On the stability problem of Nicolai with variable cross-section and compressibility effect

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Abstract

We consider the problem of Nicolai on dynamic stability of an elastic cantilever rod loaded by an axial compressive force and tangential twisting torque in continuous formulation. The rod is assumed to be non-uniform, i.e., having variable cross-section with non-equal principal moments of inertia. New linear equations and boundary conditions are derived from nonlinear governing equations. These equations form the basis for analytical and numerical studies. The important new details of this formulation include the pre-twisting effect due to the torque and compressibility of the rod. General formulae for the influence of small geometrical imperfections to the stability region are derived and numerical examples are presented.

1. Introduction

In Nicolai (1928) the problem of elastic stability of a rod loaded by a tangential (non-conservative) twisting torque and an axial force was considered. For the rod with two equal principal moments of inertia he found that, for the cantilever boundary conditions, there is no static form of equilibrium except the trivial (straight) one. Then Nicolai studied the stability of the trivial form of equilibrium using a dynamic method and came to the conclusion that it is unstable for an arbitrary small magnitude of the twisting torque. This phenomenon is called the paradox of Nicolai. For the dynamic stability study he used a model having two degrees of freedom with a lumped mass attached to the free end of rod. In his next paper Nicolai (1929) considered the stability problem of a cantilever rod with two different principal moments of inertia loaded only by a tangential twisting torque. He introduced geometrical imperfections related to non-equal principal moments of inertia and used the same discrete model for the stability study as in Nicolai (1928). Then Nicolai came to the conclusion that the rod with non-equal principal moments of inertia possesses a finite non-zero critical twisting torque, i.e., geometrical imperfections have a stabilizing effect. As it is stated in the book Bolotin (1963), the works Nicolai (1928, 1929) were the first papers on stability problems with follower (non-conservative) loads.

The problem of Nicolai was reconsidered in the recent paper Seyranian and Mailybaev (2011). For linear vibrational systems of arbitrary degrees of freedom with potential forces having a multiple eigenfrequency it was shown that the addition of arbitrary small non-conservative positional forces generally destabilizes the system. The geometrical interpretation of this effect is that the paradox of Nicolai is related to a conical singularity of the stability boundary. The formulas for the stabilizing effect of small imperfections of the moments of inertia of the rod and small internal and external damping forces were derived. However, the pre-twisting effect due to tangential torque was not taken into account.

In this paper we consider the problem of Nicolai on dynamic stability of an elastic cantilever rod loaded by an axial compressive force and twisting tangential torque in continuous formulation. The rod is assumed to be non-uniform, i.e., having variable cross-section with non-equal principal moments of inertia. New linear equations and boundary conditions are derived from nonlinear governing equations. These equations form the basis for analytical and numerical studies. The important new details of this formulation include the pre-twisting effect of the rod due to the torque and the compressibility of the rod axis. General formulae for the influence of small geometrical imperfections to the stability region are derived and numerical examples are presented. Thus, the main novelty of the paper is in the continuous formulation of the problem of Nicolai for a non-uniform rod with the compressibility effect of the rod axis. Concerning the main results we emphasize that a general formula for the influence of small geometrical imperfections of a non-uniform rod to its stability is derived and numerical examples for different cases are presented. An important
conclusion is that the pre-twisting effect does not influence the first-order instability condition.

2. Governing equations

Consider an elastic naturally straight cantilever rod having variable cross-sectional area and different principal moments of inertia $I_{22}, I_{33}$ (Fig. 1). Our intention is to derive the governing equations of motion if pre-twisting and compressibility effects are taken into account.

Let $x_{10}, x_{20}$ and $x_{30}$ be a fixed right handed rectangular Cartesian coordinate system with the origin at the clamped end $O$ (see Fig. 1). The axis $x_{10}$ coincides with the rod axis in the natural configuration and the axes $x_{20}, x_{30}$ are in the direction of the principal axes of the cross-section at $O$. The unit vectors along the $x_{10}, x_{20}$ and $x_{30}$ axes are denoted by $\mathbf{e}_{10}, \mathbf{e}_{20}$ and $\mathbf{e}_{30}$, respectively (see Fig. 1). A constant compressive force and a follower torsional torque, acting on the free end of the cantilever (see Fig. 1) are given by $P = -Pe_{10}$ and $\mathbf{M} = \mathbf{Le}_{10}$, respectively.

Let $S$ and $l$ be the arc-length of the rod axis and the length of the rod, in the natural configuration, respectively. We introduce a local right handed Cartesian coordinate system $x_{1}, x_{2}, x_{3}$ at an arbitrary point $C$ on the rod axis (see Fig. 1). The axes $x_{1}, x_{2}, x_{3}$ are chosen in such a way that the axis $x_{1}$ is oriented along the normal to the cross-section while the axes $x_{2}, x_{3}$ are oriented along the principal axes of the cross-section. The local coordinate system changes its orientation along with the cross-section.

Let $\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}$ be the unit vectors along $x_{1}, x_{2}, x_{3}$ axes, respectively. Now the configuration of the rod can be described by a vector function $\mathbf{r}(S, t) = x_{10}\mathbf{e}_{10} + x_{20}\mathbf{e}_{20} + x_{30}\mathbf{e}_{30}$ specifying the position of a point on the rod axis and the local coordinate system $x_{1}, x_{2}, x_{3}$ determining the orientation of the cross-section (see Antman, 1995; Domokos and Healey, 2005). The orientation of the system $\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}$ with respect to the unit vectors parallel with $\mathbf{e}_{10}, \mathbf{e}_{20}, \mathbf{e}_{30}$ and passing through an arbitrary point on the rod axis is given by the Euler type of angles. In this case we choose 1–3–2 Euler angles called ship angles, see Lurie, 2002; Svetlitsky, 1987 that bring $\mathbf{e}_{10}, \mathbf{e}_{20}, \mathbf{e}_{30}$ to $\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}$. The sequence of rotations starts with the rotation of an amount $\theta_{1}$ about the $x_{10}$ axis. The second rotation is about the $\xi$ axis for an amount $\theta_{2}$ (see Fig. 2). The last rotation is of an amount $\theta_{3}$ about the $x_{2}$ axis. All rotations are performed counterclockwise.

By using 1–3–2 Euler angles the relation between $\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}$ and $\mathbf{e}_{10}, \mathbf{e}_{20}, \mathbf{e}_{30}$ becomes

$$
\begin{align*}
\mathbf{e}_{1} &= \begin{bmatrix} c_{2}c_{3} & c_{1}c_{2}s_{3} + s_{1}s_{2}c_{1}c_{3} & c_{2}s_{2}s_{3} - c_{1}s_{1}s_{3}c_{1}c_{2} \end{bmatrix} \mathbf{e}_{10} \\
\mathbf{e}_{2} &= \begin{bmatrix} -s_{3} & c_{1}c_{3} & c_{2}s_{1}s_{3}c_{1} + s_{1}s_{2}s_{3}c_{1}c_{2} \end{bmatrix} \mathbf{e}_{20} \\
\mathbf{e}_{3} &= \begin{bmatrix} s_{2}s_{3} - c_{1}s_{1}s_{3}c_{1}c_{2} & c_{2}c_{3} & c_{1}s_{2}s_{3} + s_{1}s_{2}s_{3}c_{1}c_{2} \end{bmatrix} \mathbf{e}_{30}
\end{align*}
$$

(1)

where $s_{k} = \sin \theta_{k}, c_{k} = \cos \theta_{k}, \ldots, c_{3} = \cos \theta_{1}$. Next we introduce the “angular velocity vector” $\omega$ (see Love, 1944; Svetlitsky, 1987) to describe deformation. This vector, expressed in terms of 1–3–2 Euler angles, reads

$$
\omega = \dot{\theta}_{1}\mathbf{e}_{10} + \dot{\theta}_{2}\mathbf{k} + \dot{\theta}_{3}\mathbf{e}_{2},
$$

(2)

where $(\cdot)' = \partial/(\partial \xi)$ and $\mathbf{k}$ is the unit vector along $\xi$ axis. Note that the angular velocity vector in the undeformed configuration $\omega_{0}$ is zero, i.e. $\omega_{0} = 0$.

