{n}}^{-d(1-\gamma)} &= \hat{\mathbf{n}}(p+q)^{-N}p^{N-1}q\hat{\mathbf{n}}^{-1}b_{\mathbf{n}}^{-d(1-\gamma)} \\ &\leqslant (q/p)b_{\mathbf{n}}^{-d(1-\gamma)} \\ &\leqslant qs_{\mathbf{n}}(\hat{\mathbf{n}}b_{\mathbf{n}}^{d})^{-1/(2N)}b_{\mathbf{n}}^{-d(1-\gamma)} \\ &= qs_{\mathbf{n}}(\hat{\mathbf{n}}b_{\mathbf{n}}^{d(1+(1-\gamma)2N)})^{-1/(2N)}, \end{aligned} \tag{3.18}$$

which tends to zero by (3.3). The last term of (3.17) tends to zero by assumption. The proof of (3.6) is completed by (3.17) and (3.18)

Proof of (3.7). Let

$$S'_{\mathbf{n}} = T(\mathbf{n}, x, 1), \qquad S''_{\mathbf{n}} = \sum_{i=2}^{2^N} T(\mathbf{n}, x, i). \tag{3.19}$$

Then $S'_{\mathbf{n}}$ is the sum of r.v.'s $Y_{\mathbf{j}}$ in large blocks and $S''_{\mathbf{n}}$ is the sum of r.v.'s in small blocks. Lemma 2.2 implies $\hat{\mathbf{n}}^{-1}E|S_{\mathbf{n}}|^2 \to \sigma^2$. This combined with (3.6) shows $\hat{\mathbf{n}}^{-1}E|S'_{\mathbf{n}}|^2 \to \sigma^2$. Now

$$\hat{\mathbf{n}}^{-1}E|S'_{\mathbf{n}}|^2 = \hat{\mathbf{n}}^{-1}\sum_{\substack{j_k=0\\k=1,\dots,N}}^{r_k-1} E[U(1, \mathbf{n}, x, \mathbf{j})]^2 + \hat{\mathbf{n}}^{-1}$$
$$\times \sum_{\substack{j_k=0\\k=1,\dots,N}}^{r_k-1}\ \sum_{\substack{i_k=0\\k=1,\dots,N\\i_k\neq j_k \text{ for some } k}}^{r_k-1} \operatorname{cov}\{U(1, \mathbf{n}, x, \mathbf{j})\, U(1, \mathbf{n}, x, \mathbf{i})\}. \tag{3.20}$$

Observe that (3.7) follows from (3.20) if the last term of (3.20) tends to zero as $\mathbf{n} \to \infty$. By the same argument used in obtaining a bound for A_2 defined in (3.10), the last term of (3.20) is bounded by

$$Cb_{\mathbf{n}}^{-d(1-\gamma)}\sum_{\substack{i_k=0\\k=1,\dots,N\\\|i\|>q}}^{r_k-1}\{\varphi(\|\mathbf{i}\|)\}^{1-\gamma}$$
$$\leqslant Cb_{\mathbf{n}}^{-d(1-\gamma)}\sum_{i=q}^{\infty} i^{N-1}(\varphi(i))^{1-\gamma}, \tag{3.21}$$

which tends to zero by Condition (3.1).

Proof of (3.8). Clearly $|Y_{\mathbf{j}}| \leqslant Cb_{\mathbf{n}}^{-d/2}$. Therefore $|U(1, \mathbf{n}, x, \mathbf{j})| < Cp^N b_{\mathbf{n}}^{-d/2}$. Hence

$$Q_4 \leqslant Cp^{2N} b_{\mathbf{n}}^{-d} \hat{\mathbf{n}}^{-1} \sum_{\substack{j_k=0 \\ k=1,\dots,N}}^{r_k-1} P[U(1, \mathbf{n}, x, \mathbf{j}) > \varepsilon\sigma\hat{\mathbf{n}}^{1/2}]. \tag{3.22}$$

Now

$$U(1, \mathbf{n}, x, \mathbf{j})/(\sigma\hat{\mathbf{n}}^{1/2}) \leqslant Cp^N(\hat{\mathbf{n}} b_{\mathbf{n}}^d)^{-1/2} \to 0,$$

since $p = [(\hat{\mathbf{n}} b_{\mathbf{n}}^d)^{1/(2N)}/s_{\mathbf{n}}]$, where $s_{\mathbf{n}} \to \infty$. Thus $P[U(1, \mathbf{n}, x, \mathbf{j}) > \varepsilon\sigma\hat{\mathbf{n}}^{1/2}] = 0$ for all $\mathbf{j}$ for sufficiently large $\hat{\mathbf{n}}$. Thus $Q_4 = 0$ for large $\hat{\mathbf{n}}$.

