Computers and Mathematics with Applications 58 (2009) 1418–1424

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# An efficient dynamical systems method for solving singularly perturbed integral equations with noise

N.H. Sweilam [a,*], A.M. Nagy [b], M.H. Alnasr [c]

[a] Cairo University, Faculty of Science, Mathematics Department, Giza, Egypt
[b] Benha University, Faculty of Science, Mathematics Department, Benha, Egypt
[c] Qatar University, Faculty of Science, Department of Mathematics, Doha, Qatar

ARTICLE INFO

*Article history:*
Received 27 August 2008
Received in revised form 14 April 2009
Accepted 12 June 2009

*Keywords:*
Dynamical systems method (DSM)
Variational regularization method (VRM)
Ill-posed problems

ABSTRACT

In this paper we apply the dynamical systems method (DSM) proposed by A. G. Ramm, and the variational regularization method (VRM), to obtain numerical solution to some singularly perturbed ill-posed problems contaminated by noise. The results obtained by these methods are compared to the exact solution for the model problems. It is found that the dynamical systems method is preferable because it is easier to apply, highly stable, robust, and it always converges to the solution even for large size models.


In recent years, interest has substantially increased in the solution of singularly perturbed problems, see for example [1] and the references cited therein. A singularly perturbation problem is a problem that depends on a parameter in such a way that solutions behave non-uniformly as the parameter tends towards the limiting value of interest. Singularly perturbed problems arise in various fields of science and engineering such as fluid mechanics, fluid dynamics, elasticity, quantum mechanics, chemical reactor theory, convection diffusion processes, optimal control and other branches of applied mathematics. Such these problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts and varies slowly in some other parts (see for example [2,3] and the references cited therein). The analytic solution of such problems usually exhibits thin transition layers, in which the solution varies rapidly. When the standard numerical methods are used to solve such problem on a coarse mesh, large oscillations may arise and pollute the numerical solution on the entire interval of integration.

The purpose of this paper is to present a numerical study of DSM and VRM for the singularly perturbed integral equations. The singularly perturbed linear Fredholm integral equations of the second kind is of the form:

$$\epsilon y(t) = g(t) + \int_a^b K(t,s)y(s)\mathrm{d}s, \quad t \in I : (a,b), \tag{1}$$

while the singularly perturbed linear Volterra integral equation is of the form:

$$\epsilon y(t) = g(t) + \int_0^t K(t,s)y(s)\mathrm{d}s, \quad t \in I : [0,T] \tag{2}$$

where $y$ is an unknown function. Both $g$ and the kernel $K(t,s)$ are given functions and $0 < \epsilon \ll 1$. A simple model problem of Eq. (2) is given by the following equation:

$$\epsilon y(t) + \int_0^t a(s)y(s)\mathrm{d}s = g(t), \quad t \geq 0 \tag{3}$$

* Corresponding author.
*E-mail addresses:* n_sweilam@yahoo.com (N.H. Sweilam), abdelhameed_nagy@yahoo.com (A.M. Nagy), modialnasr@qu.edu.qa (M.H. Alnasr).

0898-1221/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2009.06.040

which is important because of its relationship to stiff initial value problem for linear first order differential equations where $a$ and $g$ are smooth functions, $a(t) > 0$. The derivative of (3) yields

$$\epsilon y'(t) + a(t)y(t) = g'(t), \quad t \geq 0,\ y(0) = \frac{g(0)}{\epsilon}. \tag{4}$$

Moreover the linear overdamped initial value problem is of the form

$$\epsilon y(t) + \int_0^t [a_0(s) + (t-s)a_1(s)]y(s)\mathrm{d}s = g(t), \quad t \geq 0 \tag{5}$$

where $a_0(s), a_1(s)$ and $g(t)$ are smooth functions. Eq. (5), which is special case from (2), is important because of its relationship to the stiff initial value problem for linear second order differential equations. Some typical problems of type (4) and (5) are electrical-circuit problems with large resistance and/or small induction, mechanical problems with small masses and/or large damping, and the propagation of radiation through a high absorbing medium, for more details see [4]. By using discretization techniques like the $\theta$-method, see [5], the Galerkin method with an orthonormal basis or quadrature method, see [6,7] and [8], Eqs. (1) and (2) can be written as a linear system

