Deformation theory of objects in homotopy and derived categories III: Abelian categories ☆

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Abstract

This is the third paper in a series. In Part I we developed a deformation theory of objects in homotopy and derived categories of DG categories. Here we show how this theory can be used to study deformations of objects in homotopy and derived categories of abelian categories. Then we consider examples from (noncommutative) algebraic geometry. In particular, we study noncommutative Grassmanians that are true noncommutative moduli spaces of structure sheaves of projective subspaces in projective spaces.
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Contents

1. Introduction

In our paper [5] we developed the deformation theory of a right DG module over a DG category $\mathcal{A}$ in the corresponding homotopy and derived categories. In the subsequent paper [6] we proved pro-representability of the corresponding deformation pseudo-functors. In this paper we would like to show how to apply this theory to deformations of complexes over *abelian* categories (in the corresponding homotopy and derived categories).

In the second part of the paper we discuss the example of complexes of (quasi-)coherent sheaves on a scheme. Then we give examples when our pro-representabity theorems in [6] can be applied to this geometric situation.

The third part is devoted to the example of a global noncommutative moduli space of objects in derived categories: noncommutative Grassmanians $\mathrm{NGr}(m, V)$. The noncommutative scheme $\mathrm{NGr}(m, V)$ is a true noncommutative moduli space of structure sheaves $\mathcal{O}_{\mathbb{P}(W)} \in D^b_{coh}(\mathbb{P}(V))$, where $W \subset V$ are vector subspaces of dimension $\dim W = m$. Namely, it satisfies the following properties:

1) There is a natural fully faithful functor Φ from the category of perfect complexes $\mathrm{Perf}(\mathrm{NGr}(m, V))$ (Definition 8.20) to $D^b_{coh}(\mathbb{P}(V))$. Its image is the double orthogonal to the family of objects $\mathcal{O}_{\mathbb{P}(W)}$, i.e. the full subcategory generated by objects $\mathcal{O}_{\mathbb{P}(V)}(m - n)$, $\dots, \mathcal{O}_{\mathbb{P}(V)}(-1), \mathcal{O}_{\mathbb{P}(V)}$. This is Corollary 8.22 below.

2) There is a k-point $x_W \in \mathrm{NGr}(m, V)(k) = X_{\mathcal{A}^{m,v}}(k)$ (see Section 9 below) for each subspace $W \subset V$ of dimension $\dim W = m$. Further, $(x_W)_*(k)$ lies in $\mathrm{Perf}(\mathrm{NGr}(m, V))$ and $\Phi(x_*(k)) \cong \mathcal{O}_{\mathbb{P}(W)}$. This is a part of Theorem 9.11 below.

3) The completion of the local ring of the k-point x_W (see Section 10.1) is isomorphic to $H^0(\hat{S})^{op}$, where $\hat{S}$ is dual to the bar construction of the minimal A_∞-structure on $\mathrm{Ext}^{\cdot}(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(W)})$ (Theorem 10.3). It can be shown that the DG algebra $\mathbf{R}\mathrm{Hom}^{\cdot}(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(W)})$ is formal and the graded algebra $\mathrm{Ext}^{\cdot}(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(W)})$ is quadratic Koszul, and hence the projection $\hat{S} \to H^0(\hat{S})$ is a quasi-isomorphism. Hence, the moduli space is not a DG space but just noncommutative space.

We do not have a moduli functor of our family of objects $\mathcal{O}_{\mathbb{P}(W)}$, which should be defined on the category of noncommutative affine schemes. However, the properties 1)–3) suggest that $\mathrm{NGr}(m, V)$ is a true moduli space of this family of objects, in our context of deformations of objects in derived categories.

It is remarkable that there is a natural morphism from the commutative Grassmanian $\mathrm{Gr}(m, V)$ to noncommutative one $\mathrm{NGr}(m, V)$. Moreover, the functor $\Phi : \mathrm{Perf}(\mathrm{NGr}(m, V)) \to D^b_{coh}(\mathbb{P}(V))$ above coincides with $\mathbf{L}f^*_{1,m,V}$, where $f_{1,m,V} : \mathbb{P}(V) \to \mathrm{NGr}(m, V)$ is a natural morphism. Both these statements are parts of Proposition 9.13 below.

Section 6 contains some preliminaries on $\mathbb{Z}$-algebras and the associated noncommutative schemes (or stacks) $\mathrm{Proj}(\mathcal{A})$ regarded as an abelian category of quasi-coherent sheaves together with a structure sheaf.

In Section 7 we define the noncommutative Grassmanians as Proj of certain $\mathbb{Z}$-algebras.

In Section 8 we describe the derived categories of noncommutative Grassmanians (Theorem 8.19). This is an application of the more general result for geometric $\mathbb{Z}$-algebras (Theorem 8.14) which originally appeared in [4].

In Section 9 we make an attempt to relate two different approaches to noncommutative geometry. Namely, we associate to each (positively oriented) $\mathbb{Z}$-algebra a presheaf of groupoids $X_{\mathcal{A}}$ on the category Alg^{op}_k dual to the category of associative unital k-algebras. The groupoid $X_{\mathcal{A}}(B)$ should be thought of as a groupoid of maps $Sp(B) \to \mathrm{Proj}(\mathcal{A})$. We compare our definition with maps between commutative schemes (Proposition 9.9). Then we describe the k-points of noncommutative Grassmanians (Theorem 9.11).

In the last Section 10, for any presheaf X of sets on Alg^{op}_k, and its k-point $x \in X(k)$, we define the notion of a completion of local ring $\widehat{\mathcal{O}_x}$, which cannot exist a priori. Then we prove that in the case of the noncommutative Grassmanian $\mathrm{NGr}(m, V)$ and the k-point x_W corresponding to the subspace $W \subset V$ of dimension m, the completion $\widehat{\mathcal{O}_x}$ exists and is isomorphic to $H^0(\hat{S})^{op}$ ($\cong H^0(\hat{S})$) in the above notation.

We freely use the notation and results of [5] and [6]. The reference to [5] or [6] appears in the form I, Theorem …, or II, Theorem … respectively.

Part 1. Deformations of objects in homotopy and derived categories of abelian categories

Let $\mathcal{M}$ be small a k-linear abelian category. Denote by $C(\mathcal{M})$, $\mathcal{H}(\mathcal{M})$, $D(\mathcal{M})$ the category of complexes over $\mathcal{M}$, its homotopy category and its derived category respectively. We will also consider the usual categories $C^b(\mathcal{M})$, $C^{\pm}(\mathcal{M})$ of bounded (resp. bounded above, below) complexes and the categories $\mathcal{H}^b(\mathcal{M})$, $D^b(\mathcal{M})$, $\mathcal{H}^{\pm}(\mathcal{M})$, $D^{\pm}(\mathcal{M})$ of cohomologically bounded (resp. bounded below, above) complexes. Given $E \in C(\mathcal{M})$ and an artinian DG algebra $\mathcal{R}$ there are natural notions of homotopy and derived $\mathcal{R}$-deformations (and $\mathcal{R}$-co-deformations) of E. We start by defining this deformation theory and then show (under some assumptions) how it can be interpreted as a deformation theory of a DG module over an appropriate DG category $\mathcal{A}$. This interpretation allows us to translate the previous results obtained in the DG context to the case of $C(\mathcal{M})$. Our point of view is that the deformation theory developed in [5,6] in the language of DG modules is more flexible. Hence for example in the context of abelian categories we omit the notion of pseudo-functors DEF and coDEF from the 2- dgart to **Gpd**.

2. Categories $C_{\mathcal{R}}(\mathcal{M}), C^{dg}_{\mathcal{R}}(\mathcal{M}), \mathcal{H}_{\mathcal{R}}(\mathcal{M}), D_{\mathcal{R}}(\mathcal{M})$

The category of complexes over $\mathcal{M}$ is also naturally a DG category with the Hom-complexes being the usual complexes of morphisms between objects in $C(\mathcal{M})$. We denote this DG category

by $C^{dg}(\mathcal{M})$. Then $\mathcal{H}(\mathcal{M})$ is simply $\mathrm{Ho}(C^{dg}(\mathcal{M}))$ and the category $D(\mathcal{M})$ is obtained from $\mathrm{Ho}(C^{dg}(\mathcal{M}))$ by inverting quasi-isomorphisms. Notice that this is NOT the same as $D(C^{dg}(\mathcal{M}))$ as defined in I, Section 3.1.

Definition 2.1. Let $\mathcal{R}$ be an artinian DG algebra. A right $\mathcal{R}$-complex over $\mathcal{M}$ (or simply an $\mathcal{R}^{op}$-complex) is an object $S \in C(\mathcal{M})$ together with a homomorphism of DG algebras $\mathcal{R}^{op} \to \mathrm{Hom}^{\cdot}(S, S)$. This is the same as a DG functor from the DG category $\mathcal{R}^{op}$ (with one object) to the DG category $C^{dg}(\mathcal{M})$. Thus $\mathcal{R}^{op}$-complexes over $\mathcal{M}$ naturally form a DG category $\mathrm{Fun}_{\mathrm{DG}}(\mathcal{R}^{op}, C^{dg}(\mathcal{M}))$ which we denote by $C_{\mathcal{R}}^{dg}(\mathcal{M})$. If in the category $C_{\mathcal{R}}^{dg}(\mathcal{M})$ we only consider morphisms which are degree zero cycles (i.e. DG transformations between DG functors), then we obtain an abelian category, which we denote by $C_{\mathcal{R}}(\mathcal{M})$. The homotopy category $\mathrm{Ho}(C_{\mathcal{R}}^{dg}(\mathcal{M}))$ is denoted by $\mathcal{H}_{\mathcal{R}}(\mathcal{M})$. If we invert quasi-isomorphisms in $\mathcal{H}_{\mathcal{R}}(\mathcal{M})$ we obtain the derived category $D_{\mathcal{R}}(\mathcal{M})$.

The categories $\mathcal{H}_{\mathcal{R}}(\mathcal{M})$ and $D_{\mathcal{R}}(\mathcal{M})$ are naturally triangulated. We will also consider the obvious full DG subcategories $C_{\mathcal{R}}^{b}(\mathcal{M}), C_{\mathcal{R}}^{\pm}(\mathcal{M}) \subset C_{\mathcal{R}}^{dg}(\mathcal{M})$ and the full triangulated subcategories $\mathcal{H}_{\mathcal{R}}^{b}(\mathcal{M}), \mathcal{H}_{\mathcal{R}}^{\pm}(\mathcal{M}) \subset \mathcal{H}_{\mathcal{R}}(\mathcal{M})$; $D_{\mathcal{R}}^{b}(\mathcal{M}), D_{\mathcal{R}}^{\pm}(\mathcal{M}) \subset D_{\mathcal{R}}(\mathcal{M})$.

Remark 2.2. Consider $\mathcal{M}$ as a DG category (with all morphisms being of degree zero) and let $\mathcal{R}$ be an artinian DG algebra. Notice that an $\mathcal{R}^{op}$-complex $S \in C(\mathcal{M})$ defines (by Yoneda) a DG-module over the DG category $\mathcal{M}_{\mathcal{R}}^{op} = \mathcal{M}^{op} \otimes \mathcal{R}^{op}$ (3.1, 3.3 in Part I), i.e. there is a full and faithful embedding of DG categories

$$h_{\mathcal{R}}^{\bullet} : C_{\mathcal{R}}^{dg}(\mathcal{M}) \hookrightarrow \mathcal{M}_{\mathcal{R}}^{op}\text{-mod}.$$

Using this embedding we could directly apply our machinery in Part I to obtain a deformation theory of objects in $C(\mathcal{M})$. This deformation theory however would not always give the right answer (in case of derived deformations). Our point is that there exists a natural independent deformation theory for complexes over abelian categories which we define in the next section. Eventually we will compare this theory to deformations of DG-modules as in Part I.

The next lemma is a repetition of I, Lemma 3.19 in our context.

Lemma 2.3. *Assume that $\mathcal{R} \in \mathrm{dgart}\_$. Then there exist truncation functors in $D_{\mathcal{R}}(\mathcal{M})$: for every $\mathcal{R}^{op}$-complex S there exists a short exact sequence of $\mathcal{R}^{op}$-complexes*

$$\tau_{<0}S \to S \to \tau_{\geqslant 0}S,$$

where $H^{i}(\tau_{<0}S) = 0$ if $i \geqslant 0$ and $H^{i}(\tau_{\geqslant 0}S) = 0$ for $i < 0$.

Proof. Indeed, put

$$\tau_{<0}S := \bigoplus_{i<0} S^{i} \oplus d(S^{-1}). \qquad \square$$

The next definition is the analogue of I, Definition 3.8.

Definition 2.4. Let $\mathcal{R} \in$ dgart. An $\mathcal{R}^{op}$-complex S is called graded $\mathcal{R}$-free (resp. graded $\mathcal{R}$-cofree) if there exist $M \in C^{dg}(\mathcal{M})$ and an isomorphism of graded objects (forgetting the differential) in $C^{dg}_{\mathcal{R}}(\mathcal{M})$ $M \otimes \mathcal{R} \simeq S$ (resp. $M \otimes \mathcal{R}^* \simeq S$).

Proposition 2.5. *A homomorphism $\phi : \mathcal{R} \to \mathcal{Q}$ of artinian DG algebras induces DG functors*

$$\phi^* : C^{dg}_{\mathcal{R}}(\mathcal{M}) \to C^{dg}_{\mathcal{Q}}(\mathcal{M}), \quad \phi_* : C^{dg}_{\mathcal{Q}}(\mathcal{M}) \to C^{dg}_{\mathcal{R}}(\mathcal{M}), \quad \phi^! : C^{dg}_{\mathcal{R}}(\mathcal{M}) \to C^{dg}_{\mathcal{Q}}(\mathcal{M}).$$

The DG functors (ϕ^*, ϕ_*) and $(\phi_*, \phi^!)$ are adjoint. That is for $S \in C^{dg}_{\mathcal{R}}(\mathcal{M})$ and $T \in C^{dg}_{\mathcal{Q}}(\mathcal{M})$ there are functorial isomorphisms of complexes

$$\mathrm{Hom}^{\cdot}\big(\phi^* S, T\big) = \mathrm{Hom}^{\cdot}(S, \phi_* T), \qquad \mathrm{Hom}^{\cdot}(\phi_* T, S) = \mathrm{Hom}^{\cdot}\big(T, \phi^! S\big).$$

We denote by the same symbols the induced functors between the abelian categories $C_{\mathcal{R}}(\mathcal{M})$, $C_{\mathcal{Q}}(\mathcal{M})$ and the homotopy categories $\mathcal{H}_{\mathcal{R}}(\mathcal{M})$, $\mathcal{H}_{\mathcal{Q}}(\mathcal{M})$. These induced functors are also adjoint.

Proof. The categories $\mathcal{M}$ and $C(\mathcal{M})$ are abelian and as such have all finite limits and colimits. Let $S \in C^{dg}_{\mathcal{R}}(\mathcal{M})$. We put

$$\phi^*(S) = S \otimes_{\mathcal{R}} \mathcal{Q}, \qquad \phi^!(S) = \mathrm{Hom}^{\cdot}_{\mathcal{R}^{op}}(Q, S).$$

That is $\phi^*(S)$ is defined as a colimit of a finite (since $\dim \mathcal{R} < \infty$) diagram involving the object $S \otimes_k \mathcal{Q}$; and $\phi^!(S)$ is a limit of a finite (since $\dim \mathcal{R} < \infty$) diagram involving the object $S \otimes_k Q^*$ (since $\dim Q < \infty$). So these objects are well defined. The DG functor ϕ_* is simply the restriction of scalars.

For each $M \in C^{dg}_{\mathcal{Q}}(\mathcal{M})$, $N \in C^{dg}_{\mathcal{R}}(\mathcal{M})$ we have natural functorial closed morphisms of degree zero

$$\eta_1(M) : \phi^* \phi_*(M) = \phi_*(M) \otimes_{\mathcal{R}} \mathcal{Q} \to M, \qquad \eta_2(N) : N \to \phi_*(N \otimes_{\mathcal{R}} \mathcal{Q}) = \phi_* \phi^*(N),$$

induced by the structure morphism $M \otimes \mathcal{Q} \to M$, and by the inclusion $N \to N \otimes \mathcal{Q}$. They give rise to the morphisms of functors

$$\eta_1 : \phi^* \phi_* \to \mathrm{id}, \qquad \eta_2 : \mathrm{id} \to \phi_* \phi^*.$$

It is also clear that the compositions

$$\phi^* \xrightarrow{\phi^*(\eta_2)} \phi^* \phi_* \phi^* \xrightarrow{\eta_1(\phi^*)} \phi^*, \qquad \phi_* \xrightarrow{\eta_2(\phi_*)} \phi_* \phi^* \phi_* \xrightarrow{\phi_*(\eta_1)} \phi_*$$

are equal to identity morphisms. Hence, the pair of DG functors (ϕ^*, ϕ_*) is adjoint. The same arguments hold for abelian and homotopy categories.

The adjunction $(\phi_*, \phi^!)$ is proved in the same way. □

Example 2.6. Let $i : \mathcal{R} \to k$ be the augmentation map and $p : k \to \mathcal{R}$ be the obvious inclusion. Then we obtain the corresponding functors $i^*, i_*, i^!, p^*, p_*, p^!$.

Definition 2.7. An object $S \in C_{\mathcal{R}}^{dg}(\mathcal{M})$ is called h-projective (resp. h-injective) if for every acyclic $T \in C_{\mathcal{R}}^{dg}(\mathcal{M})$ the complex $\mathrm{Hom}^{\cdot}(S, T)$ (resp. $\mathrm{Hom}^{\cdot}(T, S)$) is acyclic.

Remark 2.8. Note that the collection of h-projectives (resp. h-injectives) is closed under arbitrary (existing in $C_{\mathcal{R}}^{dg}(\mathcal{M})$) direct sums (resp. direct products).

Corollary 2.9. *The DG functor ϕ^* (resp. $\phi^!$) preserves h-projectives (resp. h-injectives).*

Proof. This follows from the adjunctions (ϕ^*, ϕ_*), $(\phi_*, \phi^!)$ and the fact that ϕ_* preserves acyclic complexes. $\quad\square$

Proposition 2.10. *Fix $\mathcal{R} \in \mathrm{dgart}\_$.*

a) *Assume that $\mathcal{M}$ has enough projectives. Then for every $S \in C_{\mathcal{R}}^-(\mathcal{M})$ there exists an h-projective $P \in C_{\mathcal{R}}^-(\mathcal{M})$ and a quasi-isomorphism $P \to S$. We may choose P to be graded $\mathcal{R}$-free.*

b) *Assume that $\mathcal{M}$ has enough injectives. Then for every $T \in C_{\mathcal{R}}^+(\mathcal{M})$ there exists an h-injective $I \in C_{\mathcal{R}}^+(\mathcal{M})$ and a quasi-isomorphism $T \to I$. We may choose I to be graded $\mathcal{R}$-cofree.*

Proof. a) Fix $S \in C_{\mathcal{R}}^-(\mathcal{M})$ and assume that $S^i = 0$ for $i > i_0$. Since $\mathcal{M}$ has enough projectives we can find (by a standard construction) an h-projective $Q_0 \in C^-(\mathcal{M})$ and a surjective quasi-isomorphism $\epsilon' : Q_0 \to p_*S$. We may and will assume that each $Q_0^j \in \mathcal{M}$ is projective. Moreover we may and will assume that $Q_0^i = 0$ for $i > i_0$. By adjunction we obtain a surjective morphism in $C_{\mathcal{R}}^-(\mathcal{M})$,

$$\epsilon : P_0 = p^*Q_0 \to S,$$

which is also surjective on cohomology. Denote $K = \mathrm{Ker}(\epsilon)$. Note that since $\mathcal{R} \in \mathrm{dgart}\_$ we have $K^i = 0$ for $i > i_0$. Now repeat the procedure with K instead of S. Finally we obtain an exact sequence

$$\cdots \to P_{-2} \to P_{-1} \to P_0 \xrightarrow{\epsilon} S \to 0$$

in $C_{\mathcal{R}}^-(\mathcal{M})$ with the following properties:

(i) P_{-n} is h-projective for every n.
(ii) $P_{-n}^i = 0$ for $i > i_0$ and all n.
(iii) The complex

$$\cdots \to H^{\cdot}(P_{-1}) \to H^{\cdot}(P_0) \to H^{\cdot}(S) \to 0$$

is exact.

Denote by P the total complex

$$P = \mathrm{Tot}(\cdots \to P_{-1} \to P_0).$$

Note that as a graded object

$$P = \bigoplus_{n \geqslant 0} P_{-n}[n],$$

so that in each degree i the contribution to P^i comes from finitely many P_{-n}'s. Thus P is a well-defined object in $C_{\mathcal{R}}^-(\mathcal{M})$ (we do not assume that infinite direct sums exist in $\mathcal{M}$). Because of the property (iii) above the morphism

$$\epsilon : P \to S$$

is a quasi-isomorphism. It remains to show that P is h-projective.

We have the standard increasing filtration by $\mathcal{R}^{op}$-subcomplexes

$$F_n P = \mathrm{Tot}(P_{-n} \to \cdots \to P_{-1} \to P_0).$$

This filtration satisfies the following properties:

(a) $P = \bigcup_{n \geqslant 0} F_n P$;
(b) each quotient $F_n P / F_{n-1} P$ is h-projective;
(c) each inclusion of graded $\mathcal{R}^{op}$-objects $(F_{n-1}P)^{gr} \hookrightarrow (F_n P)^{gr}$ splits.

It follows that P is h-projective.

b) The proof is very similar to that of a), but we present it anyway for completeness. Fix $T \in C_{\mathcal{R}}^+(\mathcal{M})$, say $T^i = 0$ for $i < i_0$. Since $\mathcal{M}$ has enough injectives we can find an h-injective $J_0 \in C^+(\mathcal{M})$ and an injective quasi-isomorphism $s' : p_* T \to J_0$. We may and will assume that J_0 consists of objects which are injective in $\mathcal{M}$. Moreover we may and will assume that $J_0^i = 0$ for $i < i_0$. By adjunction we obtain an injective morphism of objects in $C_{\mathcal{R}}^+(\mathcal{M})$,

$$s : T \to I_0 := p^! J_0,$$

which is also injective on cohomology. Denote $L = \mathrm{coker}(s)$. Note that since $\mathcal{R} \in \mathrm{dgart\_}$ we have $L^i = 0$ for $i < i_0$. Now repeat the procedure with L instead of T. Finally we obtain an exact sequence

$$0 \to T \xrightarrow{s} I_0 \to I_1 \to \cdots$$

in $C_{\mathcal{R}}^+(\mathcal{M})$ with the following properties:

(i′) I_n is h-injective for all n.
(ii′) $I_n^i = 0$ for $i < i_0$ and all n.
(iii′) The complex

$$0 \to H^{\cdot}(T) \to H^{\cdot}(I_0) \to H^{\cdot}(I_1) \to \cdots$$

is exact.

Denote by I the total complex

$$I := \mathrm{Tot}(I_0 \to I_1 \to \cdots).$$

Note that as a graded object

$$I = \bigoplus_{n \geqslant 0} I_n[-n],$$

so that in each degree i the contribution to I^i comes from finitely many I_n's. Thus I is a well-defined object in $C_{\mathcal{R}}^+(\mathcal{M})$. Because of property (iii′) above the morphism $s : T \to I$ is a quasi-isomorphism. It remains to show that I is h-injective.

We have the standard decreasing filtration by $\mathcal{R}^{op}$-subcomplexes

$$F_n I = \mathrm{Tot}(I_n \to I_{n+1} \to \cdots).$$

This filtration satisfies the following properties:

(a) $I = \varprojlim I/F_n I$;
(b) each quotient $F_n P/F_{n+1} P$ is h-injective;
(c) each inclusion of graded $\mathcal{R}^{op}$-objects $(F_{n+1} I)^{gr} \hookrightarrow (F_n P)^{gr}$ splits.

It follows that I is h-injective.
Proposition is proved. $\square$

Using the last proposition we can define derived functors of the functors ϕ^* and $\phi^!$. Namely assume that $\mathcal{M}$ has enough projectives (resp. injectives). Then given a homomorphism $\phi : \mathcal{R} \to \mathcal{Q}$ of artinian (non-positive) DG algebras we define the functor $\mathbf{L}\phi^* : D_{\mathcal{R}}^-(\mathcal{M}) \to D_{\mathcal{Q}}^-(\mathcal{M})$ (resp. $\mathbf{R}\phi^! : D_{\mathcal{R}}^+(\mathcal{M}) \to D_{\mathcal{Q}}^+(\mathcal{M})$) using h-projectives (resp. h-injectives) in the usual way. Notice that the functor ϕ_* is exact, hence it extends trivially to $\phi_* : D_{\mathcal{Q}}^\pm(\mathcal{M}) \to D_{\mathcal{R}}^\pm(\mathcal{M})$. The functors $(\mathbf{L}\phi^*, \phi_*)$ and $(\phi_*, \mathbf{R}\phi^!)$ are adjoint.

Remark 2.11. Let $\mathcal{R} \in \mathrm{dgart}_-$ and $M, N \in C_{\mathcal{R}}^{dg}(\mathcal{M})$. Assume that (i) $\mathcal{M}$ has enough projectives and $M \in D_{\mathcal{R}}^-(\mathcal{M})$ or (ii) $\mathcal{M}$ has enough injectives and $N \in D_{\mathcal{R}}^+(\mathcal{M})$. Then we can define the complex $\mathbf{R}\mathrm{Hom}^\cdot(M, N)$ and hence the vector spaces $\mathrm{Ext}^i(M, N)$. Namely, by Proposition 2.10 in the first case we may replace M by a quasi-isomorphic h-projective P and in the second case we may replace N by a quasi-isomorphic h-injective I. Then define $\mathbf{R}\mathrm{Hom}(M, N)$ as either $\mathrm{Hom}^\cdot(P, N)$ or $\mathrm{Hom}^\cdot(M, I)$.

