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ET0L Forms*

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This paper explores the notions of "tabled L form" and its "interpretations," which produce a family of structurally similar ET0L systems. Biologically this is the study of a family of organisms which are similar developmentally. In particular, the tables ensure a similarity of changing environmental conditions under which each organism develops. We demonstrate a number of normal form results for ET0L forms, which carry over in a nontrivial way from ET0L systems. The main section of the paper investigates "completeness." This leads to the surprising discovery of a normal form in which the only rules for terminals are $a \to a$.

1. Introduction

In Maurer, Salomaa and Wood [9], from now on referred to as [MSW] the notion of an E0L form and its interpretations is introduced. This is the result of an attempt to combine the notions of L systems (see for example, Herman and Rozenberg [6]) and grammar forms (Cremers and Ginsburg [2]) in a meaningful way for L systems. This results in a notion of an "interpretation" which is more restrictive than that normally used for grammar forms, the so called *strict* interpretation.

In this paper we consider ET0L forms. Since the ET0L languages form a very natural class of languages, we expected that the investigation of ET0L forms would be fruitful. This paper is intended to demonstrate that this is indeed the case. Many problems remain to be solved and we will return to some of them in later papers.

From a biological point of view an ET0L form gives rise to a class of related ET0L systems, which can be interpreted as a "family" or "species" of organisms. Each member of the family will develop under related environmental conditions defined by the tables. Therefore in this model each member of the family of organisms will develop only under similar conditions to its co-members.

This paper consists of a further four sections. The basic notions of ET0L systems and ET0L forms are introduced in Section 2. In Section 3 a review is given of some

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of the results which carry over from grammar forms, followed by some technical results including two central "simulation" results. Section 4 concerns reduction of ET0L forms, and finally Section 5 develops a theory of "completeness." This section includes many examples of ET0L-complete ET0L forms and also demonstrates the existence of an E0L-complete ET0L form in which no table is E0L-complete.

2. Basic Notation and Terminology

We briefly review the basic notion of ET0L systems (from a grammatical viewpoint, rather than a substitutional one) and introduce ET0L forms and their interpretations.

Definition. An *ET0L scheme* (or an n-ET0L scheme) T is a $(n+2)$-tuple $T = (V, \Sigma, P_1, ..., P_n)$ for some $n \geqslant 1$, where V is an alphabet, $\Sigma \subseteq V$ is the terminal alphabet, $V - \Sigma$ the nonterminal alphabet and for all i, $1 \leqslant i \leqslant n$, P_i is a finite set of pairs (α, x) with α in V and x in V^* such that for each α in V at least one such pair is in P_i. The elements $p = (\alpha, x)$ of P_i are called *rules* and are usually written $\alpha \xrightarrow{p} x$, when P_i is understood. T is said to be *deterministic* (an *EDT0L* scheme) if for all i, $1 \leqslant i \leqslant n$ and for all α in V, exactly one rule $\alpha \to x$ is in P_i, and T is *propagating* (an *EPT0L* scheme and if T is also deterministic, an *EPDT0L* scheme) if for all i, $1 \leqslant i \leqslant n$ and for all $\alpha \to x$ in P_i, $x \neq \epsilon$, the *empty word.*

Definition. Let $T = (V, \Sigma, P_1, ..., P_n)$ be an n-ET0L scheme. For words $x = \alpha_1\alpha_2 \cdots \alpha_m$ with α_i in V and $y = y_1 \cdots y_m$ with y_i in V^* we write $x \Rightarrow_{P_j} y$ if for all i, $\alpha_i \to y_i$ is a rule of P_j, for some j. We also write $x \Rightarrow_T y$. For all x in V^* and all j we write $x \Rightarrow^0_{P_j} x$ and $x \Rightarrow^0_T x$. For $m > 0$ we write $x \Rightarrow^m_T y$ if for some z in V^* and j, $x \Rightarrow_{P_j} z \Rightarrow^{m-1}_T y$. By $x \overset{*}{\Rightarrow}_T y$ ($x \Rightarrow^+_T y$) we mean $x \Rightarrow^m_T y$ for some $m \geqslant 0$ ($m > 0$). Similarly, we write $x \Rightarrow y^m_{P_j}$ ($x \overset{*}{\Rightarrow}_{P_j} y$, $x \Rightarrow^+_{P_j} y$) if each derivation step uses rules from the table P_j.

For convenience we often will not indicate the particular ET0L scheme below the arrow $\Rightarrow$ if it is understood by the context.

A sequence of words $x_0, x_1, ..., x_m$ with $x_0 \Rightarrow x_1 \Rightarrow \cdots \Rightarrow x_m$ is a *derivation* (*of length* m *leading from* x_0 *to* x_m).

Definition. An *n-ET0L system* (ET0L system) G is an $(n+3)$-tuple $G = (V, \Sigma, P_1, ..., P_n, S)$ where $(V, \Sigma, P_1, ..., P_n)$ is an n-ET0L scheme and S in $V - \Sigma$ is the *start symbol.* The notions introduced for ET0L schemes are carried over to ET0L systems in an obvious manner. The *language generated by* G, denoted by $L(G)$, is:

$$L(G) = \{x \text{ in } \Sigma^*: S \overset{*}{\Rightarrow} x\}.$$

For convenience languages which differ by at most the empty word will be considered equal. Families of languages will be considered equal if for any nonempty language in one family there is a language in the other family which differs by at most the empty word, and conversely.

DEFINITION. The class of E0L languages is simply the class of 1-ET0L languages. Denote by E0L, n-ET0L, ET0L the classes of E0L, n-ET0L and ET0L languages.

It is of course well known that E0L = 1-ET0L $\subsetneq$ 2-ET0L = n-ET0L ($n \geq 2$) = ET0L.