$$f = A_\epsilon u. \tag{6}$$

Problem (6) is called a discrete ill-posed problem if the matrix $A_\epsilon$ is ill- conditioned, that is the condition number is large and the singular values of $A_\epsilon$ decay gradually to zero. Also the inverse of $A_\epsilon$ may not exist or it may be unbounded. In practice the left hand side $f$ in (6) is measured with some error, so $f_\delta$ is known, $\|f_\delta - f\| \leq \delta$. In general, the system in (6) is known to be severely ill-posed in the sense that a small change in the data may result in a dramatic change in the solution. Due to the severely ill-posedness of the problem, numerical computation is very difficult. Our goal in this paper is to use the merits of DSM and VRM to compute a stable approximate solution to the above system.

## 1. Analysis of the DSM

The DSM analysis [9–19] (see also [20,21]) is based on a construction of a dynamical systems with the trajectory; by using the Cauchy problem for nonlinear differential equations in a Hilbert space; starting from an initial approximation point and having a solution to the problem

$$F(u) = Au - f = 0, \quad u \in H, \tag{7}$$

where $H$ is a Hilbert space and $A$ is a linear operator in $H$ which is not necessarily bounded but closed and densely defined. It is proved in [18] that if Eq. (7) is solvable and $\|f - f_\delta\| \leq \delta$, the following results hold:

**Theorem 1.** *Assume that $f = Ay$, $y \perp N(A)$, $A$ is a linear operator, closed and densely defined in $H$. Consider the problem*

$$u' = -u + T_{\epsilon(t)}^{-1}A^*f, \quad u(0) = u_0,$$

*where $N(A) := \{u : Au - f = 0\}$, $u_0 \in H$ is arbitrary, $T_\epsilon = T + \epsilon(t)$, $T = A^*A$, $\epsilon = \epsilon(t)$ is a continuous function monotonically decaying to zero at $t \to \infty$ and $\int_0^\infty \epsilon(s)\mathrm{d}s = \infty$. Then problem (7) has a unique solution $u(t)$ defined on $[0, \infty)$, and the following limit exists:*

$$\lim_{t\to\infty} u(t) := u(\infty) \quad \textit{and} \quad u(\infty) = y.$$

It is pointed out in [18] that if $f_\delta$ is given in place of $f$, calculate its solution $u_\delta(t)$ as $t = t_\delta$, it can be proved that:

$$\lim_{\delta\to 0} \|u_\delta(t_\delta) - y\| = 0.$$

If suppose that $t_\delta$ is suitable chosen. The DSM to solve Eq. (7) consists of solving the Cauchy problem [22]

$$u' = -P(Au(t) - f), \qquad u(0) = u_0, \qquad u_0 \perp N, \qquad u' = \frac{\mathrm{d}u}{\mathrm{d}t}, \tag{8}$$

and proving the existence of the limit $\lim_{t\to\infty} u(t) = u(\infty)$ and the relation $u(\infty) = y$; i.e.,

$$\lim_{t\to\infty} \|u(t) - y\| = 0.$$

Here $P$ is a bounded operator such that $T := PA \geq 0$ is selfadjoint, and denote by $y$ the unique minimal-norm solution to (7), $y \perp N := N(A) = \{u : Au = 0\}$; $Ay = f$. We can make a choice of $P$; namely, $P = (A^*A + aI)^{-1}A^*$; $a = \text{const} > 0$.

The unique solution of Eq. (8) is:

$$u(t) = u_0 \mathrm{e}^{-tT} + \mathrm{e}^{-tT} \int_0^t \mathrm{e}^{sT}\mathrm{d}sPf. \tag{9}$$

For more details on DSM see ([23,22] and [9–19]). Eq. (9) leads to the following iterative formula [22]:

$$u_{n+1} = u_n - (A^*A + aI)^{-1}(AA^*u_n - A^*f_\delta), \quad u_0 = 0. \tag{10}$$

Iteration formula (10) will terminate if $u_n$ satisfies the following condition:

$$\|Au_n - f_\delta\| \le 1.01\delta.$$

Also, as suggested in [22] we can choose $a$ that satisfy the condition

$$\delta \le \phi(a) := \|A(A^*A + aI)^{-1}A^*f_\delta - f_\delta\| \le 2\delta. \tag{11}$$

by the following iterations:

1. As an initial guess for $a$ one takes $a = \frac{\delta\|A\|^2}{3f_\delta}$.
2. Compute $\phi(a)$. If it satisfies (11), then we are done. Otherwise, we go to step 3.
3. If $\frac{\phi(a)}{\delta} = c > 3$, then one takes $a = \frac{a}{2(c-1)}$; as go back to step 2. If $2 < c \le 3$ then one takes $a = \frac{a}{2(c-1)}$ and go back to step 2. Otherwise, we go to step 4.
4. If $\frac{\phi(a)}{\delta} = c < 1$, then $a := 3a$ if the inequality $c < 1$ has occurred in an earlier iteration, we stop the iteration and use $3a$ as our choice for $a$ in iteration (10). Otherwise we go to back to step 2.

## 2. Variational regularization method

This method is the most common and well known technique for regularizing ill-posed problems (see [24,9,25] and [26]). This method attempts to provide a good estimate of the solution of (7) by a solution $u_{\alpha,\delta}$ of the problem

$$\min\{\|Au - f_\delta\|^2 + \alpha\|u\|^2\}, \tag{12}$$

where $\alpha$ is the regularization parameter, $u_{\alpha,\delta}$ is the regularization solution, $f_\delta$ is a noisy data and $\|f - f_\delta\| \le \delta$. The global minimizer of the quadratic functional (9) is the unique solution to the linear system $(A^*A + \alpha I)u_{\alpha,\delta} = A^*f_\delta$, where $I$ is the unit matrix. This system has a unique solution $u_{\alpha,\delta} = (A^*A+\alpha I)^{-1}A^*f_\delta$. To determine the suitable $\alpha$, see algorithm 1 in [23].

## 3. Application

In this section we will consider a discrete quadrature method for solving numerically the singularly perturbed linear integral Eqs. (1) and (2). In Example 1, we will apply the standard discrete quadrature method as in [6] to approximate the integrals as follows:

$$\int_a^b y(s)\mathrm{d}s \approx \sum_{j=1}^{n} w_j y(s_j),$$

where $s_1, \dots, s_n$ are abscissas and $w_1, \dots, w_n$ are corresponding weights. Then the integral in (1) is approximated by

$$\epsilon y(t) = g(t) + \sum_{j=1}^{n} w_j K(t, s_j) y(s_j).$$

The most straightforward discretization procedure is to equate (or collocate) the left and right hand side of (1) in $n$ points $t_1, t_2, \dots, t_n$:

$$\epsilon y(t_i) = g(t_i) + \sum_{j=1}^{n} w_j K(t_i, s_j) y(s_j), \quad i, j = 1, 2, \dots, n.$$

By using midpoint rule we take $w_i = \frac{b-a}{n}$ and $t_i = s_j = a + \frac{(j-0.5)(b-a)}{n}$. Hence we obtain a system of linear algebraic equations

$$\epsilon y = g + Ky, \quad \Rightarrow (\epsilon I - K)y = g, \qquad \Rightarrow A_\epsilon x = b, \tag{13}$$

where $K$ is an $n \times n$ matrix and $I$ is the identity matrix. The elements of the matrix $K$, the right-hand side $g$ and the solution vector $f$ are given by

$$k_{ij} = w_j K(t_i, s_j), \qquad g_i = g(t_i) \quad \text{and} \quad y_j = y(t_j).$$

For more details see [7]. The system of equation in (13) is in most cases an ill-conditioned system. Moreover, we will solve Eq. (13) after add random noise on vector $b$. The aim now is to solve the ill-condition system of the form

$$A_\epsilon x = b_\delta, \quad \|b - b_\delta\| \le \delta. \tag{14}$$

We will apply DSM and VRM as stable methods to solve numerically the resultant system of equations.

**Table 1**
Numerical results for Example 1 with $\delta_{rel} = 0.02$.

| $\epsilon$ | $n$ | DSM | | VRM | |
|---|---|---|---|---|---|
| | | Iteration | Rerr | Iteration | Rerr |
| $10^{-6}$ | 20 | 4 | 0.0119 | 1 | 0.0132 |
| $10^{-6}$ | 40 | 4 | 0.0071 | 1 | 0.0111 |
| $10^{-6}$ | 60 | 4 | 0.0042 | 1 | 0.0082 |
| $10^{-6}$ | 80 | 4 | 0.0021 | 1 | 0.0079 |
| $10^{-6}$ | 100 | 4 | 0.0031 | 1 | 0.0080 |
| $10^{-6}$ | 200 | 4 | 0.0029 | 1 | 0.0066 |

In Example 2 we will use $\theta$-methods as a more general discrete quadrature method for solving (2). In the $\theta$-methods we consider $h > 0$ be a fixed step size and consider the grid $t_n = nh$, $n = 0, \dots, N$ with $t_N = T$.