Proposition 2.12. *Let $\phi : \mathcal{R} \to \mathcal{Q}$ be a morphism of artinian DG algebras which is a quasi-isomorphism.*

a) *Assume that $\mathcal{M}$ has enough projectives. Then the functor $\mathbf{L}\phi^* : D_{\mathcal{R}}^-(\mathcal{M}) \to D_{\mathcal{Q}}^-(\mathcal{M})$ is an equivalence of categories.*
b) *Assume that $\mathcal{M}$ has enough injectives. Then the functor $\mathbf{R}\phi^! : D_{\mathcal{R}}^+(\mathcal{M}) \to D_{\mathcal{Q}}^+(\mathcal{M})$ is an equivalence of categories.*

Proof. a) It suffices to prove that for each h-projective $P \in C_{\mathcal{R}}^{dg}(\mathcal{M})$, $P' \in C_{\mathcal{Q}}^{dg}(\mathcal{M})$, the adjunction morphisms

$$\eta_1(P') : \phi^* \phi_*(P') \to P', \qquad \eta_2(P) : P \to \phi_* \phi^*(P)$$

are quasi-isomorphisms. As complexes in $C(\mathcal{M})$, the cones of both morphisms are of the form $P'' \otimes_{\mathcal{R}} Cone(\mathcal{R} \to \mathcal{Q})$, where P'' is P or P' respectively. So, it remains to prove the following

Lemma 2.13. *If N is an acyclic finite-dimensional $\mathcal{R}$-module and $P \in C_{\mathcal{R}}^{dg}(\mathcal{M})$ is h-projective, then the complex $P \otimes_{\mathcal{R}} N$ is null-homotopic.*

Proof. Let M be an object in $C(\mathcal{M})$. Then we have

$$\mathrm{Hom}^{\cdot}(P \otimes_{\mathcal{R}} N, M) = \mathrm{Hom}_{\mathcal{R}^{op}}^{\cdot}\big(P, \mathrm{Hom}_k^{\cdot}(N, M)\big),$$

and this complex is acyclic since the $\mathcal{R}^{op}$-complex $\mathrm{Hom}_k^{\cdot}(N, M)$ is acyclic and P is h-projective. Hence, the complex $P \otimes_{\mathcal{R}} N$ is null-homotopic. □

b) The proof is similar. It suffices to prove that for each h-injective $I \in C_{\mathcal{R}}^{dg}(\mathcal{M})$, $I' \in C_{\mathcal{Q}}^{dg}(\mathcal{M})$, the adjunction morphisms

$$\eta_1(I) : \phi_* \phi^!(I) \to I, \qquad \eta_2(I') : I' \to \phi^! \phi_*(I')$$

are quasi-isomorphisms. As complexes in $C(\mathcal{M})$, the cones of both morphisms are of the form $\mathrm{Hom}_{\mathcal{R}^{op}}^{\cdot}(Cone(\mathcal{R} \to \mathcal{Q}), I'')$ where I'' is I or I' respectively. So, it remains to prove the following:

Lemma 2.14. *If N is an acyclic finite-dimensional $\mathcal{R}^{op}$-module and $I \in C_{\mathcal{R}}^{dg}(\mathcal{M})$ is h-projective, then the complex $\mathrm{Hom}_{\mathcal{R}^{op}}^{\cdot}(N, I)$ is null-homotopic.*

Proof. Let M be an object in $C(\mathcal{M})$. Then we have

$$\mathrm{Hom}^{\cdot}\big(M, \mathrm{Hom}_{\mathcal{R}^{op}}^{\cdot}(N, I)\big) = \mathrm{Hom}_{\mathcal{R}^{op}}^{\cdot}\big(M, \mathrm{Hom}_k^{\cdot}(N, I)\big),$$

and this complex is acyclic since the $\mathcal{R}^{op}$-complex $\mathrm{Hom}_k^{\cdot}(N, M)$ is acyclic, h-injective and hence null-homotopic. Hence, the complex $\mathrm{Hom}_{\mathcal{R}}^{\cdot}(N, I)$ is null-homotopic. □

Proposition is proved. □

3. Deformation pseudo-functors

Let E be an object in $C(\mathcal{M})$. As in the DG setting we first define the homotopy deformation and co-deformation pseudo-functors

$$\mathrm{Def}^{\mathrm{h}}(E), \ \mathrm{coDef}^{\mathrm{h}}(E) : \mathbf{dgart} \to \mathbf{Gpd}.$$

The definitions are copies (in our context) of the corresponding definitions in Part I.

Definition 3.1. Fix $E \in C(\mathcal{M})$ and let $\mathcal{R} \in$ dgart. An object in the groupoid $\text{Def}^{\text{h}}_{\mathcal{R}}(E)$ is a pair (S, σ), where $S \in C_{\mathcal{R}}(\mathcal{M})$ and $\sigma : i^*S \to E$ is an isomorphism of objects in $C(\mathcal{M})$ such that the following holds: there exists an isomorphism of graded objects $\eta : (E \otimes \mathcal{R})^{\text{gr}} \to S^{\text{gr}}$ so that the composition

$$E = i^*(E \otimes \mathcal{R}) \xrightarrow{i^*\eta} i^*S \xrightarrow{\sigma} E$$

is the identity.

Given objects $(S, \sigma), (S', \sigma') \in \text{Def}^{\text{h}}_{\mathcal{R}}(E)$, a map $f : (S, \sigma) \to (S', \sigma')$ is an isomorphism $f : S \to S'$ such that $\sigma' \cdot i^*f = \sigma$. An allowable homotopy between maps f, g is a homotopy $h : f \to g$ such that $i^*h = 0$. We define morphisms in $\text{Def}^{\text{h}}_{\mathcal{R}}(E)$ to be classes of maps modulo allowable homotopies.

Note that a homomorphism of artinian DG algebras $\phi : \mathcal{R} \to \mathcal{Q}$ induces the functor $\phi^* : \text{Def}^{\text{h}}_{\mathcal{R}}(E) \to \text{Def}^{\text{h}}_{\mathcal{Q}}(E)$. This defines the pseudo-functor

$$\text{Def}^{\text{h}}(E) : \text{dgart} \to \textbf{Gpd}.$$

We refer to objects of $\text{Def}^{\text{h}}_{\mathcal{R}}(E)$ as homotopy $\mathcal{R}$-deformations of E.

Example 3.2. We call $(p^*E, \text{id}) \in \text{Def}^{\text{h}}_{\mathcal{R}}(E)$ the trivial $\mathcal{R}$-deformation of E.

Definition 3.3. Denote by $\text{Def}^{\text{h}}_{+}(E)$, $\text{Def}^{\text{h}}_{-}(E)$, $\text{Def}^{\text{h}}_{0}(E)$, $\text{Def}^{\text{h}}_{\text{cl}}(E)$ the restrictions of the pseudo-functor $\text{Def}^{\text{h}}(E)$ to subcategories dgart$_{+}$, dgart$_{-}$, art, cart respectively.

Let us give an alternative description of the same deformation problem. We will define the homotopy *co-deformation* pseudo-functor $\text{coDef}^{\text{h}}(E)$ and (eventually) show that it is equivalent to $\text{Def}^{\text{h}}(E)$. The point is that in practice one should use $\text{Def}^{\text{h}}(E)$ for an h-projective E and $\text{coDef}^{\text{h}}(E)$ for an h-injective E.

Definition 3.4. Fix $E \in C(\mathcal{M})$. Let $\mathcal{R}$ be an artinian DG algebra. An object in the groupoid $\text{coDef}^{\text{h}}_{\mathcal{R}}(E)$ is a pair (T, τ), where $T \in C_{\mathcal{R}}(\mathcal{M})$ and $\tau : E \to i^!T$ is an isomorphism of objects in $C(\mathcal{M})$ so that the following holds: there exists an isomorphism of graded objects $\delta : T^{\text{gr}} \to (E \otimes \mathcal{R}^*)^{\text{gr}}$ such that the composition

$$E \xrightarrow{\tau} i^!T \xrightarrow{i^!\delta} i^!(E \otimes \mathcal{R}^*) = E$$

is the identity.

Given objects (T, τ) and $(T', \tau') \in \text{coDef}^{\text{h}}_{\mathcal{R}}(E)$ a map $g : (T, \tau) \to (T', \tau')$ is an isomorphism $f : T \to T'$ such that $i^!f \cdot \tau = \tau'$. An allowable homotopy between maps f, g is a homotopy $h : f \to g$ such that $i^!(h) = 0$. We define morphisms in $\text{coDef}^{\text{h}}_{\mathcal{R}}(E)$ to be classes of maps modulo allowable homotopies.

Note that a homomorphism of DG algebras $\phi : \mathcal{R} \to \mathcal{Q}$ induces the functor $\phi^! : \text{Def}^{\text{h}}_{\mathcal{R}}(E) \to \text{Def}^{\text{h}}_{\mathcal{Q}}(E)$. This defines the pseudo-functor

$$\text{Def}^{\text{h}}(E) : \text{dgart} \to \textbf{Gpd}.$$

We refer to objects of $\mathrm{coDef}_{\mathcal{R}}^{\mathrm{h}}(E)$ as homotopy $\mathcal{R}$-co-deformations of E.

Example 3.5. For example we can take $T = E \otimes \mathcal{R}^*$ with the differential $d_{E,R^*} := d_E \otimes 1 + 1 \otimes d_{R^*}$ (and $\tau = \mathrm{id}$). This we consider as the *trivial* $\mathcal{R}$-co-deformation of E.

Definition 3.6. Denote by $\mathrm{coDef}_+^{\mathrm{h}}(E)$, $\mathrm{coDef}_-^{\mathrm{h}}(E)$, $\mathrm{coDef}_0^{\mathrm{h}}(E)$, $\mathrm{coDef}_{\mathrm{cl}}^{\mathrm{h}}(E)$ the restrictions of the pseudo-functor $\mathrm{coDef}^{\mathrm{h}}(E)$ to subcategories dgart_+, dgart_-, art, cart respectively.

3.1. Derived deformation pseudo-functors

Likewise we define derived deformation pseudo-functors. In view of Proposition 2.10 above we restrict ourselves to non-positive artinian DG algebras and consider two cases: enough projectives or enough injectives in $\mathcal{M}$.

Definition 3.7. Assume that $\mathcal{M}$ has enough projectives and fix $E \in D^-(\mathcal{M})$. We are going to define a pseudo-functor

$$\mathrm{Def}_-(E) : \mathrm{dgart}_- \to \mathbf{Gpd}$$

of derived deformations of E. Fix an artinian DG algebra $\mathcal{R} \in \mathrm{dgart}_-$. An object of the groupoid $\mathrm{Def}_{\mathcal{R}}(E)$ is a pair (S, σ), where $S \in D_{\mathcal{R}}^-(\mathcal{M})$ and σ is an isomorphism (in $D^-(\mathcal{M})$)

$$\sigma : \mathbf{L}i^*S \to E.$$

A morphism $f : (S, \sigma) \to (T, \tau)$ between two $\mathcal{R}$-deformations of E is an isomorphism (in $D_{\mathcal{R}}^-(\mathcal{M})$) $f : S \to T$, such that

$$\tau \cdot \mathbf{L}i^*(f) = \sigma.$$

This defines the groupoid $\mathrm{Def}_{\mathcal{R}}(E)$. A homomorphism of artinian DG algebras $\phi : \mathcal{R} \to \mathcal{Q}$ induces the functor

$$\mathbf{L}\phi^* : \mathrm{Def}_{\mathcal{R}}(E) \to \mathrm{Def}_{\mathcal{Q}}(E).$$

Thus we obtain a pseudo-functor

$$\mathrm{Def}_-(E) : \mathrm{dgart}_- \to \mathbf{Gpd}.$$

We call $\mathrm{Def}_-(E)$ the functor of derived deformations of E.

Remark 3.8. A quasi-isomorphism $\phi : \mathcal{R} \to \mathcal{Q}$ of artinian DG algebras induces an equivalence of groupoids

$$\mathbf{L}\phi^* : \mathrm{Def}_{\mathcal{R}}(E) \to \mathrm{Def}_{\mathcal{Q}}(E).$$

Indeed, $\mathbf{L}\phi^* : D_{\mathcal{R}}^-(\mathcal{M}) \to D_{\mathcal{Q}}^-(\mathcal{M})$ is an equivalence of categories (Lemma 2.12 a)) which commutes with the functor $\mathbf{L}i^*$.

Remark 3.9. A quasi-isomorphism $\delta : E_1 \to E_2$ in $D^-(\mathcal{M})$ induces an equivalence of pseudo-functors

$$\delta_* : \mathrm{Def}_-(E_1) \to \mathrm{Def}_-(E_2)$$

by the formula $\delta_*(S, \sigma) = (S, \delta \cdot \sigma)$.

Definition 3.10. Denote by $\mathrm{Def}_0(E)$, $\mathrm{Def}_{\mathrm{cl}}(E)$ the restrictions of the pseudo-functor $\mathrm{Def}_-(E)$ to subcategories art, cart respectively.

Let us now define the pseudo-functor of derived co-deformations.

Definition 3.11. Assume that $\mathcal{M}$ has enough injectives and fix $E \in D^+(\mathcal{M})$. We are going to define a pseudo-functor

$$\mathrm{coDef}_-(E) : \mathrm{dgart}_- \to \mathbf{Gpd}$$

of derived co-deformations of E. Fix an artinian DG algebra $\mathcal{R} \in \mathrm{dgart}_-$. An object of the groupoid $\mathrm{coDef}_{\mathcal{R}}(E)$ is a pair (S, σ), where $S \in D^+_{\mathcal{R}}(\mathcal{M})$ and σ is an isomorphism (in $D^+(\mathcal{M})$)

$$\sigma : E \to \mathbf{R}i^! S.$$

A morphism $f : (S, \sigma) \to (T, \tau)$ between two $\mathcal{R}$-deformations of E is an isomorphism (in $D^+_{\mathcal{R}}(\mathcal{M})$) $f : S \to T$, such that

$$\mathbf{R}i^!(f) \cdot \sigma = \tau.$$

This defines the groupoid $\mathrm{coDef}_{\mathcal{R}}(E)$. A homomorphism of artinian DG algebras $\phi : \mathcal{R} \to \mathcal{Q}$ induces the functor

$$\mathbf{R}\phi^! : \mathrm{coDef}_{\mathcal{R}}(E) \to \mathrm{coDef}_{\mathcal{Q}}(E).$$

Thus we obtain a pseudo-functor

$$\mathrm{coDef}_-(E) : \mathrm{dgart}_- \to \mathbf{Gpd}.$$

We call $\mathrm{coDef}_-(E)$ the functor of derived co-deformations of E.

Remark 3.12. A quasi-isomorphism $\phi : \mathcal{R} \to \mathcal{Q}$ of artinian DG algebras induces an equivalence of groupoids

$$\mathbf{R}\phi^! : \mathrm{coDef}_{\mathcal{R}}(E) \to \mathrm{coDef}_{\mathcal{Q}}(E).$$

Indeed, $\mathbf{R}\phi^! : D^+_{\mathcal{R}}(\mathcal{M}) \to D^+_{\mathcal{Q}}(\mathcal{M})$ is an equivalence of categories (Lemma 2.12 b)) which commutes with the functor $\mathbf{R}i^!$.

Remark 3.13. A quasi-isomorphism $\delta : E_1 \to E_2$ in $D^+(\mathcal{M})$ induces an equivalence of functors

$$\delta^* : \mathrm{coDef}(E_2) \to \mathrm{coDef}(E_1)$$

by the formula $\delta^*(S, \sigma) = (S, \sigma \cdot \delta)$.

Definition 3.14. Denote by $\mathrm{coDef}_0(E)$, $\mathrm{coDef}_{\mathrm{cl}}(E)$ the restrictions of the pseudo-functor $\mathrm{coDef}_-(E)$ to subcategories art, cart respectively.

3.2. Summary of main properties of deformation pseudo-functors

Proposition 3.15. *Let $E \in C(\mathcal{M})$. There exists a natural equivalence of pseudo-functors*

$$\mathrm{Def}^{\mathrm{h}}(E) \to \mathrm{coDef}^{\mathrm{h}}(E).$$

Consider E as an object in the DG category $C^{dg}(\mathcal{M})$ and denote by $\mathcal{B}$ the DG algebra $\mathcal{B} = \mathrm{End}(E)$. Then these pseudo-functors are also equivalent to the Maurer–Cartan pseudo-functor $\mathcal{MC}(\mathcal{B}) : \mathrm{dgart} \to \mathbf{Gpd}$ (I, Section 5).

Theorem 3.16. *Let $E, E' \in C(\mathcal{M})$ be such that the DG algebras $\mathrm{End}(E)$ and $\mathrm{End}(E')$ are quasi-isomorphic. (For example assume that E and E' are homotopy equivalent.) Then the pseudo-functors $\mathrm{Def}^{\mathrm{h}}(E)$ $(\simeq \mathrm{coDef}^{\mathrm{h}}(E))$, $\mathrm{Def}^{\mathrm{h}}(E')$ $(\simeq \mathrm{coDef}^{\mathrm{h}}(E'))$ are equivalent.*

Theorem 3.17. *Assume that $\mathcal{M}$ has enough projectives and let $P \in C^-(\mathcal{M})$ be h-projective such that $\mathrm{Ext}^{-1}(P, P) = 0$. Then there is an equivalence of deformation pseudo-functors*

$$\mathrm{Def}_-(P) \simeq \mathrm{Def}^{\mathrm{h}}_-(P).$$

Theorem 3.18. *Assume that $\mathcal{M}$ has enough injectives and let $I \in C^+(\mathcal{M})$ be h-injective such that $\mathrm{Ext}^{-1}(I, I) = 0$. Then there is an equivalence of deformation pseudo-functors*

$$\mathrm{coDef}_-(I) \simeq \mathrm{coDef}^{\mathrm{h}}_-(I).$$

Corollary 3.19. *Assume that $\mathcal{M}$ has enough projectives (resp. injectives) and $E \in D^-(\mathcal{M})$ (resp. $E \in D^+(\mathcal{M})$) is such that $\mathrm{Ext}^{-1}(E, E) = 0$. Then the deformation pseudo-functor $\mathrm{Def}_-(E)$ (resp. $\mathrm{coDef}_-(E)$) depends up to an equivalence only on the quasi-isomorphism class of the DG algebra $\mathbf{R}\,\mathrm{Hom}(E, E)$. In particular, let $\mathcal{N}$ be another abelian category with enough projectives (resp. enough injectives); assume that $F : \mathcal{M} \to \mathcal{N}$ is a functor such that $C(F) : C(\mathcal{M}) \to C(\mathcal{N})$ preserves h-projectives (resp. h-injectives) and induces an equivalence $\tilde{F} : D^-(\mathcal{M}) \to D^-(\mathcal{N})$ (resp. $\tilde{F} : D^+(\mathcal{M}) \to D^+(\mathcal{N})$), then the pseudo-functors $\mathrm{Def}_-(E)$ and $\mathrm{Def}_-(F(E))$ (resp. $\mathrm{coDef}_-(E)$ and $\mathrm{coDef}_-(F(E))$) are equivalent.*

Theorem 3.20. *Assume that $\mathcal{M}$ has enough projectives and enough injectives. Let $E \in D^b(\mathcal{M})$ be a complex with $\mathrm{Ext}^{-1}(E, E) = 0$. Then there exists an equivalence of pseudo-functors*

$$\mathrm{Def}_-(E) \simeq \mathrm{coDef}_-(E).$$

These propositions and theorems will be proved in the next two subsections.

3.3. Relation with the homotopy deformation theory of DG modules

Let $\mathcal{R}$ be an artinian DG algebra. Recall (Remark 2.2) that an $\mathcal{R}^{op}$-complex $M \in C^{dg}_{\mathcal{R}}(\mathcal{M})$ defines a DG functor from the DG category $\mathcal{M}$ to the DG category of DG $\mathcal{R}^{op}$-modules by the formula $N \mapsto \mathrm{Hom}_{C^{dg}(\mathcal{M})}(N, M)$. This may be considered also as a DG module over the DG category $\mathcal{M}^{op}_{\mathcal{R}} = (\mathcal{M} \otimes \mathcal{R})^{op}$. Thus we obtain a full and faithful (by Yoneda) DG functor

$$h^{\bullet}_{\mathcal{R}} : C^{dg}_{\mathcal{R}}(\mathcal{M}) \hookrightarrow \mathcal{M}^{op}_{\mathcal{R}}\text{-mod}.$$

Notice that this DG functor $h^{\bullet}$ does not commute with the DG functor ϕ^* in general (because the usual Yoneda functor $\mathcal{M} \to \mathcal{M}^{op}$-mod is not exact). However we have the following result which suffices for our purposes.

Lemma 3.21. *Let $\phi : \mathcal{R} \to \mathcal{Q}$ be a homomorphism of artinian DG algebras.*

a) *There is a natural isomorphism of DG functors from $C^{dg}_{\mathcal{Q}}(\mathcal{M})$ to $\mathcal{M}^{op}_{\mathcal{R}}$-mod*

$$h^{\bullet}_{\mathcal{R}} \cdot \phi_* \simeq \phi_* \cdot h^{\bullet}_{\mathcal{Q}}.$$

b) *There is a natural isomorphism of DG functors from $C^{dg}_{\mathcal{R}}(\mathcal{M})$ to $\mathcal{M}^{op}_{\mathcal{Q}}$-mod*

$$h^{\bullet}_{\mathcal{Q}} \cdot \phi^! \simeq \phi^! \cdot h^{\bullet}_{\mathcal{R}}.$$

c) *There is a natural morphism of DG functors from $C^{dg}_{\mathcal{R}}(\mathcal{M})$ to $\mathcal{M}^{op}_{\mathcal{Q}}$-mod*

$$\phi^* \cdot h^{\bullet}_{\mathcal{R}} \to h^{\bullet}_{\mathcal{Q}} \cdot \phi^*,$$

which is an isomorphism on objects $T \in C^{dg}_{\mathcal{R}}(\mathcal{M})$ such that T as a graded $\mathcal{R}^{op}$-module is isomorphic to $S \otimes_k V$, where $S \in C^{dg}(\mathcal{M})$ and V is a finite-dimensional $\mathcal{R}^{op}$-module. In particular it is an isomorphism on objects T which are graded $\mathcal{R}$-free.

Proof. a) is obvious. Let us prove b) and c). Fix $M \in C^{dg}_{\mathcal{R}}(\mathcal{M})$ and $N \in \mathcal{M}$. We have

$$\phi^! h^M_{\mathcal{R}}(N) = \mathrm{Hom}^{\cdot}_{\mathcal{R}^{op}}\big(\mathcal{Q}, \mathrm{Hom}^{\cdot}_{C^{dg}(\mathcal{M})}(N, M)\big),$$

and

$$h^{\phi^! M}_{\mathcal{R}}(N) = \mathrm{Hom}^{\cdot}_{C^{dg}(\mathcal{M})}\big(N, \mathrm{Hom}^{\cdot}_{\mathcal{R}^{op}}(\mathcal{Q}, M)\big).$$

Notice that both these complexes are naturally isomorphic to

$$\mathrm{Hom}^{\cdot}_{C^{dg}(\mathcal{M}) \otimes \mathcal{R}^{op}}(N \otimes \mathcal{Q}, M).$$

This proves b).

Now

$$\phi^* h_{\mathcal{R}}^M(N) = \mathrm{Hom}_{C^{dg}(\mathcal{M})}^{\cdot}(N, M) \otimes_{\mathcal{R}} Q,$$

and

$$h_{\mathcal{R}}^{\phi^* M}(N) = \mathrm{Hom}_{C^{dg}(\mathcal{M})}^{\cdot}(N, M \otimes_{\mathcal{R}} Q).$$

We define the morphism of complexes $\delta : \phi^* h_{\mathcal{R}}^M(N) \to h_{\mathcal{R}}^{\phi^* M}(N)$ by

$$\delta(f \otimes q)(n) = (-1)^{\bar{q}\bar{n}} f(n) \otimes q.$$

Assume that M as a graded object is isomorphic to $S \otimes V$, where $S \in C^{dg}(\mathcal{M})$ and V is a finite-dimensional $\mathcal{R}^{op}$-module. Then both complexes $\phi^* h_{\mathcal{R}}^M(N)$ and $h_{\mathcal{R}}^{\phi^* M}(N)$ are graded isomorphic to $\mathrm{Hom}_{C^{dg}(\mathcal{M})}^{\cdot}(N, S) \otimes (V \otimes_{\mathcal{R}} Q)$ and δ is the identity map. This proves c) and the lemma. $\quad\square$

Warning. In what follows we will compare deformations and co-deformations of objects E in $C(\mathcal{M})$ (or, which is the same, in $C^{dg}(\mathcal{M})$) as defined above, with deformations and co-deformations of DG modules over DG categories (such as h^E for example), as defined in [5]. These pseudo-functors are denoted by the same symbols (like Def^h), but we hope that there is no danger of confusion because we always mention the corresponding argument (such as E or h^E).

Corollary 3.22. *Fix $E \in C^{dg}(\mathcal{M})$. The collection of DG functors $\{h_{\mathcal{R}}^{\bullet}\}$ defines morphisms of pseudo-functors*

$$h^{\bullet} : \mathrm{Def}^h(E) \to \mathrm{Def}^h(h^E), \qquad h^{\bullet} : \mathrm{coDef}^h(E) \to \mathrm{coDef}^h(h^E).$$

Proof. Notice that for an artinian DG algebra $\mathcal{R}$ the graded $\mathcal{M}_{\mathcal{R}}^{op}$-modules $h^E \otimes \mathcal{R}$ and $h^{E \otimes \mathcal{R}}$ (resp. $h^E \otimes \mathcal{R}^*$ and $h^{E \otimes \mathcal{R}^*}$) are naturally isomorphic. The rest follows from Lemma 3.21. $\quad\square$

Proposition 3.23. *For every $E \in C^{dg}(\mathcal{M})$ the morphisms*

$$h^{\bullet} : \mathrm{Def}^h(E) \to \mathrm{Def}^h(h^E), \qquad h^{\bullet} : \mathrm{coDef}^h(E) \to \mathrm{coDef}^h(h^E)$$

are equivalences of pseudo-functors.

Proof. Since the DG functors $h_{\mathcal{R}}^{\bullet}$ are full and faithful it follows that the induced functor $h_{\mathcal{R}} : \mathrm{Def}_{\mathcal{R}}^h(E) \to \mathrm{Def}_{\mathcal{R}}^h(h^E)$ is full and faithful. It remains to show that $h_{\mathcal{R}}$ is essentially surjective.

Let $(\tilde{S}, \mathrm{id}) \in \mathrm{Def}_{\mathcal{R}}^h(h^E)$. Consider the DG $\mathcal{M}^{op}$-module $p_* \tilde{S}$. Notice that $\tilde{S}$ is just the DG $\mathcal{M}^{op}$-module $p_* \tilde{S}$ together with a homomorphism of DG algebras $\mathcal{R} \to \mathrm{End}(p_* \tilde{S})$. Thus it suffices to show that $p_* \tilde{S}$ is isomorphic to h^S for some $S \in C^{dg}(\mathcal{M})$. Notice that $p_* \tilde{S}$ is obtained from E by taking finite direct sums and iterated cone constructions. The DG functor $h^{\bullet}$ preserves cones of morphisms, hence $p_* \tilde{S}$ is in the essential image of the DG functor $h^{\bullet}$.

The same proof works for the pseudo-functors coDef^h. $\quad\square$

3.4. Proof of main properties of deformation pseudo-functors

Corollary 3.24. *For any $E \in C(\mathcal{M})$ the pseudo-functors $\mathrm{Def}^h(E)$ and $\mathrm{coDef}^h(E)$ from dgart to* **Gpd** *are equivalent.*

Proof. Indeed, by the last proposition we have equivalences $\mathrm{Def}^h(E) \simeq \mathrm{Def}^h(h^E)$ and $\mathrm{coDef}^h(E) \simeq \mathrm{coDef}^h(h^E)$. It remains to apply I, Proposition 4.7.   □

For $E \in C^{dg}(\mathcal{M})$ denote by $\mathcal{B}$ the DG algebra $\mathrm{End}(E)$. Recall the Maurer–Cartan pseudo-functor $\mathcal{MC}(\mathcal{B}) : \mathrm{dgart} \to$ **Gpd** (I, Definition 5.4).

Corollary 3.25. *The pseudo-functors $\mathrm{Def}^h(E)$ $(\simeq \mathrm{coDef}^h(E))$ and $\mathcal{MC}(\mathcal{B})$ are equivalent. In particular these pseudo-functors depend (up to an equivalence) only on the isomorphism class of the DG algebra $\mathrm{End}(E)$.*

Proof. This follows from Corollary 3.24 and I, Proposition 6.1.   □

Recall that for quasi-isomorphic DG algebras $\mathcal{B}$ and $\mathcal{C}$ the corresponding Maurer–Cartan pseudo-functors $\mathcal{MC}(\mathcal{B})$ and $\mathcal{MC}(\mathcal{C})$ are equivalent (I, Theorem 8.1). Hence we obtain the following corollary.