DEFINITION. Let $G = (V, \Sigma, P_1, ..., P_n, S)$ be an n-ET0L system. For all i, $1 \leq i \leq n$, let *max* $r(P_i) = \max\{|x| : \alpha \to_{P_i} x\}$, and *max* $r(G) = \max\{\max r(P_i): 1 \leq i \leq n\}$. For a word x let $Alph(x)$ be the set of all symbols occurring in x. Let $LS(G) = \{|x| : x \text{ in } L(G)\}$ be the *length set generated by* G and for a language L let $LS(L) = \{|x| : x \text{ in } L\}$ be the *length set of* L. For a set M of symbols and a set N of words $M \to N$ denotes the set of rules $\{\alpha \to x: \alpha \text{ in } M \text{ and } x \text{ in } N\}$.

DEFINITION. Let $G = (V, \Sigma, P_1, ..., P_n, S)$ be an n-ET0L system. A symbol α in V is *reachable* (from S) if $S \overset{*}{\Rightarrow} x\alpha y$ holds for some x and y.

G is *reduced* if each α in V is reachable. G is *looping* if for some reachable α in V, $\alpha \Rightarrow^+ \alpha$ holds, and G is *expansive* if for some reachable α in V, $\alpha \overset{*}{\Rightarrow} x\alpha y\alpha z$ holds for some x, y and z. A derivation $x_0 \Rightarrow_{P_j}^m x_m$ is *nonterminal* (*total nonterminal*), written $x_0 \Rightarrow_{ntP_j}^m x_m$ ($x_0 \Rightarrow_{tntP_j}^m x_m$), if for some (any) sequence of words $x_1, x_2, ..., x_{m-1}$ with $x_i \Rightarrow_{P_j} x_{i+1}$, for $i = 0, ..., m-1$,

$$S \underset{G}{\overset{*}{\Rightarrow}} y_0x_0z_0 \underset{P_j}{\Rightarrow} y_1x_1z_1 \underset{P_j}{\Rightarrow} \cdots \underset{P_j}{\Rightarrow} y_{m-1}x_{m-1}z_{m-1} \underset{P_j}{\Rightarrow} y_mx_mz_m$$

$y_ix_iz_i$ contains at least one nonterminal for each i, $1 \leq i \leq m-1$. Note that $x_0 \Rightarrow \cdots \Rightarrow x_m$ is nonterminal if each of $x_1, ..., x_{m-1}$ contains at least one nonterminal.

We are now in a position to introduce the central notions of ET0L forms and their interpretations.

DEFINITION. An *n-ET0L form* (or *ET0L form*) F is an n-ET0L system, $F = (V, \Sigma, P_1, ..., P_n, S)$. An ET0L system $F' = (V', \Sigma', P'_1, ..., P'_n, S')$ is an *interpretation of* F (*modulo* μ), $F' \lhd F(\mu)$ (or simply, $F' \lhd F$ when μ is understood), if μ is a substitution defined on V and (i)–(v) hold:

(i) $\mu(A) \subseteq V' - \Sigma'$ for each A in $V - \Sigma$,

(ii) $\mu(a) \subseteq \Sigma'$ for each a in Σ,

(iii) $\mu(\alpha) \cap \mu(\beta) = \varnothing$, for any $\alpha \neq \beta$,

(iv) for all i, $1 \leq i \leq n$, $P'_i \subseteq \mu(P_i)$, where $\mu(P_i) = \bigcup_{\alpha \to x \text{ in } P_j} \mu(\alpha) \to \mu(x)$, and

(v) S' is in $\mu(S)$.

The family of (n-)*ET0L systems generated by* F, denoted $\mathscr{G}(F)$, is: $\mathscr{G}(F) = \{F' : F' \lhd F(\mu)$ for a substitution $\mu\}$, and the *family of languages generated by* F, denoted $\mathscr{L}(F)$, is: $\mathscr{L}(F) = \{L(F') : F' \text{ is in } \mathscr{G}(F)\}$.

Notice that since an ET0L form is an ET0L system, and conversely, we use these terms interchangeably.

Remarks. It is worthwhile re-emphasizing the points of departure of ET0L forms and their interpretations from grammar forms and their interpretations as made for E0L forms in [MSW]. The formulation chosen here is similar to that in Maurer and Wood [10]. The only formal difference is that we demand $\mu(a) \subseteq \Sigma'$ rather than $\mu(a)$ being a finite subset of Σ'^* and that (iii) holds also for terminals. This modification seems natural for ET0L forms because:

(1) since rules of the form $\alpha \to x$ occur where α is a terminal, $\mu(\alpha) \subseteq \Sigma'$ prevents the appearance of the empty word on the left of a rule and the appearance of left-hand sides with length greater than one,

(2) the restriction of $\mu(\alpha) \subseteq \Sigma'$ is most natural from a mathematical point of view since when it is combined with (iii), μ^{-1} is a length preserving homomorphism, as exploited in Nivat [11] and Walter [15], and

(3) the restriction $\mu(\alpha) \subseteq \Sigma'$ and (iii) for terminals has already been used by Bertsch [1], Ginsburg and Mayer [5], and Leong and Wotschke [7], who call it *strict interpretation.*

DEFINITION. Two ET0L forms F_1 and F_2 are *equivalent* if $L(F_1) = L(F_2)$ and *form equivalent* if $\mathscr{L}(F_1) = \mathscr{L}(F_2)$.

EXAMPLES. (1) $F = (\{S, N, a, b\}, \{a, b\}, \{S \to aa; a \to b; b \to N; N \to N\}, S)$ is a 1-ET0L form, that is an E0L form. It is not synchronized (see Section 4), since there is a derivation $a \Rightarrow_F^+ x$ where x is a terminal word, for example $x = b$.

It has been proved in [MSW] that $\{aa, bb, ab\}$ is not in $\mathscr{L}(F)$. It is easy to see that $\{aa, bb\}$ is in $\mathscr{L}(F)$, since $L(F) = \{aa, bb\}$, and F is an interpretation of itself.