THEOREM 3.1. *Suppose $X_{\mathbf{n}}$ satisfies* (1.1) *and* (1.2) *with $\varphi(x) = O(x^{-\mu})$ for some $\mu > 2N$. Let $0 < \gamma < (\mu - N)\mu^{-1}$. Suppose there exists a sequence of positive integers $q = q_{\mathbf{n}} \to \infty$ such that*

$$q = o((\hat{\mathbf{n}} b_{\mathbf{n}}^{d(1+(1-\gamma)2N)})^{1/(2N)}).$$

If Assumptions 1 *and* 2 *hold and $b_{\mathbf{n}}$ tends to zero in such a manner that*

$$\hat{\mathbf{n}} q^{-\mu} \to 0, \tag{3.23}$$

$$b_{\mathbf{n}}^{-d(1-\gamma)} q^{N-\mu(1-\gamma)} \to 0, \tag{3.24}$$

then $(\hat{\mathbf{n}} b_{\mathbf{n}}^d)^{1/2}([f_{\mathbf{n}}(x) - Ef_{\mathbf{n}}(x)]/\sigma)$ has a standard normal distribution as $\mathbf{n} \to \infty$.

Proof. Note that $\sum_{i=1}^{\infty} i^{N-1}(\varphi(i))^a < \infty$ for some $N\mu^{-1} < a < \frac{1}{2}$; and

$$\hat{\mathbf{n}} \sum_{i=1}^{\infty} i^{N-1}\varphi(iq) \leqslant C\hat{\mathbf{n}} \sum_{i=1}^{\infty} i^{N-1}(iq)^{-\mu} = C\hat{\mathbf{n}} q^{-\mu}$$

which tends to zero by (3.23). Clearly $\mu(1-\gamma) > N$ and

$$\sum_{i=q}^{\infty} i^{N-1}(\varphi(i))^{1-\gamma} \leqslant C \sum_{i=q}^{\infty} i^{N-1-\mu(1-\gamma)} \leqslant Cq^{N-\mu(1-\gamma)}.$$

Hence (3.24) implies (3.1). The theorem then follows from Lemma 3.1.

Remark 3.1. Neither (3.23) implies (3.24) or vice versa. By (3.23),

$$[\hat{\mathbf{n}} q^{-\mu}]^{(\mu(1-\gamma)-N)(1/\mu)} \to 0$$

or

$$\hat{\mathbf{n}}[(\mu(1-\gamma)-N)/\mu] q^{N-\mu(1-\gamma)} \to 0.$$

The ratio, say, R of the left-hand side of (3.23) over the left-hand side of (3.24) is

$$\hat{\mathbf{n}}^{[\mu(1-\gamma)-N]/\mu} b_{\mathbf{n}}^{d(1-\gamma)}.$$

R is marginally close to $b_{\mathbf{n}}^{d(1-\gamma)}$ if $\mu(1-\gamma)-N$ is small and close to $[\hat{\mathbf{n}} b_{\mathbf{n}}^d]^{(1-\gamma)}$ if $\mu(1-\gamma) \gg N$. Hence R tends to zero in the former case and infinity in the latter case.

In the important case that $\varphi(x)$ tends to zero at an exponential rate, we have

THEOREM 3.2. *Suppose $X_{\mathbf{n}}$ satisfies (1.1), (1.2) with $\varphi(x)=O(e^{-\xi x})$ for some $\xi>0$. Suppose Assumptions 1 and 2 hold and $b_{\mathbf{n}}$ tends to zero in such a manner that*

$$(\hat{\mathbf{n}} b_{\mathbf{n}}^{d(1+(1-\gamma)2N)})^{(1/2N)} (\log \hat{\mathbf{n}})^{-1} \to \infty \tag{3.25}$$

for some $0<\gamma<1$, then $(\hat{\mathbf{n}} b_{\mathbf{n}}^d)^{1/2} ([f_{\mathbf{n}}(x)-Ef_{\mathbf{n}}(x)]/\sigma)$ has a standard normal distribution as $\mathbf{n} \to \infty$ with σ as defined in (3.2).