Corollary 3.26. *Assume that for $E, E' \in C^{dg}(\mathcal{M})$ the DG algebras $\mathrm{End}(E)$ and $\mathrm{End}(E')$ are quasi-isomorphic. Then the pseudo-functors $\mathrm{Def}^h(E)$ $(\simeq \mathrm{coDef}^h(E))$ and $\mathrm{Def}^h(E')$ $(\simeq \mathrm{coDef}^h(E'))$ are equivalent.*

The next example is a copy of I, Proposition 8.3 in our context.

Example 3.27. a) Assume that for $E, E' \in C^{dg}(\mathcal{M})$ are homotopy equivalent. Then the DG algebras $\mathrm{End}(E)$ and $\mathrm{End}(E')$ are canonically quasi-isomorphic.

b) Let $P \in C^{dg}(\mathcal{M})$ and $I \in C^{dg}(\mathcal{M})$ be h-projective and h-injective respectively. Assume that $f : P \to I$ is a quasi-isomorphism. Then the DG algebras $\mathrm{End}(P)$ and $\mathrm{End}(I)$ are canonically quasi-isomorphic.

The proof is the same as that of I, Proposition 8.3.

We will need a more precise result as in Proposition 8.5 in Part I.

Lemma 3.28. *Fix an artinian DG algebra $\mathcal{R}$.*

Let $g : E \to E'$ be a homotopy equivalence in $C^{dg}(\mathcal{M})$. Assume that $(V, \mathrm{id}) \in \mathrm{Def}^h_{\mathcal{R}}(E)$ and $(V', \mathrm{id}) \in \mathrm{Def}^h_{\mathcal{R}}(E')$ are objects that correspond to each other via the equivalence $\mathrm{Def}^h_{\mathcal{R}}(E) \simeq \mathrm{Def}^h_{\mathcal{R}}(E')$ of Corollary 3.26 and Example 3.27 a). Then there exists a homotopy equivalence $\tilde{g} : V \to V'$ which extends g, i.e. $i^*\tilde{g} = g$. Similarly for the objects of $\mathrm{coDef}^h_{\mathcal{R}}$ with $i^!$ instead of i^*.

Proof. The full and faithful Yoneda DG functor $h^\bullet$ allows us to translate the problem to DG modules over the DG category $\mathcal{M}$ (Proposition 3.23). So it remains to apply I, Proposition 8.5 a).   □

Theorem 3.29.

a) *Assume that $\mathcal{M}$ has enough projectives. Let $E \in C^-(\mathcal{M})$ be such that $\mathrm{Ext}^{-1}(E, E) = 0$. Choose an h-projective $P \in C^-(\mathcal{M})$ and a quasi-isomorphism $P \to E$. Then there exists an equivalence of pseudo-functors*

$$\mathrm{Def}_-(E) \simeq \mathrm{Def}^{\mathrm{h}}_-(P).$$

b) *Assume that $\mathcal{M}$ has enough injectives. Let $E \in C^+(\mathcal{M})$ be such that $\mathrm{Ext}^{-1}(E, E) = 0$. Choose an h-injective $I \in C^+(\mathcal{M})$ and a quasi-isomorphism $E \to I$. Then there exists an equivalence of pseudo-functors*

$$\mathrm{coDef}_-(E) \simeq \mathrm{coDef}^{\mathrm{h}}_-(I).$$

Proof. We may and will assume that each $P^j \in \mathcal{M}$ (resp. $I^j \in \mathcal{M}$) is projective (resp. injective). We need a few preliminaries.

Lemma 3.30. *Fix $\mathcal{R} \in \mathrm{dgart}_-$. In the notation of the above theorem let $(S, \sigma) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P)$ (resp. $(S, \sigma) \in \mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(I)$). Then $S \in C^{dg}_{\mathcal{R}}(\mathcal{M})$ is h-projective (resp. h-injective).*

Proof. Let $(S, \sigma) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P)$. We may and will assume that $i^*S = P$ and $\sigma = \mathrm{id}$. By definition $S^{\mathrm{gr}} \simeq (P \otimes \mathcal{R})^{\mathrm{gr}}$ and since $\mathcal{R}$ is non-positive the graded $\mathcal{R}$-submodule $\bigoplus_{j > j_0} P^j \otimes \mathcal{R}$ is actually a subcomplex for each j_0. Notice that for each j the $\mathcal{R}^{op}$-complex $P^j \otimes \mathcal{R}$ is h-projective (since P^j is projective). Hence also each $\mathcal{R}^{op}$-submodule $\bigoplus_{j > j_0} P^j \otimes \mathcal{R} \subset S$ is h-projective. Now we repeat the argument in the proof of Proposition 2.10 a) to show that S is h-projective.

The proof for $(S, \sigma) \in \mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(I)$ is similar. $\quad\square$

Lemma 3.31. *Let $\mathcal{R}$ be an artinian DG algebra and $S, T \in C^{dg}_{\mathcal{R}}(\mathcal{M})$ be graded $\mathcal{R}$-free (resp. graded $\mathcal{R}$-cofree).*

a) *There is an isomorphism of graded vector spaces $\mathrm{Hom}^{\cdot}(S, T) = \mathrm{Hom}^{\cdot}(i^*S, i^*T) \otimes \mathcal{R}$ (resp. $\mathrm{Hom}^{\cdot}(S, T) = \mathrm{Hom}^{\cdot}(i^!S, i^!T) \otimes \mathcal{R}$), which is an isomorphism of graded algebras if $S = T$. In particular, the map $i^* : \mathrm{Hom}^{\cdot}(S, T) \to \mathrm{Hom}^{\cdot}(i^*S, i^*T)$ (resp. $i^! : \mathrm{Hom}^{\cdot}(S, T) \to \mathrm{Hom}^{\cdot}(i^!S, i^!T)$) is surjective.*
b) *The $\mathcal{R}^{op}$-complex S has a finite filtration with subquotients isomorphic to i^*S as objects in $C^{dg}(\mathcal{M})$ (resp. to $i^!S$ as objects in $C^{dg}_{\mathcal{R}}(\mathcal{M})$).*
c) *The DG algebra $\mathrm{End}(S)$ has a finite filtration by DG ideals with subquotients isomorphic to $\mathrm{End}(i^*S)$.*
d) *If $f \in \mathrm{Hom}^{\cdot}(S, T)$ is a closed morphism of degree zero such that i^*f (resp. $i^!f$) is an isomorphism or a homotopy equivalence, then f is also such.*

Proof. The full and faithful Yoneda DG functor $h^{\bullet}$ allows us to translate the problem to DG modules over the DG category $\mathcal{M}$ (Lemma 3.21). So it remains to apply I, Proposition 3.12. $\quad\square$

Now we can prove the theorem. We first prove a). Using Remark 3.9 it suffices to prove that the pseudo-functors $\mathrm{Def}_-(P)$ and $\mathrm{Def}^{\mathrm{h}}_-(P)$ are equivalent. Let us define a morphism

$$\beta : \mathrm{Def}^{\mathrm{h}}_-(P) \to \mathrm{Def}_-(P).$$

Fix $\mathcal{R} \in$ dgart_ and let $(S, \sigma) \in \operatorname{Def}^{\mathfrak{h}}_{\mathcal{R}}(P)$. By Lemma 3.30 the $\mathcal{R}^{op}$-complex S is h-projective. Hence $\mathbf{L}i^*S = i^*S$ and therefore $(S, \sigma) \in \operatorname{Def}_{\mathcal{R}}(P)$. This defines a functor $\beta_{\mathcal{R}} : \operatorname{Def}^{\mathfrak{h}}_{\mathcal{R}}(P) \to \operatorname{Def}_{\mathcal{R}}(P)$ and a morphism of pseudo-functors $\beta : \operatorname{Def}^{\mathfrak{h}}_{-}(P) \to \operatorname{Def}_{-}(P)$. We need to show that $\beta_{\mathcal{R}}$ is an equivalence.

Surjective on isomorphism classes. Let $(T, \tau) \in \operatorname{Def}_{\mathcal{R}}(P)$. We may and will assume that $T \in C^{dg}_{\mathcal{R}}(\mathcal{M})$ is h-projective and graded $\mathcal{R}$-free (Proposition 2.10 a)). Thus $(T, \tau) \in \operatorname{Def}^{\mathfrak{h}}_{\mathcal{R}}(i^*T)$. Since T is h-projective, so is i^*T and hence $\tau : \mathbf{L}i^*T = i^*T \to P$ is a homotopy equivalence. It follows from Lemma 3.28 that there exists $(S, \operatorname{id}) \in \operatorname{Def}^{\mathfrak{h}}_{\mathcal{R}}(P)$ such that S and T are homotopy equivalent and (S, id) and (T, τ) are isomorphic objects in $\operatorname{Def}_{\mathcal{R}}(P)$. I.e. $\beta_{\mathcal{R}}(S, \operatorname{id}) \simeq (T, \tau)$.

Full. Let $(S, \operatorname{id}), (S', \operatorname{id}) \in \operatorname{Def}^{\mathfrak{h}}_{\mathcal{R}}(P)$. Let $f : \beta_{\mathcal{R}}(S, \operatorname{id}) \to \beta_{\mathcal{R}}(S', \operatorname{id})$ be an isomorphism in $\operatorname{Def}_{\mathcal{R}}(P)$. Since S, S' are h-projective (Lemma 3.30) this isomorphism f is a homotopy equivalence. Because P is h-projective i^*f is homotopic to id_P. Let $h : i^*f \to \operatorname{id}$ be a homotopy. Since S, S' are graded $\mathcal{R}$-free the map $i^* : \operatorname{Hom}^{\cdot}(S, S') \to \operatorname{Hom}^{\cdot}(P, P)$ is surjective (Lemma 3.31 a)). Choose a lift $\tilde{h} : S \to S'[1]$ of h and replace f by $\tilde{f} = f - d\tilde{h}$. Then $i^*\tilde{f} = id$. Since S and S' are graded $\mathcal{R}$-free $\tilde{f}$ is an isomorphism (Lemma 3.31 d)). Thus $\tilde{f} : (S, \operatorname{id}) \to (S', \operatorname{id})$ is a morphism in $\operatorname{Def}^{\mathfrak{h}}_{\mathcal{R}}(P)$ such that $\beta_{\mathcal{R}}\tilde{f} = f$.

Faithful. Let $(S, \operatorname{id}), (S', \operatorname{id}) \in \operatorname{Def}^{\mathfrak{h}}_{\mathcal{R}}(P)$ and let $g_1, g_2 : S \to S'$ be two isomorphisms (in $C^{dg}_{\mathcal{R}}(\mathcal{M})$) such that $i^*g_1 = i^*g_2 = \operatorname{id}_P$. That is g_1, g_2 are maps which represent morphisms in $\operatorname{Def}^{\mathfrak{h}}_{\mathcal{R}}(P)$. Assume that $\beta_{\mathcal{R}}(g_1) = \beta_{\mathcal{R}}(g_2)$, i.e. there exists a homotopy $s : g_1 \to g_2$. Then $d(i^*s) = i^*(ds) = 0$. Since by our assumption $H^{-1}\operatorname{Hom}^{\cdot}(P, P) = 0$ there exists $t \in \operatorname{Hom}^{-2}(P, P)$ with $dt = i^*s$. Choose a lift $\tilde{t} \in \operatorname{Hom}^{-2}(S, S')$ of t. Then $\tilde{s} := s - d\tilde{t}$ is an allowable homotopy between g_1 and g_2. This proves that $\beta_{\mathcal{R}}$ is faithful.

The proof of part b) of the theorem is similar and we omit it. $\quad\square$

Theorem 3.32. *Assume that $\mathcal{M}$ has enough projectives and enough injectives. Let $E \in D^b(\mathcal{M})$ be a complex such that $\operatorname{Ext}^{-1}(E, E) = 0$. Then there exists an equivalence of pseudo-functors*

$$\operatorname{Def}_{-}(E) \simeq \operatorname{coDef}_{-}(E).$$

Proof. Choose quasi-isomorphisms $P \to E$ and $E \to I$, where P is a bounded above h-projective and I is a bounded below h-injective (Proposition 2.10). Then by Theorem 3.29 there exist equivalences of pseudo-functors

$$\operatorname{Def}_{-}(E) \simeq \operatorname{Def}^{\mathfrak{h}}_{-}(P), \qquad \operatorname{coDef}_{-}(E) \simeq \operatorname{coDef}^{\mathfrak{h}}_{-}(I).$$

But pseudo-functors $\operatorname{Def}^{\mathfrak{h}}_{-}(P)$ and $\operatorname{coDef}^{\mathfrak{h}}_{-}(I)$ are equivalent by Example 3.27, Corollary 3.26. $\quad\square$

Corollary 3.33. *Assume that $\mathcal{M}$ has enough projectives (resp. injectives) and $E \in D^{-}(\mathcal{M})$ (resp. $E \in D^{+}(\mathcal{M})$) is such that $\operatorname{Ext}^{-1}(E, E) = 0$. Then the deformation pseudo-functor $\operatorname{Def}_{-}(E)$ (resp. $\operatorname{coDef}_{-}(E)$) depends up to an equivalence only on the quasi-isomorphism class of the DG algebra $\mathbf{R}\operatorname{Hom}(E, E)$. In particular, let $\mathcal{N}$ be another abelian category with enough projectives (resp. enough injectives); assume that $F : \mathcal{M} \to \mathcal{N}$ is a functor such that $C(F) : C(\mathcal{M}) \to C(\mathcal{N})$ preserves h-projectives (resp. h-injectives) and induces an equivalence $\tilde{F} : D^{-}(\mathcal{M}) \to D^{-}(\mathcal{N})$ (resp. $\tilde{F} : D^{+}(\mathcal{M}) \to D^{+}(\mathcal{N})$), then the pseudo-functors $\operatorname{Def}_{-}(E)$ and $\operatorname{Def}_{-}(F(E))$ (resp. $\operatorname{coDef}_{-}(E)$ and $\operatorname{coDef}_{-}(F(E))$) are equivalent.*

Proof. This follows from Theorem 3.29 and Corollary 3.26. □

3.5. Relation with the derived deformation theory of DG modules

Finally in the situation of Theorem 3.29 we want to interpret the *derived* deformation pseudo-functors Def_- and coDef_- in a context of DG modules.

Theorem 3.34. *Assume that $\mathcal{M}$ has enough projectives and let $P \in C^-(\mathcal{M})$ be h-projective such that $\mathrm{Ext}^{-1}(P, P) = 0$. Then the DG functor $h^\bullet : C^{dg}(\mathcal{M}) \to \mathcal{M}^{op}$-mod induces an equivalence of pseudo-functors $\mathrm{Def}_-(P) \simeq \mathrm{Def}_-(h^P)$.*

Proof. By Proposition 3.23 the morphism of pseudo-functors

$$h^\bullet : \mathrm{Def}^{\mathrm{h}}_-(P) \to \mathrm{Def}^{\mathrm{h}}_-(h^P)$$

is an equivalence. By Theorem 3.29 a) there exists an equivalence of pseudo-functors $\mathrm{Def}_-(P) \simeq \mathrm{Def}^{\mathrm{h}}_-(P)$. We claim that the pseudo-functors $\mathrm{Def}_-(h^P)$ and $\mathrm{Def}^{\mathrm{h}}_-(h^P)$ are also equivalent. Indeed, notice that the DG $\mathcal{M}^{op}$-module h^P satisfies property (P) (Definition 3.2 in Part I). Hence it is h-projective. Therefore

$$\mathrm{Ext}^{-1}\big(h^P, h^P\big) = H^{-1} \mathrm{Hom}^{\cdot}\big(h^P, h^P\big) \simeq H^{-1} \mathrm{Hom}^{\cdot}(P, P) = 0.$$

Clearly h^P is bounded above. Hence by I, Theorem 11.6 a) we have $\mathrm{Def}_-(h^P) \simeq \mathrm{Def}^{\mathrm{h}}_-(h^P)$.

Combining these three equivalences we obtain $\mathrm{Def}_-(P) \simeq \mathrm{Def}_-(h^P)$. □

Remark 3.35. Notice that the DG functor $h^\bullet : C^{dg}(\mathcal{M}) \to \mathcal{M}^{op}$-mod does not preserve quasi-isomorphisms in general. If $\mathcal{M}$ has enough projectives then we can consider a similar DG functor

$$^P h^\bullet : C^{dg}(\mathcal{M}) \to \mathcal{P}^{op}\text{-mod},$$

where $\mathcal{P} \subset \mathcal{M}$ is the full subcategory of projectives. This DG functor $^P h^\bullet$ has all the good properties of $h^\bullet$ (full and faithful, induces an equivalence of homotopy deformation and co-deformation pseudo-functors, etc.) and in addition it preserves quasi-isomorphisms. Thus $^P h^\bullet$ is better suited than $h^\bullet$ for comparing derived deformation pseudo-functors.

Next we want to prove the analogue of Theorem 3.34 for the pseudo-functor coDef_- in case $\mathcal{M}$ has enough injectives. We can only prove it with an extra finiteness assumption.

We are still going to work with a covariant DG functor from $C^{dg}(\mathcal{M})$ to $\mathcal{M}^{op}$-mod, but it will not be $h^\bullet$. Consider the DG functor $h^*_\bullet : C^{dg}(\mathcal{M}) \to \mathcal{M}^{op}$-mod defined by

$$h^*_M(N) := \mathrm{Hom}^{\cdot}_{C^{dg}(\mathcal{M})}(M, N)^*,$$

where $(\cdot)_*$ denotes the (graded) k-dual. Recall (I, Section 3.1) that for any $W \in \mathcal{M}^{op}$-mod, $M \in C^{dg}(\mathcal{M})$,

$$\mathrm{Hom}^{\cdot}\big(W, h^*_M\big) = W(M)^*.$$

In particular

$$\mathrm{Hom}^{\cdot}\left(h_{M_1}^*, h_{M_2}^*\right) = h_{M_1}^*(M_2)^* = \mathrm{Hom}^{\cdot}_{C^{dg}(\mathcal{M})}(M_1, M_2)^{**}.$$

Therefore the DG functor $h_{\bullet}^*$ is not full in general, but it induces a quasi-isomorphism

$$h_{\bullet}^* : \mathrm{Hom}^{\cdot}(M_1, M_2) \to \mathrm{Hom}^{\cdot}\left(h_{M_1}^*, h_{M_2}^*\right)$$

if $\dim H^i \mathrm{Hom}(M_1, M_2) < \infty$ for all i.

Also for each $M \in \mathcal{M}$ the DG $\mathcal{M}^{op}$-module h_M^* is h-injective. Hence h_M^* is h-injective for each $M \in C^+(\mathcal{M})$.

Theorem 3.36. *Assume that $\mathcal{M}$ has enough injectives and let $I \in C^+(\mathcal{M})$ be h-injective such that $\mathrm{Ext}^{-1}(I, I) = 0$. Assume that for each i $\dim \mathrm{Ext}^i(I, I) < \infty$. Then the DG functor*

$$h_{\bullet}^* : C^{dg}(\mathcal{M}) \to \mathcal{M}^{op}\text{-mod}$$

induces an equivalence of pseudo-functors $\mathrm{coDef}_-(I) \simeq \mathrm{coDef}_-(h_I^*)$.

Proof. By Theorem 3.29 b) there exists an equivalence of pseudo-functors

$$\mathrm{coDef}_-(I) \simeq \mathrm{coDef}^{\mathrm{h}}_-(I).$$

Since $\dim H^i \mathrm{Hom}(I, I) < \infty$ for each i the homomorphism of DG algebras $h_{\bullet}^* : \mathrm{End}(I) \to \mathrm{End}(h_I^*)$ is a quasi-isomorphism. Hence the pseudo-functors $\mathrm{coDef}^{\mathrm{h}}_-(I)$ and $\mathrm{coDef}^{\mathrm{h}}_-(h_I^*)$ are equivalent (Corollary 3.25 and I, Proposition 6.1, I, Theorem 8.1).

The DG $\mathcal{M}^{op}$-module h_I^* is h-injective and bounded below. Hence by I, Theorem 11.6 b)

$$\mathrm{coDef}^{\mathrm{h}}_-\left(h_I^*\right) \simeq \mathrm{coDef}_-\left(h_I^*\right).$$

Combining the above three equivalences we obtain the desired equivalence

$$\mathrm{coDef}_-(I) \simeq \mathrm{coDef}_-\left(h_I^*\right). \qquad \square$$

In case of finite injective dimension we could still use the DG functor $h^{\bullet}$ to compare the derived co-deformation pseudo-functors. Namely, we have the following result.

Theorem 3.37. *Assume that $\mathcal{M}$ has enough injectives and let $I \in C^b(\mathcal{M})$ be a (bounded) h-injective such that $\mathrm{Ext}^{-1}(I, I) = 0$. Then the DG functor*

$$h^{\bullet} : C_{dg}(\mathcal{M}) \to \mathcal{M}^{op}\text{-mod}$$

induces an equivalence of pseudo-functors

$$\mathrm{coDef}_-(I) \simeq \mathrm{coDef}_-\left(h^I\right).$$

Proof. By Proposition 3.23 we have an equivalence of pseudo-functors

$$h^{\bullet} : \mathrm{coDef}^{\mathrm{h}}(I) \to \mathrm{coDef}^{\mathrm{h}}(h^I).$$

By Theorem 3.18 there is an equivalence of pseudo-functors

$$\mathrm{coDef}_-(I) \simeq \mathrm{coDef}^{\mathrm{h}}_-(I).$$

Finally, notice that h^I is a bounded h-projective object in $\mathcal{M}^{op}$-mod. Hence by I, Theorem 11.6 b) there is an equivalence of pseudo-functors

$$\mathrm{coDef}^{\mathrm{h}}_-\big(h^I\big) \simeq \mathrm{coDef}_-\big(h^I\big).$$

This proves the theorem.   □

Part 2. Geometric examples, applications and conjectures

4. Deformations of bounded complexes on locally Noetherian schemes

Fix a locally Noetherian scheme X over k. Let E be a bounded complex of quasi-coherent sheaves on X. The abelian category Qcoh_X of quasi-coherent sheaves on X does not have enough projectives in general. Still there is a natural (classical) derived deformation pseudo-functor

$$\mathrm{Def}_{\mathrm{cl}}(E) : \mathrm{cart} \to \mathbf{Gpd},$$

which is defined using h-flat objects.

The abelian category Qcoh_X has enough injectives, so we can define the pseudo-functor $\mathrm{coDef}_-(E)$ as in Definition 3.11 above. Our main result (Theorem 4.4 below) claims that the pseudo-functors $\mathrm{Def}_{\mathrm{cl}}(E)$ and $\mathrm{coDef}_{\mathrm{cl}}(E)$ are naturally equivalent. This allows us to consider the pseudo-functor $\mathrm{coDef}_-(E)$ as a natural extension to dgart_ of the classical deformation functor $\mathrm{Def}_{\mathrm{cl}}(E)$.

Let us first introduce some notation. For a scheme Y we denote by Mod_Y, Qcoh_Y, coh_Y the abelian categories of $\mathcal{O}_Y$-modules, quasi-coherent $\mathcal{O}_Y$-modules and coherent $\mathcal{O}_Y$-modules respectively. Denote by $D(Y)$, $D(\mathrm{Qcoh}_Y)$, $D(coh_Y)$ the corresponding derived categories and by $D^{\pm}_{\mathrm{Qcoh}}(Y)$, $D^b_{coh}(Y)$, $D^{\pm}_{coh}(\mathrm{Qcoh}_Y)$, ... their usual full subcategories defined by a cohomological condition.

Note that none of the categories Mod_Y, Qcoh_Y, coh_Y has enough projectives in general. The categories Mod_Y, Qcoh_Y have enough injectives and if the scheme Y is locally Noetherian, then the natural functor

$$D^+(\mathrm{Qcoh}_Y) \to D^+_{\mathrm{Qcoh}}(Y)$$

is an equivalence of categories [8].

Definition 4.1. A complex $F \in C(\mathrm{Mod}_Y)$ is h-flat if the complex $F \otimes_{\mathcal{O}_Y} G$ is acyclic for every acyclic $G \in C(\mathrm{Mod}_Y)$.

For every $S \in C(\mathrm{Mod}_Y)$ Spaltenstein in [15] has constructed a functorial h-flat resolution. That is he defines an h-flat $F(S) \in C(\mathrm{Mod}_Y)$ and a quasi-isomorphism $F(S) \to S$. The complex $F(S)$ consists of $\mathcal{O}_Y$-modules which are direct sums of sheaves $\mathcal{O}_U$ for affine open subsets $U \subset Y$ ($\mathcal{O}_U$ is the extension by zero to Y of the structure sheaf of U). Using these h-flat resolutions we may define derived functors $\mathbf{L}f^* : D(Y) \to D(Z)$ for a morphism of schemes $f : Z \to Y$. Namely, put

$$\mathbf{L}f^*(S) := f^*\big(F(S)\big).$$

For a commutative local artinian algebra $\mathcal{R}$ and a scheme Y put

$$Y_{\mathcal{R}} = X \otimes_{\mathrm{Spec}\,k} \mathrm{Spec}\,\mathcal{R}$$

and denote by $i : Y \hookrightarrow Y_{\mathcal{R}}$ the closed embedding.

Definition 4.2. Let X be a scheme, $E \in D_{\mathrm{Qcoh}}(X)$. We define the pseudo-functor

$$\mathrm{Def}_{\mathrm{cl}}(E) : \mathrm{cart} \to \mathbf{Gpd}$$

of "classical" deformations of E as follows.

Fix a commutative local artinian algebra $\mathcal{R}$. An object of the groupoid $\mathrm{Def}_{\mathcal{R}}(E)$ is a pair (S, σ), where $S \in D_{\mathrm{Qcoh}}(X_{\mathcal{R}})$ and $\sigma : \mathbf{L}i^*S \to E$ is an isomorphism in $D_{\mathrm{Qcoh}}(X)$. A morphism between two such pairs (S, σ) and (S', σ') is an isomorphism $f : S \to S'$ such that $\sigma = \sigma' \cdot \mathbf{L}i^*(f)$.

A homomorphism $\phi : \mathcal{R} \to \mathcal{Q}$ of commutative local artinian algebras induces a morphism of schemes $\phi : X_{\mathcal{Q}} \to X_{\mathcal{R}}$ which fits in a commutative diagram

$$
\begin{array}{ccc}
X_{\mathcal{Q}} & \xrightarrow{\;\phi\;} & X_{\mathcal{R}} \\
{\scriptstyle i}\big\uparrow & & \big\uparrow{\scriptstyle i} \\
X & =\!=\!= & X.
\end{array}
$$

Hence we obtain the functor $\mathbf{L}\phi^* : \mathrm{Def}_{\mathcal{R}}(E) \to \mathrm{Def}_{\mathcal{Q}}(E)$. This defines the pseudo-functor $\mathrm{Def}_{\mathrm{cl}}(E) : \mathrm{cart} \to \mathbf{Gpd}$.

In [15] it is also shown that for every object $S \in C(\mathrm{Mod}_Y)$ there exists an h-injective $J \in C(\mathrm{Mod}_Y)$ and a quasi-isomorphism $S \to J$. As usual we define right derived functors using h-injectives.

