Primary pulse transmission in coupled steel granular chains embedded in PDMS matrix: Experiment and modeling

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ABSTRACT

We present an experimental study of primary pulse transmission in coupled ordered steel granular chains embedded in poly-di-methyl-siloxane (PDMS) elastic matrix. Two granular one-dimensional chains are considered (an ‘excited’ and an ‘absorbing’ one), each composed of 11 identical steel beads of 9.5 mm diameter with the centerline of the chain spaced at fixed distances of 0.5, 1.5 or 2.5 mm apart. We directly force one of the chains (the excited one) by a transient pulse and measure, by means of laser vibrometry, the primary transmitted pulses at the end beads of both chains and at the first bead of the absorbing chain. It is well known that the dynamics of this type of ordered granular media is strongly nonlinear due, (i) to Hertzian interactions between adjacent beads, and (ii) to possible bead separations in the absence of compressive forces and ensuing collisions between neighboring beads. Accordingly, we develop a strongly nonlinear theoretical model that takes into account the coupling of the granular chains due to the PDMS matrix, with the aim to model primary pulse transmission in this system. After validating the model with experimental measurements, we employ it in a predictive fashion to estimate energy transfer between chains as a function of the interspatial distance between chains. Furthermore, based on this model we perform predictive matrix design to achieve maximum energy transfer from the excited to the absorbing chain, and provide a theoretical explanation of the nonlinear dynamics governing energy transfer (including energy equi-partition) in this system.

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1. Introduction

The dynamics of coupled oscillators has received much attention since such systems are often encountered in engineering practice. Examples range from nonlinear optics and coherent couplers (Jensen, 1982) to targeted energy transfer in mechanical and aerospace structures for vibration and shock mitigation (Vakakis et al., 2008). Recently, applications of the theory of coupled nonlinear oscillators have been considered in the field of ordered granular media. The aim is to design a new class of strongly nonlinear and highly discontinuous acoustic metamaterials with highly adaptive and tunable properties leading to effective mitigation of propagating disturbances (Nesterenko, 2001; Daraio et al., 2006; Sen et al., 2008; Jayapракash et al., 2011; Potekin et al., 2012).

Ordered one-dimensional granular media are composed of periodic sets of elastic spherical beads (granules) in contact. These strongly nonlinear media exhibit interesting dynamics due to their strongly nonlinear (discontinuous) behavior and highly tailorable properties (Nesterenko, 2001; Daraio et al., 2006; Sen et al., 2008) owing to their capacity to change their dynamics from strongly to weakly nonlinear (or even nearly linear) with increasing applied pre-compression. It is well known that the application of pre-compression in a granular chain introduces a linear component in the force interaction law and, hence, a linear component in the acoustics of the system. In that case the system can no longer be considered as being essentially nonlinear, and this affects the speed of pulse propagation. In the limit of application of strong pre-compression the dynamics are expected to become nearly linear, since no bead separation is possible and the nonlinear effects act as perturbations of the linear dynamics; even in that case, however, for sufficiently high amplitude applied pulses, the granular system can still exhibit strong nonlinear effects.

For no applied pre-compression, the strongly nonlinear Hertzian interactions between beads lead to complete elimination of linear acoustics and zero speed of sound as defined in the classical sense in these media; hence, the characterization of un-compressed ordered granular media as ‘sonic vacua’
(Nesterenko, 2001). Of particular interest is the capacity of appropriately designed granular media (such as granular dimers) to passively attenuate propagating disturbances due to strongly nonlinear resonances (Jayaprakash et al., 2011; Potekin et al., 2012), and their use as passive mitigators of transmitted pulses.

The majority of works in ordered granular media consider one-dimensional chains (Nesterenko, 2001; Daraio et al., 2006; Sen et al., 2008; Jayaprakash et al., 2011; Potekin et al., 2012), although some recent works have considered two-dimensional granular arrays in an effort to study energy mitigation and passive redirection in these complex media (Nishida et al., 2009; Leonard and Daraio, 2012a; Leonard et al., 2012b; Ngo et al., 2012; Starosvetsky et al., 2012; Hasan et al., 2013). It was shown that strongly nonlinear granular arrays can be designed for energy localization or redirection, exhibiting interesting intrinsic dynamics, such as nonlinear beating phenomena, traveling waves and standing or propagating breathers (Starosvetsky et al., 2012). All these works considered unsupported granular media (with the exception of the experimental study in Potekin et al., 2012, where the granules were supported by flexures); yet, practical implementations of granular media in acoustic metamaterials will ultimately require the embedding these structures into some form of supporting elastic medium or matrix. The present work is a first experimental study of the dynamics of this type of embedded, highly discontinuous, ordered medium, with the aim to (i) better understand the effect of the embedding elastic matrix on the strongly nonlinear acoustics of these media; (ii) study energy transfers between closely spaced embedded granular chains; and (iii) develop predictive designs for designing such type of metamaterials for efficient energy redirection. After describing the experimental fixture and discussing the experimental tests and results, we develop a mathematical model that captures primary pulse transmission in these systems. This model is validated with experimental measurements and then employed for predictive design. We end this work with some concluding remarks.

2. Fabrication of the experimental samples

Each experimental sample was composed of two ordered granular chains composed of a series of identical spherical steel beads in contact and no applied pre-compression, embedded in an elastic matrix. As described below, special care in the fabrication process was taken to ensure a constant gap (filled by elastic matrix) between the two granular chains of each sample. In particular, two bead chains were fabricated, one designated as the excited chain and the other as the absorbing chain. Poly-di-methyl-siloxane (PDMS, Dow Corning® Sylgard® 184) elastic matrix was used to embed eleven Type 302 stainless steel beads (McMaster Carr®, 9291K31) per granular chain. The PDMS was selected as the matrix material since its mechanical properties have been well studied (Fiard, 2008; Khanafer et al., 2009). Dow Corning® Sylgard® 184 silicon elastomer came in a two-part kit, consisting of a base and a curing agent, in a weight ratio of 10:1. Since a varying ratio of the two components changes the mechanical properties of PDMS, the 10:1 ratio was fixed for all of the samples fabricated for the present study (Khanafer et al., 2009). The base and curing agents were mixed for 5 min during which air bubbles were created inside the PDMS matrix. Accordingly, the PDMS matrix was placed in a desiccator under 580 torr (0.67 atm) vacuum for 20 min in order to degas it before pouring it into the mold.