Putting $t = t_n$ in (2) we obtain

$$\epsilon y(t_n) = g(t_n) + \int_0^{t_n} K(t_n, s)y(s)\mathrm{d}s, \tag{15}$$

and replacing $K(t_n, s)y(s)$ by

$$(1-\theta)K(t_n, t_j)y(t_j) + \theta K(t_n, t_{j+1})y(t_{j+1}),$$

on the interval $[t_j, t_{j+1}]$, we obtain the following iteration formula:

$$\epsilon y_n = g_n + h\left[(1-\theta)K_{n,0}y_0 + \sum_{j=1}^{n-1} K_{n,j}y_j + \theta K_{n,n}y_n\right], \tag{16}$$

$n = 1, 2, \dots, N$. Here $y_n$ is an approximation to $y(t_n)$, $g_n := g(t_n)$ and $K_{n,j} := K(t_n, t_j)$, $y_0 = g(0)/\epsilon$, $\epsilon > 0$.

The system (16) is a lower triangular system of $N$ linear equations which can be written as

$$A_\epsilon x = b, \tag{17}$$

where

$$a_{ij} = \begin{cases} \epsilon - \theta h K_{ij}, & j = i; \\ -hK_{ij}, & 1 \le j < i \le N; \\ 0, & \text{otherwise.} \end{cases}$$

$$x = [y_1, y_2, \dots, y_N]^*;$$
$$b = [g_1 + a_1, g_2 + a_2, \dots, g_N + a_N]^*;$$
$$a_i = h(1-\theta)K_{i0}y_0, \quad i = 1, 2, \dots, N.$$

For $\theta = 0, \theta = 1/2$, and $\theta = 1$ these are direct quadrature methods based on the left rectangular rule (Forward Euler, FE), the Trapezium rule (TR), and the right rectangular rule (Backward Euler, BE), respectively. The aim now is to solve the ill-condition system of the form

$$A_\epsilon x = b_\delta, \quad \|b - b_\delta\| \le \delta.$$

**Example 1.** Consider the following integral equation

$$\epsilon f(x) = -\cosh(x) + \int_{-1}^{1} \cosh(x+y) f(y)\mathrm{d}y,$$

which has the exact solution

$$f(x) = \frac{2\cosh(x)}{2 + \sinh(2) - 2\epsilon}.$$

The condition number of, for example, of matrix $A_\epsilon$ with dimension $n \times n$, $n = 200$, is equal to $2.8134 \times 10^6$. Table 1 shows values of $n$, the number of iterations and the relative errors (Rerr) for DSM and VRM respectively (see Figs. 1 and 2).

The third and fourth columns in Table 1 give the number of iterations and relative errors for both methods, respectively, where the relative error $Rerr = \frac{\|u^{exact} - u^{approx}\|}{\|u^{exact}\|}$. The effect of the term $\epsilon$ on the singular perturbed linear Ferdholm integral equation of the second kind is given in Table 2.

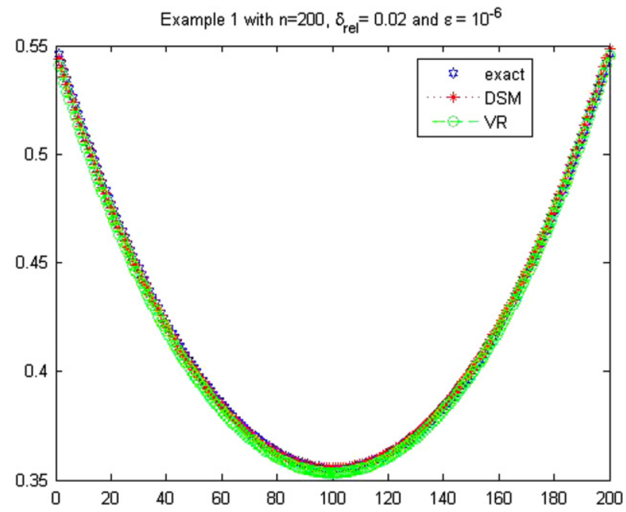


**Fig. 1.** Plot of the exact solution and approximate solution obtained by using DSM and VRM at $n = 200$, where $\epsilon = 10^{-6}$ and $\delta_{rel} = 0.02$.