For example if $\phi : \mathcal{R} \to \mathcal{Q}$ is a homomorphism of commutative artinian local algebras we obtain the functor

$$\phi^! : C(\mathrm{Mod}_{Y_{\mathcal{R}}}) \to C(\mathrm{Mod}_{Y_{\mathcal{Q}}}), \quad \phi^! T := \mathcal{H}om_{\mathcal{O}_{Y_{\mathcal{R}}}}(\mathcal{O}_{Y_{\mathcal{Q}}}, T)$$

and its derived functor $\mathbf{R}\phi^! : D(Y_{\mathcal{R}}) \to D(Y_{\mathcal{Q}})$.

In particular for a commutative local artinian algebra $\mathcal{R}$ and a scheme Y the closed embedding $i : Y \hookrightarrow Y_{\mathcal{R}}$ induces the functor

$$i^! : C(\mathrm{Mod}_{Y_{\mathcal{R}}}) \to C(\mathrm{Mod}_Y), \quad i^! T := \mathcal{H}om_{\mathcal{O}_{Y_{\mathcal{R}}}}(\mathcal{O}_Y, T),$$

and its derived functor

$$\mathbf{R}i^! : D(Y_{\mathcal{R}}) \to D(Y).$$

Definition 4.3. Let X be a scheme (over k) and $E \in D_{\mathrm{Qcoh}}(X)$. We define the pseudo-functor

$$\mathrm{coDef}_{\mathrm{cl}}(E) : \mathrm{cart} \to \mathbf{Gpd}$$

of ("classical") derived co-deformations of E as follows.

Let $\mathcal{R}$ be a commutative local artinian algebra. An object of the groupoid $\mathrm{coDef}_{\mathcal{R}}(E)$ is a pair (T, τ), where $T \in D_{\mathrm{Qcoh}}(X_{\mathcal{R}})$ and $\tau : E \to \mathbf{R}i^! T$ is a quasi-isomorphism. A morphism between two such object (T, τ) and (T', τ') is a quasi-isomorphism $f : T \to T'$ such that $\tau' = \mathbf{R}i^!(f) \cdot \tau$.

A homomorphism $\phi : \mathcal{R} \to \mathcal{Q}$ of commutative local artinian algebras induces the morphism $X_{\mathcal{Q}} \to X_{\mathcal{R}}$ and hence the functor

$$\mathbf{R}\phi^! : \mathrm{coDef}_{\mathcal{R}}(E) \to \mathrm{coDef}_{\mathcal{Q}}(E).$$

Thus we obtain the pseudo-functor $\mathrm{coDef}_{\mathrm{cl}}(E) : \mathrm{cart} \to \mathbf{Gpd}$.

The next theorem is our main result of this section.

Theorem 4.4. *Let X be a locally Noetherian scheme, $E \in D^b_{\mathrm{Qcoh}}(X)$. Then there exists an equivalence of pseudo-functors*

$$\mathrm{Def}_{\mathrm{cl}}(E) \to \mathrm{coDef}_{\mathrm{cl}}(E).$$

This theorem follows from a more precise Theorem 4.16 below.

We need a few lemmas.

Fix a commutative artinian local algebra $\mathcal{R}$. Denote as usual by $\mathcal{R}^*$ the $\mathcal{R}$-module $\mathrm{Hom}_k(\mathcal{R}, k)$. Let $F_{\mathcal{R}}$ and $I_{\mathcal{R}}$ denote the categories of free and injective $\mathcal{R}$-modules respectively.

Lemma 4.5.

a) *$\mathcal{R}^*$ is the unique (up to isomorphism) indecomposable injective $\mathcal{R}$ module.*
b) *Every injective $\mathcal{R}$-module is isomorphic to a direct sum of copies of $\mathcal{R}^*$. A direct sum of injective $\mathcal{R}$-modules is injective.*
c) *The categories $F_{\mathcal{R}}$ and $I_{\mathcal{R}}$ are equivalent. The mutually inverse equivalences are given by $\varphi : F_{\mathcal{R}} \to I_{\mathcal{R}}$, $\psi : I_{\mathcal{R}} \to F_{\mathcal{R}}$, where*

$$\varphi(M) = M \otimes_{\mathcal{R}} \mathcal{R}^*, \qquad \psi(N) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^*, N).$$

d) *The functorial diagram*

$$
\begin{array}{ccc}
F_{\mathcal{R}} & \xrightarrow{\ \varphi\ } & I_{\mathcal{R}} \\
{\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^!} \\
k\text{-mod} & =\!\!=\!\!= & k\text{-mod}
\end{array}
$$

commutes.

Proof. a) Since the ring $\mathcal{R}$ is Noetherian and has a unique prime ideal $m \subset \mathcal{R}$, the $\mathcal{R}$-module $\mathcal{R}^*$ (which is the injective hull of $\mathcal{R}/m = k$) is the unique (up to isomorphism) indecomposable injective $\mathcal{R}$-module (see [11]).

b) This follows from a) and the fact that the abelian category of $\mathcal{R}$-modules is locally Noetherian (see [8,7]).

c) Notice that the natural map of $\mathcal{R}$-modules $\mathcal{R} \to \mathrm{Hom}_{\mathcal{R}}^{\cdot}(\mathcal{R}^*, \mathcal{R}^*)$ is an isomorphism. Now everything follows from b) and the fact that the functors φ and ψ commute with direct sums.

d) Let M be an $\mathcal{R}$-module. Define a morphism of vector spaces

$$
\beta : M \otimes_{\mathcal{R}} k \to \mathrm{Hom}_{\mathcal{R}}^{\cdot}\big(k, M \otimes_{\mathcal{R}} \mathrm{Hom}_k(\mathcal{R}, k)\big), \quad \beta(m \otimes \xi)(\eta) = m \otimes \xi\eta\epsilon,
$$

where $\epsilon : \mathcal{R} \to k$ is the augmentation map. This map is an isomorphism if $M = \mathcal{R}$. Hence it is an isomorphism for a free $\mathcal{R}$-module M. $\square$

Definition 4.6. A $\mathcal{Q}_{X_{\mathcal{R}}}$-module M is called $\mathcal{R}$-free (resp. $\mathcal{R}$-injective) if every stalk M_x is free (resp. injective) as an $\mathcal{R}$-module. We call a complex $S \in C(\mathrm{Mod}_{X_{\mathcal{R}}})$ $\mathcal{R}$-free (resp. $\mathcal{R}$-injective) if every $\mathcal{O}_{X_{\mathcal{R}}}$-module S^j is such. Denote by $C_F(\mathrm{Mod}_{X_{\mathcal{R}}})$ (resp. $C_I(\mathrm{Mod}_{X_{\mathcal{R}}})$) the full subcategories of $C(\mathrm{Mod}_{X_{\mathcal{R}}})$ which consist of $\mathcal{R}$-free (resp. $\mathcal{R}$-injective) complexes.

Proposition 4.7. *Consider the functors* $\varphi, \psi : C(\mathrm{Mod}_{X_{\mathcal{R}}}) \to C(\mathrm{Mod}_{X_{\mathcal{R}}})$.

$$
\varphi(S) = S \otimes_{\mathcal{O}_{X_{\mathcal{R}}}} p^! \mathcal{O}_X = S \otimes_{\mathcal{R}} \mathcal{R}^*, \qquad \psi(T) = \mathcal{H}om_{\mathcal{O}_{X_{\mathcal{R}}}}\big(p^! \mathcal{O}_X, T\big) = \mathrm{Hom}_{\mathcal{R}}^{\cdot}\big(\mathcal{R}^*, T\big),
$$

where $p^! \mathcal{O}_X = \mathcal{O}_X \otimes_k \mathcal{R}^*$.

a) *These functors induce mutually inverse equivalences of categories*

$$
\varphi : C_F(\mathrm{Mod}_{X_{\mathcal{R}}}) \to C_I(\mathrm{Mod}_{X_{\mathcal{R}}}), \qquad \psi : C_I(\mathrm{Mod}_{X_{\mathcal{R}}}) \to C_F(\mathrm{Mod}_{X_{\mathcal{R}}}).
$$

b) *The functorial diagram*

$$
\begin{array}{ccc}
C_F(\mathrm{Mod}_{X_{\mathcal{R}}}) & \xrightarrow{\ \phi\ } & C_I(\mathrm{Mod}_{X_{\mathcal{R}}}) \\
{\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^!} \\
C(\mathrm{Mod}_X) & =\!\!=\!\!= & C(\mathrm{Mod}_X)
\end{array}
$$

commutes.

Proof. a) Let $x \in X_{\mathcal{R}}$. Then $\mathcal{O}_{X_{\mathcal{R}},x} = \mathcal{O}_{X,x} \otimes_k \mathcal{R}$. We have

$$\varphi(S)_x = S_x \otimes_{\mathcal{R}} \mathcal{R}^*$$

and

$$\psi(T)_x = \operatorname{Hom}_{\mathcal{R}}^{\cdot}(\mathcal{R}^*, T_x).$$

Now the assertion follows from part c) of Lemma 4.5.

b) For an $\mathcal{O}_{X_{\mathcal{R}}}$-module M we have

$$i^*M = M \otimes_{\mathcal{O}_{X_{\mathcal{R}}}} \mathcal{O}_X = M \otimes_{\mathcal{R}} k,$$

$$i^!M = \mathcal{H}om_{\mathcal{O}_{X_{\mathcal{R}}}}(\mathcal{O}_X, M) = \operatorname{Hom}_{\mathcal{R}}^{\cdot}(k, M).$$

Hence the assertion follows from part d) of Lemma 4.5. $\quad\square$

Proposition 4.8. *Let $F \in C(\operatorname{Mod}_{X_{\mathcal{R}}})$. Suppose that $\mathbf{L}i^*F \in D^b(X)$. Then F is quasi-isomorphic to a bounded $\mathcal{R}$-free complex.*

Before we prove the proposition let us state an immediate corollary.

Corollary 4.9. *Given $(S, \sigma) \in \operatorname{Def}_{\mathcal{R}}(E)$ there exists an isomorphic $(S', \sigma') \in \operatorname{Def}_{\mathcal{R}}(E)$ such that S' is a bounded $\mathcal{R}$-free complex.*

Proof. Let us prove the proposition. This is done in the next two lemmas.

Lemma 4.10. *For every $S \in C(\operatorname{Mod}_{X_{\mathcal{R}}})$ there exist quasi-isomorphisms $P \to S$ and $S \to J$, where P is h-flat and $\mathcal{R}$-free and J is h-injective and $\mathcal{R}$-injective.*

Proof. This is proved in [15]. Namely, the assertion about P follows from Proposition 5.6 and that about J follows from Lemma 4.3 and Theorem 4.5 in [15]. $\quad\square$

Let F be as in the proposition. By the above lemma we may and will assume that F is h-flat and $\mathcal{R}$-free. Hence $\mathbf{L}i^*F = i^*F$. The following claim implies the proposition.

Claim. *Let $K \in C(\operatorname{Mod}_{X_{\mathcal{R}}})$ be $\mathcal{R}$-free and such that $H^j(i^*K) = 0$ for $j < j_0$ and $j > j_1$. Then K is quasi-isomorphic to its truncation $\tau_{\leqslant j_1} \tau_{\geqslant j_0} F$ and moreover this truncation is $\mathcal{R}$-free.*

Our claim follows from the next lemma.

Lemma 4.11. *Let $M^{\bullet} := M^{-1} \overset{d^{-1}}{\to} M^0 \overset{d^0}{\to} M^1$ be a complex of free $\mathcal{R}$-modules. Assume that $H^0(M^{\bullet} \otimes_{\mathcal{R}} k) = 0$. Then $H^0(M^{\bullet}) = 0$ and $\operatorname{Ker} d^0$ is a free $\mathcal{R}$-module.*

Proof. We can find a finite filtration $\mathcal{R} \supset m_1 \supset m_2 \ldots$ by ideals such that $m_s/m_{s+1} \simeq k$. Consider the induced filtration on the complex $M^{\bullet}$:

$$M^{\bullet} \supset m_1 M^{\bullet} \supset m_2 M^{\bullet} \cdots.$$

Then each subquotient $m_s M^\bullet / m_{s+1} M^\bullet$ is isomorphic to the complex $M^\bullet \otimes_\mathcal{R} k$. Hence $H^0(m_s M^\bullet / m_{s+1} M^\bullet) = 0$ for each s and hence $H^0(M^\bullet) = 0$ by devissage. This proves the first assertion of the lemma.

To prove the second one we use the following fact: an $\mathcal{R}$-module N is free if (and only if) $\mathrm{Tor}_1^\mathcal{R}(N, k) = 0$ (see [3], Ch. 2, Ex. 26 and [1], Prop. 2.1.4).

Consider the exact sequence

$$M^{-1} \to M^0 \to M^1 \to \mathrm{coker}\, d^0 \to 0.$$

Then by our assumption $H^0(M^\bullet \otimes_\mathcal{R} k) = \mathrm{Tor}_1^\mathcal{R}(\mathrm{coker}\, d^0, k) = 0$. Hence $\mathrm{coker}\, d^0$ is a free $\mathcal{R}$-module. Thus $\mathrm{Im}\, d^0$ is free and hence also $\mathrm{Ker}\, d^0$ is free. This proves the lemma. $\quad\square$

Proposition 4.8 is proved. $\quad\square$

Now we want to prove the analogue of Corollary 4.9 for the co-deformation functor.

Proposition 4.12. *Let $G \in C(\mathrm{Mod}_{X_\mathcal{R}})$. Assume that $\mathbf{R}i^! G \in D^b(X)$. Then G is quasi-isomorphic to a bounded $\mathcal{R}$-injective complex.*

Proof. By Lemma 4.10 we may and will assume that G is h-injective and $\mathcal{R}$-injective. Hence $\mathbf{R}i^! G = i^! G$. Then by Proposition 4.7 the complex $\psi(G)$ is $\mathcal{R}$-free and $i^* \psi(G) = i^! G$, so we may and will assume that $H^j(i^* \psi(G)) = 0$ for $j < j_0$ and $j > j_1$. By the Claim in the proof of Proposition 4.8 the complex $\psi(G)$ is quasi-isomorphic to its truncation $\tau_{\leqslant j_1} \tau_{\geqslant j_0} \psi(G)$ which is moreover $\mathcal{R}$-free. But then this truncation is a direct summand of $\psi(G)$ as a complex (of sheaves) of free $\mathcal{R}$-modules. Applying the functor φ from Proposition 4.7 we find that $G = \varphi(\psi(G))$ is quasi-isomorphic to its truncation $\varphi(\tau_{\leqslant j_1} \tau_{\geqslant j_0} \psi(G)) = \tau_{\leqslant j_1} \tau_{\geqslant j_0} \varphi(\psi(G))$ which is moreover $\mathcal{R}$-injective. This proves the proposition. $\quad\square$

We obtain the immediate corollary.

Corollary 4.13. *Given $(T, \tau) \in \mathrm{coDef}_\mathcal{R}(E)$ there exists an isomorphic $(T', \tau') \in \mathrm{coDef}_\mathcal{R}(E)$ such that T' is a bounded $\mathcal{R}$-injective complex.*

Proposition 4.14.

a) *Let $F \in C^b(\mathrm{Mod}_{X_\mathcal{R}})$ be a bounded $\mathcal{R}$-free complex. Then F is acyclic for the functors i^* and φ. That is $i^* F = \mathbf{L}i^* F$ and $\varphi(F) = \mathbf{L}\varphi(F)$.*

b) *Let $G \in C^b(\mathrm{Mod}_{X_\mathcal{R}})$ be a bounded $\mathcal{R}$-injective complex. Then G is acyclic for the functors $i^!$ and ψ. That is $i^! G = \mathbf{R}i^! G$, $\psi(G) = \mathbf{R}\psi(G)$.*

c) *If F (resp. G) has quasi-coherent cohomology then so do $i^* F$ and $\varphi(F)$ (resp. $i^! G$ and $\psi(G)$).*

Proof. a) We have $i^* F = F \otimes_\mathcal{R} k$ and $\varphi(F) = F \otimes_\mathcal{R} \mathcal{R}^*$. Now use the fact that a bounded complex of free $\mathcal{R}$-modules is h-projective.

b) We have $i^! G = \mathrm{Hom}_\mathcal{R}^\cdot(k, G)$ and $\psi(G) = \mathrm{Hom}_\mathcal{R}^\cdot(\mathcal{R}^*, G)$. Now use the fact that a bounded complex of injective $\mathcal{R}$-modules is h-injective.

c) This follows from a), b) and Propositions 3.3 and 4.3 in [8]. $\quad\square$

Theorem 4.15. *Let X be a locally Noetherian scheme and $E \in D^b_{\mathrm{Qcoh}}(X)$. Then there exist mutually inverse equivalences of pseudo-functors*

$$\mathbf{L}\varphi : \mathrm{Def}_{\mathrm{cl}}(E) \to \mathrm{coDef}_{\mathrm{cl}}(E), \qquad \mathbf{R}\psi : \mathrm{coDef}_{\mathrm{cl}}(E) \to \mathrm{Def}_{\mathrm{cl}}(E),$$

such that for a commutative artinian local algebra $\mathcal{R}$ and $S = (S, \sigma) \in \mathrm{Def}_{\mathcal{R}}(E)$, $T = (T, \tau) \in \mathrm{coDef}_{\mathcal{R}}(E)$,

$$\mathbf{L}\phi(S) = S \overset{\mathbf{L}}{\otimes}_{\mathcal{R}} \mathcal{R}^*, \qquad \mathbf{R}\psi(T) = \mathbf{R}\mathcal{H}om_{\mathcal{R}}\big(\mathcal{R}^*, T\big).$$

Proof. Fix an artinian commutative local algebra $\mathcal{R}$. By Proposition 4.8 (resp. Proposition 4.12) the category $\mathrm{Def}_{\mathcal{R}}(E)$ (resp. $\mathrm{coDef}_{\mathcal{R}}(E)$) is equivalent to its full subcategory consisting of objects (S, σ) (resp. (T, τ)) such that $S \in C^b(\mathrm{Mod}_{X_{\mathcal{R}}})$ is $\mathcal{R}$-free (resp. $T \in C^b(\mathrm{Mod}_{X_{\mathcal{R}}})$ is $\mathcal{R}$-injective). Moreover by Proposition 4.14 $\mathbf{L}i^*S = i^*S$, $\mathbf{R}i^!T = i^!T$ and $\mathbf{L}\phi(S) = S \otimes_{\mathcal{R}} \mathcal{R}^*$, $\mathbf{R}\psi(T) = \mathrm{Hom}_{\mathcal{R}}^{\cdot}(\mathcal{R}^*, T)$. Now the theorem follows from Proposition 4.7 and Proposition 4.14 c). This also proves Theorem 4.4. $\square$

The above theorem allows us to apply general (classical) pro-representability results to the classical deformation functor $\mathrm{Def}_{\mathrm{cl}}(E)$. The point is that since the abelian category Qcoh_X does not have enough projectives we cannot directly compare the pseudo-functor $\mathrm{Def}_{\mathrm{cl}}(E)$ to the analogous deformation pseudo-functor for a DG module over a DG category. But since Qcoh_X has enough injectives this can be done for the pseudo-functor $\mathrm{coDef}_{\mathrm{cl}}(E)$. Namely, we have the following corollary.

Corollary 4.16. *Let X be a locally Noetherian scheme over k and $E \in D^b_{\mathrm{Qcoh}}(X)$. Choose a bounded below complex I of injective quasi-coherent sheaves on X which is quasi-isomorphic to E. Assume that the minimal A_∞-model A of a DG algebra $\mathcal{C} = \mathrm{End}(I)$ is admissible (II, Definition 4.1) finite dimensional Koszul (II, Definition 16.5) A_∞-algebra. Put $\hat{S} = (B\bar{A})^*$, where $B\bar{A}$ is the bar construction of the augmentation (A_∞-)ideal $\bar{A}$. (Thus $\hat{S}$ is a local complete DG algebra which is acyclic except in degree zero.) Then*

a) *there exist equivalences of pseudo-functors from* cart *to* **Gpd***,*

$$\mathrm{Def}_{\mathrm{cl}}(E) \simeq \mathrm{coDef}_{\mathrm{cl}}(E) \simeq \mathrm{coDef}_{\mathrm{cl}}(\mathcal{C}) \simeq \mathrm{Def}_{\mathrm{cl}}(\mathcal{C});$$

b) *there exists an isomorphism of functors from* cart *to* Sets*,*

$$h_{H^0(\hat{S})} \simeq \pi_0 \cdot \mathrm{Def}_{\mathrm{cl}}(E).$$

Proof. The first and the last equivalences in a) follow from Theorem 4.4 and II, Theorem 13.5 respectively. Also b) follows from II, Theorem 16.7 b) and the middle equivalence in a). Thus it suffices to prove the equivalence $\mathrm{coDef}_{\mathrm{cl}}(E) \simeq \mathrm{coDef}_{\mathrm{cl}}(\mathcal{C})$. Clearly, $\mathrm{coDef}_{\mathrm{cl}}(E) \simeq \mathrm{coDef}_{\mathrm{cl}}(I)$.

Denote by $\mathcal{M}$ the abelian category of quasi-coherent sheaves on X. We may consider $\mathcal{M}$ as a DG category and then denote by $\mathcal{M}^{op}$-mod the DG category of DG modules over the opposite DG category $\mathcal{M}^{op}$. Also let $C^{dg}(\mathcal{M})$ be the DG category of complexes over $\mathcal{M}$. Consider the covariant DG functor $h^*_\bullet : C^{dg}(\mathcal{M}) \to \mathcal{M}^{op}$ defined by

$$h_M^*(N) := \mathrm{Hom}_{C^{dg}(\mathcal{M})}^{\cdot}(M, N)^*,$$

where $(\cdot)^*$ denotes the (graded) k-dual. Then by Theorem 3.36 this DG functor establishes an equivalence of pseudo-functors $\mathrm{coDef}_{\mathrm{cl}}(I) \simeq \mathrm{coDef}_{\mathrm{cl}}(h_I^*)$, where the second pseudo-functor is defined in I, Definitions 10.8, 10.14. (Notice that the homomorphism of DG algebras $h_I^*: \mathrm{End}(I) \to \mathrm{End}(h_I^*)$ is a quasi-isomorphism.) Finally, since the DG $\mathcal{M}^{op}$-module h_I^* is h-injective (I, Section 3.1) and is bounded below (I, Definition 11.5) we may apply II, Proposition 9.10 to find an equivalence of pseudo-functors $\mathrm{coDef}_{\mathrm{cl}}(h_I^*) \simeq \mathrm{Def}_{\mathrm{cl}}(\mathcal{C})$. $\quad\square$

4.1. Explicit description of the equivalence $\mathrm{Def}_{\mathrm{cl}}(E) \simeq \mathrm{Def}_{\mathrm{cl}}(\mathcal{C})$

Assume in the last corollary that the A_∞-algebra A satisfies the condition $(*)$ in II, Definition 15.3, i.e. the canonical morphism

$$k \to \mathbf{R}\,\mathrm{Hom}_{\bar{A}^{op}}\big(\mathbf{R}\,\mathrm{Hom}_{\bar{A}}(k, A), A\big)$$

is a quasi-isomorphism. Then we can make explicit the equivalence $\mathrm{Def}_{\mathrm{cl}}(E) \simeq \mathrm{Def}_{\mathrm{cl}}(\mathcal{C})$.

Namely, consider the A_∞ $\bar{A}_{\hat{S}}$-module k. It was shown in II, Section 15 that the DG $(\mathcal{C} \otimes \hat{S})^{op}$-module

$$\mathcal{E} = \mathrm{Hom}_{\bar{A}}^{\cdot}(k, \mathcal{C})$$

is the universal pro-deformation of the DG $\mathcal{C}^{op}$-module $\mathcal{C}$. In particular, given a (commutative) local artinian algebra $\mathcal{R} \in \mathrm{cart}$ and an object $(T, \tau) \in \mathrm{Def}_{\mathcal{R}}(\mathcal{C})$ there exists a homomorphism of DG algebras $g : \hat{S} \to \mathcal{R}$ such that the object $(\mathcal{E} \otimes_g \mathcal{R}, \mathrm{id})$ in $\mathrm{Def}_{\mathcal{R}}(\mathcal{C})$ is isomorphic to (T, τ) (it follows from II, Lemma 8.10). (Notice that $\mathcal{E}$ as a graded $(\mathcal{C} \otimes \hat{S})^{op}$-module is isomorphic to $\mathcal{C} \otimes \hat{S}$, so actually the graded $\mathcal{C}^{op}$-module $\mathcal{E} \otimes_g \mathcal{R}$ is free of finite rank.)

The complex $(\mathcal{E} \otimes_g \mathcal{R}) \otimes_{\mathcal{C}}^{\mathbf{L}} I$ of quasi-coherent sheaves on $X_{\mathcal{R}}$ is an object in $\mathrm{Def}_{\mathcal{R}}(I) = \mathrm{Def}_{\mathcal{R}}(E)$ corresponding to (T, τ).

5. Deformation of points objects on a smooth variety and other examples

Definition 5.1. Let X be a scheme and $E \in D^b(\mathrm{coh}_X)$. We call E a point object of dimension d if the DG algebra $\mathbf{R}\,\mathrm{Hom}^{\cdot}(E, E)$ is formal, i.e. it is quasi-isomorphic to its cohomology algebra $\mathrm{Ext}^{\cdot}(E, E)$, and this algebra is isomorphic to the exterior algebra of dimension d.

Let E be a point object and put $\mathcal{C} = \mathrm{Ext}^{\cdot}(E, E)$. By II, Theorem 15.2 the deformation pseudo-functor $\mathrm{DEF}_-(E)$ is pro-representable by the DG algebra $\hat{S} = (B\bar{\mathcal{C}})^*$. This DG algebra is quasi-isomorphic to its zero cohomology algebra $H^0(\hat{S})$, which is a commutative power series ring. Thus the formal DG moduli space of point objects is an ordinary (concentrated in degree zero) commutative regular scheme.

The following proposition justifies our term "point object".

Proposition 5.2. *Let X be a scheme of finite type over k and let $p \in X$ be a smooth k-point. Then the structure sheaf $\mathcal{O}_p \in D^b(\mathrm{coh}_X)$ is a point object of dimension* $\dim \mathcal{O}_{X,p}$.

Proof. Denote by $j : \operatorname{Spec} \mathcal{O}_{X,p} \hookrightarrow X$ the canonical morphism of schemes. It induces an exact functor $j_* : \mathcal{O}_{X,p}\text{-mod} \to \operatorname{Mod}_X$ which preserves injective objects (being the right adjoint to the exact functor $j^* : \operatorname{Mod}_X \to \mathcal{O}_{X,p}\text{-mod}$). The functor j_* maps k to $\mathcal{O}_p$ and hence induces a quasi-isomorphism of DG algebras

$$j_* : \mathbf{R}\operatorname{Hom}_{\mathcal{O}_{X,p}}(k,k) \to \mathbf{R}\operatorname{Hom}_{D(X)}(\mathcal{O}_p, \mathcal{O}_p).$$

So it suffices to show that the DG algebra $\mathbf{R}\operatorname{Hom}_{\mathcal{O}_{X,p}}(k,k)$ is quasi-isomorphic to the exterior algebra.

Denote the local ring $\mathcal{O}_{X,p} = A$ and let $m \subset A$ be the maximal ideal. Consider A as an augmented DG algebra concentrated in degree zero.