Let $c, \omega_{1}, \omega_{2}, \omega_{3}$ be the strain of the rod axis and the components of $\omega$, in the local system $\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}$, respectively. Since we analyze a compressible unsharable rod these quantities are strains (see Antman, 1995; Eliseyev, 1988; Atanackovic, 1997). In scalar form (2) becomes

$$
\begin{align*}
\omega_{1} &= \theta_{1}c_{2}c_{3} - \theta_{2}s_{2}, \\
\omega_{2} &= \theta_{2}c_{3} - \theta_{1}s_{3}, \\
\omega_{3} &= \theta_{1}c_{2}s_{2} + \theta_{2}s_{2}c_{1},
\end{align*}
$$

(3)

or

$$
\begin{align*}
\theta_{1}' &= \frac{c_{2}}{c_{3}}\omega_{1} + \frac{s_{2}}{c_{3}}\omega_{3}, \\
\theta_{2}' &= \frac{c_{3}}{c_{3}}\omega_{1} + \frac{s_{2}}{c_{3}}\omega_{2} + \frac{s_{3}}{c_{3}}\omega_{3}, \\
\theta_{3}' &= -s_{2}\omega_{1} + c_{2}\omega_{3}.
\end{align*}
$$

(4)

If we denote the contact force by $\mathbf{F} = F_{1}\mathbf{e}_{1} + F_{2}\mathbf{e}_{2} + F_{3}\mathbf{e}_{3}$ and the contact moment by $\mathbf{M} = M_{1}\mathbf{e}_{1} + M_{2}\mathbf{e}_{2} + M_{3}\mathbf{e}_{3}$ the constitutive equations can be assumed in the form see Simites and Hodges, 2006; Atanackovic, 1997
where $A_{22}$ and $A_{33}$ are the bending rigidities, $A_{11}$ is the torsional rigidity and $B_{11}$ is the axial rigidity. In what follows we assume that all the rigidities are greater than zero. In usual engineering notation we have $A_{11} = G_1$, $A_{22} = E_2$, $A_{33} = E_3$, $B_{11} = E_1$ where $A$ is the cross-sectional area, $G$ is the shear modulus, $E$ is the elasticity modulus, $I_1$ is the torsional moment of inertia of the cross-section, $I_{22}$ and $I_{33}$ are the principal moments of inertia of the cross-section. Note that the strain of the rod axis is defined by $\varepsilon = ds/ds - 1$, where $s$ is the arc-length of the rod axis in the current position. Now, since there is no shear effect, the following geometrical relation also holds

$$r' = (1 + \varepsilon) e_1.$$  

In order to complete the analysis we need the equations of motion. Let $\rho_e$ and $\rho$ be masses per unit length of the rod in undeformed (unloaded) and current configuration of the rod, respectively. Then the law of conservation of mass implies

$$\rho_d \dot{d} + \rho d = \rho ds.$$  

Using (7) and the D’Alembert Principle for the infinitesimal part of the rod we get

$$F' - \rho_e r' = 0, \quad M' = -r' \times F,'$$  

where $t$ is time and $(\cdot) = \partial/\partial t(\cdot)$. Using (1), (2), (6), (8), (5) and the relation for the "angular velocity vector" $\omega$

$$\frac{d\omega}{dt} = \omega \times e_1,$$

we get the following system of equations

$$F_1 + F_2 M_{22} - F_2 M_{33} = \rho_0 [(c_2 c_3 x_{10} + (c_1 c_2 s_3 + s_1 s_2)) y_{20} + (c_2 s_3 s_1 - c_1 s_2) y_{30}],$$

$$F_2 - F_2 M_{11} + F_1 M_{33} = \rho_0 [-s_2 y_{10} + c_1 c_2 y_{30} + c_1 s_2 y_{30}],$$

$$F_3 + F_2 M_{11} - F_1 M_{22} = \rho_0 (c_2 y_{10} + (c_1 y_{20} - c_2 s_3 s_1) y_{30}) + (s_1 s_3 s_2 + c_1 c_2) y_{30},$$

$$M'_1 + M'_2 \frac{M_{33}}{A_{33}} - M'_2 \frac{M_{33}}{A_{33}} = 0,$$

$$M'_2 - M'_1 \frac{M_{33}}{A_{33}} + M'_3 \frac{M_{11}}{A_{11}} - F_3 \left(1 + \frac{F_1}{B_{11}}\right),$$

$$y'_2 = (c_1 c_2 s_3 + s_1 s_2) y_{20} + (c_2 s_3 s_1 - c_1 s_2) y_{30} = x_{20},$$

$$y'_3 = (s_1 c_2 s_3 - c_1 s_2) y_{30} = x_{30},$$

subject to the boundary conditions

$$x_{10}(0, t) = x_{20}(0, t) = x_{20}(0, t) = 0, \quad \theta_1(0, t) = \theta_2(0, t) = 0,$$

$$\theta_1(0, t) = 0, M_1(0, t) = M_2(0, t) = M_3(0, t) = 0,$$

$$F(l, t) = -P \cdot e_{10} \Rightarrow \begin{cases} F_1 = -P c_2 c_3, \\ F_2 = P s_3, \\ F_3 = -P c_2 s_3. \end{cases}$$

Note that (1), (2), (6), (8), (5) are the same as the Eqs. (3), (4), (7), (8), (6) from Glavardanov et al. (2009) if inertial forces and undeformed curvatures are neglected and $A_{22} \neq A_{33}$. However, we note that the problem treated in Glavardanov et al. (2009) significantly differs from this one. Namely, in Glavardanov et al. (2009) the cross-section of the rod is not variable and the principal moments of inertia are equal, the boundary conditions are different, the applied twisting torque is a conservative load and stability is investigated by the use of the adjacent equilibrium method.

In order to simplify stability analysis we transform the nonlinear system (9), (10) by introducing the notation

$$\theta_1^0 = \int_0^s \frac{L}{A_{11}} \, d\xi,$$

and the new variables

$$y_2 = x_{20} \cos \theta_1^0 + x_{30} \sin \theta_1^0, \quad y_3 = -x_{30} \sin \theta_1^0 + x_{30} \cos \theta_1^0$$

In that way we get the nonlinear system

$$F_1 + F_2 M_{22} - F_2 M_{33} = \rho_0 [(c_2 c_3 x_{10} + (c_1 c_2 s_3 + s_1 s_2)) y_{20} + (c_2 s_3 s_1 - c_1 s_2) y_{30}],$$

$$F_2 - F_2 M_{11} + F_1 M_{33} = \rho_0 [-s_2 y_{10} + c_1 (c_1 y_{20} + s_1 s_2) y_{30}],$$

$$F_3 + F_2 M_{11} - F_1 M_{22} = \rho_0 (c_2 y_{10} + (c_1 y_{20} - c_2 s_3 s_1) y_{30}) + (s_1 s_3 s_2 + c_1 c_2) y_{30},$$

$$M'_1 + M'_2 \frac{M_{33}}{A_{33}} - M'_2 \frac{M_{33}}{A_{33}} = 0,$$

$$M'_2 - M'_1 \frac{M_{33}}{A_{33}} + M'_3 \frac{M_{11}}{A_{11}} - F_3 \left(1 + \frac{F_1}{B_{11}}\right),$$

$$\theta'_1 = c_1 c_2 s_3 + s_1 s_2,$$

$$\theta'_2 = c_1 c_2 s_3 + s_1 s_2,$$

$$\theta'_3 = -s_2 y_{10} + c_1 c_2 y_{30},$$

$$x'_2 = (c_1 c_2 s_3 + s_1 s_2) y_{20} + (c_2 s_3 s_1 - c_1 s_2) y_{30} = x_{20},$$

$$x'_3 = (s_1 c_2 s_3 - c_1 s_2) y_{30} = x_{30},$$

where $x_{10}(0, t) = y_{20}(0, t) = y_{30}(0, t) = 0, \quad \theta_1(0, t) = \theta_2(0, t) = \theta_3(0, t) = 0,$

$$M_1(l, t) = L, \quad M_2(l, t) = M_3(l, t) = 0,$$

$$F(l, t) = -P \cdot e_{10} \Rightarrow \begin{cases} F_1 = -P c_2 c_3, \\ F_2 = P s_3, \\ F_3 = -P c_2 s_3. \end{cases}$$

Next, we note that a solution to (9), (10) reads

$$y_2^0 = y_3^0 = \theta_1^0 = \theta_2^0 = \theta_3^0 = 0, \quad M_2^0 = M_3^0 = 0, \quad F_1^0 = -P, \quad x_{10}^0 = x_{20}^0 = 0, \quad x_{30}^0 = 0, \quad \theta_1^0 = 0, \quad \theta_2^0 = 0, \quad \theta_3^0 = 0,$$

$$F_1^0 = -P, \quad x_{10}^0 = \int_0^L \left(1 - \frac{P}{B_{11}}\right) \, d\xi, \quad \theta_1^0 = \int_0^L \frac{L}{A_{11}} \, d\xi, \quad M_2^0 = M_3^0 = 0.$$