For each chain of each sample, its 11 stainless beads were aligned and fixed in the one-dimensional mold. As the stainless steel beads were conductive, contact between all the beads in the mold was verified by measuring the resistance of each chain with a digital multi meter. The degassed PDMS matrix was then poured into the mold and cured at room temperature for 48 h. Higher curing temperatures were avoided as stainless steel beads were expected to expand at high temperature, which would result in subsequent misalignment in the mold, and, hence, the embedded granular chains.

After the exciting and absorbing chains were fabricated separately, they were joined laterally. Three separate samples were prepared corresponding to lateral gaps (filled by PDMS matrix) between the two chains equaling 0.5, 1.5 and 2.5 mm, respectively. The uncured PDMS adhesive method was adopted in order to fill the lateral gaps between the cured PDMS chains and provide the highest bond strength without recourse to any special surface treatments (Eddings et al., 2008; Koh et al., 2012). The PDMS adhesive layers filling the separation gaps were cured also in room temperature for 48 hours, again in order to avoid thermal expansion and misalignment of the stainless steel beads. Finally, the sample is wrapped with cello tape and multi meter measurements of both the excited and absorbing chains again verified that each bead of each chain of every sample remained in contact with its neighboring beads.

3. Experimental tests and results

A schematic diagram of the experimental sample is depicted in Fig. 1a. As mentioned in the previous section three samples were fabricated with lateral gaps equaling 0.5, 1.5 and 2.5 mm. The sample with 1.5 mm lateral gap between the two granular chains is presented in Fig. 1b; all beads are embedded in the PDMS matrix with the exception of the first bead of the excited chain (bottom left bead) which is left partially uncovered to allow the application of the forcing pulse as described below.

The experimental fixture used to perform the experiments described in this work is shown in Fig. 2a. Each sample was placed on top of a Teflon sheet and apart for its bottom face in all other faces of the sample there exist traction-free boundary conditions. At the bottom face friction forces between the PDMS enclosing matrix and the Teflon base are generated. An impulsive excitation was applied to the first bead of one of the chains, which is referred to as the ‘excited chain,’ whereas the other chain was initially at rest and designated as ‘absorbing chain.’ The excitation mechanism consisted of a spring-flexure cantilever system with a bead at its end. This provides the required input force pulse to the embedded excited chain. The entire fixture was clamped to an optical table and two laser vibrometers were used to measure the velocities in the axial direction of the end beads of the excited and absorbed chains, as well as the velocity of the bead of the excitation system (see Fig. 2b). Laser vibrometer 1 was used to measure the velocity of the end bead of the excited granular chain, whereas laser vibrometer 2 measured the end velocity of the end bead of the absorbing one. Moreover, we also measured the velocity of the first bead of the absorbing chain, but this was not possible with the first bead of the excited chain due to the presence of the impacting bead of the excitation mechanism. However, by means of laser vibrometry we can directly measure the velocity of the impacting bead of the excitation mechanism (flexure), and based on this measurement we can infer the applied velocity pulse to the excited chain of the sample; this is important for comparing the experimental measurements with the theoretical model derived below. These last two measurements were performed by placing a laser vibrometer on the opposite side of the sample compared to their positions shown in Fig. 2b. We note that due to the nature of the problem these are the only possible non-contacting measurements that can be performed in the samples under consideration. This implies that pulse propagation in these matrix-
identified mass to each of the beads of the embedded granular chains. Two levels of excitations were considered, designated as ‘low’ and ‘high’ excitations. In any case, the levels of input excitations are sufficiently small to avoid rigid-body motion of the sample, i.e., relative motion of the sample with respect to its Teflon base. From direct experimental measurement of the velocity of the impacting bead through laser vibrometry, the low input velocity impulse was measured at approximately 0.29 m/s, and the high input level at 0.41 m/s. Typical measurements are depicted in Fig. 3. We note that the low and high impulse excitations can be reproduced reliably through the described excitation mechanism: In each case we simply displace the impacting cantilevered bead by a fixed amount (which is larger in the case of the high impulse excitation).

Applying an initial impulse to the first particle of the excited chain, it is expected that a pulse will propagate through the excited granular chain. In this study we will be interested only in primary pulse transmission, i.e., only in studying the propagation of the generated primary pulses through the excited and absorbing granular chains and not on secondary scattering of these pulses once they encounter the right boundaries (end beads) of the chains. Moreover, we expect that, in spite of the fact that the impulsive excitation is applied to only one of the granular chains (the excited one), there will be energy transfer and pulse propagation in the other chain as well (the absorbing one) due to the coupling between chains provided by the PDMS elastic matrix; such energy transfers between coupled granular chains have been theoretically predicted in previous works (Starosvetsky et al., 2011). In particular, it was shown that a weakly coupled array of granular chains is a medium in which an initially localized excitation may become gradually defocused resulting in drastic reduction of propagating pulses as they “spread” among the granular chains. As mentioned previously, to measure the responses of the two granular chains in the consider samples, we employ two laser vibrometers (see Fig. 2b). Because of the limitations of our experimental equipment, we can measure the responses of only two beads at the same time. Hence, when testing each of the three samples, initially we measure the velocity profiles of the end beads of each of the two granular chains. Two levels of excitations were considered, designated as ‘low’ and ‘high’ excitations. In any case, the levels of input excitations are sufficiently small to avoid rigid-body motion of the sample, i.e., relative motion of the sample with respect to its Teflon base. From direct experimental measurement of the velocity of the impacting bead through laser vibrometry, the low input velocity impulse was measured at approximately 0.29 m/s, and the high input level at 0.41 m/s. Typical measurements are depicted in Fig. 3. We note that the low and high impulse excitations can be reproduced reliably through the described excitation mechanism: In each case we simply displace the impacting cantilevered bead by a fixed amount (which is larger in the case of the high impulse excitation).