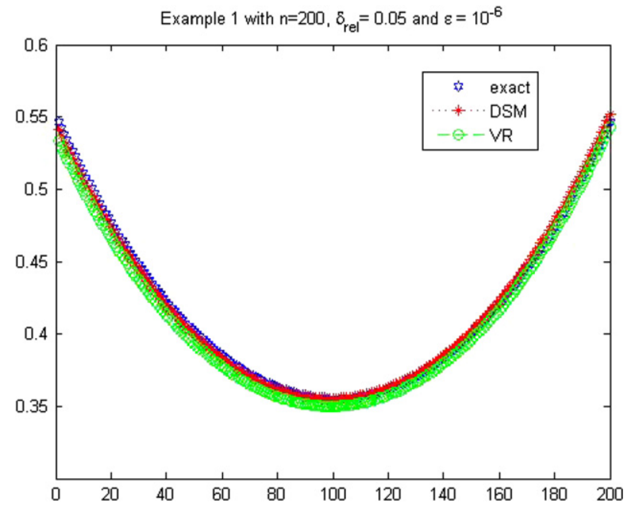


**Fig. 2.** Plot of the exact solution and approximate solution obtained by using DSM and VRM at $n = 200$, where $\epsilon = 10^{-6}$ and $\delta_{rel} = 0.05$.

**Table 2**
Numerical results for Example 1 with $\delta_{rel} = 0.02$.

| $\epsilon$ | $n$ | DSM | | VRM | |
|---|---|---|---|---|---|
| | | Iteration | Rerr | Iteration | Rerr |
| $10^{-2}$ | 200 | 4 | 0.0056 | 1 | 0.0125 |
| $10^{-3}$ | 200 | 4 | 0.0029 | 1 | 0.0067 |
| $10^{-4}$ | 200 | 4 | 0.0029 | 1 | 0.0066 |
| $10^{-5}$ | 200 | 4 | 0.0029 | 1 | 0.0066 |

**Example 2.** Consider the problem

$$\epsilon y(t) = \int_0^t (1 + t - s)[1 + s - y(s)]\mathrm{d}s$$

which has the exact solution

$$y(t) = t + 1 + \frac{1}{\gamma_1 - \gamma_2}\left[\left(\gamma_2 - 1 + \frac{1}{\epsilon}\right)\exp(\gamma_1 t) - \left(\gamma_1 - 1 + \frac{1}{\epsilon}\right)\exp(\gamma_2 t)\right],$$

**Table 3**
Numerical results for Example 2 at $\epsilon = h = \frac{1}{16}$ and $\delta_{rel} = 0.01$.

| $n$ | Trapezium rule (TR), $\theta = 1/2$. | | | | Backward Euler method (BE), $\theta = 1$. | | | |
|---|---|---|---|---|---|---|---|---|
| | DSM | | VRM | | DSM | | VRM | |
| | Iteration | Rerr | Iteration | Rerr | Iteration | Rerr | Iteration | Rerr |
| 20 | 4 | 0.1002 | 1 | 0.1316 | 4 | 0.0855 | 1 | 0.1056 |
| 40 | 5 | 0.1487 | 2 | 0.1633 | 5 | 0.1404 | 2 | 0.1437 |
| 60 | 6 | 0.1726 | 2 | 0.2065 | 6 | 0.1711 | 2 | 0.1850 |
| 80 | 6 | 0.1745 | 2 | 0.2169 | 6 | 0.1650 | 2 | 0.1995 |
| 100 | 7 | 0.1929 | 2 | 0.2504 | 7 | 0.1824 | 2 | 0.2354 |