Choose a subspace $V \subset m$ which maps isomorphically to m/m^2. Consider the exterior coalgebra $\bigwedge^\bullet V$, where $\deg V = -1$, $\Delta(v) = v \otimes 1 + 1 \otimes v$ for $v \in V$ and $d = 0$. Then the identity map $V \to m$ is an admissible twisting cochain $\tau \in \operatorname{Hom}_k^\cdot(\bigwedge^\bullet V, A)$ (II, Definition 2.2). The corresponding DG A^{op}-module $\bigwedge^\bullet V \otimes_\tau A$ (II, Example 2.6) is just the usual Koszul complex for A, hence it is quasi-isomorphic to k. Thus

$$\mathbf{R}\operatorname{Hom}_{A^{op}}(k,k) = \operatorname{Hom}_{A^{op}}^\cdot\left(\bigwedge{}^{\!\bullet} V \otimes_\tau A, \bigwedge{}^{\!\bullet} V \otimes_\tau A \right).$$

Define a map of complexes

$$\theta : \operatorname{Hom}_k^\cdot\left(\bigwedge{}^{\!\bullet} V, k \right) \to \operatorname{Hom}_{A^{op}}^\cdot\left(\bigwedge{}^{\!\bullet} V \otimes_\tau A, \bigwedge{}^{\!\bullet} V \otimes_\tau A \right)$$

by the formula $\theta(f)(a \otimes b) = f(a_{(1)})a_{(2)} \otimes b$, where $\Delta(a) = a_{(1)} \otimes a_{(2)}$. Now exactly as in the proof of II, Lemma 3.8 one can show that θ is a homomorphism of DG algebras, which is a quasi-isomorphism. $\square$

Another example of a point object is provided by a line bundle on an abelian variety.

On the other hand let E be a line bundle on a (smooth projective) curve X of genus g. Then the DG algebra $\mathbf{R}\operatorname{Hom}(E,E)$ is formal and $\operatorname{Ext}^0(E,E) = k$, $\operatorname{Ext}^1(E,E) = W$—a vector space of dimension g, and $\operatorname{Ext}^i(E,E) = 0$ for $i > 1$. By II, Theorem 8.2 the pseudo-functor $\operatorname{DEF}_-(E)$ is pro-representable by the DG algebra $\hat{S} = (B\overline{\operatorname{Ext}^\cdot(E,E)})^*$. This DG algebra is concentrated in degree zero and is isomorphic to a noncommutative power series ring of dimension g.

Remark 5.3. The above two examples of line bundles show that the Picard variety of an abelian variety is (at least locally) the "true" moduli space of line bundles, whereas the Picard variety of a curve (of genus $g > 1$) is not (!). Indeed, the above argument shows that in the case of a curve the Picard variety (at least locally) has a natural noncommutative structure.

Part 3. Noncommutative Grassmanians

6. Preliminaries on $\mathbb{Z}$-algebras

In this section we define the notion of a $\mathbb{Z}$-algebra and associate to it an abelian category which should be thought of as a category of quasi-coherent sheaves on the corresponding noncommutative stack. We also define Koszul $\mathbb{Z}$-algebras.

Definition 6.1. A $\mathbb{Z}$-algebra $\mathcal{A}$ over the field k is a k-linear category with the set of objects $\mathbb{Z}$. For $i, j \in \mathbb{Z}$, we write $\mathcal{A}_{ij}$ instead of $\mathrm{Hom}_{\mathcal{A}}(i, j)$. Sometimes we will identify a $\mathbb{Z}$-algebra $\mathcal{A}$ with the corresponding ordinary non-unital algebra $\mathrm{Alg}_{\mathcal{A}}$.

Further, if $\mathcal{A}$ is a $\mathbb{Z}$-algebra, then we define the abelian category Mod-$\mathcal{A}$ as the category $\mathrm{Fun}(\mathcal{A}^{op}, k\text{-Vect})$ of contravariant functors from $\mathcal{A}$ to k-vector spaces. Equivalently Mod-$\mathcal{A}$ is the full subcategory of Mod-$\mathrm{Alg}_{\mathcal{A}}$ which consists of right $\mathrm{Alg}_{\mathcal{A}}$-modules M such that

$$M = \bigoplus_{i \in \mathbb{Z}} M \cdot \mathbf{1}_i$$

(quasi-unital modules). We call the objects of Mod-$\mathcal{A}$ $\mathcal{A}^{op}$-modules. For each $i \in \mathbb{Z}$ put

$$P_i := \mathrm{Hom}(-, i) = \mathbf{1}_i \, \mathrm{Alg}_{\mathcal{A}} \in \text{Mod-}\mathcal{A}.$$

By Ioneda Lemma, for each $M \in \text{Mod-}\mathcal{A}$ we have

$$\mathrm{Hom}_{\mathcal{A}^{op}}(P_i, M) = M(i),$$

hence P_i are projectives. Clearly, each $M \in \text{Mod-}\mathcal{A}$ can be covered by a direct sum of P_i's, hence the abelian category Mod-$\mathcal{A}$ has enough projectives.

Definition 6.2. Let $M \in \text{Mod-}\mathcal{A}$ be an $\mathcal{A}^{op}$-module. An element $x \in M(i)$ is called torsion if we have $x\mathcal{A}_{ji} = 0$ for $j \ll i$. Torsion elements form a submodule of M which we denote by $\tau(M)$. An $\mathcal{A}^{op}$-module M is called torsion if we have $M = \tau(M)$. We denote by $\mathrm{Tors}(\mathcal{A})$ the full subcategory of Mod-$\mathcal{A}$ which consists of torsion $\mathcal{A}^{op}$-modules.

The category $\mathrm{QMod}(\mathcal{A})$ is defined as the quotient category Mod-$\mathcal{A}/\mathrm{Tors}(\mathcal{A})$. We denote by $\pi : \text{Mod-}\mathcal{A} \to \mathrm{QMod}(\mathcal{A})$ the projection functor.

If M, N are $\mathcal{A}^{op}$-modules then

$$\mathrm{Hom}_{\mathrm{QMod}(\mathcal{A})}\big(\pi(M), \pi(N)\big) = \varinjlim \mathrm{Hom}_{\mathcal{A}^{op}}\big(M', N/\tau(N)\big),$$

where M' runs over the quasi-directed category of submodules $M' \subset M$ such that M/M' is torsion.

The category $\mathrm{QMod}(\mathcal{A})$ should be thought of as the category $\mathrm{QCoh}(\mathrm{Proj}(\mathcal{A}))$ of quasi-coherent sheaves on the noncommutative projective stack $\mathrm{Proj}(\mathcal{A})$. Furthermore, the object $\pi(P_0) \in \mathrm{QMod}(\mathcal{A})$ should be thought of as the structure sheaf $\mathcal{O}_{\mathrm{Proj}(A)}$.

Remark 6.3. Let $A = \bigoplus_{i \in \mathbb{Z}} A^i$ be a $\mathbb{Z}$-graded (unital) algebra. Then one can associate to it a $\mathbb{Z}$-algebra $\mathcal{A}$ with $\mathcal{A}_{ij} = A^{j-i}$ so that the composition in $\mathcal{A}$ comes from the multiplication in A. Recall that in [2] there defined the category $\mathrm{QGr}(A)$ as the quotient category $\mathrm{Gr}\, A/\mathrm{Tors}$ of the category $\mathrm{Gr}\, A$ of graded A-modules by the subcategory Tors of torsion modules. It is clear that the categories $\mathrm{Gr}\, A$, Tors, $\mathrm{QGr}\, A$ are equivalent to Mod-$\mathcal{A}$, $\mathrm{Tors}(\mathcal{A})$, $\mathrm{QMod}(\mathcal{A})$ respectively.

Notice that it can happen that the graded algebras A_1 and A_2 are not isomorphic but the associated $\mathbb{Z}$-algebras are equivalent. Thus, it is more reasonable to consider $\mathbb{Z}$-algebras but not graded algebras.

The projection functor $\pi : \text{Mod-}\mathcal{A} \rightarrow \text{QMod}(\mathcal{A})$ admits a right adjoint functor $\omega : \text{QMod}(\mathcal{A}) \rightarrow \text{Mod-}\mathcal{A}$ defined by the formula

$$\omega(X)(i) = \text{Hom}\big(\pi(P_i), X\big).$$

The adjunction morphism $\pi\omega \rightarrow \text{id}$ is an isomorphism.

Definition 6.4. A $\mathbb{Z}$-algebra $\mathcal{A}$ is called:

a) positively (resp. negatively) oriented if $\mathcal{A}_{ij} = 0$ for $i > j$ (resp. for $i < j$);
b) connected if $\mathcal{A}_{ii} \cong k$ for each $i \in \mathbb{Z}$;
c) locally finite if $\dim \mathcal{A}_{ij} < \infty$ for any $i, j \in \mathbb{Z}$.

Let $\mathcal{A}$ be a positively oriented $\mathbb{Z}$-algebra. We denote by $\mathcal{A}_{\leqslant i}$ the full subcategory of $\mathcal{A}$ such that $Ob(A_{\leqslant i}) = \{j: \; j \leqslant i\}$. Clearly, we also have the categories $\text{Mod-}\mathcal{A}_{\leqslant i}$ and $\text{Tors}(\mathcal{A}_{\leqslant i})$. It is easy to see that the quotient category $\text{Mod-}\mathcal{A}_{\leqslant i}/\text{Tors}(\mathcal{A}_{\leqslant i})$ is equivalent to $\text{QMod}(\mathcal{A})$. We denote by $\pi_{\leqslant i} : \text{Mod-}\mathcal{A}_{\leqslant i} \rightarrow \text{QMod}(\mathcal{A})$ and $\omega_{\leqslant i} : \text{QMod}(\mathcal{A}) \rightarrow \text{Mod-}\mathcal{A}_{\leqslant i}$ the projection functor and its right adjoint respectively.

If $\mathcal{A}$ is a positively oriented $\mathbb{Z}$-algebra then we put

$$T_{ij} = P_j/(P_j)_{\leqslant i},$$

where

$$(P_j)_{\leqslant i} = \bigoplus_{k \leqslant i} \mathcal{A}_{kj}.$$

Clearly, the $\mathcal{A}^{op}$-modules T_{ij} are torsion.

If $\mathcal{A}$ is a positively or negatively oriented connected $\mathbb{Z}$-algebra then we denote by S_n the simple $\mathcal{A}^{op}$-modules defined by the formula

$$S_n(i) = \begin{cases} k & \text{for } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that if $\mathcal{A}$ is positively oriented then $S_n = T_{n,n}$.

Definition 6.5. A connected positively (resp. negatively) oriented $\mathbb{Z}$-algebra is called quadratic if the algebra $\text{Alg}_{\mathcal{A}}$ is generated by the subspaces $\mathcal{A}_0 = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_{ii}$ and $\mathcal{A}_1 = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_{i,i+1}$ (resp. $\mathcal{A}_{-1} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_{i+1,i}$) and is determined by the quadratic relations $I_{i,i+2} \in \mathcal{A}_{i+1,i+2} \otimes \mathcal{A}_{i,i+1}$ (resp. $I_{i+2,i} \in \mathcal{A}_{i+1,i} \otimes \mathcal{A}_{i+2,i+1}$).

For a locally finite positively (resp. negatively) oriented quadratic $\mathbb{Z}$-algebra $\mathcal{A}$ one can define the dual quadratic $\mathbb{Z}$-algebra $\mathcal{A}^!$ with the opposite orientation. It is defined by the dual generators $\mathcal{A}^!_{i+1,i} = \mathcal{A}^*_{i,i+1}$ (resp. $\mathcal{A}^!_{i,i+1} = \mathcal{A}^*_{i+1,i}$) and the dual quadratic relations $S(I^\perp_{i,i+2}) \subset \mathcal{A}^*_{i+1,i} \otimes \mathcal{A}^*_{i+1,i+2}$ (resp. $S(I^\perp_{i+2,i}) \subset \mathcal{A}^*_{i+2,i+1} \otimes \mathcal{A}^*_{i+1,i}$), where $I^\perp_{i,i+2} \subset \mathcal{A}^*_{i+1,i+2} \otimes \mathcal{A}^*_{i,i+1}$ (resp. $I^\perp_{i+2,i} \subset \mathcal{A}^*_{i+1,i} \otimes \mathcal{A}^*_{i+2,i+1}$) is the dual subspace and S is the transposition of factors.

Further, one can define a Koszul complex

$$K := \mathcal{A}^{!*} \otimes_{\mathcal{A}_0} \mathcal{A} = \bigoplus \mathcal{A}_{kj}^{!*} \otimes_k \mathcal{A}_{ij}.$$

Here $\mathcal{A}^{!*} = \bigoplus_{i,j} \mathcal{A}_{ij}^{!*}$ is a bounded dual of $\mathcal{A}^{!}$. It is an $\mathcal{A}^{!}$-bimodule.

The differential $d : K \to K$ is defined as follows. Suppose that $\mathcal{A}$ is positively oriented. We have the natural maps

$$\mathcal{A}_{j,j+1} \otimes \mathcal{A}_{ij} \to \mathcal{A}_{i,j+1} \quad \text{and} \quad \mathcal{A}_{j,j+1}^{*} \otimes \mathcal{A}_{kj}^{!*} \to \mathcal{A}_{k,j+1}^{!*}.$$

In particular, we have the maps

$$\psi_{ijk} : \mathcal{A}_{j,j+1}^{*} \otimes \mathcal{A}_{j,j+1} \to \mathrm{Hom}_k\big(\mathcal{A}_{kj}^{!*} \otimes_k \mathcal{A}_{ij}, \mathcal{A}_{k,j+1}^{!*} \otimes_k \mathcal{A}_{i,j+1}\big).$$

The non-zero components of d are the maps $d_{ijk} = \psi_{ijk}(\mathbf{1}_{A_{jj+1}})$. Note that d is $\mathcal{A}_0$-linear and $\mathcal{A}^{!} \otimes \mathcal{A}^{op}$-linear. Thus, $K_n = \mathbf{1}_n K$ and $K_n^m = K_n \mathbf{1}_m$ are d-invariant. The complex K_n is of the form

$$\cdots \to \mathcal{A}_{n,n-2}^{!*} \otimes P_{n-2} \to \mathcal{A}_{n,n-1}^{!*} \otimes P_{n-1} \to P_n \to 0,$$

and the complex K_n^m is of the form

$$\cdots \to \mathcal{A}_{n,n-2}^{!*} \otimes \mathcal{A}_{m,n-2} \to \mathcal{A}_{n,n-1}^{!*} \otimes \mathcal{A}_{m,n-1} \to \mathcal{A}_{m,n} \to 0.$$

In particular, $K_n^n \cong k$.

Analogously for negatively oriented $\mathbb{Z}$-algebras.

For the rest of this section we assume that $\mathcal{A}$ is positively oriented.

Definition 6.6. A quadratic locally finite $\mathbb{Z}$-algebra is called Koszul if the complex K_n^m is acyclic for $n \neq m$, or, equivalently, K_n is a resolution of S_n.

We refer to [4] for the definition of co-Koszul and Gorenstein $\mathbb{Z}$-algebras. We will not need these definitions but we will need the following proposition:

Proposition 6.7. *(See [4].) Let $\mathcal{A}$ be a Koszul (positively oriented) $\mathbb{Z}$-algebra of finite homological dimension n. Then the following conditions are equivalent:*

(i) $\mathcal{A}$ *is co-Koszul;*
(ii) $\mathcal{A}$ *is Gorenstein;*
(iii) $\mathcal{A}^{!}$ *is Frobenious, i.e. $\mathcal{A}_{i+n,i}^{!} = k$ for all i, and the multiplication $\mathcal{A}_{ji} \otimes \mathcal{A}_{i+n,j} \to k$ is a perfect pairing for all i and j.*

Now we define the notion of a coherent $\mathbb{Z}$-algebra and the category $\mathrm{qmod}(\mathcal{A})$ for a coherent $\mathbb{Z}$-algebra $\mathcal{A}$.

Definition 6.8. Let $\mathcal{A}$ be a $\mathbb{Z}$-algebra. An $\mathcal{A}^{op}$-module M is called finitely generated if there exists a surjective morphism (in Mod-$\mathcal{A}$)

$$\bigoplus_{j=1}^{m} P_{i_j} \to M,$$

where $i_1, \ldots, i_m \in \mathbb{Z}$. Further, a finitely generated $\mathcal{A}^{op}$-module M is called coherent if for each (not necessarily surjective) morphism (in Mod-$\mathcal{A}$)

$$\phi : \bigoplus_{j=1}^{m} P_{i_j} \to M$$

the $\mathcal{A}^{op}$-module $\ker(\phi)$ is finitely generated. A $\mathbb{Z}$-algebra $\mathcal{A}$ is called coherent if for each $i \in \mathbb{Z}$ the module P_i is coherent.

If $\mathcal{A}$ is coherent then we denote by $\mathrm{qmod}(\mathcal{A})$ the full (abelian) subcategory of $\mathrm{QMod}(\mathcal{A})$ which consists of the images of coherent $\mathcal{A}^{op}$-modules.

The category $\mathrm{qmod}(\mathcal{A})$ should be thought of as the category $\mathrm{Coh}(\mathrm{Proj}(\mathcal{A}))$. By definition, we have that the structure sheaf $\mathcal{O}_{\mathrm{Proj}(\mathcal{A})}$ is coherent.

7. The definition of noncommutative Grassmanians

Let V be a finite-dimensional k-vector space of dimension $n > 0$. Let m be an integer such that $1 \leqslant m \leqslant n - 1$. We define the noncommutative Grassmanians by the formula

$$\mathrm{NGr}(m, V) := \mathrm{Proj}\big(\mathcal{A}^{m,V}\big),$$

where $\mathcal{A}^{m,V}$ is the following quadratic $\mathbb{Z}$-algebra:

$$\mathcal{A}^{m,V}_{i,i+1} = \begin{cases} V^* & \text{for } (n-m+1) \nmid i, \\ \Lambda^{n-m} V & \text{otherwise,} \end{cases}$$

and the quadratic relations are defined by the natural exact sequences

$$\begin{cases} \Lambda^2 V^* \to \mathcal{A}^{m,V}_{i+1,i+2} \otimes \mathcal{A}^{m,V}_{i,i+1} \to \mathcal{A}^{m,V}_{i,i+2} \to 0 & \text{for } (n-m+1) \nmid i, i+1, \\ \Lambda^{n-m-1} V \to \mathcal{A}^{m,V}_{i+1,i+2} \otimes \mathcal{A}^{m,V}_{i,i+1} \to \mathcal{A}^{m,V}_{i,i+2} \to 0 & \text{otherwise.} \end{cases}$$

Notice that if we fix a volume form $\omega \in \Lambda^n V$, then the $\mathcal{A}^{1,V}$ is naturally equivalent to the $\mathbb{Z}$-algebra associated to the symmetric algebra $\bigoplus_{l \geqslant 0} S^l V^*$, where $\deg(V^*) = 1$. Hence, the stack $\mathrm{NGr}(1, V)$ is isomorphic to the commutative projective space $\mathbb{P}(V)$.

We claim that $\mathrm{NGr}(m, V)$ is a true noncommutative moduli space of structure sheaves $\mathcal{O}_{\mathbb{P}(W)} \in D^b_{coh}(\mathbb{P}(V))$, where $W \subset V$ are vector subspaces of dimension $\dim W = m$. Namely, it satisfies the following properties:

1) There is a natural fully faithful functor Φ from the category of perfect objects $\mathrm{Perf}(\mathrm{NGr}(m, V))$ (Definition 8.20) to $D^b_{coh}(\mathbb{P}(V))$. Its image is the double orthogonal to

the family of objects $\mathcal{O}_{\mathbb{P}(W)}$, i.e. the full subcategory generated by objects $\mathcal{O}_{\mathbb{P}(V)}(m - n)$, $\ldots, \mathcal{O}_{\mathbb{P}(V)}(-1), \mathcal{O}_{\mathbb{P}(V)}$. This is Corollary 8.22 below.

2) There is a k-point $x_W \in \mathrm{NGr}(m, V)(k) = X_{\mathcal{A}^{m,V}}(k)$ (see Section 9 below) for each subspace $W \subset V$ of dimension $\dim W = m$. Further, $(x_W)_*(k)$ lies in $\mathrm{Perf}(\mathrm{NGr}(m, V))$ and $\Phi(x_*(k)) \cong \mathcal{O}_{\mathbb{P}(W)}$. This is a part of Theorem 9.11 below.

3) The completion of the local ring of the k-point x_W (see Section 10.1) is isomorphic to $H^0(\hat{S})$, where $\hat{S}$ is dual to the bar construction of the minimal A_∞-structure on $\mathrm{Ext}^{\cdot}(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(W)})$ (Theorem 10.3). It can be shown that the DG algebra $\mathbf{R}\,\mathrm{Hom}(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(W)})$ is formal and the graded algebra $\mathrm{Ext}^{\cdot}(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(W)})$ is quadratic Koszul, and hence the projection $\hat{S} \to H^0(\hat{S})$ is a quasi-isomorphism. Hence, the moduli space is not a DG space but just noncommutative space.

Furthermore, we do not have a moduli functor of our family of objects $\mathcal{O}_{\mathbb{P}(W)}$, which should be defined on the category of noncommutative affine schemes. However, the properties 1)–3) suggest that $\mathrm{NGr}(m, V)$ is a true moduli space of this family of objects, in our context of deformations of objects in derived categories.

It is remarkable that there is a natural morphism from the commutative Grassmanian $\mathrm{Gr}(m, V)$ to noncommutative one $\mathrm{NGr}(m, V)$. Moreover, the functor $\Phi : \mathrm{Perf}(\mathrm{NGr}(m, V)) \to D^b_{coh}(\mathbb{P}(V))$ above coincides with $\mathbf{L}f^*_{1,m,V}$, where $f_{1,m,V} : \mathbb{P}(V) \to \mathrm{NGr}(m, V)$ is a natural morphism. Both these statements are parts of Proposition 9.13 below.

8. The derived categories of noncommutative Grassmanians

Before we formulate and prove results on the derived categories of quasi-coherent sheaves on noncommutative Grassmanians $\mathrm{NGr}(m, V)$ we need to remind some notions and results from [4].

Let D be a k-linear enhanced triangulated category.

Definition 8.1. An object $E \in Ob(D)$ is called exceptional if $\mathrm{Hom}^i(E, E) = 0$ for $i \neq 0$, and $\mathrm{Hom}^0(E, E) = k$.

Definition 8.2. A collection $(E_1, \ldots, E_m)$ of exceptional objects in D is called exceptional if $\mathrm{Hom}^*(E_i, E_j) = 0$ for $i > j$.

Definition 8.3. A full exceptional collection of objects in the category D is a collection which generates D as triangulated category.

Definition 8.4. An exceptional collection $(E_1, \ldots, E_n)$ is called strong exceptional if it satisfies the additional assumption $\mathrm{Hom}^i(E_k, E_l) = 0$ for $i \neq 0$ and all k and l.

Let (E, F) be an exceptional pair. Define the objects $L_E F$ and $R_F E$ by the exact triangles

$$L_E F \to \mathrm{Hom}^{\cdot}(E, F) \otimes E \to F, \qquad E \to \mathrm{Hom}^{\cdot}(E, F)^{\vee} \otimes F \to R_F E.$$

Let $\sigma = (E_1, \ldots, E_n)$ be an exceptional collection. If $1 \leqslant i \leqslant n - 1$ (resp. $2 \leqslant i \leqslant n$), then the right (resp. left) mutation of the object E_i in this collection is the object $R^1 E_i = R_{E_{i+1}} E_i$ (resp. $L^1 E_i = L_{E_{i-1}} E_i$); the corresponding mutated collection

$$R^1_{E_i}\sigma := (E_1, \ldots, E_{i-1}, E_{i+1}, R_{E_{i+1}}E_i, E_{i+2}, \ldots, E_n)$$

(and the analogous collection $L^1_{E_i}\sigma$) is exceptional. The multiple mutations of the objects and of the collection are defined inductively:

$$R^k E_i = R_{E_{i+k}} R^{k-1} E_i, \qquad R^k_{E_i}\sigma = R^1_{R^{k-1}E_i}\left(R^{k-1}_{E_i}\sigma\right), \quad k \leqslant n - i$$

(and in the same way for left mutations).

Definition 8.5. A helix of the period n is an infinite sequence $\{E_i\}_{i \in \mathbb{Z}}$ such that for each $i \in \mathbb{Z}$ the collection $(E_i, \ldots, E_{i+n})$ is exceptional, and moreover $R^{n-1}E_i = E_{i+n}$.

If $\sigma = (E_1, \ldots, E_n)$ is an exceptional collection then it naturally extends to a helix by the conditions

$$E_{i+n} = R^{n-1}E_i, \quad i \geqslant 1, \qquad E_{i-n} = L^{n-1}E_i, \quad i \leqslant n.$$

In this case the helix is said to be generated by the collection σ.

If the helix is generated by the full exceptional collection then it satisfies the property of the partial periodicity: $\Phi(E_i) \cong E_{i-n}$, where $\Phi = F[1 - n]$ is the composition of the Serre functor F and the multiple shift $[1 - n]$.

Definition 8.6. (See [4].) A helix $\mu = \{E_i\}$ is called geometric if for each pair $(i, j) \in \mathbb{Z}^2$ such that $i \leqslant j$ one has

$$\mathrm{Hom}^k(E_i, E_j) = 0 \quad \text{for } k \neq 0.$$

Definition 8.7. (See [4].) An exceptional collection is called geometric if it generates a geometric helix.

Proposition 8.8. *(See [4].) Each sub-collection of a geometric exceptional collection is again geometric.*

Proposition 8.9. *(See [4].) A full exceptional collection of the length m of coherent sheaves on a smooth projective variety X of dimension n is geometric iff $m = n + 1$.*

Definition 8.10. (See [4].) The endomorphism $\mathbb{Z}$-algebra $\mathcal{A} = \mathrm{End}(S)$ of a helix $S = \{E_i\}$ is defined by the formula

$$\mathcal{A}_{ij} = \mathrm{Hom}(E_i, E_j)$$

with natural composition.

Theorem 8.11. *(See [4].) If S is a geometric helix generated by an exceptional collection of length n then the endomorphism $\mathbb{Z}$-algebra $\mathcal{A}$ of S is Koszul, co-Koszul and of finite global homological dimension n.*

Definition 8.12. (See [4].) Koszul co-Koszul $\mathbb{Z}$-algebra of finite homological dimension n is called a geometric $\mathbb{Z}$-algebra of the period n.

Let $\mathcal{A}$ be a geometric $\mathbb{Z}$-algebra of the period n. Let $K \subset D(\text{Mod-}\mathcal{A})$ be the full triangulated subcategory generated by the modules P_i. Note that $S_i \in K$. Let $F \in K$ be the full triangulated subcategory generated by the modules S_i.

Theorem 8.13. *(See [4].) Let $\mathcal{A}$ be a geometric $\mathbb{Z}$-algebra of the period n. Then F is a thick subcategory in K; the images of modules P_i in K/F form a geometric helix S of the period n, and moreover the $\mathbb{Z}$-algebra of S is equivalent to $\mathcal{A}$.*

Now we prove the main theorem of this section. It is closely related to the previous one but unfortunately cannot be deduced from it.