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Four different sets of experimental tests were carried out, one with a sample consisting of just the single embedded excited granular chain with no coupling to an absorbing one; and three additional tests with the three samples corresponding to coupled chains with the three different lateral gaps described previously. To ensure repeatability of the experimental results, at least 10 experimental trials were conducted for each of the two input levels of applied impulse excitation. As will be discussed below, these sets of experiments were carried out with the primary aim to study primary pulse transmission and nonlinear energy exchanges between the embedded excited and absorbing granular chains. The
realization of energy transfers between the two granular chains was inferred by directly measuring the velocities of their end beads and applying appropriate theoretical modeling. Moreover, by examining three samples with different lateral gaps between chains we aim to study the dependence of primary pulse transmission and energy transfer on the different level of coupling between chains provided by the PDMS elastic matrix.

In Fig. 4, we depict the velocity profile of the end bead (i.e., the 11th bead) of the single one-dimensional excited granular chain embedded in a PDMS matrix for low and high levels of impulse excitation. The formation of a solitary-like pulse propagating through the embedded granular chain is clearly observed, followed by secondary pulses of smaller magnitude. In the initial phase of primary pulse propagation the strongly nonlinear Hertzian interactions between beads dominate the dynamics, whereas in the lateral phase of secondary pulses the effects of the PDMS matrix become dominant (with the specimen oscillating as a whole predominantly in its axial vibration mode). Indeed, the primary pulse can be regarded as perturbation by the PDMS matrix effects of the well-known solitary wave observed in the corresponding ‘dry’ homogeneous granular chain (with no matrix) theoretically predicted (Nesterenko, 1983) and experimentally verified (Lazaridi and Nesterenko, 1985; Nesterenko, 2001) in previous works. As the one-dimensional granular chain was embedded in PDMS matrix and the experiment was performed by placing the sample on top of a Teflon-sheet, the PDMS matrix is acting, in effect, as an elastic foundation so the solitary wave is perturbed. The effect of this effective elastic foundation is deduced from the negative velocity attained by the end bead immediately following the arrival of the primary pulse (see Fig. 4a). We also carried out several other experiments (not shown here), where we experimented with the effect of the elastic foundation on the primary pulse. In one set of experiments, we enhanced the effective elastic foundation by more rigidly attaching the sample to the Teflon-sheet base with the use of double-sided cello tape; in that case we observed a sharp reduction of the primary pulse amplitude due to the stronger effects of the elastic foundation on the dynamics.

The test involving the uncoupled single one-dimensional embedded granular chain served as a base-line for the main series of experimental tests which was carried out with the three samples consisting of pairs of excited/absorbing granular chains. The experiments were carried out for both low and high excitation impulses for all three samples. Fig. 5 depicts the experimental results for the 0.5 mm gap sample for low impulse excitation. Two separate tests were performed to derive that results presented in Fig. 5a–c, respectively. In the second test we also measured the response of the last bead of the absorbing chain and confirmed that it matched exactly the response shown in Fig. 5b.

Focusing on the velocity waveforms of Fig. 5a and b we note that the response of the end bead of the excited chain is higher than that of the end bead of the absorbing chain. Comparing, however, the magnitude of the primary pulse in Fig. 5a to the corre-
sponding pulse in Fig. 4a, we deduce that a significant reduction of the pulse in the sample occurs; this is partly attributed to energy transfer from the excited chain to the absorbing one, and partly to energy scattering of the primary pulse in the embedding PDMS elastic matrix. Considering now the responses of Fig. 5b and c depicting the velocity waveforms of the first and end beads of the absorbing chain, we can clearly note the build-up of the primary pulse in the absorbing chain, which as discussed in the next section is in agreement with previous studies on primary pulse transmissions in coupled granular chains (Starosvetsky et al., 2011, 2012) (this result will also be confirmed theoretically in the next section). From these results we also observe that there is a time delay between the responses of end beads of the excited and absorbing chains; this is expected given the well-known dependence of the speed of a pulse propagating in a homogeneous granular chain on its magnitude (Nesterenko, 2001; Daraio et al., 2006; Sen et al., 2008). We mention at this point that due to the discrete nature of the considered granular systems, defining a phase velocity (as in continuum systems) is not possible. However, an analogous quantity that can be related to the speed of propagation of a pulse in these discrete systems is the mentioned time delay (or shift). This can be defined as the time difference between the velocity peaks of successive beads of a chain. From a theoretical perspective, the time delay is a valid measure that describes the speed of a propagating solitary wave. By estimating the total time delay of the velocity peaks between the first and last beads of the experimental granular chain, we can get a measure of the speed of pulse propagation in that chain.

These results indicate that initially the first bead of the absorbing chain gains energy with some time delay from the first bead of the excited chain. Then, the next bead of the absorbing chain gains energy both from its neighboring bead and also from the excited chain through the coupling induced by matrix, and so on until the developed primary pulse reaches the end bead. These energy exchange phenomena are discussed in detail in Section 4.