If we perturb the solution (15) we get

$$F_1 = -P + \Delta F_1, \quad F_2 = \Delta F_2, \quad F_3 = \Delta F_3, \quad M_1 = L + \Delta M_1, \quad M_2 = \Delta M_2, \quad M_3 = \Delta M_3, \quad \theta_1 = \theta_1^0 + \Delta \theta_1, \quad \theta_2 = \Delta \theta_2, \quad \theta_3 = \Delta \theta_3,$$

$$x_{10} = x_{10}^0 + \Delta x_{10}, \quad y_2 = \Delta y_2, \quad y_3 = \Delta y_3,$$
Substituting (16) into (13), (14), omitting $\Delta$ in front of $y_2, y_3, \theta_2, \theta_3, M_2, M_3, F_2, F_3$ and linearizing we obtain
\[
\Delta F'_1 = \rho_0 \Delta x_{11}, \\
F'_2 - F_3 \frac{L}{A_{11}} M'_3 = \rho_0 \Delta y_2, \\
F'_3 + F_2 \frac{L}{A_{11}} M'_2 = \rho_0 \Delta y_3, \\
\Delta M'_1 = 0, \\
M'_2 + \left( \frac{1}{A_{33}} - \frac{1}{A_{11}} \right) LM_3 = F_3 (1 - \frac{P}{B_{11}}), \\
M'_4 - \left( \frac{1}{A_{22}} - \frac{1}{A_{11}} \right) LM_2 = -F_2 (1 - \frac{P}{B_{11}}), \\
\Delta \theta'_1 = \Delta M_1, \\
\theta'_2 = \frac{L}{A_{11}} \theta_2 + M_3, \\
\theta'_3 = -\frac{L}{A_{11}} \theta_2 + M_3 \\
\Delta x_{10} = \Delta M_1, \\
y'_2 = \theta_2 \left( 1 - \frac{P}{B_{11}} \right) + \frac{L}{A_{11}} y_3, \\
y'_3 = -\theta_2 \left( 1 - \frac{P}{B_{11}} \right) - \frac{L}{A_{11}} y_2.
\]

The boundary conditions corresponding to (17) read
\[
\Delta x_{10}(0, t) = y_2(0, t) = y_3(0, t) = 0, \\
\Delta \theta_1(0, t) = \theta_2(0, t) = \theta_3(0, t) = 0, \\
\Delta M_1(l, t) = M_2(l, t) = M_3(l, t) = 0, \\
\Delta F_1(l, t) = 0, \\
F_2(l, t) = P \theta_3(t, l), \\
F_3(l, t) = -P \theta_2(l, t).
\]

From Eqs. (17) we would like to derive differential equations on deflection functions $y_2$ and $y_3$. For this reason from (17) we express
\[
F_3 = \frac{M'_2 + \left( \frac{1}{A_{33}} - \frac{1}{A_{11}} \right) LM_3}{1 - \frac{P}{B_{11}}}, \\
F_2 = -\frac{M'_4 - \left( \frac{1}{A_{22}} - \frac{1}{A_{11}} \right) LM_2}{1 - \frac{P}{B_{11}}},
\]
where it is supposed that $1 - \frac{P}{B_{11}} > 0$. This restriction is satisfied due to the assumption of small strain. Substituting (19) into (17) we obtain
\[
\rho_0 \Delta y_2 = \left[ M'_2 - \left( \frac{1}{A_{33}} - \frac{1}{A_{11}} \right) LM_3 \right] + \frac{L}{A_{11}} M'_4 - \left( \frac{1}{A_{22}} - \frac{1}{A_{11}} \right) LM_2, \\
\rho_0 \Delta y_3 = \left[ M'_4 - \left( \frac{1}{A_{22}} - \frac{1}{A_{11}} \right) LM_2 \right] - \frac{L}{A_{11}} M'_2 + \left( \frac{1}{A_{33}} - \frac{1}{A_{11}} \right) LM_3 + \frac{p M_4}{A_{33}},
\]
\[
\theta_2 = -\frac{y'_2 + \frac{P}{B_{11}} y_2}{1 - \frac{P}{B_{11}}}, \\
\theta_3 = \frac{y'_3 - \frac{P}{B_{11}} y_3}{1 - \frac{P}{B_{11}}}.
\]

Thus, using (17) and (21) we get
\[
M_2 = A_{22} \left( \frac{L}{A_{11}} \theta_2 \right) = -A_{22} \left[ \left( y'_2 + \frac{P}{B_{11}} y_2 \right) - \frac{L}{A_{11}} \left( y'_3 - \frac{P}{B_{11}} y_3 \right) \right], \\
M_3 = A_{33} \left( \frac{L}{A_{11}} \theta_3 \right) = A_{33} \left[ \left( y'_3 - \frac{P}{B_{11}} y_3 \right) - \frac{L}{A_{11}} \left( y'_2 + \frac{P}{B_{11}} y_2 \right) \right],
\]
and then from (19) and (22) it follows
\[
F_3 = \left\{ A_{33} \left[ \frac{y'_3 + \frac{P}{B_{11}} y_3}{1 - \frac{P}{B_{11}}} \right] + \frac{L}{A_{11}} \left( \frac{y'_3 - \frac{P}{B_{11}} y_3}{1 - \frac{P}{B_{11}}} \right) \right\}, \\
F_2 = -A_{22} \left[ \frac{y'_2 + \frac{P}{B_{11}} y_2}{1 - \frac{P}{B_{11}}} \right].
\]

Substituting (22) into (20) and taking only terms of first order in $L$ we obtain the differential equations expressed in terms of the deflection functions $y_2$ and $y_3$
\[
\left\{ -A_{33} \left[ \frac{y'_3 + \frac{P}{B_{11}} y_3}{1 - \frac{P}{B_{11}}} \right] + \frac{L}{A_{11}} \left( \frac{y'_3 - \frac{P}{B_{11}} y_3}{1 - \frac{P}{B_{11}}} \right) \right\} + \frac{L}{A_{11}} \left[ \frac{y'_2 + \frac{P}{B_{11}} y_2}{1 - \frac{P}{B_{11}}} \right] = \rho_0 \Delta y_2, \\
\left\{ -A_{22} \left[ \frac{y'_2 + \frac{P}{B_{11}} y_2}{1 - \frac{P}{B_{11}}} \right] + \frac{L}{A_{11}} \left( \frac{y'_2 - \frac{P}{B_{11}} y_2}{1 - \frac{P}{B_{11}}} \right) \right\} + \frac{L}{A_{11}} \left[ \frac{y'_3 + \frac{P}{B_{11}} y_3}{1 - \frac{P}{B_{11}}} \right] = \rho_0 \Delta y_3.
\]

Using (21) (23), the corresponding boundary conditions (18) now become
\[
y_2(0, t) = y_3(0, t) = 0, \\
y_2(0, t) = y_3(0, t) = 0, \\
A_{22} \left[ \frac{y'_2 + \frac{P}{B_{11}} y_2}{1 - \frac{P}{B_{11}}} \right] + \frac{L}{A_{11}} \left( \frac{y'_2 - \frac{P}{B_{11}} y_2}{1 - \frac{P}{B_{11}}} \right) (l, t) = 0, \\
A_{33} \left[ \frac{y'_3 + \frac{P}{B_{11}} y_3}{1 - \frac{P}{B_{11}}} \right] - \frac{L}{A_{11}} \left( \frac{y'_3 - \frac{P}{B_{11}} y_3}{1 - \frac{P}{B_{11}}} \right) (l, t) = 0, \\
A_{22} \left[ \frac{y'_2 + \frac{P}{B_{11}} y_2}{1 - \frac{P}{B_{11}}} \right] + \frac{L}{A_{11}} \left( \frac{y'_2 - \frac{P}{B_{11}} y_2}{1 - \frac{B_{11}}{11}} \right) (l, t) = 0, \\
A_{33} \left[ \frac{y'_3 + \frac{P}{B_{11}} y_3}{1 - \frac{P}{B_{11}}} \right] + \frac{L}{A_{11}} \left( \frac{y'_3 - \frac{P}{B_{11}} y_3}{1 - \frac{P}{B_{11}}} \right) (l, t) = 0.
\]