Theorem 8.14. *Let $\mathcal{A}$ be a geometric helix of the period n. Put $B = \mathcal{A}_{[1,n]} = \bigoplus_{1 \leqslant i, j \leqslant n} \mathcal{A}_{ij}$. Then there is an equivalence of categories $D^*(\text{QMod}(\mathcal{A})) \cong D^*(\text{Mod-}B)$.*

Proof. The proof is in two main steps. First we prove that the category $D^*(\text{QMod}(\mathcal{A}))$ is naturally equivalent to the quotient category of $D^*(\text{Mod-}\mathcal{A}_{\leqslant n})$ by the full thick triangulated subcategory $D^*_{\text{Tors}}(\mathcal{A}_{\leqslant n})$ which consists of complexes with torsion cohomology. Then we construct mutually inverse exact equivalences between the categories $D^*(\text{Mod-}\mathcal{A}_{\leqslant n})/D^*_{\text{Tors}}(\mathcal{A}_{\leqslant n})$ and $D^*(\text{Mod-}\mathcal{B})$ given by DG bimodules.

Lemma 8.15. *The categories $D^*(\text{QMod}(\mathcal{A}))$ and $D^*(\text{Mod-}\mathcal{A}_{\leqslant n})/D^*_{\text{Tors}}(\mathcal{A}_{\leqslant n})$ are naturally equivalent.*

Proof. First recall the functor $\omega_{\leqslant n} : \text{QMod}(\mathcal{A}) \to \text{Mod-}\mathcal{A}_{\leqslant n}$. It induces a fully faithful functor

$$K^*(\omega_{\leqslant n}) : K^*\big(\text{QMod}(\mathcal{A}_{\leqslant n})\big) \to K^*(\text{Mod-}\mathcal{A}_{\leqslant n})$$

between homotopy categories, which is right adjoint to the functor $K^*(\pi_{\leqslant n})$. It follows that $K^*(\text{QMod}(\mathcal{A}_{\leqslant n}))$ is equivalent to the quotient category $K^*(\text{Mod-}\mathcal{A}_{\leqslant n})/K^*(\pi_{\leqslant n})^{-1}(0)$. Let $K^*_{\text{Tors}}(\mathcal{A}_{\leqslant n}) \subset K^*(\text{Mod-}\mathcal{A}_{\leqslant n})$ be the full subcategory that consists of all complexes with torsion cohomology. It is easy to see that acyclic complexes in the category $K^*(\text{QMod}(\mathcal{A}_{\leqslant n}))$ correspond to the classes of complexes with torsion cohomology in $K^*(\text{Mod-}\mathcal{A}_{\leqslant n})/K^*(\pi_{\leqslant n})^{-1}(0)$. Thus, $D^*(\text{QMod}(\mathcal{A}_{\leqslant n}))$ is equivalent to the quotient of $K^*(\text{Mod-}\mathcal{A}_{\leqslant n})/K^*(\pi_{\leqslant n})^{-1}(0)$ by $K^*_{\text{Tors}}(\mathcal{A}_{\leqslant n})/K^*(\pi_{\leqslant n})^{-1}(0)$. This quotient is further equivalent to $D^*(\text{Mod-}\mathcal{A}_{\leqslant n})/D^*_{\text{Tors}}(\mathcal{A}_{\leqslant n})$. The lemma is proved. $\quad\square$

Denote by $(Q_1, \ldots, Q_n)$ the exceptional collection of indecomposable projective B^{op}-modules. By Theorem 8.13 the helix $\{Q_i\}_{i \in \mathbb{Z}}$ generated by $(Q_1, \ldots, Q_n)$ is geometric. It follows from its partial periodicity that for $i \leqslant 0$ we have $Q_i \cong H^{n-1}(Q_i)[1-n]$. Thus, we may and will assume that Q_i is concentrated in degree $n-1$ for $i \leqslant 0$. Put $M_0 = \bigoplus_{i \leqslant 0} Q_i[n-1]$. Since $\text{Hom}_B(Q_i, Q_j) = \mathcal{A}_{ij} = \mathcal{A}_{(i+n)(j+n)}$, M_0 is naturally an $\mathcal{A}_{\leqslant n} \otimes B^{op}$-module. Further, the functor $\Phi^{-1} = F^{-1}[1-n]$ can be given by the object $B^![1-n] \in D(B \otimes B^{op})$, where $B^! = \mathbf{R}\,\text{Hom}_{B^{op} \otimes B}(B, B \otimes B)$. Since $Q_i \otimes^{\mathbf{L}}_B B^![1-n]$ are pure modules for $i = 1, \ldots, n$,

$B^![1-n]$ is a pure bimodule. We define the object $M_1 \in D^b(\mathcal{A}_{\leqslant n} \otimes B^{op})$ by the formula

$$M_1 := M_0 \overset{\mathbf{L}}{\otimes}_B B^![2-2n].$$

We have that

$$P_i \overset{\mathbf{L}}{\otimes}_{\mathcal{A}_{\leqslant n}} M_1 \cong Q_i$$

for $i \leqslant n$. Since $\mathcal{A}_{\leqslant n}$ has finite left homological dimension, we have a well-defined functor

$$- \overset{\mathbf{L}}{\otimes}_{\mathcal{A}_{\leqslant n}} M_1 : D^*(\text{Mod-}\mathcal{A}_{\leqslant n}) \to D^*(\text{Mod-}B).$$

Lemma 8.16. *For each* $K^{\cdot} \in D_{\text{Tors}}(\mathcal{A}_{\leqslant n}^{op})$ *we have*

$$K^{\cdot} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}_{\leqslant n}} M_1 = 0.$$

Proof. Clearly, it suffices to prove the lemma for M_0 instead of M_1. We have that the complex K_l from Section 6 is a projective resolution of S_l for $l \leqslant n$. Further, $S_l \otimes^{\mathbf{L}}_{\mathcal{A}_{\leqslant n}} M_0 \cong K_l \otimes_{\mathcal{A}_{\leqslant n}} M_0$ and the last complex is up to shift of the following form

$$\cdots 0 \to A^{!*}_{l-n,l-2n} \otimes Q_{l-2n}[n-1] \to \cdots$$
$$\to A^{!*}_{l-n,l-n-1} \otimes Q_{l-n-1}[n-1] \to Q_{l-n}[n-1] \to 0 \to \cdots.$$

This complex is acyclic since it corresponds to the image of K_{l-n} in K/F under the equivalence of Theorem 8.13, and the image of K_{l-n} in K/F is zero.

Further, the torsion modules T_{km}, $k \leqslant m \leqslant n$ have finite filtrations with subquotients being direct sums of S_l. Thus, we have $T_{km} \otimes^{\mathbf{L}}_{\mathcal{A}_{\leqslant n}} M_0 = 0$.

Since each torsion $\mathcal{A}_{\leqslant n}^{op}$-modules has a left resolution by the direct sums of T_{km}, it follows that the statement of the lemma holds if $K^{\cdot}$ is a pure torsion $\mathcal{A}_{\leqslant n}^{op}$-module. Finally, since M_0 is quasi-isomorphic to a finite complex of bimodules which are projective as left $\mathcal{A}$-modules, the statement of the lemma holds for each $K^{\cdot} \in D_{\text{Tors}}(\mathcal{A}_{\leqslant n}^{op})$. $\square$

By the previous lemma, the formula $- \otimes^{\mathbf{L}}_{\mathcal{A}_{\leqslant n}} M_1$ defines a functor

$$\Phi : D^*(\text{Mod-}\mathcal{A}_{\leqslant n})/D^*_{\text{Tors}}(\mathcal{A}_{\leqslant n}) \to D^*(\text{Mod-}B).$$

Further, $M_2 := \bigoplus_{1 \leqslant i \leqslant n} P_i$ is naturally an $\mathcal{A}_{\leqslant n}^{op} \otimes B$-module. Consider the functor

$$\Psi : D^*(\text{Mod-}B) \to D^*(\text{Mod-}\mathcal{A}_{\leqslant n})/D^*_{\text{Tors}}(\mathcal{A}_{\leqslant n})$$

defined by the formula

$$\Psi(-) = \pi_{tors}\Big(- \overset{\mathbf{L}}{\otimes}_B M_2\Big),$$

where $\pi_{tors} : D^*(\text{Mod-}\mathcal{A}_{\leqslant n}) \to D^*(\text{Mod-}\mathcal{A}_{\leqslant n})/D^*_{\text{Tors}}(\mathcal{A}_{\leqslant n})$ is the projection.

Lemma 8.17. *The functors Φ and Ψ are mutually inverse equivalences.*

Proof. First, the isomorphism

$$M_2 \overset{\mathbf{L}}{\otimes}_{\mathcal{A}_{\leqslant n}} M_1 \to B$$

in $D(B \otimes B^{op})$ induces the isomorphism of functors $\Phi \circ \Psi \cong \text{Id}$.

Further, we claim that $H^0(M_1 \otimes^{\mathbf{L}}_B M_2) \cong \mathcal{A}_{\leqslant n}$, $H^{n-1}(M_1 \otimes^{\mathbf{L}}_B M_2)$ is torsion as $\mathcal{A}^{op}$-module and $H^i(M_1 \otimes^{\mathbf{L}}_B M_2) = 0$ for $i \neq 0, n-1$. Indeed, since K_l is a resolution of S_l it follows from Lemma 8.16 by decreasing induction on $l \leqslant n$ that

$$H^i\Big(P_l \overset{\mathbf{L}}{\otimes}_{\mathcal{A}_{\leqslant n}} M_1 \overset{\mathbf{L}}{\otimes}_B M_2\Big) = \begin{cases} P_l & \text{for } i = 0, \\ \text{is torsion} & \text{for } i \geqslant n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, it remains to note that $H^k(M_1) = 0$ for $k \geqslant n$ and M_2 is pure.

Finally, we have the natural morphism $\mathcal{A}_{\leqslant n} \to M_1 \otimes^{\mathbf{L}}_B M_2$ in $D(A_{\leqslant n} \otimes A^{op}_{\leqslant n})$ and for each $K^{\cdot} \in D(\text{Mod-}\mathcal{A}_{\leqslant n})$ we have that

$$Cone\Big(K^{\cdot} \to K^{\cdot} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}_{\leqslant n}} M_1 \overset{\mathbf{L}}{\otimes}_B M_2\Big) \cong K^{\cdot} \overset{\mathbf{L}}{\otimes}_{\mathcal{A}_{\leqslant n}} Cone\Big(\mathcal{A}_{\leqslant n} \to M_1 \overset{\mathbf{L}}{\otimes}_B M_2\Big) \in D_{\text{Tors}}\big(\mathcal{A}^{op}_{\leqslant n}\big).$$

Thus, $\Psi \circ \Phi \cong \text{Id}$. The lemma is proved. $\quad\square$

The theorem follows from Lemmas 8.17 and 8.15. $\quad\square$

Now we apply the above theorem to noncommutative Grassmanians introduced in Section 7. By Propositions 8.8 and 8.9 we have that the exceptional collection

$$\sigma = \big(\mathcal{O}_{\mathbb{P}(V)}(m-n), \dots, \mathcal{O}_{\mathbb{P}(V)}(-1), \mathcal{O}_{\mathbb{P}(V)}\big)$$

of coherent sheaves on $\mathbb{P}(V)$ is geometric. Let $S = \{E_i\}$ be the helix generated by σ, so that $E_i = \mathcal{O}_{\mathbb{P}(V)}(i)$ for $i = m - n, \dots, -1, 0$.

Proposition 8.18. *The endomorphism $\mathbb{Z}$-algebra $\mathcal{A}$ of the helix S is equivalent to $\mathcal{A}^{m,V}$.*

Proof. Note that both $\mathcal{A}$ and $\mathcal{A}^{m,V}$ are quadratic and $(n-m+1)$-periodic. It remains to show that the space $\mathcal{A}_{i,i+1}$ is isomorphic to $\mathcal{A}^{m,V}_{i,i+1}$ for $i = m-n, \dots, -1, 0$, and the quadratic relations $I_{i,i+2} \in \mathcal{A}_{i+1,i+2} \otimes \mathcal{A}_{i,i+1}$ coincide with that of $\mathcal{A}^{m,V}$ for $i = m-n, \dots, 0$.

All of this is clear for $i = m-n, \dots, -1$. Further, the object E_1 is isomorphic to the complex

$$\cdots \to 0 \to \mathcal{O}_{\mathbb{P}(V)}(m-n) \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(m-n+1) \to \cdots \to \Lambda^{n-m} V \otimes \mathcal{O}_{\mathbb{P}(V)} \to 0 \to \cdots,$$

where the last non-zero term is in degree zero. It follows that

$$\mathcal{A}_{0,1} = \mathrm{Hom}(E_0, E_1) = \Lambda^{n-m} V = \mathcal{A}_{0,1}^{m,V}.$$

Furthermore, the quadratic relation $I_{-1,1} \subset \mathcal{A}_{0,1} \otimes \mathcal{A}_{-1,0}$ coincides with the subspace $\Lambda^{n-m-1} V \subset \Lambda^{n-m} V \otimes V^*$, as in $\mathcal{A}^{m,V}$. Finally, E_2 is the convolution of the complex

$$\cdots \to 0 \to E_{m-n+1} \to V \otimes E_{m-n+2} \to \cdots \to \Lambda^{n-m-1} V \otimes E_0 \to V^* \otimes E_1 \to 0 \to \cdots,$$

where the last non-zero term is in degree zero. It follows that the quadratic relation $I_{0,2} \subset \mathcal{A}_{1,2} \otimes \mathcal{A}_{0,1}$ coincides with the subspace $\Lambda^{n-m-1} V \subset V^* \otimes \Lambda^{n-m} V$, as in $\mathcal{A}^{m,V}$. □

Let $B^{m,V}$ be the endomorphism algebra of σ. As a corollary of the above results we obtain the following:

Theorem 8.19. *The derived category $D^*(\mathrm{QMod}(\mathcal{A}^{m,V}))$ is equivalent to the derived category $D^*(B^{m,V})$. The objects $\pi(P_i)$ in $D^*(\mathrm{QMod}(\mathcal{A}^{m,V}))$ form a geometric helix of the period $\dim V - m + 1$.*

Proof. Indeed, by Proposition 8.18 and Theorem 8.11 the $\mathbb{Z}$-algebra $\mathcal{A}^{m,V}$ is geometric. Thus, the first statement follows from Theorem 8.14. After that, the second statement follows from Theorem 8.13. □

Now we introduce the perfect derived category.

Definition 8.20. Let $\mathcal{A}$ be a positively oriented $\mathbb{Z}$-algebra. The category of perfect objects $\mathrm{Perf}(\mathrm{QMod}(\mathcal{A}))$ is the minimal full thick triangulated subcategory of $D(\mathrm{QMod}(\mathcal{A}))$ which contains the objects $\pi(P_i)$. We will call the objects of $\mathrm{Perf}(\mathrm{QMod}(\mathcal{A}))$ perfect complexes.

We will also write below $\mathrm{Perf}(\mathrm{NGr}(m, V))$ instead of $\mathrm{Perf}(\mathrm{QMod}(\mathcal{A}^{m,V}))$.

Proposition 8.21. *Let $\mathcal{A}$ be a geometric $\mathbb{Z}$-algebra of the period n, and $B = \bigoplus_{1-n \leqslant i, j \leqslant 0} \mathcal{A}_{ij}$. Then the category $\mathrm{Perf}(\mathrm{QMod}(\mathcal{A}))$ is equivalent to $D^b(\mathrm{mod}_{finite}\text{-}B)$.*

Proof. By Theorem 8.14 $(\pi(P_{1-n}), \ldots, \pi(P_{-1}), \pi(P_0))$ is full strong exceptional collection in $\mathrm{Perf}(\mathrm{QMod}(\mathcal{A}))$. Further, the category $\mathrm{Perf}(\mathrm{QMod}(\mathcal{A}))$ is enhanced and $\mathrm{End}(\bigoplus_{1-n \leqslant i \leqslant 0} \pi(P_i)) = B$. Hence $\mathrm{Perf}(\mathrm{QMod}(\mathcal{A}))$ is equivalent to $D^b(\mathrm{mod}_{finite}\text{-}B)$. □

Corollary 8.22. *The category $\mathrm{Perf}(\mathrm{QMod}(\mathcal{A}^{m,V}))$ is equivalent to the full triangulated subcategory $\mathcal{T}^{m,V} \in D^b_{coh}(\mathbb{P}(V))$ generated by the exceptional collection*

$$\left(\mathcal{O}_{\mathbb{P}(V)}(m - n), \ldots, \mathcal{O}_{\mathbb{P}(V)}(-1), \mathcal{O}_{\mathbb{P}(V)} \right).$$

Under this equivalence the exceptional collection $(\pi(P_{m-n}), \ldots, \pi(P_0))$ corresponds to the exceptional collection $(\mathcal{O}_{\mathbb{P}(V)}(m - n), \ldots, \mathcal{O}_{\mathbb{P}(V)}(-1), \mathcal{O}_{\mathbb{P}(V)})$.

Remark 8.23. Notice that by [12] the $\mathbb{Z}$-algebra $\mathcal{A}^{\dim V-1,V}$ is coherent. Further, the $\mathbb{Z}$-algebra $\mathcal{A}^{1,V}$ is Noetherian and hence is coherent. It should be plausible that all the $\mathbb{Z}$-algebras $\mathcal{A}^{m,V}$ (and, more generally, all geometric $\mathbb{Z}$-algebras) are coherent, but it is not clear how to prove this statement. However, if $\mathcal{A}$ and B are as in Theorem 8.14, and $\mathcal{A}$ is coherent, then the subcategory $\mathrm{Perf}(\mathrm{QMod}(\mathcal{A})) \subset D(\mathrm{QMod}(\mathcal{A}))$ coincides with the subcategory $D^b_{\mathrm{qmod}}(\mathrm{QMod}(\mathcal{A}))$ which consists of complexes with cohomology lying in qmod. This category is further equivalent to $D^b(\mathrm{qmod}(\mathcal{A}))$. Therefore, in this case we also have an equivalence

$$D^b\big(\mathrm{qmod}(\mathcal{A})\big) \cong D^b(\mathrm{mod}_{\mathit{finite}}\text{-}B).$$

The coherence of geometric $\mathbb{Z}$-algebra $\mathcal{A}$ of period n is equivalent to some statement about t-structures. Namely, let $(\tau_{\leqslant 0}, \tau_{\geqslant 1})$ be a t-structure on $D^b(\mathrm{Mod}\text{-}\mathcal{A}_{[1,n]})$ induced by the equivalence of Theorem 8.14. It can be shown that $\mathcal{A}$ is coherent iff the t-structure $(\tau_{\leqslant 0}, \tau_{\geqslant 1})$ induces a t-structure on $D^b(\mathrm{mod}\text{-}_{\mathit{finite}}(\mathcal{A}_{[1,n]}))$.

9. The k-points of noncommutative Grassmanians

To discuss the k-points of noncommutative Grassmanians defined above we should first relate the following two approaches to noncommutative geometry.

The first one is to think of noncommutative stacks as of $\mathrm{Proj}(\mathcal{A})$, where $\mathcal{A}$ is a $\mathbb{Z}$-algebra. The special case of graded algebras, i.e. 1-periodic $\mathbb{Z}$-algebras is studied in [10,16,17,2] and other papers. However, it seems to be more reasonable to consider $\mathbb{Z}$-algebras. Note that our noncommutative Grassmanians are naturally defined as Proj of a $\mathbb{Z}$-algebra but not a graded algebra.

The other approach is to think of a noncommutative stacks as of (equivalence classes of) presheaves of (small) groupoids X on the category Alg_k^{op} opposite to the category of unital associative algebras. Morally the groupoid $X(A)$ should be thought of as the groupoid of maps from the affine noncommutative scheme $Sp(A)$ to X. This approach is studied in [13] in the case of sets (trivial groupoids). In this case we have the category of quasi-coherent sheaves which is not always abelian (it always has cokernels but may not admit kernels), and the structure sheaf.

In the second approach we obviously have the groupoid of k-points $X(k)$.

From this moment we assume that the $\mathbb{Z}$-algebra $\mathcal{A}$ is positively oriented but not necessarily connected. We will make an attempt to define the presheaf of groupoids of morphisms $Sp(A) \to \mathrm{Proj}(\mathcal{A})$, $A \in \mathrm{Alg}_k$. First note that a morphism $f : Sp(A) \to \mathrm{Proj}(\mathcal{A})$ must give a k-linear additive functor $f^* : \mathrm{QMod}(\mathcal{A}) \to \mathrm{Mod}\text{-}A$ together with an isomorphism $f^*(\pi(P_0)) \cong A$. Moreover, f^* must commute with colimits.

Notice that if $\mathcal{C}$ is a k-linear abelian category with infinite direct sums and $\mathcal{F} : \mathcal{A} \to \mathcal{C}$ is a k-linear functor then we have the tensor product functor

$$- \otimes_{\mathcal{A}} \mathcal{F} : \mathrm{Mod}\text{-}\mathcal{A} \to \mathcal{C}$$

given by the formula

$$M \otimes_{\mathcal{A}} \mathcal{F} = \mathrm{coker}(b : M \otimes \mathcal{A} \otimes \mathcal{F} \to M \otimes \mathcal{F}),$$

where $b = \mu_M \otimes \mathbf{1}_{\mathcal{A}} - \mathbf{1}_M \otimes \mu_{\mathcal{F}}$ (we identify $\mathcal{F}$ with $\bigoplus_{i \in \mathbb{Z}} \mathcal{F}(i) \in \mathcal{C}$). Clearly, the functor $- \otimes_{\mathcal{A}} \mathcal{F}$ commutes with colimits. We denote by $\mathrm{Tor}_i^{\mathcal{A}}(-, \mathcal{F})$ its left derived functors.

Definition 9.1. Let $\mathcal{A}$ be a positively oriented $\mathbb{Z}$-algebra and let $\mathcal{C}$ be a k-linear abelian category with infinite direct sums together with a distinguished object $Y \in \mathcal{C}$.

We denote by $\mathcal{G}_1(\mathcal{A}, \mathcal{C}, Y)$ the groupoid of pairs (f^*, θ), where $f^* : \mathrm{QMod}(\mathcal{A}) \to \mathcal{C}$ is a k-linear functor commuting with colimits and $\theta : f^*(\pi(P_0)) \to Y$ is an isomorphism.

We denote by $\mathcal{G}_2(\mathcal{A}, \mathcal{C}, Y)$ the groupoid of pairs $(\mathcal{F}, \sigma)$, where $\mathcal{F} : \mathcal{A} \to \mathcal{C}$ is a k-linear functor such that $\mathrm{Tor}_0^{\mathcal{A}}(T, \mathcal{F}) = \mathrm{Tor}_1^{\mathcal{A}}(T, \mathcal{F}) = 0$ for each torsion $\mathcal{A}^{op}$-module T, and $\sigma : \mathcal{F}(0) \to Y$ is an isomorphism.

Theorem 9.2. *Let $\mathcal{A}$ be a positively oriented $\mathbb{Z}$-algebra and let $\mathcal{C}$ be a k-linear abelian category with infinite direct sums together with a distinguished object $Y \in \mathcal{C}$. Then the groupoids $\mathcal{G}_1(\mathcal{A}, \mathcal{C}, Y)$ and $\mathcal{G}_2(\mathcal{A}, \mathcal{C}, Y)$ are equivalent.*

Proof. We define the functor $\Phi : \mathcal{G}_1(\mathcal{A}, \mathcal{C}, Y) \to \mathcal{G}_2(\mathcal{A}, \mathcal{C}, Y)$ as follows. Let $(f^*, \theta) \in \mathcal{G}_1(\mathcal{A}, \mathcal{C}, Y)$. The functor $\Phi(f^*) : \mathcal{A} \to \mathcal{C}$ is defined by the formulas

$$\Phi\big(f^*\big)(i) = f^*\big(\pi(P_i)\big),$$

and for $x \in \mathcal{A}_{ij}$,

$$\Phi\big(f^*\big)(x) = f^*\big(\pi(x)\big).$$

We claim that the pair $(\Phi(f^*), \theta)$ is an object of $\mathcal{G}_2(\mathcal{A}, \mathcal{C}, Y)$. Indeed, let $T \in \mathrm{Tors}(\mathcal{A})$. Since the sequence

$$\pi(T \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}) \to \pi(T \otimes \mathcal{A} \otimes \mathcal{A}) \to \pi(T \otimes \mathcal{A}) \to 0$$

is exact in $\mathrm{QMod}(\mathcal{A})$ it follows that the sequence

$$T \otimes \mathcal{A} \otimes \mathcal{A} \otimes \Phi\big(f^*\big)(\mathcal{A}) \to T \otimes \mathcal{A} \otimes \Phi\big(f^*\big)(\mathcal{A}) \to T \otimes \Phi\big(f^*\big)(\mathcal{A}) \to 0$$

is exact in $\mathcal{C}$, i.e.

$$\mathrm{Tor}_0^{\mathcal{A}}\big(S_i, \Phi\big(f^*\big)\big) = \mathrm{Tor}_0^{\mathcal{A}}\big(S_i, \Phi\big(f^*\big)\big) = 0.$$

Thus, the functor Φ is defined on objects. It obviously extends to morphisms.

Now we define the functor $\Psi : \mathcal{G}_2(\mathcal{A}, \mathcal{C}, Y) \to \mathcal{G}_1(\mathcal{A}, \mathcal{C}, Y)$ as follows. Let $(\mathcal{F}, \sigma) \in \mathcal{G}_2(\mathcal{A}, \mathcal{C}, Y)$. We claim that the formula

$$\Psi(\mathcal{F})\big(\pi(M)\big) = M \otimes_{\mathcal{A}} \mathcal{F}$$

well defines a functor $\Psi(\mathcal{F}) : \mathrm{QMod}(\mathcal{A}) \to \mathcal{C}$ which is right exact and commutes with infinite direct sums. Indeed it follows from the condition

$$\mathrm{Tor}_0^{\mathcal{A}}(T, \mathcal{F}) = \mathrm{Tor}_1^{\mathcal{A}}(T, \mathcal{F}) = 0$$

for torsion $\mathcal{A}^{op}$-modules T.

Hence, the pair $(\Psi(\mathcal{F}), \sigma)$ is an object of $\mathcal{G}_1(\mathcal{A}, \mathcal{C}, Y)$. This defines the functor Ψ on objects and it obviously extends to morphisms.

It is clear that the composition $\Phi \circ \Psi$ is isomorphic to the identity functor. To see this for the composition $\Psi \circ \Phi$, it remains to note that each functor f^* from the pair in $\mathcal{G}_1(\mathcal{A}, \mathcal{C}, Y)$ can be reconstructed from the functor $\Phi(f^*)$ using exact sequences

$$\pi(M \otimes \mathcal{A} \otimes \mathcal{A}) \to \pi(M \otimes \mathcal{A}) \to \pi(M) \to 0. \qquad \square$$

Notice that it follows from the above theorem that each functor $f^* : \mathrm{QMod}(\mathcal{A}) \to \mathcal{C}$ commuting with direct sums and right exact has the right adjoint $f_* : \mathcal{C} \to \mathrm{QMod}(\mathcal{A})$ given by the formula $f_*(X) = \pi(\widetilde{f_*(X)})$,

$$\widetilde{f_*(X)}(i) = \mathrm{Hom}_{\mathcal{C}}\big(f^*\big(\pi(P_i)\big), X\big),$$

and for $\phi \in \mathrm{Hom}_{\mathcal{C}}(f^*(\pi(P_j)), X)$, $x \in \mathcal{A}_{ij}$,

$$\phi \cdot x = \phi \cdot f^*\big(\pi(x)\big).$$

Indeed, this follows from the formula

$$f^*\big(\pi(M)\big) = M \otimes_{\mathcal{A}} \mathcal{F}.$$

It is clear that $X_i(\mathcal{A}) = (A \mapsto \mathcal{G}_i(\mathcal{A}, \mathrm{Mod}\text{-}A, A))$, $i = 1, 2$, are presheaves of groupoids on the category Alg_k^{op}, and the equivalence from the above theorem extends to the equivalence of these presheaves.