In Figs. 6–8, we compare the responses of the three samples for the same high level of impulse excitation. Comparing these responses we note that the magnitude of the primary pulse measured at the end bead of the excited chain increases as the gap between the chains increases, whereas the corresponding magnitude of the pulse at the end bead of the absorbing chain decreases. These findings indicate that, as the lateral gap between the two chains increases, the effective coupling between the two chains decreases, and, hence, energy transfer from the excited chain to the absorbing chain becomes less profound. Now, if we compare the measured velocity primary pulse at the end bead of the excited chain for all four different samples considered (i.e., the 2.5, 1.5, 0.5 mm gap chains and the single one-dimensional chain), for the high level of impulse excitation (i.e., comparing Figs. 4b, 6a, 7a and 8a), it is clear that the response of the end bead of the excited chain is higher for the case of the single one-dimensional chain compared to all three lateral-gap samples. This clearly demonstrates the occurrence of energy transfer from the excited to the absorbing chain due to the coupling forces between chains induced by the PDMS elastic matrix; in fact, the closer the two chains are located, the higher the energy transfer is.

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Fig. 4. Experimental velocity profile of the end bead of the single one-dimensional granular chain embedded in PDMS matrix: (a) low excitation, (b) high excitation; at least 10 experimental trials were made for each input level to ensure consistency and repeatability of the results.
A study of primary pulse propagation in the granular chains and its dependence on the lateral gap between chains is provided in the next section where a set of strongly nonlinear coupled oscillators theoretically modeling primary pulse transmission in the coupled granular chains is developed. After validating this model, by matching the theoretical and experimental bead responses, we employ it for studying energy transfers between the chains and for predictive design of this type of material systems based on coupled granular media for pulse transmission and redirection.

4. Theoretical modeling of primary pulse propagation in the granular chains

For the theoretical treatment of the problem, we assume that initially all beads are in point contact with their neighboring beads and that no pre-compression exists in either granular chain. We assume that the characteristic time scale of bead to bead Hertzian interaction under compression is significantly higher than the characteristic time scale of elastic stress wave propagation inside...
a bead, so that the later effect can be ignored due to time scale separation, allowing for the application of a quasistatic bead interaction law with constant value of elastic modulus. Moreover, we focus mainly on primary pulse transmission in the granular medium; therefore, the theoretical model should be capable of accounting and modeling for the strongly nonlinear bead interactions that dominate the dynamics during primary pulse propagation, but not necessarily of secondary scattering of the primary pulse after it encounters the end of the medium or of axial vibrations of the sample when it oscillates as a whole after the primary pulse has scattered.

Accordingly, we consider the model of nonlinear coupled oscillators depicted in Fig. 9. Each chain consists of a number of identical, linearly elastic, spherical granular beads (11 beads for this case) undergoing essentially nonlinear Hertzian interactions, with coupling between beads due to the embedding PDMS elastic matrix being accounted by additional stiffness and damping elements. It is well known that the local contact model accounting for dynamic bead-to-bead interactions plays an important role for the dissipation and dispersion of energy of propagating pulses in homogeneous granular chains. Indeed, the energy dispersion effect depends on the intrinsic nature of the interaction law between two

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**Fig. 6.** Velocity profiles of beads in the sample with 2.5 mm lateral-gap for the case of high impulse excitation: (a) end bead of the excited chain, (b) end bead of the absorbing chain, (c) first bead of the absorbing chain; at least 10 repeated tests are depicted in each result.
particles during impacts (Falcon et al., 1998; Zhao et al., 2008). In homogeneous granular chains, spatially localized waves propagate without distortion due to the counterbalancing of two effects, namely dispersion and strong nonlinearity due to Hertzian law interaction between beads. Proceeding with our theoretical model, we note that effects of bead rotation and dry friction are neglected. Furthermore, we assume that all beads of both chains are constrained to move in the horizontal direction, and that each bead within a chain is coupled (and, hence, exchanges energy) with only three adjacent beads of the other chain through linear shear springs (see Fig. 9). Moreover, we assume that each granular chain is supported by a weak linear shear elastic foundation modeling the interaction of the experimental sample with its Teflon base. Accordingly, we denote by $k_1$, $k_2$, and $b_{k_2}$ (see below for the definition of parameter $b_1$) the stiffness coefficients of the linear elastic foundations and the linear coupling elements between the two granular chains, respectively; and by $\gamma$ the coefficient of the axial coupling element between neighboring beads within each chain due to the effect of the surrounding PDMS elastic matrix. As shown in the optimization study performed in the next section, it holds

Fig. 7. Velocity profiles of beads in the sample with 1.5 mm lateral-gap for the case of high impulse excitation: (a) end bead of the excited chain, (b) end bead of the absorbing chain, (c) first bead of the absorbing chain; at least 10 repeated tests are depicted in each result.
that $k_1, k_2, \beta k_2 \ll 1$, so the chains are weakly coupled and there are weak elastic foundations. Hence, in the model the strongly nonlinear Hertzian interactions between beads are expected to dominate the dynamics, so primary pulse propagation can be accurately modeled; moreover, by its very structure, the dynamics of the model is strongly nonlinear (Jayaprakash et al., 2011; Starosvetsky et al., 2011, 2012). Finally, we designate the lower chain as the excited chain, and the upper chain as the absorbing one.

Dissipative effects in bead to bead interactions and damping effects in the elastic matrix are modeled by linear viscous damping elements; this damping approximation has been found to be quite valid in modeling dissipative effects in primary pulse transmission in homogeneous granular chains (Potekin et al., 2012). In other studies, velocity relaxation of a viscous one-dimensional granular gas (Rosas et al., 2005), was modeled by introducing viscous drag terms which depended on the velocities of individual particles. A dissipative term based on the relative velocities between granular particles was also introduced in Herbold and Nesterenko (2007), Rosas et al. (2007), Carretero-González et al. (2009) and Potekin et al. (2012). To represent viscous dissipation in a one-dimensional lattice of mass points connected by nonlinear springs (Duvall et al., 1969) dissipative dashpots were used to represent thus as mass–spring–dashpot elements.