Defining matrices
\[
N = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\
J = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \\
q = \begin{bmatrix} y_2 \\ y_3 \end{bmatrix},
\]
Eqs. (24) and (25) yield
\[
\begin{bmatrix} \frac{y'_2 + \frac{P}{B_{11}} y_2}{1 - \frac{P}{B_{11}}} \left( \frac{y'_3 - \frac{P}{B_{11}} y_3}{1 - \frac{P}{B_{11}}} \right) \right] + L \left( \frac{y'_3 - \frac{P}{B_{11}} y_3}{1 - \frac{P}{B_{11}}} \right) (l, t) = 0, \\
- L \left( \frac{y'_2 - \frac{P}{B_{11}} y_2}{1 - \frac{B_{11}}{11}} \right) (l, t) = 0.
\]
subject to
\[
\begin{align*}
q(0, t) &= 0, \\
q'(0, t) &= 0,
\end{align*}
\]
\[
\left\{ \left( \frac{q' - \frac{L}{A_{11}} Nq'}{1 - \frac{L}{B_{11}}} \right)' - \frac{L}{A_{11}} \frac{Nq'}{1 - \frac{L}{B_{11}}} \right\}(l, t) = 0,
\]
\[
\left\{ \left[ J \left( \frac{q' - \frac{L}{A_{11}} Nq'}{1 - \frac{L}{B_{11}}} \right)' - \frac{L}{A_{11}} \frac{Nq'}{1 - \frac{L}{B_{11}}} \right]' + P \left( \frac{q' - \frac{L}{A_{11}} Nq'}{1 - \frac{L}{B_{11}}} \right)' \right\}(l, t) = 0.
\]
(28)

In concluding this section we note that (27) and (28) are the new linear equations of a non-uniform pre-twisted compressible cantilever rod loaded by an axial force and tangential torque. It is important to emphasize that these equations are derived from nonlinear governing equations. In what follows the derived equations are the basis of stability analysis.

3. Stability of non-uniform inextensible pre-twisted rod loaded by axial force and tangential torque

Our main goal in this section is to analyze the influence of small geometric imperfections on the stability of a pre-twisted cantilever rod. The governing equation for a non-uniform inextensible pre-twisted rod follows from (27) and (28) if we set \(1/B_{11} = 0\). Thus, it reads
\[
\rho_s q + \left\{ \left[ J \left( \frac{q' - \frac{L}{A_{11}} Nq'}{1 - \frac{L}{B_{11}}} \right)' - \frac{L}{A_{11}} \frac{Nq'}{1 - \frac{L}{B_{11}}} \right]' + L Nq'' - \frac{L}{A_{11}} Njq' \right\}'
\]
\[
- \frac{L}{A_{11}} Njq'' + P \left( \frac{q' - \frac{L}{A_{11}} Nq'}{1 - \frac{L}{B_{11}}} \right)' - \frac{L}{A_{11}} Nq'' = 0.
\]
(29)

subject to
\[
q(0, t) = 0, \\
q'(0, t) = 0,
\]
\[
\left\{ \left( \frac{q' - L Nq'}{A_{11}} \right)' - \frac{L}{A_{11}} Nq' \right\}(l, t) = 0.
\]
(30)

It is worth noting that the above Eq. (29) is a generalization of Eq. (4.1) from Seyranian and Mailybaev (2011). The difference comes from the pre-twisting (torsional) effect. In order to analyze the stability of the trivial solution \(u = 0\) recall that this is a non-conservative problem. For this reason we substitute \(q = u(S)e^{\text{rot}} = (u_1(S), u_2(S))^T e^{\text{rot}}\) into (29), (30) to get the eigenvalue problem
\[
\rho_s \frac{d}{dS} \left( \frac{\frac{d}{dS} u(S)e^{\text{rot}}}{A_{11}} \right) + \left[ \left( \frac{\frac{d}{dS} u(S)e^{\text{rot}}}{A_{11}} \right)' - \frac{L}{A_{11}} \frac{u(S)e^{\text{rot}}}{1 - \frac{L}{B_{11}}} \right]' + L \frac{u(S)e^{\text{rot}}''}{1 - \frac{L}{B_{11}}} - \frac{L}{A_{11}} \frac{u(S)e^{\text{rot}}'}{1 - \frac{L}{B_{11}}} = \mu \rho_s u,
\]
(31)

subject to
\[
u(S) = 0, \\
u'(S) = 0,
\]
\[
\left\{ \left( \frac{\frac{d}{dS} u(S)e^{\text{rot}}}{A_{11}} \right)' - \frac{L}{A_{11}} \frac{u(S)e^{\text{rot}}}{1 - \frac{L}{B_{11}}} \right\}(l) = 0,
\]
\[
\left\{ \left[ J \left( \frac{\frac{d}{dS} u(S)e^{\text{rot}}}{A_{11}} \right)' - \frac{L}{A_{11}} \frac{u(S)e^{\text{rot}}}{1 - \frac{L}{B_{11}}} \right]' + P \left( \frac{\frac{d}{dS} u(S)e^{\text{rot}}}{A_{11}} \right)' \right\}(l) = 0,
\]
(32)

where \(\mu = \omega^2\) is an eigenvalue.

Next, we analyze a uniform rod with equal principal moments of inertia of the cross-section \((A_{12} = A_{13} = \bar{J}_0 = \text{const})\). The rod is supposed to be loaded by the force \(P\) whose magnitude does not exceed the critical Euler value, i.e. \(P < \frac{\pi^2 E I}{L^2}\). The corresponding eigenvalue problem follows from (31) and (32) and reads
\[
\left[ J u'' \right]' + Pu'' = \mu \rho_s u,
\]
subject to
\[
u(0) = 0, \quad \nu'(0) = 0, \quad \nu''(l) = 0,
\]
\[
\left\{ \left[ J u'' \right]' + Pu' \right\}(l) = 0
\]
(33)

where
\[
J = J_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = J_0 \mathbf{I}, \quad A_{11} = \bar{A}_{11} = \text{const}, \quad \rho_s = \bar{\rho}_s = \text{const}, \quad \omega = \Omega_c.
\]
(35)

In order to find eigenvalues and eigenmodes corresponding to the eigenvalue problem (33)–(35) one solves
\[
J_0 w'' + Pw'' = \omega^2 \bar{\rho}_s w,
\]
subject to
\[
w(0) = w'(0) = w''(l) = 0, \quad J_0 w''(l) + Pw'(l) = 0.
\]
(37)

It is known (see Seyranian and Mailybaev, 2011) that the eigenvalues in the eigenvalue problem (33)–(35), are double semi-simple (for terminology see Seyranian and Mailybaev, 2003) while the eigenmodes are given by
\[
u_1 = \begin{bmatrix} w \\ 0 \end{bmatrix}, \quad \nu_2 = \begin{bmatrix} 0 \\ w \end{bmatrix}.
\]
(38)

Similar problem with the analysis of a double zero eigenvalue was studied in a recent paper (see Luongo and D’Annibale, 2013). If we introduce the following dimensionless quantities
\[
p = \frac{p L^2}{J_0}, \quad \Omega = \frac{\omega}{\sqrt{\frac{E A}{J_0}}}, \quad r_1 = \sqrt{\frac{P^2}{2J_0} + 4\Omega^2}, \quad r_2 = -\sqrt{\frac{P^2}{2J_0} + 4\Omega^2},
\]
(39)
a solution to (36) and (37) reads
\[
w(S) = C \left\{ r_1^2 \cos r_1 + r_2^2 \cosh r_2 \left[ r_2 \sin \left( \frac{S}{T} \right) - r_1 \sinh \left( \frac{S}{T} \right) \right] - r_1 r_2 \left( r_1 \sin r_1 + r_2 \sinh r_2 \right) \cos \left( \frac{S}{T} r_1 \right) - r_2 \sinh r_2 \right\},
\]
(40)

where \(C\) is an arbitrary constant and the following characteristic equation holds
\[
2\Omega^2 + \left( P^2 + 2\Omega^2 \right) \cos r_1 \cosh r_2 - \omega p \sin r_1 \sinh r_2 = 0.
\]
(41)

The first frequency \(\Omega\) against \(p\) is presented in Fig. 3.

In order to simplify further calculations we understand that the constant \(C\) is determined by the condition
\[
\int_0^l \nu'(S) u(S) dS = \int_0^l w^2 \delta_0 dS = \frac{1}{P_0} \delta_i,
\]
(42)

where \(\delta_i\) is the Kronecker delta symbol. Now the general solution to (33)–(35) reads
\[
u = x_1 \nu_1 + x_2 \nu_2 = x_1 u_1 + x_2 u_2,
\]
(43)
where the summation convention is used and \( \alpha_1, \alpha_2 \) are unknown constants.