**Table 4**
Numerical results for Example 2 at $\epsilon = \frac{1}{64}$, $h = \frac{1}{16}$ and $\delta_{rel} = 0.01$.

| $n$ | Trapezium Rule (TR), $\theta = 1/2$ | | | | Backward Euler method (BE), $\theta = 1$ | | | |
|---|---|---|---|---|---|---|---|---|
| | DSM | | VRM | | DSM | | VRM | |
| | Iteration | Rerr | Iteration | Rerr | Iteration | Rerr | Iteration | Rerr |
| 20 | 5 | 0.1488 | 1 | 0.2154 | 4 | 0.1256 | 1 | 0.1595 |
| 40 | 5 | 0.1855 | 2 | 0.2134 | 5 | 0.1578 | 2 | 0.1782 |
| 60 | 6 | 0.1980 | 2 | 0.2481 | 6 | 0.1773 | 2 | 0.2194 |
| 80 | 7 | 0.1922 | 2 | 0.2504 | 7 | 0.1747 | 2 | 0.2272 |
| 100 | 8 | 0.2095 | 2 | 0.2759 | 8 | 0.1919 | 2 | 0.2585 |

**Table 5**
Numerical results for Example 2 at $\epsilon = \frac{1}{256}$, $h = \frac{1}{16}$ and $\delta_{rel} = 0.01$.

| $n$ | Trapezium Rule (TR), $\theta = 1/2$ | | | | Backward Euler method (BE), $\theta = 1$ | | | |
|---|---|---|---|---|---|---|---|---|
| | DSM | | VRM | | DSM | | VRM | |
| | Iteration | Rerr | Iteration | Rerr | Iteration | Rerr | Iteration | Rerr |
| 20 | 5 | 0.1768 | 1 | 0.2472 | 5 | 0.1365 | 1 | 0.1764 |
| 40 | 5 | 0.2056 | 2 | 0.2319 | 5 | 0.1645 | 2 | 0.1893 |
| 60 | 6 | 0.2120 | 2 | 0.2602 | 6 | 0.1824 | 2 | 0.2296 |
| 80 | 7 | 0.2038 | 2 | 0.2600 | 7 | 0.1789 | 2 | 0.2355 |
| 100 | 8 | 0.2187 | 2 | 0.2829 | 8 | 0.1973 | 2 | 0.2648 |

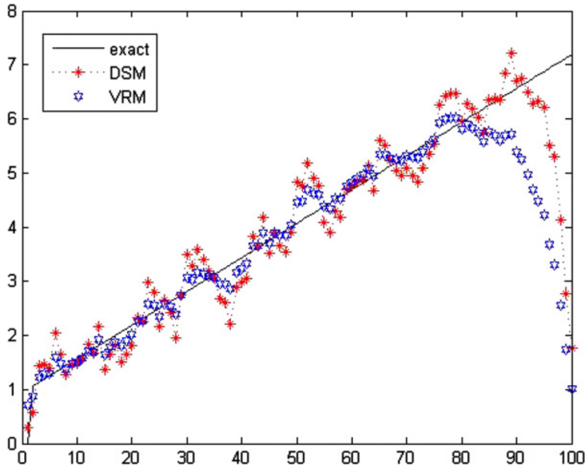


**Fig. 3.** Plot of solutions with $\epsilon = \frac{1}{256}$, $h = \frac{1}{16}$, $\theta = 1$ and $\delta_{rel} = 0.01$.

where the quantities $\gamma_1$ and $\gamma_2$ are defined as

$$\gamma_1 = \frac{1}{2\epsilon}(-1 + \sqrt{1 - 4\epsilon}), \qquad \gamma_2 = \frac{1}{2\epsilon}(-1 - \sqrt{1 - 4\epsilon}).$$

The numerical treatment are the followings:

1- Trapezium rule (TR), $\theta = \frac{1}{2}$.
2- Backward Euler Method (BE), $\theta = 1$.

In the following tables (Tables 3–5) we will consider two cases for solving Example 2 (see Fig. 3).

## Concluding remarks

In this paper numerical solutions to some singularly perturbed linear integral equations contaminated by noise using DSM and VRM are presented. It is clear from the numerical experimental that both methods are converges to the stable solution. The obtained solutions by DSM are relatively accurate than that of VRM method. The computational time is relatively smaller using VRM, for example in test Example 1, the elapsed time by DSM is 3.74 s and by VRM is 1.21 s, where the matrix in this case is of dimension 200. In test Example 2, the elapsed time using DSM is 2.12 second and by VRM is 1.91 second, where the matrix in this case is of dimension 100. All calculations are computed by MATLAB 7, on a computer machine with CPU 2.40 GHz CPU and 225 MB RAM.