Fig. 8. Velocity profiles of beads in the sample with 0.5 mm lateral-gap for the case of high impulse excitation: (a) end bead of the excited chain, (b) end bead of the absorbing chain, (c) first bead of the absorbing chain; at least 10 repeated tests are depicted in each result.
Main nonlinear equations of motion of this system of coupled oscillators can be expressed in the following dimensionless form after some appropriate normalizations are introduced with respect by diving both equations with the mass of an individual bead,

\[
\begin{align*}
\dot{x}_n + k_1 x_n + k_2 (x_n - y_n) + \beta k_2 (x_n - y_{n-1}) + \beta k_2 (x_n - y_{n+1}) + \lambda_1 x_n + \lambda_2 (x_n - y_n) + \lambda_2 (x_n - y_{n+1}) + \lambda_2 (x_n - y_{n-1}) =& \\
= \left[ (x_{n-1} - x_n)^{\frac{1}{2}} - (x_n - x_{n+1})^{\frac{1}{2}} \right] \\
+ \gamma [\text{sgn}(x_{n-1} - x_n)|x_{n-1} - x_n|^m - \text{sgn}(x_n - x_{n+1})|x_n - x_{n+1}|^m] \\
+ \lambda_3 (x_{n-1} - x_n) H(x_{n-1} - x_n) + \lambda_3 (x_{n+1} - x_n) H(x_{n} - x_{n+1})
\end{align*}
\]

\[ (1a) \]

\[
\begin{align*}
\dot{y}_n + k_1 y_n + k_2 (y_n - x_n) + \beta k_2 (y_n - x_{n-1}) + \beta k_2 (y_n - x_{n+1}) + \lambda_1 y_n + \lambda_2 (y_n - x_n) + \lambda_2 (y_n - x_{n+1}) + \lambda_2 (y_n - x_{n-1}) =& \\
= \left[ (y_{n-1} - y_n)^{\frac{1}{2}} - (y_n - y_{n+1})^{\frac{1}{2}} \right] \\
+ \gamma [\text{sgn}(y_{n-1} - y_n)|y_{n-1} - y_n|^m - \text{sgn}(y_n - y_{n+1})|y_n - y_{n+1}|^m] \\
+ \lambda_3 (y_{n-1} - y_n) H(y_{n-1} - y_n) + \lambda_3 (y_{n+1} - y_n) H(y_{n} - y_{n+1})
\end{align*}
\]

\[ (1b) \]

where \( n = 1, 2, \ldots, N, x_0 = y_0 = x_{N+1} = y_{N+1} = 0, \) and \( N = 11. \)

In Eq. (1), the variables \( x_n \) and \( y_n \) denote the displacements of the \( n \)th bead for the excited and absorbing chains, respectively; \( m \) denotes the exponent of the contact stiffness between beads, \( \beta \) is a parameter defining the strength of the diagonal coupling between beads of different chains. The subscripts \((\ast)\) in (1) indicate that only nonnegative values in the parentheses should be taken into account, with zero values being assigned otherwise; this accounts for possible separations between beads that may occur in the absence of compression. Moreover, \( H(\cdot) \) denotes the Heaviside function. Based on these observations, we have incorporated dissipation in the embedded granular-chain system. Finally, to model dissipative effects in the embedded granular chains due to inherent internal structural damping within the beads, small plastic deformations, frictional effects during bead-to-bead dynamical interactions or due to viscoelastic effects in the PDMS matrix we introduced three types of damping terms with coefficients \( \lambda_1, \lambda_2 \) and \( \lambda_3. \) The viscous damping coefficients \( \lambda_1 \) represents the interaction of the granular bead with the embedding matrix, and \( \lambda_2 \) represents energy loss due to the coupling matrix between the two chains, and \( \lambda_3 \) models dissipation during Hertzian interactions between neighboring beads. Moreover all dissipation coefficients are much smaller than unity so the system 1 is weakly dissipative.

![Figure 9](image1.png)

**Fig. 9.** Theoretical model for primary pulse propagation in two weakly coupled granular chains mounted on weak linear elastic foundations.

![Figure 10](image2.png)

**Fig. 10.** Points on the experimental and simulated primary pulses used for the definition of the objective function \( f \) for the optimization study.

![Figure 11](image3.png)

**Fig. 11.** Evolution of objective function value.

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<td>( \lambda_3 )</td>
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<td>0.00022</td>
</tr>
</tbody>
</table>

![Table 1](image4.png)

**Table 1**

Bounds for the four parameters of the optimization study (single embedded chain).

![Table 2](image5.png)

**Table 2**

Starting and final (optimized) normalized parameter values (single embedded chain).
We assume that an impulsive excitation is applied to the first bead of the lower chain whereas the upper chain is initially at rest. In accordance to the experiment, the impulsive excitation was chosen as an instantaneous velocity applied to the first bead of the excited chain in order to simulate the velocity pulse induced by the bead of the excitation mechanism as it impacts the excited chain. In agreement with the experimental work two levels of velocity impulse excitation were considered, a low level impulse corresponding to $x_1(0^+)=0.29$ m/s, and a high level impulse with $x_1(0^+)=0.41$ m/s; moreover all other initial conditions of the beads of the two chains of the model 1 are assumed to be zero at $t=0$. A vectorized, fourth-order Runge–Kutta time integration scheme was used to numerically compute the dynamics of system 1, and the results obtained from the numerical simulations were compared to the experimental measurements after appropriate rescalings were introduced. Our aim for constructing the computational model and performing the numerical analysis was threefold: (i) First, to validate the model by demonstrating that it can accurately predict and reliably reproduce the experimental measurements of primary pulse transmission in the tested samples; (ii) based on the validation achieved in (i) to provide quantitative estimates of energy exchanges between the excited and absorbing chains in the experimental tests, after accounting for energy dispersion in the elastic matrix and energy loss due to the various dissipative mechanisms present in the experiment; (iii) and finally, to use the validated model to predict matrix design for enhanced energy exchanges between the excited and absorbing chains, with the eventual aim of pulse equi-partition or even redirection between the coupled chains. We emphasize that our study concerns of primary pulse transmission in the two chains, given that secondary scattering of the transmitted pulse by the boundaries introduces additional complex dynamics and flexible vibrations of the sample as a whole that cannot be captured by the simplified model of Fig. 9. Hence, in the next sections we will exclusively focus on primary pulse transmission in the coupled granular system and ignore the dynamics following this primary pulse propagation.