Since our intention is to analyze the influence of small geometric imperfections on the stability of a pre-twisted rod we introduce them by changing the shape of the cross-section area. This implies the following changes in the rigidities and density

\[
J = J_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \delta A_{13} \\ \delta A_{22} \end{bmatrix} = J_0 + \delta J, \quad A_{11} = A_{11} + \delta A_{11},
\]

\[
\rho_o = \rho_o + \delta \rho_o.
\]  

(44)

The next step is to solve the eigenvalue problem (31), (32) with (44), i.e.,

\[
\left\{ \begin{array}{l}
\{ (J_0 + \delta J) \left( u' - \frac{LNu}{A_{11} + \delta A_{11}} \right) - \frac{LNu}{A_{11} + \delta A_{11}} \} \\
- \frac{LNu}{A_{11} + \delta A_{11}}
\end{array} \right\} - L \left( \frac{LNu}{A_{11} + \delta A_{11}} \right) + \mu(\bar{T}_x + \delta \bar{T}_x)u, \quad (45)
\]

subject to

\[
u(0) = 0, \quad \nu(0) = 0.
\]  

(46)

Following Seyranian and Mailybaev (2011) we find variations of the eigenmode and eigenvalue of (45), (46) in the form

\[
u = \lambda \nu_0 + \mu + \ldots, \quad \mu = \omega_o^2 + \mu + \ldots
\]  

(47)

Substituting (47) into (45), (46), taking only terms of first order and using the fact that \( \nu_0 \) and \( \nu_0 \) satisfy (33) and (34) we obtain

\[
(\partial \nu_0 \nu_0')' + J_0 \partial \nu_0' + \left( 1 - \frac{4J_0}{A_{11}} \right) L N \nu_0 u'' - \frac{2\mu N}{A_{11}} N \nu_0 u'' = \omega_o^2 \nu_0 + (\omega_o^2 \partial \nu_0 + \delta \partial \bar{T}_x) x_1 u_0,
\]  

(48)

subject to

\[
\partial \nu_0(0) = 0, \quad \partial \nu_0(0) = 0.
\]

(49)

Next, we multiply (48) by \( \nu_0^* \) on the left and then integrate with respect to \( S \) to get

\[
\int_0^l \nu_0^* \{ (\partial \nu_0 \nu_0')' + \left( 1 - \frac{4J_0}{A_{11}} \right) L N \nu_0 u'' - \frac{2\mu N}{A_{11}} N \nu_0 u'' \} dS + \int_0^l \{ J_0 \nu_0' + \partial \nu_0' \} dS = \int_0^l \nu_0^* \{ \omega_o^2 \nu_0 + (\omega_o^2 \partial \nu_0 + \delta \partial \bar{T}_x) x_1 u_0 \} dS.
\]  

(50)

Integration by parts and the boundary conditions (34), (49) lead to

\[
\int_0^l \nu_0'^* \partial \nu_0' dS = \left\{ \nu_0'^* \partial \nu_0' - \left( \nu_0' \right)^2 \partial \nu_0' \right\} + \int_0^l \left( \nu_0' \right)^2 \partial \nu_0' dS,
\]

\[
\int_0^l \nu_0'^* \partial \nu_0' dS = \left\{ \nu_0'^* \partial \nu_0' - \left( \nu_0' \right)^2 \partial \nu_0' - \left( \nu_0' \right)^2 \partial \nu_0' \right\} + \int_0^l \left( \nu_0' \right)^2 \partial \nu_0' dS.
\]  

(51)

Substituting (51) into (50) and using (33), (34), (49), we obtain

\[
\int_0^l \left\{ u_0^* \{ (\partial \nu_0 \nu_0')' + \left( 1 - \frac{4J_0}{A_{11}} \right) L N \nu_0 u'' - \frac{2\mu N}{A_{11}} N \nu_0 u'' \} \right\} dS + \int_0^l \{ J_0 \nu_0' + \partial \nu_0' \} dS = \int_0^l \nu_0^* \{ \omega_o^2 \nu_0 + (\omega_o^2 \partial \nu_0 + \delta \partial \bar{T}_x) x_1 u_0 \} dS.
\]  

(52)

Applying partial integration to the integral on the left hand side of (52) and then using the boundary condition (49), we transform (52) into

\[
\int_0^l \left\{ u_0^* \{ (\partial \nu_0 \nu_0')' + \left( 1 - \frac{4J_0}{A_{11}} \right) L N \nu_0 u'' - \frac{2\mu N}{A_{11}} N \nu_0 u'' \} \right\} dS = \int_0^l \nu_0^* \{ \omega_o^2 \nu_0 + (\omega_o^2 \partial \nu_0 + \delta \partial \bar{T}_x) x_1 u_0 \} dS.
\]  

(53)

From (34), (38) it follows that

\[
\int_0^l \nu_0^* N \nu_0 dS = \frac{\nu_0^* N \nu_0}{2}, \quad \int_0^l \nu_0^* N \nu_0' dS = \frac{\nu_0^* N \nu_0'}{2}
\]  

(54)

and (53) becomes

\[
x_1 \int_0^l \left\{ u_0^* \{ (\partial \nu_0 \nu_0')' + \left( 1 - \frac{4J_0}{A_{11}} \right) L N \nu_0 u'' - \frac{2\mu N}{A_{11}} N \nu_0 u'' \} \right\} dS = \delta \mu \int_0^l \bar{T}_x x_1 u_0 dS.
\]  

(55)

Using (42) on the right hand side of (55) we get the following form of (55)

\[
a_{0x} x_1 = \delta \mu x_1.
\]

where

\[
a_{0x} = \int_0^l \left\{ u_0^* \{ (\partial \nu_0 \nu_0')' + \left( 1 - \frac{4J_0}{A_{11}} \right) L N \nu_0 u'' - \frac{2\mu N}{A_{11}} N \nu_0 u'' \} \right\} dS.
\]  

(56)

Thus, the increment \( \delta \mu \) is an eigenvalue of the matrix \( a_{0x} \) so that \( \delta \mu \) satisfies the characteristic equation

\[
(\delta \mu)^2 - (a_{11} + a_{22}) \delta \mu + a_{11} a_{22} - a_{12} a_{21} = 0.
\]  

(57)

In order to transform matrix \( a_{0x} \) into a more suitable form for application we use (38) to get
\[ |a_i| = \int_0^1 \delta J (\ddot{w})^2 dS - \frac{1}{2} LN \int_0^1 \phi (\dot{w})^2 dS - \omega_0 \int_0^1 \delta \rho_0 \omega^2 dS. \]  
(58)

Using the boundary condition (37) we get
\[ \int_0^1 \phi (\dot{w})^2 dS = \frac{1}{2} |w(l)|^2, \]
so that (58) becomes
\[ |a_i| = \int_0^1 \delta J (\ddot{w})^2 dS - \frac{1}{2} LN |w(l)|^2 - \omega_0 \int_0^1 \delta \rho_0 \omega^2 dS. \]  
(59)

or explicitly
\[ a_{11} = \int_0^1 \delta A_{11} (\dot{w})^2 dS - \omega_0^2 \int_0^1 \delta \rho_0 \omega^2 dS, \]
\[ a_{22} = \int_0^1 \delta A_{22} (\dot{w})^2 dS - \omega_0^2 \int_0^1 \delta \rho_0 \omega^2 dS, \]
\[ a_{12} = -a_{21} = -\frac{L}{2} |w(l)|^2. \]

According to Seyranian and Mailybaev (2011) the trivial solution of (29), (30) is stable if both eigenvalues of the matrix \( |a_i| \) are real and distinct. Also the trivial solution gets unstable when the discriminant of the characteristic Eq. (57) is negative, i.e.
\[ \left( \frac{a_{11} - a_{22}}{2} \right)^2 + a_{12} a_{21} < 0. \]  
(60)

Now using (59) and (60) the first-order instability condition reads
\[ L^2 > b^2, \]  
(61)

where
\[ b = \int_0^1 (\delta A_{22} - \delta A_{11}) (\dot{w})^2 dS \]  
(62)

From (61) it follows that a pre-twisted rod with equal moments of inertia (\( \delta A_{22} = 0, \delta A_{11} = 0 \)) is destabilized by an arbitrary small tangential torque \( L \). This effect is known as the paradox of Nicolai (1928, 1929). However by introducing small geometric imperfections the rod can be stabilized at an arbitrary force \( P \) less than the critical Euler value. It is worth noting that (61) and (62) cover the results of Seyranian and Mailybaev (2011), where the pre-twisting effect was not taken into account. Also an interesting and new result is obtained. Namely, since \( A_{11} \) and \( \delta A_{11} \) do not appear in (61) and (62) it can be concluded that the pre-twisting effect does not influence the first-order instability condition.