However, not all functors f^* commuting with colimits should come from true morphisms $f : Sp(A) \to \mathrm{Proj}(\mathcal{A})$. Although a true presheaf of groupoids should be defined as a full (small) subpresheaf of $X_2(\mathcal{A})$. We are going to make an attempt in this direction. Our motivation is the following Proposition.

Proposition 9.3. *Let $\mathcal{A}$ be a $\mathbb{Z}$-algebra. Further, let $\mathcal{C}$ be a k-linear abelian category with infinite direct sums and with the distinguished object Y. Let $(f^*, \theta) \in \mathcal{G}_1(\mathcal{A}, \mathcal{C}, Y)$ and $(\mathcal{F}, \sigma) \in \mathcal{G}_2(\mathcal{A}, \mathcal{C}, Y)$ be objects which correspond to each other under the equivalence of Theorem* 9.2. *The following conditions are equivalent:*

(i) *there exists a left derived functor $\mathbf{L}f^* : D^-(\mathrm{QMod}(\mathcal{A})) \to D^-(\mathcal{C})$, and $\mathbf{L}^i f^*(\pi(P_j)) = 0$ for $i \neq 0$ and all j;*
(ii) *we have $\mathrm{Tor}_i^{\mathcal{A}}(T, \mathcal{F}) = 0$ for all $i \geqslant 0$, $T \in \mathrm{Tors}(\mathcal{A})$.*

Proof. Prove that (i) implies (ii). We have that the functor $f^* \cong - \otimes \mathcal{F}$ maps acyclic right bounded complexes of direct sums of $\pi(P_i)$ to acyclic complexes. Applying this to the projection of the free resolution of a torsion module T, we obtain that $\mathrm{Tor}_i^{\mathcal{A}}(T, \mathcal{F}) = 0$ for $i \geqslant 0$.

Prove that (ii) implies (i). Since each object in $\mathrm{QMod}(\mathcal{A})$ can be covered by a direct sum of $\pi(P_i)$, it suffices to prove that f^* maps right bounded acyclic complexes of direct sums of $\pi(P_i)$ to acyclic complexes.

Since the kernel and the cokernel of the morphism $P_j \to \omega \pi(P_j)$ are torsion, we have $\mathrm{Tor}_i^{\mathcal{A}}(\omega \pi(P_j), \mathcal{F}) = 0$ for $i > 0$. Further, if $K^{\cdot}$ is a right bounded acyclic complex of direct sums of $\pi(P_i)$ then $\omega(K^{\cdot})$ has torsion cohomology. Therefore,

$$f^*(K^{\cdot}) = \omega(K^{\cdot}) \otimes_{\mathcal{A}} \mathcal{F} = \omega(K^{\cdot}) \overset{\mathbf{L}}{\otimes}_{\mathcal{A}} \mathcal{F},$$

and the last complex is acyclic since $\mathrm{Tor}_i^{\mathcal{A}}(T, \mathcal{F}) = 0$ for $i \geqslant 0$, $T \in \mathrm{Tors}(\mathcal{A})$. $\quad \square$

Definition 9.4. For each $\mathbb{Z}$-algebra $\mathcal{A}$ we define the presheaf $X_{\mathcal{A}}$ of groupoids on the category Alg_k^{op} as follows. It is a full subpresheaf of $X_2(\mathcal{A})$ and the groupoid $X_{\mathcal{A}}(A) \subset X_2(\mathcal{A})(A)$ consists of pairs $(\mathcal{F}, \sigma) \in X_2(\mathcal{A})(A)$ such that:

1) we have $\mathrm{Tor}_i^{\mathcal{A}}(T, \mathcal{F}) = 0$ for all i and $T \in \mathrm{Tors}(\mathcal{A})$;
2) the A^{op}-modules $\mathcal{F}(i)$ are flat.

It is clear that $X_{\mathcal{A}}$ is indeed a subpresheaf of $X_2(\mathcal{A})$. For $f = (\mathcal{F}, \sigma) \in X_{\mathcal{A}}(A)$ we denote by $f^* : \mathrm{QMod}(\mathcal{A}) \to \mathrm{Mod}\text{-}A$ the corresponding functor $- \otimes_{\mathcal{A}} \mathcal{F}$. We also regard the objects $f \in X_{\mathcal{A}}(A)$ as maps from $Sp(A)$ to $\mathrm{Proj}(\mathcal{A})$, where $Sp(A)$ is a noncommutative affine scheme corresponding to A.

The following lemma simplifies the complicated condition on Tor_i. Recall the torsion $\mathcal{A}^{op}$-modules $T_{p,q}$ from Section 6.

Lemma 9.5. *Let $\mathcal{A}$ be a positively oriented connected $\mathbb{Z}$-algebra and let $\mathcal{C}$ be a k-linear abelian category with infinite direct sums. Let $\mathcal{F} : \mathcal{A} \to \mathcal{C}$ be a k-linear functor.*
Suppose that $\mathrm{Tor}_i^{\mathcal{A}}(T_{j,j}, \mathcal{F}) = 0$ for all i and j. Then $\mathrm{Tor}_i^{\mathcal{A}}(T, \mathcal{F}) = 0$ for all i and all torsion $\mathcal{A}^{op}$-modules T.

Proof. First note that if $T = T\mathbf{1}_j$, then T has a left resolution by direct sums of $T_{j,j}$. Hence, lemma holds for such T.

Further, the torsion modules $T_{p,q}$, $p \leqslant q$, have finite filtrations with subquotients T_m such that $T_m = T_m \mathbf{1}_m$, $p \leqslant m \leqslant q$. Hence $\mathrm{Tor}_i^{\mathcal{A}}(T_{p,q}, \mathcal{F}) = 0$ for all i, and $p \leqslant q$. Now lemma follows from the observation that each torsion module has a left resolution by direct sums of modules $T_{p,q}$. $\quad \square$

Definition 9.6. We say that a positively oriented $\mathbb{Z}$-algebra $\mathcal{A}$ satisfies the condition $(**)$ if the following hold:

(i) the algebra $\mathcal{A}$ is generated by its subspaces $\mathcal{A}_0$ and $\mathcal{A}_1$;
(ii) for each $i \in \mathbb{Z}$, the object $\pi(P_i)$ has a finite right resolution by direct sums of $\pi(P_j)$ with $j > i$.

The next proposition motivates the condition $(**)$.

Proposition 9.7. *Let $\mathcal{A}$ be a $\mathbb{Z}$-algebra satisfying $(**)$, and $A \in \mathrm{Alg}_k$. Then for each $f = (\mathcal{F}, \sigma) \in X_{\mathcal{A}}(A)$ we have $\mathrm{Aut}(f) = \{1\}$.*

Proof. Let $g \in \text{Aut}(f)$. Clearly, $g(0) : \mathcal{F}(0) \to \mathcal{F}(0)$ is the identity morphism. Further, for each $i \in \mathbb{Z}$ the surjection $\mathcal{A}_{i,i+1} \otimes \pi(P_i) \to \pi(P_{i+1})$ is mapped by f^* to the surjection $\mathcal{A}_{i,i+1} \otimes \mathcal{F}(i) \to \mathcal{F}(i+1)$. Hence, we obtain by increasing induction over i that $g(i) : \mathcal{F}(i) \to \mathcal{F}(i)$ is the identity for $i \geqslant 0$.

Finally, since $\mathcal{A}$ satisfies $(\ast\ast)$, it follows from Proposition 9.3 that there exists an injection of the form $\pi(P_i) \to \bigoplus_\alpha \pi(P_{j_\alpha})$ with $j_\alpha > i$ which is mapped by f^* to the injection $\mathcal{F}(i) \to \bigoplus_\alpha \mathcal{F}(j_\alpha)$. Hence, we obtain by decreasing induction on i that $g(i) : \mathcal{F}(i) \to \mathcal{F}(i)$ is the identity for all $i \in \mathbb{Z}$. $\quad\square$

Therefore, if $\mathcal{A}$ satisfies $(\ast\ast)$ we may and will replace $X_\mathcal{A}$ by the equivalent presheaf of trivial groupoids $\pi_0(X_\mathcal{A})$. It is easily seen from the proof of the above Proposition that $\pi_0(X_\mathcal{A}(A))$ is a set. Thus, $X_\mathcal{A}$ is a presheaf of sets.

Now we would like to compare our definition of morphisms from $Sp(A)$ to $\text{Proj}(\mathcal{A})$ with the morphisms from commutative Noetherian k-schemes to commutative projective k-schemes.

Note that we can restrict the presheaf $X_\mathcal{A}$ to the full subcategory of Alg_k which consists of commutative Noetherian k-algebras. Further, we can extend this restricted presheaf onto the category $Noeth_k$ of all commutative Noetherian k-schemes.

Definition 9.8. Let $\mathcal{A}$ be a positively oriented $\mathbb{Z}$-algebra. We define the presheaf $\mathcal{X}_\mathcal{A} : Noeth_k^{op} \to$ **Gpd** as follows. The groupoid $\mathcal{X}_\mathcal{A}(Y)$ is a full sub-groupoid of $\mathcal{G}_2(\mathcal{A}, \text{QCoh}(Y), \mathcal{O}_Y)$ which consists of objects $(\mathcal{F}, \sigma)$ such that the following conditions hold:

1) we have $\text{Tor}_i^\mathcal{A}(T, \mathcal{F}) = 0$ for all i and $T \in \text{Tors}(\mathcal{A})$;
2) the sheaves $\mathcal{F}(i)$ are locally flat.

We also regard the objects of the groupoid $\mathcal{X}_\mathcal{A}(Y)$ as maps from Y to $\text{Proj}(\mathcal{A})$. The analogue of Proposition 9.7 obviously holds for Noetherian k-schemes instead of associative algebras. For each commutative Noetherian k-scheme Y we denote by $Y^\vee : Noeth_k^{op} \to$ Sets the presheaf of sets represented by Y.

Now let $Z \subset \mathbb{P}(V)$ be a closed subscheme and let $\mathcal{A}$ be a $\mathbb{Z}$-algebra associated to its homogeneous coordinate ring $\bigoplus_{d \geqslant 0} S^d V^*/I$.

Proposition 9.9. *The $\mathbb{Z}$-algebra $\mathcal{A}$ satisfies the condition $(\ast\ast)$. The presheafs of sets $Z^\vee$ and $\mathcal{X}_\mathcal{A} : Noeth_k^{op} \to$ Sets on the category $Noeth_k^{op}$ are isomorphic.*

Proof. Recall that the category $\text{QCoh}(Z)$ is equivalent to $\text{Proj}(\mathcal{A})$ by Serre Theorem. The sheaves $\mathcal{O}_Z(i)$ correspond under this equivalence to $\pi(P_i)$.

Let $f : Y \to Z$ be a morphism. Then the sheaves $f^*(\mathcal{O}_Z(i))$ are invertible and hence are locally flat. Further, f^* maps acyclic right bounded complexes of direct sums of $\mathcal{O}_Z(i)$ to acyclic complexes. Finally, we have an isomorphism $f^*(\mathcal{O}_Z) \cong \mathcal{O}_Y$. Thus, we have a morphism of presheaves $Z^\vee \to \mathcal{X}_\mathcal{A}$.

Conversely, let $Y \in Noeth_k$ and $g \in \mathcal{X}_\mathcal{A}(Y)$. Notice that for each $i \in \mathbb{Z}$ we have an acyclic Koszul complex on $\mathbb{P}(V)$ twisted by $\mathcal{O}_{\mathbb{P}(V)}(i)$, and we can restrict it to Z:

$$0 \to \mathcal{O}_Z(i) = \Lambda^n V^* \otimes \mathcal{O}_Z(i) \to \Lambda^{n-1} V^* \otimes \mathcal{O}_Z(i+1) \to \cdots \to \mathcal{O}_Z(i+n) \to 0.$$

In particular, the $\mathbb{Z}$-algebra $\mathcal{A}$ satisfies the condition $(\ast\ast)$.

Hence, we have acyclic complexes

$$0 \to g^*\big(\pi(P_i)\big) = \Lambda^n V^* \otimes g^*\big(\pi(P_i)\big) \to \Lambda^{n-1} V^* \otimes g^*\big(\pi(P_{i+1})\big) \to \cdots$$
$$\to g^*\big(\pi(P_{i+n})\big) \to 0. \tag{9.1}$$

In particular, we have surjections $V^* \otimes g^*(\pi(P_i)) \to g^*(\pi(P_{i+1}))$ and injections $g^*(\pi(P_i)) \to V \otimes g^*(\pi(P_{i+1}))$. Since $g^*(\pi(P_0)) \cong \mathcal{O}_Y$, we obtain by increasing and decreasing inductions on i that all the sheaves $g^*(\pi(P_i))$ are coherent and are non-zero on each connected component of Y. Since they are locally flat, they are locally free.

Further, put $\mathcal{L} = g^*(\pi(P_1))$. Clearly, g can be reconstructed from the surjective morphism $\phi : V^* \otimes A \to \mathcal{L}$ using the exact sequences

$$\Lambda^2 V^* \otimes g^*\big(\pi(P_{i-1})\big) \to V^* \otimes g^*\big(\pi(P_i)\big) \to g^*\big(\pi(P_{i+1})\big) \to 0$$

and

$$0 \to g^*\big(\pi(P_{i-1})\big) \to V \otimes g^*\big(\pi(P_i)\big) \to \Lambda^2 V \otimes g^*\big(\pi(P_{i+1})\big)$$

from complexes (9.1). Suppose that $\mathrm{rank}(\mathcal{L}_{|Y_0}) \geqslant 2$ on some connected component $Y_0 \subset Y$. Then it is easy to see that the morphism $\Lambda^{n-1} V^* \otimes \mathcal{O}_{Y_0} \to \Lambda^{n-2} V^* \otimes \mathcal{L}_{|Y_0}$ is injective. Hence $g^*(\pi(P_{-1}))_{|Y_0} = 0$, a contradiction. Thus, $\mathcal{L}$ is an invertible sheaf.

The surjective morphism ϕ above defines a morphism $\tilde{g} : Y \to \mathbb{P}(V)$. It follows that $\tilde{g}^* \cong g^* \iota^*$, where $\iota : Z \to \mathbb{P}(V)$ is the embedding. Hence, $g^*(\pi(P_i)) \cong \mathcal{L}^{\otimes i}$. Further, the induced morphism of graded algebras

$$\bigoplus_{d \geqslant 0} S^d V^* \to \bigoplus_{i \geqslant 0} H^0\big(\mathcal{L}^{\otimes i}\big)$$

passes through $\bigoplus_{d \geqslant 0} S^d V^*/I$, thus $\tilde{g}$ passes through Z, and we obtain a morphism $Y \to Z$. Hence we have a morphism of presheaves $\mathcal{X}_\mathcal{A} \to Z^\vee$.

The constructed morphisms of presheaves are inverse to each other. Proposition is proved. $\quad\square$

Now we want to describe the k-points of noncommutative Grassmanians.

Lemma 9.10. *Let $\mathcal{A}$ be a geometric $\mathbb{Z}$-algebra of period n. Then it satisfies the condition* $(**)$.

Proof. By definition, $\mathrm{Alg}_\mathcal{A}$ is generated by $\mathcal{A}_0$ and $\mathcal{A}_1$. Further, the projections $\pi(K_i)$ of Koszul complexes are acyclic. The first non-zero term of $\pi(K_{i+n})$ equals to $\pi(P_i)$ (since $\mathcal{A}^!$ is Frobenious). Therefore, each $\pi(P_i)$ has the required right resolution. $\quad\square$

Denote by $\mathrm{pr}_r^{m,V} : D_{coh}^b(\mathbb{P}(V)) \to \mathcal{T}^{m,V}$ the functor which is right adjoint to the inclusion $\iota : \mathcal{T}^{m,V} \to D_{coh}^b(\mathbb{P}(V))$. The next theorem describes the k-points of noncommutative Grassmanians and the objects in $\mathcal{T}^{m,V}$ corresponding to their structure sheaves under the equivalence of Corollary 8.22.

Theorem 9.11. *The k-points of the noncommutative Grassmanian $\mathrm{NGr}(m, V)$, i.e. the elements $f \in X_{\mathcal{A}^{m,V}}(k)$, naturally correspond to vector subspaces $W \subset V$ of dimension $1 \leqslant \dim W \leqslant m$. Further, if f corresponds to W then $f_*(k)$ is a perfect complex, and it corresponds to the object $\mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)}) \in \mathcal{T}^{m,V}$ under the equivalence $\mathrm{Perf}(\mathrm{QMod}(\mathcal{A})) \cong \mathcal{T}^{m,V}$.*

Proof. Let $f \in X_{\mathcal{A}}(k)$. For each $i \in \mathbb{Z}$ we have the natural acyclic complex

$$0 \to f^*\big(\pi(P_i)\big) \cong \mathcal{A}_{i+n,i}^{m,V!*} \otimes f^*\big(\pi(P_i)\big) \to \mathcal{A}_{i+n,i+1}^{m,V!*} \otimes f^*\big(\pi(P_{i+1})\big) \to \cdots$$
$$\to f^*\big(\pi(P_{i+n})\big) \to 0.$$

In particular, we have surjective map $V^* \otimes f^*(\pi(P_i)) \to f^*(\pi(P_{i+1}))$ and injective map $f^*(\pi(P_i)) \to V \otimes f^*(\pi(P_{i+1}))$. Since $f^*(\pi(P_0)) \cong k$, we obtain by increasing and decreasing inductions over i that all the spaces $f^*(\pi(P_i))$ are non-zero and finite-dimensional. Further, using the exact sequences

$$\Lambda^2 V^* \otimes f^*\big(\pi(P_{i-1})\big) \to V^* \otimes f^*\big(\pi(P_i)\big) \to f^*\big(\pi(P_{i+1})\big) \to 0$$

and

$$0 \to f^*\big(\pi(P_{i-1})\big) \to V \otimes f^*\big(\pi(P_i)\big) \to \Lambda^2 V \otimes f^*\big(\pi(P_{i+1})\big),$$

one can reconstruct f from the injection $f^*(\pi(P_{-1})) \hookrightarrow V \otimes f^*(\pi(P_0)) \cong V$. Thus, we can associate a non-zero vector subspace $W \subset V$ to each $f \in X_{\mathcal{A}}(k)$ and f can be reconstructed from the subspace W. We will show that $W \subset V$ gives a k-point iff $1 \leqslant \dim W \leqslant m$.

First suppose that $\dim W > m$ and W gives a k-point f. Then

$$f^*\big(\pi(P_1)\big) = \mathrm{coker}\big(\Lambda^{n-m-1} V \otimes W \to \Lambda^{n-m} V\big),$$

and the last space is zero since $\dim W > m$. But $f^*(\pi(P_1)) \neq 0$, a contradiction.

Now let $1 \leqslant \dim W = d \leqslant m$. Let $S = \{E_i\}$ be a geometric helix in $\mathcal{T}^{m,V}$ of period $n - m + 1$ such that $E_j = \mathcal{O}_{\mathbb{P}(V)}(j)$ for $m - n \leqslant j \leqslant 0$. Then the endomorphism $\mathbb{Z}$-algebra of S is equivalent to $\mathcal{A}^{m,V}$. We define the functor $\mathcal{F} : \mathcal{A} \to k\text{-Vect}$ by the formula

$$\mathcal{F}(i) = \mathrm{Hom}(E_i, \mathcal{O}_{\mathbb{P}(W)})^\vee = \mathrm{Hom}\big(E_i, \mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})\big)^\vee.$$

We put $f = (\mathcal{F}, \mathrm{id})$.

Now we prove that $f \in X_{\mathcal{A}^{m,V}}(k)$. By Lemma 9.5, it suffices to show that $\mathrm{Tor}_i(S_j, \mathcal{F}) = 0$ for $i > 0$, $j \in \mathbb{Z}$. Since the complexes

$$0 \to E_i \cong \mathcal{A}_{i+n,i}^{m,V!*} \otimes E_i \to \mathcal{A}_{i+n,i+1}^{m,V!*} \otimes E_{i+1} \to \cdots \to E_{i+n} \to 0$$

of objects in $D_{coh}^b(\mathbb{P}(V))$ have zero convolutions, it suffices to prove that

$$\mathrm{Hom}^i\big(E_j, \mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})\big) = \mathrm{Hom}^i\big(E_i, \mathcal{O}_{\mathbb{P}(W)}\big) = 0$$

for $i \neq 0$, $j \in \mathbb{Z}$. It is clear that this holds for $m - n \leqslant j \leqslant 0$. Further, we have a Koszul resolution of the sheaf $\mathcal{O}_{\mathbb{P}(W)}$ on $\mathbb{P}(V)$:

$$0 \to \Lambda^{n-d}(V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(d-n) \to \cdots \to (V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathcal{O}_{\mathbb{P}(V)}.$$

Thus, $\mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})$ is isomorphic to the complex

$$\cdots \to 0 \to \Lambda^{n-m}(V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(m-n) \to \cdots$$
$$\to (V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathcal{O}_{\mathbb{P}(V)} \to 0 \to \cdots.$$

Since the helix $\{E_i\}$ is geometric, we have $\mathrm{Hom}^i(E_j, \mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})) = 0$ for $i > 0$, $j \leqslant 0$.

Recall that $E_{j+n-m+1} = \Phi^{-1}(E_j)$, where $\Phi = F[m-n]$, and F is a Serre functor on $\mathcal{T}^{m,V}$. We have that

$$F^{-1}(K^\cdot) \cong \mathrm{pr}_l^{m,V}(K^\cdot \otimes \mathcal{O}_{\mathbb{P}(V)}(n)[1-n]), \tag{9.2}$$

where $\mathrm{pr}_l^{m,V} : D^b_{coh}(\mathbb{P}(V)) \to \mathcal{T}^{m,V}$ is the functor which is left adjoint to the inclusion $\iota : \mathcal{T}^{m,V} \to D^b_{coh}(\mathbb{P}(V))$.

Lemma 9.12. *The functor* $\mathrm{pr}_l^{m,V} : D^b_{coh}(\mathbb{P}(V)) \to \mathcal{T}^{m,V}$ *maps* $Ob(D^{\geqslant i}_{coh}(\mathbb{P}(V)))$ *to* $Ob(D^{\geqslant i-m+1}_{coh}(\mathbb{P}(V)) \cap \mathcal{T}^{m,V})$. *The functor* Φ^{-1} *preserves* $Ob(D^{\geqslant i}_{coh}(\mathbb{P}(V)) \cap \mathcal{T}^{m,V})$.

Proof. The second statement follows from the first one by the isomorphism (9.2). To prove the first statement, it suffices to note that

$$\mathrm{pr}_l^{m,V}(X) \cong L_{\mathcal{O}_{\mathbb{P}(V)}(1)} \cdot \cdots \cdot L_{\mathcal{O}_{\mathbb{P}(V)}(m-1)}(X)[m-1]. \qquad \square$$

Since $\mathrm{pr}_l^{m,V}(\mathcal{O}(i)) = 0$ for $i = 1, \ldots, m-1$, we have

$$\Phi^{-1}(\mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})) = \mathrm{pr}_l^{m,V}(\mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)}) \otimes \mathcal{O}_{\mathbb{P}(V)}(n))[1-m]$$
$$= \mathrm{pr}_l^{m,V}(\mathcal{O}_{\mathbb{P}(W)}(n))[1-m],$$

and the last object belongs to $Ob(D^{\geqslant 0}_{coh}(\mathbb{P}(V)) \cap \mathcal{T}^{m,V})$ by Lemma 9.12. Again by Lemma 9.12 we have that $\Phi^{-l}(\mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)}))$ lies in $Ob(D^{\geqslant 0}_{coh}(\mathbb{P}(V)) \cap \mathcal{T}^{m,V})$ for $l > 0$. Thus, we have

$$\mathrm{Hom}^i(E_{j-(n-m+1)k}, \mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})) = \mathrm{Hom}^i(E_j, \Phi^{-k}(\mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)}))) = 0$$

for $i < 0$, $n - m \leqslant j \leqslant 0$, and $k > 0$. Therefore,

$$\mathrm{Hom}^i(E_j, \mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})) = 0$$

for $i \neq 0$, $j \leqslant 0$.

To prove the same for $j > 0$, note that $\mathrm{Hom}^i(E_j, \mathrm{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)}))$ is the i-th cohomology of the complex

$$\cdots \to 0 \to \Lambda^{n-m}(V/W)^* \otimes \operatorname{Hom}^{n-m}\big(E_j, \mathcal{O}_{\mathbb{P}(V)}(m-n)\big) \to \cdots$$
$$\to (V/W)^* \otimes \operatorname{Hom}^{n-m}\big(E_j, \mathcal{O}_{\mathbb{P}(V)}(-1)\big) \to \operatorname{Hom}^{n-m}\big(E_j, \mathcal{O}_{\mathbb{P}(V)}\big) \to 0 \to \cdots,$$

where the left non-zero term is in degree zero. This complex is dual to the complex

$$\cdots \to 0 \to \operatorname{Hom}\big(\mathcal{O}_{\mathbb{P}(V)}, E_{j-n+m-1}\big) \to (V/W) \otimes \operatorname{Hom}\big(\mathcal{O}_{\mathbb{P}(V)}(-1), E_{j-n+m-1}\big) \to \cdots$$
$$\to \Lambda^{n-m}(V/W) \otimes \operatorname{Hom}\big(\mathcal{O}_{\mathbb{P}(V)}(m-n), E_{j-n+m-1}\big) \to 0 \to \cdots,$$

and the last one is isomorphic to the complex

$$\cdots \to 0 \to \operatorname{Hom}\big(E_{1-j}, \mathcal{O}_{\mathbb{P}(V)}(m-n)\big) \to (V/W) \otimes \operatorname{Hom}\big(E_{1-j}, \mathcal{O}_{\mathbb{P}(V)}(m-n+1)\big) \to \cdots$$
$$\to \Lambda^{n-m}(V/W) \otimes \operatorname{Hom}(E_{1-j}, \mathcal{O}_{\mathbb{P}(V)}) \to 0 \to \cdots. \tag{9.3}$$

The complex (9.3) calculates $\operatorname{Hom}^i(E_{1-j}, X_0)$, where X_0 is the complex

$$\cdots 0 \to \mathcal{O}_{\mathbb{P}(V)}(m-n) \to (V/W) \otimes \mathcal{O}_{\mathbb{P}(V)}(m-n+1) \to \cdots$$
$$\to \Lambda^{n-m}(V/W) \otimes \mathcal{O}_{\mathbb{P}(V)} \to 0 \to \cdots.$$

Thus, it remains to show that $\operatorname{Hom}^i(E_j, X_0) = 0$ for $i < 0$, $j \leqslant 0$.

If we prove this for $n - m \leqslant j \leqslant 0$, then the rest of the proof will be analogous to the proof of the same vanishing for $\operatorname{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})$ instead of X_0. So let $i < 0$, $n - m \leqslant j \leqslant 0$. We have the chain of isomorphisms

$$\operatorname{Hom}^i\big(\mathcal{O}_{\mathbb{P}(V)}(j), X_0\big)$$
$$\cong \operatorname{Hom}^{n-m+i}\big(\operatorname{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)}), \mathcal{O}_{\mathbb{P}(V)}(m-n-j)\big) \cong \operatorname{Ext}^{n-m+i}\big(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(V)}(m-n-j)\big)$$
$$\cong \operatorname{Ext}^{m-i-1}\big(\mathcal{O}_{\mathbb{P}(V)}(m-j), \mathcal{O}_{\mathbb{P}(W)}\big)^\vee = H^{m-i-1}\big(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(j-m)\big)^\vee,$$

and the last space is zero since $m - i - 1 > d - 1 = \dim \mathbb{P}(W)$.