We have carried out an optimization study to identify the parameters of model 1 from the experimental time series measurements, both for the case of one chain embedded in matrix and the two coupled embedded chains. From system 1 it is clear that there are a total of eight parameters, namely, $k_1, k_2, \beta, \gamma, m, \lambda_1, \lambda_2, \lambda_3$. Considering first the one chain embedded in matrix, we note that due to the absence of coupling with an absorbing chain it holds that $k_3 = \lambda_2 = 0$. Moreover, to ease the optimization procedure, we assign the value of the exponent $m = 1$, i.e., we assume that there are linear coupling terms between beads due to coupling provided by the embedding PDMS matrix, and also assign the strength of the

![Fig. 12. Comparison between the experimental and simulated velocity response of the end bead of the single embedded chain for (a) low velocity impulse 0.29 m/s, and (b) high velocity impulse 0.41 m/s; the simulations were carried out for the optimized normalized parameters listed in Table 2 and $m = 1, \beta = 0.05$.](image-url)
diagonal coupling as
\[
\beta = \frac{\text{(Lateral gap)}}{\sqrt{\text{(Bead Dia)}^2 + \text{(Lateral gap)}^2}}.
\]

The experimental data consist of the time series of the velocity of the end bead for two impulse levels, as depicted in Fig. 4, but as mentioned previously we restrict our focus to matching only primary pulse propagation in this granular medium. In fact, the primary pulse carries most of the energy of the dynamics and is dominated by the strongly nonlinear bead-to-bead interactions in the excited chain. In our study we optimize the parameters \((k_1, \gamma_1, \lambda_1, \lambda_2)\) of model 1 by minimizing the difference of an objective function \(f\) characterizing the discrepancies between the numerical simulation carried out for low and high levels of velocity impulse excitations and the corresponding experimental time series measurements. Moreover, given that we are focusing on primary pulse propagation, for each of the two levels of impulse excitations we define the objective function as,
\[
f = f(k_1, \gamma_1, \lambda_1, \lambda_2)
\]
\[
= \left\{ \sum_{i=1}^{3} \|e_i - s_i\| \right\} \text{High excitation} + \left\{ \sum_{i=1}^{3} \|e_i - s_i\| \right\} \text{Low excitation}
\]
where \(e_1\) and \(s_1\) are the points of first arrival of the primary pulse measured in the experiment and computed in the simulation, respectively; \(e_2\) and \(s_2\) are the maxima of the experimental and simulated velocity primary pulses, respectively; and \(e_3\) and \(s_3\) are the corresponding minima of the experimental and simulated velocity primary pulses, respectively (see Fig. 10). Moreover, both low and high level impulsive excitations were considered.

The minimization of the objective function is subject to bounds on the four parameters. This problem was solved using \textit{Matlab fmincon} function for constrained minimization. To evaluate the objective function \(f\) numerically, system 1 was integrated using the Runge–Kutta integration routine \textit{Matlab ode45} for every choice of the model parameters \((k_1, \gamma_1, \lambda_1, \lambda_2)\). Several simulations were run for nominal values of the model parameters. These preliminary simulations provided a basis on which to establish upper and lower bounds for the parameters, and helped us select the initial sets of values from which to start the optimization. These bounds are listed in Table 1. Initial values from which to start the optimization, were chosen from a uniform random distribution and are listed in Table 2. The optimization proceeded from these starting values, converging to the respective solutions of the final parameters which are also listed in Table 2. The convergence of the objective function for this optimization study is presented in Fig. 11.

We numerically solved system 1 with the optimized values of the parameters listed in Table 2 for both levels of velocity impulses and compared the simulated velocity profile of the end bead with the corresponding experimental time series presented in Fig. 4.
Fig. 12 we depict the comparisons of the experimental (solid line) and optimized numerical (dashed line) results. It is clear that the optimized computational model 1 is capable of accurately capturing the arrival of the velocity primary pulse in the single embedded chain, but not the secondary pulses, which as discussed previously involve axial vibrations of the sample as a whole and is not dominated by the granular dynamics; hence, the granular media-based model 1 is not expected to accurately model this type of dynamics. However, since the primary pulse transmission is dominated by the strongly nonlinear (Hertzian-type) granular dynamics, it is accurately modeled by the model 1. Moreover, considering the magnitude of the velocity primary pulse that arrives at the end bead of the 11-bead granular chain we note that for the low impulse excitation the initial velocity pulse in this medium reduces in magnitude from 0.29 m/s to ~0.21 m/s, whereas for the high impulse excitation the reduction of the velocity primary pulse is from 0.41 m/s to ~0.30 m/s.

From the previous optimization study, we were able to identify the parameters for each individual embedded granular chain. Proceeding now to the three samples of coupled granular chains we note that the remaining parameters to be identified are the coefficients of the (linear) coupling shear springs and viscous dampers modeling the coupling between chains provided by the PDMS elastic matrix. For each lateral gap between chains we need to identify two coupling parameters, namely, the coupling coefficient $k_2$ and the associated viscous damping coefficient $\lambda_2$. To estimate these two parameters for each of the three samples discussed in Section 3, we consider the velocity time series of the end beads of the excited and absorbing chains and apply an optimization scheme similar to that described previously. To perform this second optimization we defined objective functions similar to (2) and carried out series of numerical simulations to determine the coupling parameters necessary to match the numerical simulations with the corresponding experimental measurements for the primary pulses of the velocity profiles of the end beads of the excited and absorbing chains for each sample. The final (optimized) values of the coupling parameters are listed in Table 3.