As an example of a non-uniform rod we consider one with circular cross-section having the constant radius \( R \) which becomes slightly elliptic with the semi axes \( R \) and \( R + \delta R \). The increment of bending rigidities in the first approximation will become \( \delta A_{22} = 3J_0 \frac{\pi}{t} \) and \( \delta A_{11} = J_0 \frac{\pi}{t} \). Then according to (62) we obtain
\[ b = \frac{2J_0 \int_0^1 (\dot{w})^2 dS}{|w(l)|^2}, \]  
(63)

Using the dimensionless quantities
\[ \bar{L} = \frac{L}{J_0}, \quad \bar{\zeta} = \frac{S}{7}, \quad \bar{w}(\xi) = \frac{w(\zeta)}{L}, \]  
(64)

and (61), (63) we obtain the instability region in the form
\[ \bar{L}^2 > \bar{b}^2, \quad \bar{b} = 2 \int_0^1 \frac{\delta R}{R} \left( \frac{\phi (\dot{w})^2}{(\phi (\dot{w})^2)} \right) d\zeta, \]  
(65)

In order to study influence of variable cross-section on the stability we assume that \( \delta R \) changes along the rod axis according to:
(a) \( \frac{\delta R}{\delta} = \epsilon \zeta (1 - \zeta) \), (b) \( \frac{\delta R}{\delta} = \epsilon \zeta / 4 \), (c) \( \frac{\delta R}{\delta} = \epsilon (1 - \zeta) / 4 \) where \( \epsilon \) is an arbitrary small parameter. Note that maximum of \( \bar{b} \) along the rod axis for these three cases is the same and is equal to \( \epsilon / 4 \).

Numerical results for three cases are shown in Fig. 4.

As we can see in Fig. 4 case (c) is the best and case (b) is the worst from the point of view of stability, while case (a) is neutral. Note that the stability boundary in cases (a) and (c) decreases with the increase of \( P \). The dependence of stability boundary on axial load \( P \) in case (c) is characterized by the higher slope in absolute value.

### 4. Stability of uniform compressible pre-twisted rod loaded by axial force and tangential torque

In this section we investigate the stability of a uniform compressible pre-twisted cantilever rod. Taking into account that the rigidities \( A_{11}, A_{22}, A_{33}, B_{11} \) are constant the governing equation given by (27) yields
\[ \rho_0 \dot{q} + \frac{\mathbf{J} \dot{w}}{L} + \frac{1}{1 - \frac{p}{\pi}} \left( \frac{1 - 2 \delta A_{11}}{A_{11}} \right) \mathbf{N} \dot{w}^2 + \frac{P}{1 - \frac{p}{\pi}} \left( \dot{w}^2 - \frac{2 \mathbf{N} \dot{w}}{A_{11}} \right) = 0, \]  
(66)

where the relation \( \mathbf{N} + \mathbf{J} = \mathbf{N}(A_{22} + A_{33}) \) is used. The corresponding boundary conditions read
\[ \mathbf{q}(0, t) = 0, \]
\[ \mathbf{q}^T(0, t) = 0, \]
\[ \left\{ \mathbf{q}^T - \frac{2 \mathbf{N} \dot{w}}{A_{11}} \right\}(l, t) = 0, \]  
(67)

Note that \( \mathbf{q}^T \) can be expressed through the boundary condition (67) and substituted into (67). Then the term
\[ \frac{1}{1 - \frac{p}{\pi}} \left( \frac{1 - 2 \delta A_{11}}{A_{11}} \right) \mathbf{N} \dot{w}^2 \]  
(68)

Following the procedure presented in the preceding section we substitute a solution \( \mathbf{q} = \mathbf{u} \mathbf{e}^{i \omega t} = (u_1(S), u_2(S)) \mathbf{e}^{i \omega t} \) into (66), (67), (68) to get the eigenvalue problem
\[ \mathbf{J u}^T + L \left( 1 - 2 \delta A_{11} \right) \mathbf{N} \mathbf{u}^T + \frac{P}{1 - \frac{p}{\pi}} \left( \mathbf{u}^T - \frac{2 \mathbf{N} \mathbf{u}}{A_{11}} \right) = \rho \mu \mathbf{u} \]  
(69)

subject to
\[ \mathbf{u}(0) = 0, \]
\[ \mathbf{u}'(0) = 0, \]  
(70)

\[ \left\{ \mathbf{u}'^T - \frac{2 \mathbf{N} \mathbf{u}}{A_{11}} \right\}(l) = 0, \]
\[ \left\{ \frac{1}{1 - \frac{p}{\pi}} \mathbf{u}'^T + P \left( \mathbf{u}' - \frac{\mathbf{N} \mathbf{u}}{A_{11}} \right) \right\}(l) = 0, \]  
(71)

where \( \mu = \omega^2 \) is an eigenvalue.

First, as a special case we analyze a uniform compressible rod \( L = 0 \) with equal principal moments of inertia of cross-section \( A_{22} = A_{33} = J_0 = \text{const} \). The corresponding eigenvalue problem follows from (69), (70) and reads
\[ \frac{1}{1 - \frac{p}{\pi}} \mathbf{u}'^T + \frac{P}{1 - \frac{p}{\pi}} \mathbf{u}' = \mu \rho \mathbf{u} \]  
(71)
subject to
\[ \mathbf{u}(0) = 0, \quad \mathbf{u}'(0) = 0, \quad \mathbf{u}''(l) = 0, \]
\[ J \left[ \frac{1}{1 - \frac{\rho_0}{\rho_1}} \mathbf{u}'' + \mathbf{p} \mathbf{u}' \right](l) = 0. \] (72)

where
\[ J = J_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_{11} = \bar{A}_{11} = \text{const}, \]
\[ B_{11} = \bar{B}_{11} = \text{const} \quad \rho_0 = \rho_0 = \text{const}, \quad \mu = \omega_0^2. \] (73)
The eigenvalues and eigenmodes corresponding to the eigenvalue problem (71), (72) and (73) follow from
\[ J_0 \mathbf{w}'' + p \left[ 1 - \frac{P}{\bar{B}_{11}} \right] \mathbf{w}'' = \omega_0^2 \left( 1 - \frac{P}{\bar{B}_{11}} \right) \mathbf{p}_0 \mathbf{w}, \] (74)
subject to
\[ \mathbf{w}(0) = \mathbf{w}'(0) = 0, \quad J_0 \mathbf{w}'' + p \left( 1 - \frac{P}{\bar{B}_{11}} \right) \mathbf{w}(l) = 0. \] (75)

As in the preceding section we assume that the force \( P \) is less than its critical value \( P_{cr} \). In order to obtain \( P_{cr} \) we use the adjoint equilibrium method. This means that we solve (74), (75) with \( \omega_0 = 0 \). This way we find
\[ \cos \sqrt{\frac{P_{cr}^2}{J_0} \left( 1 - \frac{P}{\bar{B}_{11}} \right)} = 0. \] (76)

From (76) we get
\[ \frac{\sqrt{P_{cr}^2}}{J_0} \left( 1 - \frac{P}{\bar{B}_{11}} \right) = \frac{\pi^2}{4} (2n + 1)^2, \quad n = 0, 1, 2, \ldots \] (77)
The characteristic Eq. (77) reveals that buckling can occur if
\[ P \in \left\{ \bar{B}_{11} \left[ 1 \pm \sqrt{1 - \frac{\pi^2 (2n + 1)^2}{\eta}} \right] : n \in \mathbb{N} \cup \{0\}, \right\} \]
\[ \eta = \bar{B}_{11}/J_0 \cap \mathbb{R}_+, \] (78)

This means that there is only a finite number of buckling loads. In engineering, the dimensionless quantity \( \eta \) is known as slenderness ratio (see Atanackovic, 1997; Magnusson et al., 2001). For example, for a circular cross-section of the rod we have \( \eta = 2l/r \) where \( r \) is the radius of the cross-section. From (78) we conclude that the critical value \( P_{cr} \) (the lowest value of \( P \) that leads to a nontrivial solution of (74) and (75) with \( \omega_0 = 0 \)) reads
\[ P_{cr} = \frac{1}{2} \bar{B}_{11} \left[ 1 - \sqrt{1 - \left( \frac{\pi}{\eta} \right)^2} \right], \] (79)
as well as that for \( \eta < \pi \) there exists no buckling load (see Mazzilli, 2009). In what follows we assume that \( \eta \geq \pi \).