Thus, f is indeed a k-point. Furthermore, we have that the complex

$$\cdots \to 0 \to \Lambda^{n-m}(V/W)^* \otimes \pi(P_{m-n}) \to \cdots \to (V/W)^* \otimes \pi(P_{-1}) \to \pi(P_0) \to 0 \to \cdots$$

is a resolution of $f_*(k)$. Thus, $f_*(k)$ is a perfect complex and it corresponds to the object $\operatorname{pr}_r^{m,V}(\mathcal{O}_{\mathbb{P}(W)})$ under the equivalence $\operatorname{Perf}(\operatorname{QMod}(\mathcal{A}^{m,V})) \cong \mathcal{T}^{m,V}$. The theorem is proved. $\quad\square$

It turns out that the embedding of k-points $\operatorname{Gr}(d, V)(k) \hookrightarrow \operatorname{NGr}(m, V)(k)$ for $1 \leqslant d \leqslant m$ can be extended to a morphism $\operatorname{Gr}(d, V) \to \operatorname{NGr}(m, V)$.

Proposition 9.13. *Let V be a finite-dimensional vector space and let $1 \leqslant d \leqslant m \leqslant \dim V = n$. Then there exists a natural morphism $f_{d,m,V} : \operatorname{Gr}(d, V) \to \operatorname{NGr}(m, V)$ such that the derived inverse image functor $\mathbf{L}f_{d,m,V}^*$ induces a full embedding*

$$\operatorname{Perf}(\operatorname{NGr}(m, V)) \to D_{coh}^b\big(\operatorname{Gr}(d, V)\big).$$

Proof. For each $W \in \mathrm{Gr}(d, V)$ denote by f_W the corresponding k-point of $\mathrm{NGr}(m, V)$. It is clear that there exist vector bundles $\mathcal{F}(i)$ on $\mathrm{Gr}(d, V)$ such that the fiber of $\mathcal{F}(i)$ over the point corresponding to W is naturally identified with $f_W^*(\pi(P_i))$ (in particular, $\mathcal{F}(-1)$ is a tautological bundle). So we have a natural functor $\mathcal{F} : \mathcal{A} \to \mathrm{QCoh}(\mathrm{Gr}(m, V))$. Also by Theorem 9.11 we have that the complexes of vector bundles

$$0 \to \mathcal{F}(i) \cong \mathcal{A}_{i+n,i}^{m,V!*} \otimes \mathcal{F}(i) \to \mathcal{A}_{i+n,i+1}^{m,V!*} \otimes \mathcal{F}(i+1) \to \cdots \to \mathcal{F}(i+n) \to 0$$

are acyclic in the fibers over closed points (if the residue field of a point is greater than k we can make an extension of scalars). Hence, these complexes are acyclic themselves. It follows from Lemma 9.5 that the pair $(\mathcal{F}, \mathrm{id})$ defines a map $f_{d,m,V} : \mathrm{Gr}(d, V) \to \mathrm{NGr}(m, V)$.

Further, for $m - n \leqslant j \leqslant 0$ we have that $\mathbf{L} f_{d,m,V}^*(\pi(P_j)) = f_{d,m,V}^*(\pi(P_j)) = S^{-j}E$, where E is a tautological bundle. The collection $(S^{n-m}E, \ldots, E, \mathcal{O}_{\mathrm{Gr}(d,V)})$ is a sub-collection of the full strong exceptional collection on $\mathrm{Gr}(d, V)$ constructed by Kapranov [9]. Moreover, the functor $\mathbf{L} f_{d,m,V}^*$ induces isomorphisms

$$\mathrm{Hom}\big(\pi(P_i), \pi(P_j)\big) \to \mathrm{Hom}\big(S^{-i}E, S^{-j}E\big)$$

for $m - n \leqslant i \leqslant j \leqslant 0$. Thus, the induced functor

$$\mathbf{L} f_{d,m,V}^* : \mathrm{Perf}\big(\mathrm{NGr}(m, V)\big) \to D_{coh}^b\big(\mathrm{Gr}(d, V)\big)$$

is a full embedding. $\square$

Notice that the full embedding $\mathbf{L} f_{1,m,V}^* : \mathrm{Perf}(\mathrm{NGr}(m, V)) \to D_{coh}^b(\mathbb{P}(V))$ coincides with the composition of the equivalence of Corollary 8.22 with the tautological embedding $\mathcal{T}^{m,V} \hookrightarrow D_{coh}^b(\mathbb{P}(V))$.

10. Completions of local rings of k-points

Let X be a presheaf of sets on the category Alg_k^{op} of noncommutative affine schemes. Let $x \in X(k)$ be a k-point. Define the functor $F_{X,x} : \mathrm{art} \to \mathrm{Sets}$ by the formula

$$F_{X,x}(\mathcal{R}) = \big\{ f \in X(\mathcal{R}) \mid X(\iota)(f) = x \big\},$$

where $\iota : \mathcal{R} \to k = \mathcal{R}/m$ is the projection.

Definition 10.1. Let X be a presheaf of sets on the category Alg_k^{op} of noncommutative affine schemes, and $x \in X(k)$ be a k-point. The completion of the local ring $\widehat{\mathcal{O}_x}$, if it exists, is defined as a pro-artinian algebra which pro-represents the functor $F_{X,x} : \mathrm{art} \to \mathrm{Sets}$.

We would like to describe the local rings of a k-point x_W of the noncommutative Grassmanian $\mathrm{NGr}(m, V)$ which correspond to a subspace $W \subset V$ of dimension m. Recall that by Lemma 9.10 and by Proposition 9.7 we have that $X_{\mathcal{A}^{m,V}}$ is (equivalent to) a presheaf of sets (trivial groupoids). Thus, the above definition is applicable to $X_{\mathcal{A}^{m,V}}$.

Proposition 10.2. *Let $W \subset V$ be a vector subspace, and $\dim V = n$, $1 \leqslant m = \dim W \leqslant n - 1$. Then the DG algebra $\mathbf{R}\operatorname{Hom}^{\cdot}(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(W)})$ is formal, and the graded algebra $\operatorname{Ext}^{\cdot}(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(W)})$ is isomorphic to the graded algebra*

$$C^{W,V} = \bigoplus_{d=0}^{n-m} \Lambda^d (V/W) \otimes S^d W^*.$$

Proof. Denote by $K_W^{\cdot}$ the Koszul resolution

$$0 \to \Lambda^{n-m}(V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(m-n) \to \cdots \to (V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathcal{O}_{\mathbb{P}(V)} \to 0$$

of the sheaf $\mathcal{O}_{\mathbb{P}(W)}$. Since

$$\operatorname{Ext}^k \big(\mathcal{O}_{\mathbb{P}(V)}(i), \mathcal{O}_{\mathbb{P}(V)}(j) \big) = 0$$

for $k > 0$, $m - n \leqslant i, j \leqslant 0$, the DG algebra $\mathbf{R}\operatorname{Hom}(\mathcal{O}_{\mathbb{P}(W)}, \mathcal{O}_{\mathbb{P}(W)})$ is quasi-isomorphic to the DG algebra $\operatorname{Hom}^{\cdot}_{\mathcal{O}_{\mathbb{P}(V)}}(K_W^{\cdot}, K_W^{\cdot})$.

Further, we also have that

$$\operatorname{Ext}^k \big(\mathcal{O}_{\mathbb{P}(V)}(i), \mathcal{O}_{\mathbb{P}(W)} \big) = 0$$

for $k > 0$, $m - n \leqslant i \leqslant 0$. Thus, we have the chain of isomorphisms of graded vector spaces:

$$H^{\cdot} \big(\operatorname{Hom}^{\cdot}_{\mathcal{O}_{\mathbb{P}(V)}} \big(K_W^{\cdot}, K_W^{\cdot} \big) \big) \cong H^{\cdot} \big(\operatorname{Hom}^{\cdot}_{\mathcal{O}_{\mathbb{P}(V)}} \big(K_W^{\cdot}, \mathcal{O}_{\mathbb{P}(W)} \big) \big) \cong \operatorname{Hom}^{\cdot}_{\mathcal{O}_{\mathbb{P}(V)}} \big(K_W^{\cdot}, \mathcal{O}_{\mathbb{P}(W)} \big)$$

$$\cong \bigoplus_{d=0}^{n-m} \Lambda^d (V/W) \otimes S^d W^* = C^{W,V}. \tag{10.1}$$

To prove the lemma, it suffices to construct a morphism of DG algebras

$$\varphi : C^{W,V} \to \operatorname{Hom}^{\cdot}_{\mathcal{O}_{\mathbb{P}(V)}} \big(K_W^{\cdot}, K_W^{\cdot} \big),$$

which induces the identity in cohomology (under the isomorphisms 10.1). To define φ, one needs to define its components

$$\varphi_{d,i} : \Lambda^d (V/W) \otimes S^d W^*$$
$$\to \operatorname{Hom} \big(\Lambda^{-i}(V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(i), \Lambda^{-i-d}(V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(i+d) \big)$$

for $0 \leqslant d \leqslant n - m$, $m - n \leqslant i \leqslant -d$. To do that, choose a decomposition $V = W \oplus U$. Then we have natural maps

$$\psi_{d,i} : \Lambda^d U \otimes S^d W^* \otimes \Lambda^{-i} U^* \to \Lambda^{-i-d} U^* \otimes S^d V^*.$$

We define $\varphi_{d,i}$ to be the maps corresponding to $\psi_{d,i}$ via the isomorphisms

$$\mathrm{Hom}\big(\Lambda^{-i}(V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(i), \Lambda^{-i-d}(V/W)^* \otimes \mathcal{O}_{\mathbb{P}(V)}(i+d)\big)$$
$$\cong \Lambda^{-i}(V/W) \otimes \Lambda^{-i-d}(V/W)^* \otimes S^d V^*,$$

and

$$\mathrm{Hom}_k\big(\Lambda^d(V/W) \otimes S^d W^*, \Lambda^{-i}(V/W) \otimes \Lambda^{-i-d}(V/W)^* \otimes S^d V^*\big)$$
$$\cong \mathrm{Hom}_k\big(\Lambda^d U \otimes S^d W^* \otimes \Lambda^{-i} U^*, \Lambda^{-i-d} U^* \otimes S^d V^*\big).$$

A straightforward checking shows that the map φ with components $\varphi_{d,i}$ satisfies the required properties. $\square$

Note that the graded algebra $C^{W,V} = \bigoplus_{d=0}^{n-m} \Lambda^d(V/W) \otimes S^d W^*$ is quadratic Koszul. Indeed, it coincides with "white" Manin product $\bigoplus_{d=0}^{n-m} \Lambda^d(V/W) \circ \bigoplus_{d \geqslant 0} S^d W^*$, and according to [14] the white product of quadratic Koszul algebras is again Koszul.

Thus, if we denote by $\hat{S}$ the dual of its (augmented) bar construction, then we have that the projection $\hat{S} \to H^0(\hat{S})$ is a quasi-isomorphism, and the algebra $H^0(\hat{S})$ is the completion of $C^{W,V!}$ with omitted grading. For convenience we will write C instead of $C^{W,V}$. We denote by $\mathfrak{A}$ the completion $\widehat{C^!}$ of the algebra $C^!$ with omitted grading.

Theorem 10.3. *Let $W \subset V$ be a subspace of dimension $1 \leqslant m \leqslant n-1$. Let $x_W \in X_{\mathcal{A}^{m,V}}(k)$ be the k-point of noncommutative Grassmanian $\mathrm{NGr}(m, V)$ corresponding to the subspace $W \subset V$. Then the algebra $\mathfrak{A}^{op} = \widehat{C}^{!\,op}$ coincides with the completion of the local ring of the k-point x_W.*

Proof. We will construct some morphism $u_W : Sp(\mathfrak{A}^{op}) \to \mathrm{NGr}(m, V)$ and then prove that it is the universal one. For convenience, we will write $\mathcal{A}$ instead of $\mathcal{A}^{m,V}$.

The construction of the morphism $u_W : Sp(\mathfrak{A}^{op}) \to \mathrm{NGr}(m, V)$. First we define an object $u_{W*}(\mathfrak{A})$ in the category $\mathrm{QMod}(\mathcal{A})$, together with a morphism of algebras $f : \mathfrak{A}^{op} \to \mathrm{End}(u_{W*}(\mathfrak{A}))$. Denote by $M_W^{\cdot}$ the complex of $\mathcal{A}^{op}$-modules

$$\cdots \to 0 \to \Lambda^{n-m}(V/W)^* \otimes P_{m-n} \to \cdots \to (V/W)^* \otimes P_{-1} \to P_0 \to 0 \to \cdots,$$

where P_0 is placed in degree zero. As in the proof of Lemma 10.2, the DG algebra $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(M_W^{\cdot})$ is quasi-isomorphic to C; moreover, each decomposition $V = W \oplus U$ gives a quasi-isomorphism $C \to \mathcal{B}$. The DG algebra $\mathcal{B}$ is naturally augmented: the augmentation sends each $\phi \in \mathcal{B}^0$ to its component $\phi_0 \in \mathrm{End}_{\mathcal{A}}(P_0) = k$. As usual, we have a natural element $\alpha \in \mathcal{MC}((B\bar{\mathcal{B}})^* \otimes \mathcal{B})$. We put

$$M_W^{\cdot} \otimes_\alpha (B\bar{\mathcal{B}})^* = \big(\mathcal{B} \otimes_\alpha (B\bar{\mathcal{B}})^*\big) \otimes_{\mathcal{B}} M_W^{\cdot}.$$

This is a DG $(\mathcal{A} \otimes (B\bar{\mathcal{B}})^*)^{op}$-module. Thus, $H^0(M_W^{\cdot} \otimes_\alpha (B\bar{\mathcal{B}})^*)$ is an $(\mathcal{A} \otimes \mathfrak{A})^{op}$-module.

We put

$$u_{W*}(\mathfrak{A}) = \pi\big(H^0\big(M_W^{\cdot} \otimes_\alpha (B\bar{\mathcal{B}})^*\big)\big) \in \mathrm{QMod}(\mathcal{A}).$$

The map $f : \mathfrak{A}^{op} \to \operatorname{End}(u_{W*}(\mathfrak{A}))$ is the projection by π of the $\mathfrak{A}^{op}$-action on $H^0(M_W^{\cdot} \otimes_\alpha (B\bar{\mathcal{B}})^*)$.

Define the functor $\mathcal{F} : \mathcal{A} \to \mathfrak{A}\text{-Mod}$ by the formula

$$\mathcal{F}(i) = \operatorname{Hom}_{\mathfrak{A}^{op}}\big(\operatorname{Hom}_{\operatorname{QMod}(\mathcal{A})}\big(\pi(P_i), u_{W*}(\mathfrak{A})\big), \mathfrak{A}\big).$$

Further, note that we have an isomorphism of DG $(B\bar{\mathcal{B}})^*$-modules

$$(B\bar{\mathcal{B}})^* \cong \operatorname{Hom}_{\mathcal{A}^{op}}^{\cdot}\big(P_0, M_W^{\cdot} \otimes_\alpha (B\bar{\mathcal{B}})^*\big).$$

Passing to H^0, we obtain an isomorphism

$$\mathfrak{A} \to \operatorname{Hom}_{\mathcal{A}^{op}}\big(P_0, H^0\big(M_W^{\cdot} \otimes_\alpha (B\bar{\mathcal{B}})^*\big)\big).$$

Composing it with the projection by π, we obtain the map

$$\sigma' : \mathfrak{A} \to \operatorname{Hom}_{\operatorname{QMod}(\mathcal{A})}\big(\pi(P_0), u_{W*}(\mathfrak{A})\big).$$

Applying the functor $\operatorname{Hom}_{\mathfrak{A}^{op}}(-, \mathfrak{A})$ to the map σ' we obtain the map

$$\sigma : \mathcal{F}(0) \to \mathfrak{A}.$$

Lemma 10.4.

a) *The map σ' (and hence σ) is an isomorphism, and $H^i(M_W^{\cdot} \otimes_\alpha (B\bar{\mathcal{B}})^*) = 0$ for $i \neq 0$.*
b) *The pair $(\mathcal{F}, \sigma)$ defines an object of $X_\mathcal{A}(\mathfrak{A}^{op})$, i.e. a morphism $\operatorname{Sp}(\mathfrak{A}^{op}) \to \operatorname{Proj}(\mathcal{A}) = \operatorname{NGr}(m, V)$.*

Proof. a) Choose a decomposition $V = W \oplus U$. As we already mentioned above, such a decomposition gives a quasi-isomorphism of DG algebras $C \to \mathcal{B}$. Composing the Koszul dual morphism $(B\bar{\mathcal{B}})^* \to \hat{S}$ with the projection $\hat{S} \to H^0(\hat{S}) = \mathfrak{A}$, we obtain the quasi-isomorphism $\beta : (B\bar{\mathcal{B}})^* \to \mathfrak{A}$. Thus, we may replace $M_W^{\cdot} \otimes_\alpha (B\bar{\mathcal{B}})^*$ by $M_W^{\cdot} \otimes_{\beta_*(\alpha)} \mathfrak{A}$. The last object is the complex of projective $(\mathcal{A} \otimes \mathfrak{A})^{op}$-modules. Furthermore, we have the isomorphism of complexes of $\mathcal{A}^{op}$-modules

$$\big(M_W^{\cdot} \otimes_{\beta_*(\alpha)} \mathfrak{A}\big) \otimes_{\mathfrak{A}} k \cong M_W^{\cdot}.$$

Further, according to the proof of Theorem 9.11 $H^i(M_W^{\cdot}) = 0$ for $i \neq 0$ It follows that $H^i(M_W^{\cdot} \otimes_{\beta_*(\alpha)} \mathfrak{A}) = 0$ for $i \neq 0$.

The space $\operatorname{Hom}_{\operatorname{QMod}(\mathcal{A})}(\pi(P_0), u_{W*}(\mathfrak{A}))$ is thus the zeroth cohomology group of the complex $\operatorname{Hom}_{\operatorname{QMod}(\mathcal{A})}^{\cdot}(\pi(P_0), \pi(M_W^{\cdot} \otimes_{\beta_*(\alpha)} \mathfrak{A}))$. It follows that σ' is an isomorphism.

b) First prove that $\operatorname{Ext}^k(\pi(P_i), u_{W*}(\mathfrak{A})) = 0$ for $k > 0$, $i \in \mathbb{Z}$, and $\operatorname{Hom}(\pi(P_i), u_{W*}(\mathfrak{A}))$ is a free finitely generated $\mathfrak{A}^{op}$-module for $i \in \mathbb{Z}$.

For $i \leqslant 0$, according to a), we have that $\operatorname{Ext}^k(\pi(P_i), u_{W*}(\mathfrak{A}))$ is the $k - th$ cohomology of the complex $\operatorname{Hom}^{\cdot}(\pi(P_i), \pi(M_W^{\cdot} \otimes_{\beta_*(\alpha)} \mathfrak{A}))$. It is concentrated in non-positive degrees, hence $\operatorname{Ext}^k(\pi(P_i), u_{W*}(\mathfrak{A})) = 0$ for $k > 0$. Further, it is bounded below complex of free finitely

generated $\mathfrak{A}^{op}$-modules with the only cohomology in degree zero. Thus, this cohomology $\mathrm{Hom}(\pi(P_i), u_{W*}(\mathfrak{A}))$ is free and finitely generated.

For $i > 0$, according to a), $\mathrm{Ext}^k(\pi(P_i), u_{W*}(\mathfrak{A}))$ is the $k - th$ cohomology of the complex $\mathrm{Ext}^{n-m}(\pi(P_i), \pi(M_{\check{W}} \otimes_{\beta^*(\alpha)} \mathfrak{A})[m - n])$. This is a complex of free finitely generated $\mathfrak{A}^{op}$-modules concentrated in degrees $0, 1, \ldots, n - m$. If we multiply it by the left $\mathfrak{A}$-module k, we will obtain the complex $\mathrm{Ext}^{n-m}(\pi(P_i), \pi(M_{\check{W}})[m - n])$. This complex computes $\mathrm{Ext}^k(\pi(P_i), x_{W*}(k))$. Thus, the only cohomology of the source complex is in degree zero and is a free finitely generated $\mathfrak{A}^{op}$-module.

Now we obtain that all $\mathcal{F}(i)$ are free (and hence flat) $\mathfrak{A}$-modules, and the complexes

$$\cdots \to 0 \to \mathcal{A}^{!*}_{k,k-n+m} \otimes \mathcal{F}(k-n+m) \to \cdots \to \mathcal{A}^{!*}_{k,k-1} \otimes \mathcal{F}(k-1) \to \mathcal{F}(k) \to 0 \to \cdots$$

are acyclic. By Lemma 9.5, it follows that the pair $(\mathcal{F}, \sigma)$ defines an object of $X_{\mathcal{A}}(\mathfrak{A}^{op})$ □

We define u_W as the pair $(\mathcal{F}, \sigma)$. If $\pi : \mathfrak{A} \to k$ is the projection, then by the very construction of u_W, we have $X_{\mathcal{A}}(\pi)(u_W) = x_W$.

Universality. Now we prove that the constructed morphism u_W is universal. More precisely, the pair $(\mathcal{F}, \sigma)$ gives the morphism of functors

$$\Phi : h_{\mathfrak{A}} \to F_{X_{\mathcal{A}}, x_W},$$

such that for each $f : \mathfrak{A} \to \mathcal{R}$, $\Phi(f) = X_{\mathcal{A}}(f)(u_W)$. And we prove that Φ is an isomorphism of functors.

In the proof of Theorem 9.11 we have already seen that each element $f \in X_{\mathcal{A}}(k)$ is uniquely determined by the injection $f^*(\pi(P_{-1})) \hookrightarrow V \otimes f^*(\pi(P_0)) = V$. The same observation evidently holds for arbitrary algebras R instead of k.

Choose again a decomposition $V = W \oplus U$. Choose some bases $(e_1, \ldots, e_m)$ of the vector space W, and $(e_{m+1}, \ldots, e_n)$ of the vector space U. Let R be some complete local augmented algebra with the (maximal) augmentation ideal $\mathfrak{m}$, and let $\pi : R \to k$ be the projection. Let $f = (\mathcal{F}'', \sigma'') \in X_{\mathcal{A}}(R)$ be some element such that $X_{\mathcal{A}}(\pi)(f) = x_W$. In particular, we have a natural isomorphism $\mathcal{F}''(-1) \otimes_A k \cong W$. Since the module $\mathcal{F}''(-1)$ is flat, it is free. Let $I : \mathcal{F}''(-1) \hookrightarrow V \otimes R$ be the structure injection. Clearly, there is a unique lift $(\widetilde{e_1}, \ldots, \widetilde{e_m})$ onto $\mathcal{F}''(-1)$ of the basis $(e_1, \ldots, e_m)$ such that

$$I(\widetilde{e_j}) = e_j \otimes 1 + \sum_{i=m+1}^{n} e_i \otimes y_{ij},$$

where $y_{ij} \in \mathfrak{m}$. Thus, for each $(\mathcal{F}'', \sigma'')$ as above we have associated a matrix (y_{ij}), $m + 1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, of elements in $\mathfrak{m}$. Moreover, $(\mathcal{F}'', \sigma'')$ can be reconstructed (up to a natural isomorphism) from this matrix.

If $g : R \to S$ is a morphism of complete local augmented algebras and (y_{ij}) is the matrix associated to $f \in X_{\mathcal{A}}(R)$, then the matrix associated to $X_{\mathcal{A}}(g)(f)$ equals to $(g(y_{ij}))$.

In the case $R = \mathfrak{A}^{op}$, and $f = u_W = (\mathcal{F}, \sigma)$, the associated matrix is the following:

$$x_{ij} = e_i^* \otimes e_j \in (V/W)^* \otimes W \subset \mathfrak{A}^{op}.$$

The elements x_{ij} are topological generators of the algebra $\mathfrak{A}$. The quadratic relations on them are the following:

$$[x_{ij}, x_{lj}] = 0 \quad \text{for } m+1 \leqslant i < l \leqslant n,\ 1 \leqslant j \leqslant m, \tag{10.2}$$

$$[x_{ij} + x_{ik}, x_{lj} + x_{lk}] = 0 \quad \text{for } m+1 \leqslant i < l \leqslant n,\ 1 \leqslant j < k \leqslant m. \tag{10.3}$$

It follows that the morphism Φ is injective. To prove the surjectivity and the theorem, it suffices to prove the following lemma.

Lemma 10.5. *Let R be a local complete augmented algebra and let $f = (\mathcal{F}'', \sigma'')$ be as above. Let (y_{ij}), $m+1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, be the associated matrix, $y_{ij} \in \mathfrak{m}$. Then the relations* (10.2), (10.3) *are satisfied for y_{ij} instead of x_{ij}.*

Proof. Using bases changes, we can reduce the problem to the only relation $[y_{n-1,m}, y_{n,m}] = 0$. We have

$$\mathcal{F}''(1) \cong \operatorname{coker}\big(\phi : \Lambda^{n-m-1} V \otimes \mathcal{F}''(-1) \to \Lambda^{n-m} V \otimes R\big).$$

Choose the basis of $(\widetilde{e_1}, \ldots, \widetilde{e_m})$ of the R^{op}-module $\mathcal{F}''(-1)$ as above. Then the explicit form of the map ϕ is the following:

$$\phi\big((e_{i_1} \wedge \cdots \wedge e_{i_{n-m-1}}) \otimes \widetilde{e}_j\big) = (e_{i_1} \wedge \cdots \wedge e_{i_{n-m-1}} \wedge e_j) \otimes 1$$
$$+ \sum_{i=m+1}^{n} (e_{i_1} \wedge \cdots \wedge e_{i_{n-m-1}} \wedge e_i) \otimes y_{ij}.$$

It is clear that the vector space $\mathcal{F}''(1) \otimes_R (R/\mathfrak{m})$ is one-dimensional and generated by the projection of the element $(e_{k+1} \wedge \cdots \wedge e_n) \otimes 1$. According to Nakayama lemma and the condition on $\mathcal{F}''(1)$ to be flat, the R^{op}-module $\mathcal{F}''(1)$ is freely generated by the projection of the same element.

Finally, we notice that

$$\phi\big((e_{k+1} \wedge \cdots \wedge e_{n-1}) \otimes \widetilde{e_m}\, y_{n-1,m} + (e_{k+1} \wedge \cdots \wedge e_{n-2} \wedge e_n) \otimes \widetilde{e_m}\, y_{n,m}$$
$$+ (e_{k+1} \wedge \cdots \wedge e_{n-2} \wedge e_m) \otimes \widetilde{e_m}\big)$$
$$= (e_{k+1} \wedge \cdots \wedge e_n) \otimes (y_{n,m}\, y_{n-1,m} - y_{n-1,m}\, y_{n,m}).$$

Therefore, the image of the RHS in $\mathcal{F}''(1)$ is zero, and hence $[y_{n-1,m}, y_{n,m}] = 0$. $\quad\square$

The theorem is proved. $\quad\square$

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