However if we synchronize F in the usual way to obtain $\bar{F} = (\{S, N, A, B, a, b\}, \{a, b\}, \{S \to AA; A \to a \mid B; B \to b; a \to N; b \to N; N \to N\}, S)$ then $\mathscr{L}(\bar{F})$ contains $\{aa, bb, ab\}$ since $\bar{F}' = (\{S, N, A, B, a, b\}, \{a, b\}, \{S \to AA \mid AB \mid BB;\ A \to a;\ B \to b; a \to N;\ b \to N;\ N \to N\},\ S) \lhd \bar{F}$ and $L(\bar{F}') = \{aa, bb, ab\}$. This example shows that synchronizing a form F does not always give a form equivalent form. In [MSW] it is indeed proved that for the F of this example no synchronized form equivalent form exists.

(2) $F = (\{S, a\}, \{a\}, \{S \to a \mid S \mid SS; a \to S\}, \{S \to S; a \to S\}, S)$ is a 2-ET0L form.

Now letting $M = \{b\}^* \{a\}\{b\}^*$ we define $L = \bigcup_{n \geq 0} M^{2^n}$, a well known ET0L language which is not E0L. Define an ET0L system generating L, $G = (\{S, B, N, a, b\}, \{a, b\}, P_1, P_2, S)$ where $P_1 = \{S \to SS \mid a;\ B \to b \mid B;\ a \to N;\ b \to N;\ N \to N\}$, and $P_2 = \{S \to S \mid BS \mid SB; B \to B; a \to N; b \to N; N \to N\}$.

Now G is not an interpretation of F, however we can construct $\bar{G} = (\{S, S_1, S_2, B, B_1, B_2, N, a, b\}, \{a, b\}, \bar{P}_1, \bar{P}_2, S_1)$ where $\bar{P}_1 = \{S_1 \to SS \mid a;\ B_1 \to b \mid B;\ a \to N;$

$b \to N$; $N \to N$; $S_2 \to S \mid BS \mid SB$; $B_2 \to B\}$ and $\bar{P}_2 = \{S \to S_1$; $S_1 \to S_2$; $S_2 \to N$; $B \to B_1$; $B_1 \to B_2$; $B_2 \to N$; $N \to N$; $a \to N$; $b \to N\}$.

It is not difficult to show that $L(\bar{G}) = L(G)$ and $\bar{G} \lhd F$. It should be noted that we will prove $\mathscr{L}(F) =$ ETOL (Theorem 5.2) and that the construction of $\bar{G}$ from G ensures $\mathscr{L}(G) = \mathscr{L}(\bar{G})$ (Theorem 4.6).

3. Preliminary Lemmas

In this section we mention those results that trivially carry over from grammar forms (Cremers and Ginsburg [2]) and the technical results that carry over with slight modifications from E0L forms [MSW].

THEOREM 3.1. (i) *The relation $\lhd$ for ETOL forms is decidable and transitive.*

(ii) *Let F_1 and F_2 be ETOL forms: $\mathscr{G}(F_1) \subseteq \mathscr{G}(F_2)$ iff $F_1 \lhd F_2$.*

(iii) *It is decidable for arbitrary ETOL forms F_1 and F_2 whether or not $\mathscr{G}(F_1) = \mathscr{G}(F_2)$.*

Note that for two ETOL forms F_1 and F_2, $F_1 \lhd F_2$ implies $\mathscr{L}(F_1) \subseteq \mathscr{L}(F_2)$ but the converse is certainly not true. It remains an open problem to determine the decidability of the question: given two ETOL (E0L, grammar) forms F_1 and F_2 is $\mathscr{L}(F_1) \subseteq \mathscr{L}(F_2)$?, but for certain special cases results have been established (Maurer *et al.* [8]; Culik *et al.* [3]).

We now continue by presenting a number of technical results similar to those proved in [MSW] for E0L forms.

LEMMA 3.2. *Let F be an ETOL form, and F' be an interpretation of F, $F' \lhd F(\mu)$, then*

(i) *for each derivation $x_0 \Rightarrow_{F'} x_1 \Rightarrow_{F'} \cdots \Rightarrow_{F'} x_m$ in F', $\mu^{-1}(x_0) \Rightarrow_F \mu^{-1}(x_1) \Rightarrow_F \cdots \Rightarrow_F \mu^{-1}(x_m)$ is a derivation in F,*

(ii) *if F' is looping then F is looping,*

(iii) *if F' is expansive then F is expansive.*

Proof. Clear.

We could prove a stronger version of part (i), namely, that if $x_i \Rightarrow_{P'_{j_i}} x_{i+1}$, $0 \leqslant i \leqslant m$ then $\mu^{-1}(x_i) \Rightarrow_{P_{j_i}} \mu^{-1}(x_{i+1})$, that is the table used at each step of the derivation is preserved.

We now prove simulation results analogous to the First and Second Simulation Lemmas for E0L forms in [MSW]. Both are "table-by-table" simulations.

LEMMA 3.3. *Let $F = (V, \Sigma, P_1, ..., P_n, S)$ and $\bar{F} = (\bar{V}, \bar{\Sigma}, \bar{P}_1, ..., \bar{P}_n, S)$ be two ETOL forms and let $k_1, ..., k_n$ be integers such that $\alpha \to x$ in P_i implies $\alpha \Rightarrow_{\bar{P}_i}^{k_i} x$. Then for each $F' = (V', \Sigma', P'_1, ..., P'_n, S') \lhd F(\mu)$ an ETOL system $\bar{F}' = (\bar{V}, \bar{\Sigma}', \bar{P}'_1, ..., \bar{P}'_n, S') \lhd \bar{F}(\bar{\mu})$ can be constructed such that: $\alpha' \to x'$ is in P'_i iff α' is in $\mu(V)$ and $\alpha' \Rightarrow_{\bar{P}'_i}^{k_i'} x'$.*