Next, we introduce the dimensionless force \( p \) and eigenfrequency \( \Omega \) as in (39) and using them define new dimensionless quantities
\[ \mathbf{p} = \left( 1 - \frac{P}{\eta^2} \right) p = \left( 1 - \frac{P}{\bar{B}_{11}} \right) \frac{p l^2}{J_0}, \]
\[ \Omega = \left( 1 - \frac{P}{\eta^2} \right) \Omega = \left( 1 - \frac{P}{\bar{B}_{11}} \right) \omega_0 \sqrt{\frac{p l^2}{J_0}}. \] (80)

Using (39), (80) the critical force, given by (79), can be put into dimensionless form
\[ p_{cr} = \frac{1}{2} \left( \eta^2 - \eta^2 - \eta^2 \pi^2 \right), \] (81)
while a solution to (74) and (75) reads
\[ \mathbf{w}(s) = \mathbf{C} \left\{ a_1 \sin \tau_1 \sin \tau_2 + a_2 \tau_1 \sin \tau_1 \cos \tau_2 + a_3 \cos \tau_1 \cos \tau_2 \right\}, \] (82)
where \( \mathbf{C} \) is an arbitrary constant. The corresponding characteristic equation takes the form
\[ 2\Omega^2 + (\mathbf{p}^2 + 2\Omega^2) \cos \tau_1 \cosh \tau_2 - p\Omega \sin \tau_1 \sinh \tau_2 = 0. \] (83)

It is interesting to see that (82) and (83) are now of the same form as (40) and (41), respectively. However, the quantities \( p, \Omega, \tau_1, \tau_2 \) are replaced with the modified ones \( \mathbf{p}, \Omega, \tau_1, \tau_2 \). Also, (80) implies that when \( \eta \to \infty \) (the classical Bernoulli–Euler theory) the quantities \( \mathbf{p}, \Omega, \tau_1, \tau_2 \) tend to the \( p, \Omega, \tau_1, \tau_2 \).

Next we determine the influence of the slenderness ratio on the dimensionless eigenfrequency \( \Omega \) and force \( p \). First, we choose a value of the slenderness ratio \( \eta \). Then, using (80) and the physical restriction \( 1 - \frac{P}{\bar{B}_{11}} = 1 - \frac{P}{\bar{B}_{11}} > 0 \) we obtain the following form of (83)
\[ 2\Omega^2 \left( \mathbf{p}^2 + 2\Omega^2 \right) \cos \left( \frac{1 - \eta}{\eta} \right) \left( \mathbf{p} + \sqrt{\mathbf{p}^2 + 4\Omega^2} \right) \]
\[ \times \cosh \left( \frac{1 - \eta}{\eta} \right) \left( \mathbf{p} + \sqrt{\mathbf{p}^2 + 4\Omega^2} - \mathbf{p} \right), \]
\[ - \mathbf{p} \Omega \sin \left( \frac{1 - \eta}{\eta} \right) \left( \mathbf{p} + \sqrt{\mathbf{p}^2 + 4\Omega^2} - \mathbf{p} \right), \]
\[ \times \sinh \left( \frac{1 - \eta}{\eta} \right) \left( \mathbf{p} + \sqrt{\mathbf{p}^2 + 4\Omega^2} - \mathbf{p} \right) = 0, \]
which is solved numerically for the pairs \( (p, \Omega) \). In that way we obtain Fig. 5. This figure implies that the decrease in the slenderness ratio leads to the increase in the dimensionless eigenfrequency \( \Omega \) if the value of the dimensionless force \( p \) is kept constant.
The solution to the unperturbed problem (71)–(73) now reads

$$u_1 = \begin{bmatrix} \mathbf{W} \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ \mathbf{W} \end{bmatrix},$$  

(84)

while the constant $\mathcal{C}$ in (82) is determined in the way as before

$$\int_0^{\pi} u_1', u_1, dS = \int_0^{\pi} \mathbf{W}' \delta y dS = \frac{1}{\rho_o} \delta y.$$  

(85)

As in the preceding section we introduce small geometric imperfections in the shape of cross-section area. Thus we get

$$J = J_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \delta A_{13} & 0 \\ 0 & \delta A_{22} \end{bmatrix} = J_0 + \delta J.$$  

(86)

$$A_{11} = \mathcal{A}_{11} + \delta A_{11}, \quad B_{11} = \mathcal{B}_{11} + \delta B_{11}, \quad \rho_o = \mathcal{\rho}_o + \delta \rho_o.$$  

Next, we substitute (86) into the eigenvalue problem (69), (70) to get

$$\left( J_0 + \delta J \right) u'' + I \left( 1 - 2 \frac{\delta A_{13} + \delta A_{22} + \delta A_{11}}{\mathcal{A}_{11} + \delta A_{11}} \right) \mathbf{Nu}^\gamma$$

$$+ \frac{1}{1 - \frac{\mathcal{A}_{11}}{\mathcal{\rho}_o}} \left( \mathbf{u}'' - \frac{2L\mathbf{Nu}^\gamma}{\mathcal{A}_{11} + \delta A_{11}} \right) = \left( \mathcal{\rho}_o + \delta \mathcal{\rho}_o \right) \mu \mathbf{u}$$  

(87)

subject to

$$u(0) = 0,$$

$$u'(0) = 0.$$  

(88)

Then, variations of the eigenmode and eigenvalue take the form

$$u = \zeta u_1 + \delta u + \ldots, \quad \mu = \mu_o + \delta \mu + \ldots$$  

(89)

and we substitute them into (87), (88). Using the fact that $u_1, u_2$ satisfy (71), (72) with $\mu = \mu_o^2$ and taking only terms of first order we get

$$J_0 \omega^{\omega''} - 2J_0 \rho \beta B_{11} \mathbf{Z}_u^{\omega}$$

$$+ \frac{L}{(1 - \frac{\rho_o}{\mathcal{\rho}_o})} \mathbf{Nu}^\gamma u'' + \frac{\delta \mathbf{Z}_u^{\omega}}{(1 - \frac{\rho_o}{\mathcal{\rho}_o})}$$

$$+ \frac{L}{(1 - \frac{\rho_o}{\mathcal{\rho}_o})} \mathbf{Nu}^\gamma - \frac{2L \mathbf{Nu}^\gamma}{\mathcal{A}_{11} + \delta A_{11}}$$

$$= \mu \omega \mathcal{\rho}_o \mathbf{u} + (\omega \mathcal{\rho}_o + \delta \mathcal{\rho}_o) \zeta \mathbf{u}.$$  

(90)

subject to

$$\delta u(0) = 0,$$

$$\delta u'(0) = 0.$$  

(91)

Next, we multiply (90) by $u_j'$ on the left, integrate with respect to $S$ and use (71), (72), (51) to get

$$\int_0^\pi u_j'' \left( \frac{\delta J}{(1 - \frac{\rho_o}{\mathcal{\rho}_o})} - \frac{L}{(1 - \frac{\rho_o}{\mathcal{\rho}_o})} \mathbf{Nu}^\gamma \right) \mathbf{u}' dS$$

$$+ \int_0^\pi \mathbf{u}' \left( \frac{2J_0 \rho \beta B_{11}}{1 - \frac{\rho_o}{\mathcal{\rho}_o}} \mathbf{Z}_u^{\omega} + \mathbf{P} \left( \delta \mathbf{u}' \right) \right) dS = \int_0^\pi \mathbf{u}' \left( \omega \mathcal{\rho}_o + \delta \mathcal{\rho}_o \right) \zeta \mathbf{u} dS.$$  

(92)

Applying again integration by parts and using the boundary conditions (72), (91) we obtain

$$\frac{1}{(1 - \frac{\rho_o}{\mathcal{\rho}_o})} \int_0^\pi \left( \mathbf{u}' \right)^\gamma \delta \mathbf{Z}_u^{\omega} dS = \frac{2J_0 \rho \beta B_{11}}{1 - \frac{\rho_o}{\mathcal{\rho}_o}} \int_0^\pi \left( \mathbf{u}' \right)^\gamma \mathbf{Z}_u^{\omega} dS$$

$$- \int_0^\pi \left( \mathbf{u}' \right)^\gamma \mathbf{Nu}^\gamma dS + \frac{\mathbf{P}^2 \rho \beta B_{11}}{1 - \frac{\rho_o}{\mathcal{\rho}_o}} \int_0^\pi \left( \mathbf{u}' \right)^\gamma \mathbf{Z}_u^{\omega} dS$$

$$\times \frac{L}{(1 - \frac{\rho_o}{\mathcal{\rho}_o})} \int_0^\pi \left( \mathbf{u}' \right)^\gamma \mathbf{Z}_u^{\omega} dS$$

$$= \omega \mathcal{\rho}_o \mathbf{u} \mathbf{u}_o + \delta \mathcal{\rho}_o \mathbf{Z}_u^{\omega} \mathbf{Z}_u^{\omega} \mathbf{u}.$$  

(93)