Proof. By (3.25), there exists a positive function $g(\mathbf{n})$ increasing to infinity that $(\hat{\mathbf{n}} b_{\mathbf{n}}^{d(1+(1-\gamma)2N)})^{(1/2N)} (g(\mathbf{n}) \log \hat{\mathbf{n}})^{-1} \to \infty$. Let $q = (\hat{\mathbf{n}} b_{\mathbf{n}}^{d(1+(1-\gamma)2N)})^{(1/2N)} (g(\mathbf{n}))^{-1}$. For abitrarily $C>0$, $q \geqslant C \log \hat{\mathbf{n}}$ for sufficiently large $\hat{\mathbf{n}}$. Thus

$$\begin{aligned}
\hat{\mathbf{n}} \sum_{i=1}^{\infty} i^{N-1} \varphi(iq) &\leqslant C \hat{\mathbf{n}} \sum_{i=1}^{\infty} i^{N-1} e^{-\xi iq} \\
&= C \hat{\mathbf{n}} e^{-\xi q} \sum_{i=0}^{\infty} (i+1)^{N-1} e^{-\xi iq} \\
&\leqslant C \hat{\mathbf{n}} \exp(-C\xi \log \hat{\mathbf{n}}) = C \hat{\mathbf{n}}^{-C\xi+1},
\end{aligned}$$

which tends to zero by choosing $C>1/\xi$. Next for $\xi'<\xi$,

$$\begin{aligned}
\sum_{i=q}^{\infty} i^{N-1} \varphi(i)^{1-\gamma} &\leqslant C \sum_{i=q}^{\infty} i^{N-1} e^{-\xi i(1-\gamma)} \\
&\leqslant C \sum_{i=q}^{\infty} e^{-\xi' i(1-\gamma)} \leqslant C e^{-\xi' q(1-\gamma)}.
\end{aligned}$$

Note that $b_{\hat{\mathbf{n}}}^d \geqslant C \hat{\mathbf{n}}^{-1}$ and $q > C \log \hat{\mathbf{n}}$. It is easily verified that (iii) of Lemma 3.2 is satisfied.

Remark 3.2. In the case $N=1$ and for γ close to 1, Condition (3.25) is marginally close to the condition tht $nb_n^d \to \infty$ of the independent case.

If $Ef_{\mathbf{n}}(x)-f(x)\to 0$ sufficiently fast, then $(\hat{\mathbf{n}}b_{\mathbf{n}}^{d})^{1/2}[f_{\mathbf{n}}(x)-f(x)]/\sigma \to N(0,1)$ under the conditions of Theorems 3.1 and 3.2, e.g., under the following additional assumption:

Assumption 3. Suppose K is a probability density function on R^d and for any x, $y\in R^d$ and some constant $\rho>0$

$$|f(x)-f(y)|\leqslant \rho\,\|x-y\|.$$

THEOREM 3.3. *Assume the conditions of Theorems* 3.1 *and* 3.2 *are satisfied except with* (2.2) *replaced by the stricter condition that* $\int_{R^d}\|x\|\,|K(x)|\,dx<\infty$. *Suppose in addition that Assumption* 3 *is satisfied and* $\hat{\mathbf{n}}b_{\mathbf{n}}^{d+2}\to 0$. *Then* $(\hat{\mathbf{n}}b_{\mathbf{n}}^{d})^{1/2}([f_{\mathbf{n}}(x)-f(x)]/\sigma)$ *has a standard normal distribution as* $\mathbf{n}\to\infty$.

Proof. Following the proof of Lemma 3.1 of Roussas [19] and using Assumption 3 and the fact that $\int_{R^d}\|x\|\,|K(x)|\,dx<\infty$,

$$\begin{aligned}|Ef_{\mathbf{n}}(x)-f(x)| &= \left|\int_{R^d}K(z)\,f(x)\,dz-\int_{R^d}K(z)\,f(x-b_{\mathbf{n}}z)\,dz\right| \\ &\leqslant \rho b_{\mathbf{n}}\int_{R^d}\|z\|\,|K(z)|\,dz=Cb_{\mathbf{n}}. \end{aligned}\tag{3.26}$$

From (3.26)

$$(\hat{\mathbf{n}}b_{\mathbf{n}}^{d})^{1/2}\frac{[Ef_{\mathbf{n}}(x)-f(x)]}{\sigma}\leqslant C(\hat{\mathbf{n}}b_{\mathbf{n}}^{d})^{1/2}\,b_{\mathbf{n}}\to 0,\tag{3.27}$$

since $\hat{\mathbf{n}}b_{\mathbf{n}}^{d+2}\to 0$. The theorem follows from Theorems 3.1 and 3.2 and (3.27).