Proof. Let $F_i = (V, \Sigma, P_i, S)$, $\bar{F}_i = (\bar{V}, \bar{\Sigma}, \bar{P}_i, S)$ and $F'_i = (V', \Sigma', P'_i, S')$. Since $F'_i \lhd F_i(\mu)$ and $\alpha \to x$ in F_i implies $\alpha \Rightarrow_{\bar{F}_i}^{k_i} x$ then by Lemma 3.2 of [MSW] an E0L system $\bar{F}'_i = (\bar{V}'_i, \Sigma', \bar{P}'_i, S') \lhd \bar{F}_i(\bar{\mu})$ can be constructed such that $\alpha' \to x'$ is in P'_i iff α' is in $\mu(V)$ and $\alpha' \Rightarrow^{k_i}_{P'_i} x'$. Extending $\bar{\mu}$ in the obvious way to give $\bar{F}' \lhd \bar{F}(\bar{\mu})$ the result follows immediately.

We now have:

LEMMA 3.4. *If $F = (V, \Sigma, P_1, \ldots, P_n, S)$ and $\bar{F} = (\bar{V}, \bar{\Sigma}, \bar{P}_1, \ldots, \bar{P}_n, \bar{S})$ are ET0L forms and for some integers $k_1, \ldots, k_n$, $\alpha \to x$ in P_i implies $\alpha \Rightarrow^{k_i}_{nt\bar{P}_i} x$, then $\mathscr{L}(F) \subseteq \mathscr{L}(\bar{F})$.*

Proof. Let F' be an arbitrary interpretation of F, $F' \lhd F(\mu)$. By Lemma 3.3 an ET0L form $\bar{F}' \lhd \bar{F}(\bar{\mu})$ can be constructed with $L(F') = L(\bar{F}') \cap \mu(V)^*$. From this the result follows readily.

Corresponding to Lemma 3.4 in [MSW] we have the following weak converse:

LEMMA 3.5. *Let $F = (V, \Sigma, P_1, \ldots, P_n, S)$ and $\bar{F} = (V \cup \hat{V}, \Sigma, \bar{P}_1, \ldots, \bar{P}_n, S)$ be ET0L forms with $V \cup \hat{V} = \varnothing$. Suppose for some integers $k_1, k_2, \ldots, k_n$ the conditions α is in V and $\alpha \Rightarrow^{k_j}_{\bar{P}_j} x$ imply:*

(i) $\alpha \underset{tnt\bar{P}_j}{\overset{k_j}{\Rightarrow}} x$,

(ii) $\alpha \to x$ *is in* P_j,

then $\mathscr{L}(\bar{F}) \subseteq \mathscr{L}(F)$.

Proof. This can be reduced to the proof of the Second Simulation Lemma, Lemma 3.4 for E0L forms given in [MSW].

4. REDUCTIONS OF ET0L FORMS

Given an arbitrary ET0L form we show how form equivalent simpler ET0L forms can be constructed.

DEFINITION. Let $F = (V, \Sigma, P_1, \ldots, P_n, S)$ be an ET0L form. F is *separated*, if for all P_i, $\alpha \to x$ is in P_i implies x is in $\Sigma \cup (V - \Sigma)^*$ and x is not in Σ if α is in Σ. F is *synchronized* if, for each a in Σ, $a \Rightarrow^+ x$ implies x is not in Σ^*. F is *short*, if for all P_i, $\alpha \to x$ in P_i implies $|x| \leqslant 2$, and F is *binary* if each rule in each table is of one of the types: $A \to \epsilon$, $A \to a$, $A \to B$, $A \to BC$, $a \to A$, where a is in Σ and A, B, C are in $V - \Sigma$.

Through a sequence of lemmas, which are of interest in themselves, we approach the two main results of this section, namely, for each synchronized ET0L form F there exists a form equivalent ET0L form $\bar{F}$ such that (1) $\bar{F}$ is synchronized, binary and propagating, and (2) $\bar{F}$ has two tables and is synchronized, binary and propagating.

LEMMA 4.1. *For every ET0L form F a form equivalent reduced ET0L form $\bar{F}$ can be constructed.*

Proof. Clear.

Henceforth we will assume ETOL forms are reduced.

LEMMA 4.2. *For every ETOL form* $F = (V, \Sigma, P_1, ..., P_n, S)$ *a form equivalent separated ETOL form* $\bar{F} = (\bar{V}, \Sigma, \bar{P}_1, ..., \bar{P}_n, S)$ *can be constructed.*

Proof. Let $\bar{V} = V \cup V'$ where $V' = \{\bar{\alpha}: \alpha \text{ in } V\} \cup \{\bar{\epsilon}\}$. Define $\overline{xy} = \bar{x}\bar{y}$ for x, y in V^+ and for all i, $1 \leqslant i \leqslant n$, let $\bar{P}_i = \{\alpha \to \bar{x}: \alpha \to x \text{ in } P_i\} \cup \{\bar{\alpha} \to \alpha: \bar{\alpha} \text{ in } V'\}$.

Now using the simulation lemmas of Section 3, namely Lemmas 3.4 and 3.5, it is straightforward to show that $\bar{F}$ is form equivalent to F.

It was originally thought possible to synchronize any arbitrary ETOL form to give a form equivalent ETOL form but as Example 1 in Section 2 demonstrates there exist ETOL forms which cannot be synchronized. However synchronization is necessary in the following and we will therefore usually deal with synchronized forms.

LEMMA 4.3. *For every separated and synchronized ETOL form* $F = (V, \Sigma, P_1, ..., P_n, S)$ *a form equivalent separated, synchronized and short ETOL form* $\bar{F} = (\bar{V}, \Sigma, \bar{P}_1, ..., \bar{P}_n, S)$ *can be constructed.*

Proof. Carry out a construction based on Lemma 4.4 of [MSW] for all tables simultaneously. The simulation lemmas Lemmas 3.4 and 3.5 are then used to prove form equivalence.