As in the preceding Section (54) holds so that (93) becomes

$$\frac{1}{(1 - \frac{\rho_o}{\mathcal{\rho}_o})} \int_0^\pi \left( \mathbf{u}' \right)^\gamma \delta \mathbf{Z}_u^{\omega} dS = \frac{2J_0 \rho \beta B_{11}}{1 - \frac{\rho_o}{\mathcal{\rho}_o}} \int_0^\pi \left( \mathbf{u}' \right)^\gamma \mathbf{Z}_u^{\omega} dS$$

$$- \int_0^\pi \left( \mathbf{u}' \right)^\gamma \mathbf{Nu}^\gamma dS + \frac{\mathbf{P}^2 \rho \beta B_{11}}{1 - \frac{\rho_o}{\mathcal{\rho}_o}} \int_0^\pi \left( \mathbf{u}' \right)^\gamma \mathbf{Z}_u^{\omega} dS$$

$$\times \frac{L}{(1 - \frac{\rho_o}{\mathcal{\rho}_o})} \int_0^\pi \left( \mathbf{u}' \right)^\gamma \mathbf{Z}_u^{\omega} dS$$

$$= \omega \mathcal{\rho}_o \mathbf{u} \mathbf{u}_o + \delta \mathcal{\rho}_o \mathbf{Z}_u^{\omega} \mathbf{Z}_u^{\omega} \mathbf{u}.$$  

(94)

Introducing notation for the coefficients

$$a_{ij} = \frac{1}{(1 - \frac{\rho_o}{\mathcal{\rho}_o})} \int_0^\pi \left( \mathbf{u}' \right)^\gamma \delta \mathbf{Z}_u^{\omega} dS - \frac{2J_0 \rho \beta B_{11}}{1 - \frac{\rho_o}{\mathcal{\rho}_o}} \int_0^\pi \left( \mathbf{u}' \right)^\gamma \mathbf{Z}_u^{\omega} dS$$

$$- \int_0^\pi \left( \mathbf{u}' \right)^\gamma \mathbf{Nu}^\gamma dS + \frac{\mathbf{P}^2 \rho \beta B_{11}}{1 - \frac{\rho_o}{\mathcal{\rho}_o}} \int_0^\pi \left( \mathbf{u}' \right)^\gamma \mathbf{Z}_u^{\omega} dS$$

$$- \omega \mathcal{\rho}_o \mathbf{u} \mathbf{u}_o + \delta \mathcal{\rho}_o \mathbf{Z}_u^{\omega} \mathbf{Z}_u^{\omega} \mathbf{u}.$$  

(94)
Eq. (94) yields
\[ a_0 \dot{x}_i = \delta \mu \ddot{x}_i. \]

Note that after using (84) the matrix \([a_0]\) becomes
\[ [a_0] = \left[ \begin{array}{c} \frac{\partial \mathbf{J}}{\partial \mathbf{r}} - \frac{2Jp \delta B_{11}}{\partial \mathbf{r}} \end{array} \right] \int_0^l \left( \mathbf{w}' \right)^2 dS - \frac{1}{2} \int_0^l \frac{\partial \mathbf{N} \mathbf{w}}{\partial \mathbf{l}}^2 \left( \mathbf{w}' \right)^2 dS + \frac{P^2 \delta B_{11}}{2} \int_0^l \left( \mathbf{w}' \right)^2 dS - \omega \ddot{\rho} \int_0^l \mathbf{w}^2 dS, \]

or explicitly
\[ a_{11} = \left[ \frac{\delta A_{33}}{B_{11}} - \frac{2Jp \delta B_{11}}{B_{11}} \right] \int_0^l \left( \mathbf{w}' \right)^2 dS + \frac{P^2 \delta B_{11}}{B_{11}} \int_0^l \left( \mathbf{w}' \right)^2 dS - \omega \ddot{\rho} \int_0^l \mathbf{w}^2 dS, \]
\[ a_{22} = \left[ \frac{\delta A_{22}}{B_{11}} - \frac{2Jp \delta B_{11}}{B_{11}} \right] \int_0^l \left( \mathbf{w}' \right)^2 dS + \frac{P^2 \delta B_{11}}{B_{11}} \int_0^l \left( \mathbf{w}' \right)^2 dS - \omega \ddot{\rho} \int_0^l \mathbf{w}^2 dS, \]
\[ a_{12} = -a_{21} = -\frac{L}{2} \frac{\partial \mathbf{w}}{\partial \mathbf{l}} \left( \mathbf{w}' \right)^2. \]

Following the argumentation in the preceding section we get the instability condition (60) with (96) yields
\[ L^2 > c^2 \]
where
\[ c = \left( \frac{\partial A_{22} - \partial A_{33}}{\partial \mathbf{l}} \right) \int_0^l \left( \mathbf{w}' \right)^2 dS. \]

In the case of uniform rod (97), (98) lead to a generalization of (61), (62). We note that the relations (97), (98) look similar with (61), (62). However, the difference is that the function \(w(S)\) depends on the slenderness ratio. Apart from this it is worth noting that \(\partial A_{22}\) and \(\partial A_{33}\) do not appear in the first-order instability condition (97). In order to present the influence of the compressibility effect on the instability condition we introduce the dimensionless quantity \(Q\)
\[ Q = \int_0^l \left( \frac{\partial \mathbf{w}}{\partial \mathbf{l}} \right)^2 dS = \frac{\left( \frac{\partial \mathbf{w}}{\partial \mathbf{l}} \right)^2}{\left( \frac{\partial \mathbf{w}}{\partial \mathbf{l}} \right)^2}, \]

Now, the instability condition becomes
\[ L^2 > \left( \frac{\partial A_{22} - \partial A_{33}}{l} \right) Q. \]

Calculating \(Q\) for different values of slenderness ratio \(\eta\) we get the influence of compressibility. The results, presented in Fig. 6, show that if the value of dimensionless force \(p\) is kept constant then the decrease in the slenderness ratio \(\eta\) leads to the increase in the critical stability torque \(L\).

Also, if the value of slenderness ratio \(\eta\) is kept constant then the critical stability torque decreases as the dimensionless force \(p\) increases from 0 to \(p_c\) (see Fig. 6). The critical dimensionless force \(p_c\) is determined by (81). It is worth noting that 1.2337 \(\leq Q \leq 1.631\) for all values of slenderness ratio.

5. Conclusions

We have considered the problem of Nicolai on dynamic stability of an elastic cantilever, loaded by a tangential (non-conservative) twisting torque and an axial force, in continuous formulation. The rod is assumed to be non-uniform, i.e., having variable cross-section with non-equal principal moments of inertia, and compressibility effect is taken into account. The main results are the following:

(1) The non-linear governing Eqs. (13), (14) describing the motion of a rod are presented. From these equations we derived new linear equations and boundary conditions (27), (28), which form the basis for analytical and numerical stability studies. An important detail of this formulation is that the compressibility of the rod axis and the pre-twisting effect due to the torque are taken into account.

(2) In Section 3 the first-order instability condition describing the influence of small geometrical imperfections of a non-uniform pre-twisted inextensible rod to its stability is derived (see (61), (62)). This condition shows that a pre-twisted uniform rod with equal principal moments of inertia is destabilized by an arbitrary small tangential torque \(L\), and that by introducing small geometric imperfections the rod can be stabilized. It is worth noting that the instability condition covers the results of Seyranian and Mailybaev (2011), where the pre-twisting effect was not taken into account. By analyzing the first-order instability it is observed that the pre-twisting effect and the perturbation \(\partial A_{22}\) do not influence the stability region. This is an interesting and new result. As an example a non-uniform rod with slightly elliptic cross-section is analyzed. Three different distributions of material along the rod axis are chosen (see Fig. 4). The results show that adding more material to the clamped end of the rod is more efficient from the point of view of stability compared with placing material to the free end of the rod. Also, this example suggests that the influence of distribution of material is decreasing when the axial force \(p\) is increasing.

(3) In Section 4 the influence of small geometrical imperfections on stability of a uniform pre-twisted compressible rod is presented. In particular, the first-order instability condition determining the stabilization region due to small geometrical imperfections is derived (see (97), (98)). It is important to note that this instability condition depends on the compressibility effect but it does not depend on the pre-twisting effect due to the torque.
effect and the perturbations $\delta A_{11}$ and $\delta B_{11}$. The influence of compressibility is presented by (99) and Fig. 6. In particular, it is shown that if the value of the axial force is kept constant then decreasing the slenderness ratio of the rod leads to the increase in the critical stability torque. On the other hand, if the value of slenderness ratio is kept constant then the critical stability torque decreases as the axial force increases.

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References