4. ASYMPTOTIC NORMALITY OF $f_{\mathbf{n}}(x)$ UNDER (1.1) AND (1.3)

LEMMA 4.1. *Suppose* $X_{\mathbf{n}}$ *satisfies* (1.1) *and* (1.3) *with* $\sum_{i=1}^{\infty} i^{N-1}(\varphi(i))^{a}<\infty$ *for some* $0<a<\frac{1}{2}$. *Suppose also that Assumptions* 1 *and* 2 *and the following conditions hold*:

(i) *The bandwith* $b_{\mathbf{n}}$ *tends to zero in a manner such that* $\hat{\mathbf{n}}b_{\mathbf{n}}^{d(1+(1-\gamma)2N)}\to\infty$ *for some* $0<\gamma<1$.

(ii) *There exists a sequence of positive integers* $q=q_{\mathbf{n}}\to\infty$ *with* $q=o((\hat{\mathbf{n}}b_{\mathbf{n}}^{d(1+(1-\gamma)2N)})^{1/(2N)})$ *such that* $\hat{\mathbf{n}}(\hat{\mathbf{n}}b_{\mathbf{n}}^{d})^{(k-1)/2}\sum_{i=1}^{\infty} i^{N-1}\varphi(iq)\to 0$, *where* $\tilde{k}$ *is the constant of* (1.3).

(iii) $b_{\mathbf{n}}$ *tends to zero as* $\mathbf{n} \to \infty$ *in such a manner that*

$$b_{\mathbf{n}}^{-d(1-\gamma)} \sum_{i=q}^{\infty} i^{N-1} (\varphi(i))^{-1-\gamma} \to 0 \qquad as \quad \mathbf{n} \to \infty. \tag{4.1}$$

Let σ^2 *be as defined in* (3.2). *Then* $(\hat{\mathbf{n}} b_{\mathbf{n}}^d)^{1/2}([f_{\mathbf{n}}(x) - Ef_{\mathbf{n}}(x)]/\sigma)$ *has a standard normal distribution as* $\mathbf{n} \to \infty$.

Proof. The proof is a slight variation of the argument of Lemma 3.2. The only significant difference is the verification of (3.5). Let $\tilde{U}_1, ..., \tilde{U}_M$ be as in Lemma 3.2. Note that now

$$|E(\exp[iu\tilde{U}_k] - 1)(\exp[iu\tilde{U}_j] - 1) - E(\exp[iu\tilde{U}_k] - 1) E(\exp[iu\tilde{U}_j] - 1)|$$
$$\leqslant C\varphi(\hat{d}(\tilde{I}_j, \tilde{I}_k))[p^N + p^N + 1]^{\bar{k}} \leqslant C\varphi(\hat{d}(\tilde{I}_j, \tilde{I}_k)) p^{N\bar{k}}.$$

Thus

$$Q_1 \leqslant Cp^{N\bar{k}} M \sum_{i=1}^{\infty} i^{N-1} \varphi(iq) \leqslant C\hat{\mathbf{n}} p^{N(\bar{k}-1)}$$
$$\leqslant C\hat{\mathbf{n}}[(\hat{\mathbf{n}} b_{\mathbf{n}}^d)^{1/(2N)}/s_{\mathbf{n}}]^{N(\bar{k}-1)} \sum_{i=1}^{\infty} i^{N-1} \varphi(iq),$$

which tends to zero by Condition (ii) of the lemma.

THEOREM 4.1. *Suppose* $X_{\mathbf{n}}$ *satisfies* (1.1) *and* (1.3) *with* $\varphi(x) = O(x^{-\mu})$ *for some* $\mu > 2N$. *Let* $0 < \gamma < (\mu - N)\mu^{-1}$. *Suppose there exists a sequence of positive integers* $q = q_{\mathbf{n}} \to \infty$ *such that* $q = o((\hat{\mathbf{n}} b_{\mathbf{n}}^{d(1+(1-\gamma)2N)})^{1/(2N)})$. *If Assumptions* 1 *and* 2 *hold and* $b_{\mathbf{n}}$ *tends to zero in such a manner that*

$$\hat{\mathbf{n}}(\hat{\mathbf{n}} b_{\mathbf{n}}^d)^{(\bar{k}-1)/2} q^{-\mu} \to 0, \tag{4.2}$$

$$b_{\mathbf{n}}^{-d(1-\gamma)} q^{N-\mu(1-\gamma)} \to 0, \tag{4.3}$$

then $(\hat{\mathbf{n}} b_{\mathbf{n}}^d)^{1/2} [f_{\mathbf{n}}(x) - Ef_{\mathbf{n}}(x)]/\sigma$ *has a standard normal distribution as* $\mathbf{n} \to \infty$.