LEMMA 4.4. *For every separated, synchronized and short ETOL form* $F = (V, \Sigma, P_1, ..., P_n, S)$ *a form equivalent separated, synchronized and binary ETOL form* $\bar{F} = (\bar{V}, \Sigma, \bar{P}_1, ..., \bar{P}_n, S)$ *can be constructed.*

Proof. The only non-binary rules occurring in $P_1, ..., P_n$ are either $a \to \epsilon$ or $a \to BC$, where a is in Σ. The first, $a \to \epsilon$, cannot occur since F is synchronized and the second can be replaced by $a \to N$ for some new nonterminal N, where $\bar{V} = V \cup \{N\}$, also adding $N \to N$ to each P_i. Since F is synchronized this does not affect the family of languages generated by F, hence the result.

THEOREM 4.5. *For every synchronized ETOL form* $F = (V, \Sigma, P_1, ..., P_n, S)$ *a form equivalent synchronized, binary and propagating ETOL form* $\bar{F} = (\bar{V}, \Sigma, \bar{P}_1, ..., \bar{P}_n, \bar{S})$ *can be constructed.*

Proof. We can assume F is reduced and binary as well as synchronized. Hence we may assume there is some special nonterminal N in $V - \Sigma$ for which the only rule for N in each P_i is $N \to N$. Now the rules of each P_i have the following form:

(1) $A \to B$,
(2) $A \to BC$, $\quad$ A, B, C in $V - \Sigma$,
(3) $A \to \epsilon$,
(4) $a \to N$, $\quad$ for every a in Σ,
(5) $A \to a$, $\quad$ A in $V - \Sigma$.

We now use a modification of the construction given in Salomaa [14] and Rozenberg and Wood [13]. A detailed proof will be omitted since it follows that given in [MSW] for E0L forms.

For $X \subseteq V - \Sigma$ define $P_i(X) = \{Y: Y \subseteq V - \Sigma$, there exists x, y in $(V - \Sigma)^*$ such that $x \Rightarrow_{P_i} y$, $X = \mathrm{Alph}(x)$ and $Y = \mathrm{Alph}(y)\}$.

Now construct a propagating ET0L form $\bar{F} = (\bar{V}, \Sigma, \bar{P}_1, ..., \bar{P}_n, \bar{S})$ where $\bar{V} = \Sigma \cup \{A[X]$: A is in $V - \Sigma$ and $X \subseteq V - \Sigma\}$, $\bar{S} = S[\phi]$ and for all i, $1 \leqslant i \leqslant n$, $\bar{P}_i$ contains:

(1′) $A[X] \to B[Y]$, where $A \to B$ is in P_i — for all $X \subseteq V - \Sigma$ and Y in $P_i(X)$

(2′) $A[X] \to B[Y]\, C[Y]$, — where $A \to BC$ is in P_i, for all $X \subseteq V - \Sigma$ and Y in $P_i(X)$

(2″) $A[X] \to B[Y \cup \{C\}]$ if $C \overset{*}{\Rightarrow}_F \epsilon$, — where $A \to BC$ is in P_i, for all $X \subseteq V - \Sigma$ and Y in $P_i(X)$

(2‴) $A[X] \to C[Y \cup \{B\}]$ if $B \overset{*}{\Rightarrow}_F \epsilon$, — where $A \to BC$ is in P_i, for all $X \subseteq V - \Sigma$ and Y in $P_i(X)$

(4′) $a \to N[\phi]$ if $a \to N$ is in P_i,

(5′) $A[X] \to a$ for all $X \subseteq V - \Sigma$, if ϕ is in $P_i(X)$, and $A \to a$ is in P_i.

It is straightforward to show that $\mathscr{L}(F) \subseteq \mathscr{L}(\bar{F})$, the reverse conclusion involves the construction of an $F' \lhd F(\mu)$ such that $L(F') = L(\bar{F}')$ where $\bar{F}'$ is an arbitrary interpretation $\bar{\mu}$ of $\bar{F}$. F' consists of all the rules of $\bar{F}'$ which are interpretations of (1′), (2′), (4′) and (5′). Rules which are interpretations of (2″) and (2‴) are simulated in F' by appropriate interpretations of rules of types (1), (2) and (3). A complete proof for the E0L case can be found in [MSW].

THEOREM 4.6. *For every synchronized n-ET0L form F with $n > 2$ a form equivalent synchronized 2-ET0L form $\bar{F}$ can be constructed.*

Proof. Let $F = (V, \Sigma, P_1, ..., P_n, S)$. We shall construct $\bar{F} = (\bar{V}, \Sigma, \bar{P}_1, \bar{P}_2, S)$ as follows: We can assume there is a special nonterminal N in $V - \Sigma$ which only has the rule $N \to N$ in each table P_i. Let $\bar{V} = V \cup \{[\alpha, i]$: α in V, $1 \leqslant i \leqslant n\}$, $\bar{P}_1 = \{[\alpha, i] \to x$: $\alpha \to x$ in P_i, $1 \leqslant i \leqslant n\} \cup \{\alpha \to N$: α in $V\}$ and $\bar{P}_2 = \{\alpha \to [\alpha, 1], [\alpha, n] \to N, N \to N$: α in $V\} \cup \{[\alpha, i] \to [\alpha, i+1]$: α in V and $1 \leqslant i < n\}$.

The inclusion $\mathscr{L}(F) \subseteq \mathscr{L}(\bar{F})$ is straightforward and is left to the reader. Notice that the construction given here is different from that given in Rozenberg [12] where $\bar{P}_1$ only contains rules which rewrite symbols of the form $[\alpha, i]$ as words from $\{[\alpha, i]$: α in $V\}^*$. Although for Rozenberg's construction it can be shown that $\mathscr{L}(F) \subseteq \mathscr{L}(\bar{F})$, it is not clear that the reverse inclusion obtains. The present construction is pleasing because it enables an $F' \lhd F(\mu)$ to be constructed easily when given an arbitrary $\bar{F}' \lhd \bar{F}(\bar{\mu})$, such that $L(F') = L(\bar{F}')$.