Fig. 15. Numerical simulation of the experimentally validated theoretical model 1 for the sample with 0.5 mm lateral gap, depicting the velocity profiles of all beads of the two chains: (a) excited chain, (b) absorbing chain; left and bottom primary axes correspond to dimensional (physical) units, whereas right and top secondary axes correspond to normalized (dimensionless) units.
In Fig. 13 we depict comparisons between the optimized numerical simulations and the corresponding experimental measurements for the three samples consisting of coupled 11-bead homogeneous granular chains embedded in PDMS elastic matrix and low velocity impulse excitation. The numerical results were obtained by numerically integrating the theoretical model incorporating all parameters obtained so far: The normalized parameters of each of the two granular chains were obtained by the first optimization study based on the single embedded chain, 
\[ k_1 = 0.000133, \quad c = 0.002317; \]
\[ k_2 = k_3 = 0.00583; \quad m = 1, \quad k_2 = k_3 = 0; \]
whereas the normalized coupling parameters were obtained by the second optimization study, as listed in Table 3. In performing the comparisons with the experimental measurements appropriate rescalings were performed in order to assign physical (un-normalized) values to the system parameters. It is clear from the results of Fig. 13 that the theoretical model is capable of accurately capturing primary pulse propagation in the three samples, but is not valid in modeling secondary pulses resulting from later scattering of the primary pulse. Together with the results depicted in Fig. 12, the results of Fig. 13 validate the optimized model in its capacity to accurately capture primary pulse transmission in the excited and absorbed chains of the three samples tested in Section 3.

From these results it is evident that energy exchanges between granular chains in this material system take place, with the intermediate PDMS elastic matrix playing the role of the coupling element. Clearly, the strength of this effective coupling is dependent on the lateral gap between the two chains, as it is logical to assume that the further apart the chains are located the less the coupling provided by the elastic matrix is. This dependency can be quantitatively studied by the optimized normalized coupling parameters listed in Table 3 and plotting them as functions of the lateral gap of each of the three samples. This is performed in Fig. 14, which provides the relationship between the coupling stiffness \( k_2 \) and the corresponding lateral gap between the chains, indicating that the coupling stiffness varies nonlinearly with the lateral gap.

In the next section we employ the theoretical model 1 to study energy transfer between the excitation and absorbing granular chains corresponding to primary pulse propagation for each of the tested samples. Then, we use the model in a predictive fashion, that is, to design the elastic matrix for enhancement of energy transfer between chains. As shown in previous works (Starosvetsky et al., 2011, 2012) this type of strong energy exchanges can lead to primary pulse equi-partition or redirection of the primary pulse in granular networks similar to the weakly coupled granular media considered herein.

5. Study of energy transfer in the granular system and predictive design

In the previous section, we validated the theoretical model 1 for primary pulse propagation, and determined its optimal parameters for each of the three tested samples. As a result, we were able to capture theoretically primary pulse propagation in both the excited and absorbing chains. Moreover, in contrast to the experimental tests where we were only able to measure the velocity responses of the end beads of the chains and the first bead of the absorbing chain, with the theoretical model we are able to fully reconstruct primary pulse transmission through the medium, i.e., the full development of the primary pulse in the system. In addition, the theoretical model enables us to compute quantitative measures of energy transfer between chains and to study in detail how impulsive energy initially applied to the excited chain is partitioned between the chains and the embedding elastic matrix. Hence, the validated model allows for predictive design of primary pulse transmission in this embedded granular medium. A typical
theoretical result is presented in Fig. 15, where we numerically plot the responses (in physical and normalized units) of all beads of both chains of the sample with 0.5 mm lateral-gap sample. The development of the primary pulse in both chains can be clearly deduced, as well as the arrival of the primary pulse at the end beads of the chains after which the model is not expected to yield reliable results.

In Fig. 16 we depict the relative displacements in dimensionless form of the neighboring beads for both the excited and absorbing chains for the same sample with 0.5 mm lateral gap. In these plots we consider exclusively the propagation of the primary pulse in the excited and absorbing chains, and consider in detail the dynamics of bead to bead interactions during this initial strongly nonlinear phase of the dynamics. We clearly observe that the separation of the end (11th) bead from the previous (10th) bead occurs at different time instants in the two chains; namely, this separation occurs at normalized time 75 for the excited chain and normalized time 100 for the absorbing one. This separation signifies the end of primary pulse transmission in the corresponding chain. It follows that at time instant 75, although the primary pulse reaches the end of the excited chain, the primary pulse still propagates within the absorbing chain until the time instant equaling 100. These results are important for estimating the energy transfer from the exciting to the absorbing chain during primary pulse transmission as outlined below.

Based on the previous discussion we now proceed to calculate the ratio of energy transfer from the excited to the absorbing chain in the phase of primary pulse propagation. To this end, we define the energy transfer ratio as the ratio of total (kinetic and potential) energy in the absorbing chain until normalized time = 100 to the total energy in the excited chain until normalized time 75, i.e., until the corresponding time instants of separation of the end beads from their neighboring beads. The total energy of each chain is computed as the summation of the total energies of the individual beads of the chain in the time interval of primary pulse propagation in that chain (as discussed above). Based on this calculation we construct the plots of Fig. 17 that depict the energy ratios for the three samples, for low and high impulse excitations.

We note that the energy transfer ratio is higher for the case of low excitation, and that, as expected, it is much higher for the chain with the smallest lateral gap. One possible explanation for this trend is that for low excitation case, there is sufficient time for the propagating pulse to transfer energy from the excited to the absorbing chain; moreover, the energy transfer is enhanced as the later gap decreases and the effective coupling between chains increases (see Fig. 14). A general conclusion is that all three samples energy transfer from the excited to the absorbing chain during primary pulse propagation is small, below 5%. However, it is possible to use the validated theoretical model in a predictive capacity, in order to design the elastic matrix (and the effective coupling between the two granular chains) that enhances energy transfer. This is performed in the remainder of this Section.