Using Lemma 4.1, Theorem 4.1 can be obtained by an argument similar to the proof of Theorem 3.1. Analogues of Theorems 3.2 and 3.3 can also be obtained under (1.1) and (1.3). The details are omitted for brevity.

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REFERENCES

[1] BOLTHAUSEN, E. (1982). On the central limit theorem for stationary random fields. *Ann. Probab.* **10** 1047–1050.

[2] DEO, C. M. (1973). A note on empirical processes of strong mixing sequences. *Ann. Probab.* **1** 870–875.

[3] DEVROYE, L., AND GYORFI, L. (1985). *Nonparametric Density Estimation: The L1 View*, Wiley, New York.

[4] GUYON, X., AND RICHARDSON, S. (1984). Vitesse de convergence du théoreme de la limite centrale pour des champs faiblement dépendants. *Z. Wahrsch. Verw. Gebiete* **66** 297–314.

[5] GUYON, X. (1987). Estimation d'un champ par pseudo-varisemblable conditionelle: Etude asymptotique et application au cas Markovien. *Proc.* 6th *Franco–Belgian Meeting of Statisticians.*

[6] IBRAGIMOV, I. A., AND LINNIK, YU. V. (1971). *Independent and Stationary Sequences of Random Variables.* Wolters–Noordhoff, Groningen.

[7] IOANNIDES, D. A., AND ROUSSAS, G. G. (1987). Note on the uniform convergence of density estimates for mixing random variables. *Statist. Probab. Lett.* **5** 279–285.

[8] MASRY, E. (1983). Probability density estimation from sampled data. *IEEE Trans. Inform. Theory* **IT-29** 696–709.

[9] MASRY, E. (1986). Recursive probability density estimation from weakly dependent processes. *IEEE Trans. Inform. Theory* **IT-32** 254–267.

[10] NAPAHETIAN, B. S. (1980). The central limit theorem for random fields with mixing conditions. *Adv. in Probability.* Vol. 6. *Multicomponent Systems* (R. L. Dobrushin and Ya. G. Sinai, Eds.), pp. 531–548, Dekker, New York.

[11] NAKHAPETYAN, B. S. (1987). An approach to proving limit theorems for dependent random variables. *Theory Probab. Appl.* **32** 535–539.

[12] NEADERHOUSER, C. C. (1980). Convergence of blocks spins defined on random fields. *J. Statist Phys.* **22** 673–684.

[13] PARZEN, E. (1962). On estimation of a probability density function and mode. *Ann. Math. Statist.* **33** 1065–1076.

[14] PHAM, T. D. (1986). The mixing properties of bilinear and generalized random coefficient autoregressive models. *Stochastic Process. Appl.* **23** 291–300.

[15] PHAM, D. T., AND TRAN, L. T. (1985). Some strong mixing properties of time series models. *Stochastic Process. Appl.* **19** 297–303.

[16] ROSENBLATT, M. (1956). Remarks on some nonparametric estimates of a density function. *Ann. Math. Statist.* **27** 832–837.

[17] ROSENBLATT, M. (1985). *Stationary Sequences and Random Fields.* Birkhauser, Boston.

[18] ROUSSAS, G. G. (1969). Nonparametric estimation of the distribution function of a Markov process. *Ann. Math. Statist.* **40** 1386–1400.

[19] ROUSSAS, G. G. (1988). Nonparametric estimation in mixing sequences of random variables. *J. Statist. Plann. Inference* **18** 135–149.

[20] TAKAHATA, H. (1983). On the rates in the central limit theorem for weakly dependent random fields. *Z. Wahrsch. Verw. Gebiete* **64** 445–456.

[21] WEGMAN, E. J. (1972). Nonparametric probability density estimation. I. A summary of available methods. *Technometrics* **14** 533–546.

[22] WITHERS, C. S. (1981). Conditions for linear processes to be strong mixing. *Z. Wahrsch. Verw. Gebiete* **57** 477–480.