We prove $\mathscr{L}(\bar{F}) \subseteq \mathscr{L}(F)$ as follows: Consider an arbitrary interpretation $\bar{F}' \lhd \bar{F}(\bar{\mu})$. Letting $\bar{F}' = (\bar{V}', \Sigma', \bar{P}'_1, \bar{P}'_2, S')$ construct $F' = (V', \Sigma', P'_1, ..., P'_n, S')$ such that $F' \lhd F(\mu)$, where $\mu(A) = \bar{\mu}(A)$ for A in V. The following notation is useful.

For x, y in $\mu(V)^*$ and an integer k, $1 \leqslant k \leqslant n$, we write $x \Rightarrow_{F'}^{[k]} y$ if there is a derivation

$$x = x_0 \underset{\bar{P}'_2}{\Rightarrow} x_1 \underset{\bar{P}'_2}{\Rightarrow} \cdots \underset{\bar{P}'_2}{\Rightarrow} x_k \underset{\bar{P}'_1}{\Rightarrow} x_{k+1} = y$$

where x_i is in $\bar{\mu}(\bar{V} - V)^*$, $1 \leqslant i \leqslant k$. The rules of $\bar{P}_2$ ($\bar{P}'_2$) perform a cyclic renaming of symbols such that after k renaming steps, the symbols which enable the original table P_k (P'_k) to be simulated are obtained. Now we are in a position to define P'_i, namely, $P'_i = \{\alpha \to x: \alpha \Rightarrow_{F'}^{[k]} x, \alpha \text{ in } \bar{\mu}(V)\}$.

Clearly

$$S' = x_0 \underset{P'_{i(1)}}{\Rightarrow} x_1 \underset{P'_{i(2)}}{\Rightarrow} \cdots \underset{P'_{i(m)}}{\Rightarrow} x_m \qquad \text{in } \Sigma'^*$$

iff

$$S' = x_0 \overset{[i(1)]}{\underset{F'}{\Rightarrow}} x_1 \overset{[i(2)]}{\underset{F'}{\Rightarrow}} \cdots \overset{[i(m)]}{\underset{F'}{\Rightarrow}} x_m \qquad \text{in } \Sigma'^*;$$

hence $L(F') = L(\bar{F}')$. Further it is clear that F' is an interpretation of F under μ.

5. Completeness of ET0L Forms

We first need:

Definition. For X a type of form and Y a family of languages we say an X *form* F is Y-complete if $\mathscr{L}(F) = Y$.

In this section we investigate E0L- and ET0L-complete ET0L forms.

Note first that the form $F = (\{S, a\}, \{a\}, P_1, P_2, S)$ with $P_1 = P_2 = \{S \to a \mid S \mid SS, a \to S\}$ is ET0L complete. (For consider any ET0L language L. It is well known that L can be generated by a binary, synchronized and propagating 2-ET0L system G; and clearly $G \lhd F$.)

The above example might suggest the following idea: If an ET0L form $F = (V, \Sigma, P,..., P_n, S)$ is ET0L-complete then is it true that at least one of the E0L forms $F_i = (V, \Sigma, P_i, S)$ is E0L complete? This intuitively satisfying conjecture is shown to be false by the next theorem. (That $(\{S, a\}, \{a\}, \{S \to a \mid SS; a \to S\}, S)$ is not E0L complete is shown in e.g. (Culik *et al.* [4]).

Theorem 5.1. *$F = (\{S, a\}, \{a\}, P_1, P_2, P_3, P_4, S)$ where $P_1 = \{S \to a \mid SS; a \to S\}$, $P_2 = P_3 = P_4 = \{S \to S \mid a; a \to S \mid a\}$ is ET0L-complete.*

Proof. Consider $L \subseteq \Sigma^*$ in ET0L then there exists an ET0L system $G = (V, \Sigma, T_1, T_2, Q)$ which is both synchronized and binary, that is, the rules of T_1 and T_2 have the "form": T_1: $A \to BC \mid B \mid a$; $a \to A$ and T_2: $A \to B$; $a \to A$. There always exists such an ET0L system for each ET0L language because of the constructions Theorems 4.5 and 4.6 which also produce *equivalent* systems and because an ET0L system can always be synchronized.

We prove, in the following, that cLd is in $\mathscr{L}(F)$, for any c and d. This is sufficient since any L in ET0L can be partitioned as $L = L_{\text{fin}} \bigcup_{a,b \text{ in } \Sigma} aL_{ab}b$ where L_{ab} is the result of taking the left quotient of L with a, the right quotient of the result with b and subtracting the empty word, i.e. $L_{ab} = (a \backslash L/b) - \{\epsilon\}$, and L_{fin} is the finite set of words of length 1 or 2 which remain. Since ET0L is closed under left and right quotient L_{ab} is in ET0L, clearly, L_{fin} is in $\mathscr{L}(F)$ and $\mathscr{L}(F)$ is closed under finite union, hence the result.

Number the rules in T_1 and T_2 uniquely from 1 to l.

Construct $F' = (V', \Sigma', P'_1, P'_2, P'_3, P'_4, S') \lhd F(\mu)$, for some μ, where $V' = V \cup \{\alpha^1, \alpha^2\colon \alpha \text{ in } V\} \cup \{N, K, M, S', S_1, S_2\} \cup \{n, k, c, d\} \cup \{\alpha_p\colon \alpha \text{ in } V \text{ and } 1 \leqslant p \leqslant l\}$ and $\Sigma' = \Sigma \cup \{a^1, a^2\colon a \text{ in } \Sigma\} \cup \{a_p\colon a \text{ in } \Sigma \text{ and } 1 \leqslant p \leqslant l\}$, note that c and/or d may already be in V.