Reconsidering system 1, omitting all dissipative terms by setting $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and setting $m = 1$ we derive the following normalized simplified Hamiltonian model of nonlinear coupled oscillators containing only three stiffness parameters,

$$\dot{x}_n + k_1 x_n + k_2 (x_n - y_n) + 2\beta (x_n - y_{n-1}) - \beta y_{n+1}$$

$$= \left[ \frac{1}{2} (x_{n-1} - x_n)^2 - (x_n - x_{n-1}) \right]$$

$$+ \gamma \left[ |\text{sgn}(x_{n-1} - x_n)| |x_{n-1} - x_n| - |\text{sgn}(x_{n} - x_{n-1})| |x_n - x_{n-1}| \right] y_n$$

$$+ k_1 y_n + k_2 (y_n - x_n + 2\beta y_n - \beta x_{n-1} - \beta x_{n+1})$$

$$= \left[ \frac{1}{2} (y_{n-1} - y_n)^2 - (y_n - y_{n-1}) \right]$$

$$+ \gamma \left[ |\text{sgn}(y_{n-1} - y_n)| |y_{n-1} - y_n| - |\text{sgn}(y_n - y_{n-1})| |y_n - y_{n-1}| \right]$$

where $n = 1, 2, \ldots, N$, $x_0 = y_0 = x_{N+1} = y_{N+1} = 0$, and $N = 11$. In Fig. 18 we present the responses of the Hamiltonian system (3) for low impulse excitation, and comparing them to the responses of the corresponding damped system in Fig. 15 we conclude that the effect of damping is not significant in the regime of primary pulse propagation, in the sense that the dissipative terms do not appear to significantly change the dynamics during this high energy regime (although they significantly affect the lower energy dynamics following the propagation of the primary pulse). Hence, the following parametric study will be performed on the basis of the simplified system (3) with the aim to study the effects of coupling stiffness, foundation stiffness and contact stiffness on the energy transfer in the embedded coupled granular chains.

First we study the effect of coupling stiffness on energy transfer, keeping the other stiffness parameters of model (3) fixed to the optimized values estimated in the previous section (see Table 2). In Starosvetsky et al. (2011) the dynamics of weakly coupled granular chains without elastic foundations was studied, and it was shown there that, irrespective of the lateral coupling, pulse equi-partition between the two granular chains (excited and absorbing) occurs. However, it was found that with stronger lateral coupling, the period of initial transients significantly reduced and a smaller number of beads were required to achieve energy equi-partition
among the granular chains. To obtain a similar energy equi-partition phenomenon in the considered embedded granular chain, we performed a parametric study by only varying the coupling stiffness $k_2$. As the simulations of Fig. 19 indicate, such energy equi-partition is achievable in this embedded granular medium for appropriate selection of $k_2$. These responses correspond to low impulse excitation, lateral gap of 0.5 mm and coupling stiffness equal to 11 times the corresponding coupling system identified for the experimental system and listed in Table 3 (i.e., normalized value $k_2 = 11 \times 1.56 \times 10^{-4}$). In Fig. 19c we depict the transient evolutions of the total energies of the excited and absorbing chains, and conclude that energy equi-partition between the two chains occurs at the end of the regime of primary pulse propagation. Further increase of the coupling stiffness will lead to faster energy equi-partition, so this particular value of the coupling stiffness can be considered as the minimal value of the coupling stiffness necessary for energy equi-partition during primary pulse propagation. Extending the parametric study to the other two tested samples and considering both low and high impulse excitations, we computed the minimal value of the coupling stiffness for energy equi-partition for each case. The results are listed in Table 4, confirming that energy transfer is stronger for low impulsive excitation and smaller lateral gap between chains. The studied phenomenon of energy equi-partition between chains is associated with pulse redirection between chains, so our results can find application in designs of granular networks for passive wave redirection (Starosvetsky et al., 2011, 2012).

Additional parametric studies were performed to study the effect of foundation stiffness and contact stiffness on the energy exchanges in the embedded granular chain. The results show that higher energy transfer (including energy equi-partition) is achieved for lower foundation and contact stiffness values.

6. Concluding remarks

In this work we experimentally tested a system of two coupled granular chains embedded in PDMS matrix with varying lateral gap between them. Based on the experimental measurements we con-
constructed a strongly nonlinear theoretical model of coupled oscillators that accurately predicted primary pulse propagation in the experimental system. Utilizing this model we studied energy transfers in the tested samples and determined their dependence on the later gap between the granular chains. In addition, we showed that the constructed theoretical model can be used in a predictive capacity, in designing the embedded granular system for stronger energy exchanges and even primary pulse equi-partition between chains. Currently, we are fabricating samples with stiffer elastic matrices that provide stronger coupling between the embedded granular chains in order to experimentally verify the theoretically predicted energy equi-partition results.

Table 4
Minimal coupling stiffness for energy equi-partition during primary pulse propagation.

<table>
<thead>
<tr>
<th>Excitation</th>
<th>0.5 mm lateral gap</th>
<th>1.5 mm lateral gap</th>
<th>2.5 mm lateral gap</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>11</td>
<td>14</td>
<td>16</td>
</tr>
<tr>
<td>High</td>
<td>12</td>
<td>16</td>
<td>18</td>
</tr>
</tbody>
</table>

Fig. 19. Numerical simulation of the Hamiltonian model (3) for the sample with 0.5mm lateral gap with $k_2 = 11 \times 10^{-6}, k_1 = 0.000133$ and $\gamma = 0.002317$, depicting the velocity profiles of all beads of the two chains: (a) excited chain, (b) absorbing chain, (c) transient evolution of the energies in the two chains.

Table 4

To the authors’ best knowledge, this work represents one of the first systematic studies of the dynamics of ordered granular media embedded in elastic matrix, and, hence, adds to the body of existing works concerning ‘dry’ granular chains. Moreover, the reported results can contribute towards the design of practical acoustic metamaterials with embedded granular media.

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References