## References

[1] C. Shubin, Singularly perturbed integral equations, J. Math. Anal. Appl. 313 (2006) 234–250.
[2] M.K. Kadalbajoo, K.C. Patidar, A survey of numerical techniques for solving singularly-perturbed ordinary differential equations, Appl. Math. Comput. 130 (2002) 457–510.
[3] Ikram A. Tirmizi, Fazal-i-Haq, Siraj-ul-Islam, Non-polynomial spline solution of singularly perturbed boundary value problems, J. Comput. Appl. Math. 196 (2008) 6–16.
[4] O.R. Smith, Singularly Perturbed Theory, Cambridge University Press, New York, 1985.
[5] M.H. Alnasr, Modified multilag methods for singularly perturbed Volterra integral equations, Int. J. Comput. Math. (2000) 221–233.
[6] C.T.H. Baker, The Numerical Treatment of Integral Equations, Clarendon Press, Oxford, 1977.
[7] L.M. Delves, J.L. Mohamed, Computational Methods for Integral Equations, Cambridge University Press, 1985.
[8] L.M. Delves, J. Walsh (Eds.), Numerical Solution of Integral Equations, Clarendon Press, Oxford, 1974.
[9] A.G. Ramm, Dynamical systems method for solving nonlinear operator equations, Int. J. Appl. Math. Sci. 1 (N1) (2004) 97–110.
[10] A.G. Ramm, Dynamical systems method for solving operator equations, Commun. Nonlinear Sci. Numer. Simul. 9 (N2) (2004) 383–402.
[11] A.G. Ramm, Dynamical systems method (DSM) and nonlinear problems, in: J. Lopez-Gomez (Ed.), In Spectral Theory and Nonlinear Analysis, World Sci. Publishers, Singapore, 2005, pp. 201–228.
[12] A.G. Ramm, Dynamical systems method (DSM) for nonlinear equations in Banach spaces, Nonlinear Sci. Numer. Simul. 11 (N3) (2006) 306–310.
[13] A.G. Ramm, Dynamical systems method (DSM) for unbounded operators, Proc. Amer. Math. Soc. 134 (N4) (2006) 1059–1063.
[14] A.G. Ramm, Ill-posed problems with unbounded operators, J. Math. Anal. Appl. 325 (2007) 490–495.
[15] A.G. Ramm, Dynamical systems method (DSM) for selfadjoint operators, J. Math. Anal. Appl. 328 (2) (2007) 1290–1296.
[16] A.G. Ramm, On unbounded operators and applications, Appl. Math. Lett. 21 (2008) 377–382.
[17] A.G. Ramm, Discrepancy principle for DSM II, Comm. Nonlin. Sci. Numer. Simul. 13 (2008) 1256–1263.
[18] A.G. Ramm, Inverse Problems, Springer-Verlag, New York, 2005.
[19] A.G. Ramm, Dynamical Systems Method for Solving Operator Equations, Elsevier, Amsterdam, 2007.
[20] N.H. Sweilam, On the numerical solution for deconvolution problems with noise, Aust. J. Math. Anal. Appl. 4 (2) (2007) 1–8.
[21] N.H. Sweilam, A.M. Nagy, Numerical studies on dynamical system method for solving ill-posed problems with noise, Aust. J. Math. Anal. Appl. 4 (1) (2007) 1–7.
[22] N.S. Hoang, A.G. Ramm, Dynamical systems method for solving linear finite-rank operator equations, Ann. Polon. Math. 95 (2009) 77–93.
[23] N.S. Hoang, A.G. Ramm, Solving ill-conditioned linear algebraic systems by the dynamical systems method (DSM), J. Inverse Problems Sci. Eng. 16 (N5) (2008) 617–630.
[24] C.W. Groetsch, The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind, in: Research Notes in Mathematics, vol. 105, Pitman, Boston, 1984.
[25] A.N. Tikhonov, V.Y. Arsenin, Solutions of Ill Posed Problems, Winston, Washington, DC, 1977.
[26] A.N. Tikhonov, V. Goncharsky (Eds.), Ill Posed Problems in Natural Sciences, Mir, Moscow, 1987.