Define P'_1 and P'_2 as follows:

(i) $p\colon A \to BC$ in T_1, then $A^1 \to A_p$ in P'_2, $A_p \to B^1C^1$ in P'_1,

(ii) $p\colon A \to B$ in T_1, then $A^1 \to a_p$ in P'_2, $a_p \to B^1$ in P'_1,

(iii) $p\colon A \to a$ in T_1, then $A^1 \to A_p$ in P'_2, $A_p \to a^1$ in P'_1,

(iv) $p\colon a \to A$ in T_1, then $a^1 \to a_p$ in P'_2, $a_p \to A^1$ in P'_1,

(v) $p\colon A \to B$ in T_2, then $A^2 \to a_p$ in P'_2, $a_p \to B^1$ in P'_1,

(vi) $p\colon a \to A$ in T_2, then $a^2 \to a_p$ in P'_2, $a_p \to A^1$ in P'_1,

(vii) $n \to N$, $K \to k$ in P'_2, $N \to n$, $k \to K$ in P'_1,

(viii) $S' \to S_1S_2$, $S_1 \to n$, $S_2 \to Q^1K$ are in P_1', and

(ix) all other symbols have blocking rules $A \to M$ and $M \to M$.

Let

$$P'_3 \text{ contain } \alpha^1 \to \alpha^2,\ n \to n,\ K \to K \text{ for all } \alpha \text{ in } V,$$

$$P'_4 \text{ contain } \alpha^1 \to \alpha,\ n \to c,\ K \to d \text{ for all } \alpha \text{ in } V,$$

and in both cases all other symbols have blocking rules.

Clearly F' is an interpretation of F. $L(F') = cLd = cL(G)\,d$ can be seen by noting that in any intermediate step of a derivation the leftmost and rightmost symbols are always out of phase except when P'_4 is applied. This ensures that no spurious terminal words are generated. Secondly P'_2 and P'_1 simulate T_1 and T_2, superscripts on symbols indicating which table is being simulated and subscripts indicating the unique rule that is being simulated. Finally P'_3 switches from T_1 to T_2.

Hence we have shown that any ET0L language can be obtained from some interpretation of F, in other words F is ET0L-complete.

We do not know whether a complete 2-ET0L form both of whose tables are not E0L complete exists. However, by the next theorem we establish complete 2-ET0L forms in which only one table is E0L complete.

THEOREM 5.2. *$\tilde{F} = (\{S, a\}, \{a\}, \tilde{P}_1, \tilde{P}_2, S)$ where $\tilde{P}_1 = \{S \to a \mid S \mid SS;\ a \to S\}$ and $\tilde{P}_2 = \{S \to S;\ a \to S\}$ and $\hat{F} = (\{S, a\}, \{a\}, \hat{P}_1, \hat{P}_2, S)$, where $\hat{P}_1 = \tilde{P}_1$ and $\hat{P}_2 = \{S \to S;\ a \to a\}$ are ET0L-complete.*

Proof. We have shown in Theorem 5.1 that $F = (\{S, a\}, \{a\}, P_1, P_2, P_3, P_4, S)$ is ET0L-complete. We construct a form-equivalent 2-ET0L form $\bar{F} = (\bar{V}, \{a\}, \bar{P}_1, \bar{P}_2, S)$ by applying Theorem 4.6 and Lemma 5.3.

Let

$$\bar{P}_1 = \{[S, 1] \to a \mid SS; [a, 1] \to S; [S, 2] \to a \mid S; \\ [a, 2] \to a \mid S; [S, 3] \to a \mid S; [a, 3] \to a \mid S; [S, 4] \to a \mid S; \\ [a, 4] \to a \mid S; S \to N; a \to N; N \to N\}$$

and

$$\bar{P}_2 = \{S \to [S, 1]; [S, 1] \to [S, 2]; [S, 2] \to [S, 3]; [S, 3] \to [S, 4]; [S, 4] \to N; \\ a \to [a, 1]; [a, 1] \to [a, 2]; [a, 2] \to [a, 3]; [a, 3] \to [a, 4]; [a, 4] \to N; \\ N \to N\},$$

and

$$\bar{V} = \{S, a, N\} \cup \{[S, i], [a, i]: 1 \leqslant i \leqslant 4\}.$$

Notice that $\bar{F} \lhd \tilde{F}$ where $\tilde{F} = (\{S, a\}, \{a\}, \tilde{P}_1, \tilde{P}_2, S)$, $\tilde{P}_1 = \{S \to a \mid S \mid SS; a \to S\}$ and $\tilde{P}_2 = \{S \to S; a \to S\}$ hence $\tilde{F}$ is ET0L-complete.

Modify $\bar{F}$ above to obtain $\bar{\bar{F}} = (\bar{\bar{V}}, \{a\}, \bar{\bar{P}}_1, \bar{\bar{P}}_2, S)$ where

$$\bar{\bar{P}}_1 = \{[S, 1] \to [a, 1] \mid [S, 1][S, 1] \mid a; [a, 1] \to a \mid [a, 2] \mid [a, 1]; \\ [S, 2] \to [a, 1] \mid [S, 1] \mid a; [a, 2] \to [a, 1] \mid [S, 1] \mid a; \\ [S, 3] \to [a, 1] \mid [S, 1] \mid a; [a, 3] \to [a, 1] \mid [S, 1] \mid a; \\ [S, 4] \to [a, 1] \mid [S, 1] \mid a; [a, 4] \to [a, 1] \mid [S, 1] \mid a; \\ S \to N; a \to N; N \to N\}$$

and

$$\bar{\bar{P}}_2 = \{S \to [S, 1]; [S, 1] \to [S, 2]; [S, 2] \to [S, 3]; [S, 3] \to [S, 4]; [S, 4] \to N; \\ [a, 1] \to [a, 2]; [a, 2] \to [a, 3]; [a, 3] \to [a, 4]; [a, 4] \to N; \\ a \to a; N \to N\},$$

then clearly $\mathscr{L}(\bar{F}) \subseteq \mathscr{L}(\bar{\bar{F}})$ and further $\mathscr{L}(\bar{\bar{F}}) \subseteq \mathscr{L}(\bar{F})$ since $\bar{F}$ is ET0L-complete. Finally notice that $\bar{\bar{F}} \lhd \hat{F}$ hence the result.

LEMMA 5.3. *Given an ET0L form F a synchronized ET0L form $\bar{F}$ can be constructed such that $\mathscr{L}(F) \subseteq \mathscr{L}(\bar{F})$. Hence $\bar{F}$ is ET0L-complete if F is ET0L-complete.*

Proof. Consider $F = (V, \Sigma, P_1, \ldots, P_n, S)$ and construct $\bar{F} = (\bar{V}, \Sigma, \bar{P}_1, \ldots, \bar{P}_n, S)$ where $\bar{V} = V \cup \{N\} \cup \{\bar{\alpha}: \alpha \text{ in } \Sigma\}$ and $P_i = \{bar(\alpha) \to bar(x): \alpha \to x \text{ in } P_i\} \cup \{\bar{\alpha} \to \alpha: \bar{\alpha} \text{ in } \bar{\Sigma}\} \cup \{\alpha \to N: \alpha \text{ in } \Sigma\} \cup \{N \to N\}$, $1 \leqslant i \leqslant n$, where $bar(x)$ replaces terminals in x by their barred versions.

(i) $\mathscr{L}(F) \subseteq \mathscr{L}(\bar{F})$. Consider an $F' \lhd F(\mu)$, it should be clear that a synchronized $\bar{F}'$ can be obtained such that $L(F') = L(\bar{F}')$ using the construction and further that

$\bar{F}' \lhd \bar{F}(\bar{\mu})$ for some $\bar{\mu}$. (This is the usual construction and proof for an ET0L system.) The reverse inclusion does not hold in general as already pointed out in Section 4.

(ii) If F is ET0L-complete then because $\text{ET0L} = \mathscr{L}(F) \subsetneq \mathscr{L}(\bar{F})$, $\bar{F}$ is ET0L-complete.

COROLLARY 5.4. *$F_i = (\{S, a, N\}, \{a\}, P_1{}^i, P_2{}^i, S)$, $1 \leqslant i \leqslant 3$ are ET0L-complete, where $P_1{}^1 = P_1{}^2 = \{S \to a \mid S \mid SS; a \to N; N \to N\}$, $P_1{}^3 = \{S \to a \mid S \mid SS; a \to S; N \to N\}$, $P_2{}^1 = \{S \to S; a \to S; N \to N\}$ and $P_2{}^2 = P_2{}^3 = \{S \to S; a \to N; N \to N\}$.*

Proof. $F = (\{S, a\}, \{a\}, P_1, P_2, S)$ where $P_1 = \{S \to a \mid S \mid SS; a \to S\}$ and $P_2 = S \to S$; $a \to S$ is ET0L-complete (Theorem 5.2). Construct $F_4 = (\{S, \bar{a}, N, a\}, \{a\}, P_1{}^4, P_2{}^4, S)$ where $P_1{}^4 = \{S \to \bar{a} \mid S \mid SS; \bar{a} \to a \mid S; a \to N; N \to N\}$ and $P_2{}^4 = \{S \to S; \bar{a} \to S; a \to N; N \to N\}$. F_4 is ET0L-complete by the previous lemma. Consider $F_5 = (\{S, a, M, N\}, \{a\}, P_1{}^5, P_2{}^5, S)$ where $P_1{}^5 = \{S \to a \mid S \mid SS; a \to M; a \to N; M \to M; N \to N\}$ and $P_2{}^5 = \{S \to S; a \to N; a \to M; N \to N; M \to M\}$. $F_4 \lhd F_5$ so F_5 is ET0L-complete. Now it is easy to show that $F_5 \lhd F_i$, $1 \leqslant i \leqslant 3$, hence each of the F_i is ET0L-complete.

We have shown "partial" synchronization preserves ET0L-completeness for the specific forms detailed above. This result could be proved in general.

In [MSW] a number of E0L-complete forms were given, in particular, $\{S \to a \mid S \mid SS; a \to S\}$ was shown to be E0L-complete, hence we obtain:

THEOREM 5.5. *Each of the following forms $F_i = (V^i, \Sigma^i, P_1{}^i, P_2{}^i, S)$ where $P_2{}^i = \{S \to S; a \to S\}$ and $P_1{}^i$ is given below are ET0L-complete:*

$P_1{}^1$: $S \to a \mid S \mid SS; a \to S$,

$P_1{}^2$: $S \to a \mid S \mid Sa; a \to S$,

$P_1{}^3$: $S \to a \mid S \mid aS; a \to S$,

$P_1{}^4$: $S \to a \mid S \mid SS; a \to SS$,

$P_1{}^5$: $S \to a \mid \epsilon \mid S \mid SSS; a \to S$,

$P_1{}^6$: $S \to A; A \to a \mid S \mid SS; a \to A$,
$P_1{}^7$: $S \to a \mid S \mid SSA; A \to \epsilon; a \to S$, } with $A \to A$ added to second table,

$P_1{}^8$: $S \to a \mid S \mid SS; N \to N; a \to N$,
$P_1{}^9$: $S \to a \mid S \mid SS; N \to NN; a \to N$, } with $N \to N$ added to second table,

P_1^{10} $S \to a \mid S \mid SS; a \to N; N \to N$ with P_2^{10}: $S \to S; a \to N; N \to N$.

Proof. F_1 is ET0L-complete by Theorem 5.2.

Since $S \Rightarrow_{nt}^2 a$, $S \Rightarrow_{nt}^2 S$, $S \Rightarrow_{nt}^2 SS$ and $a \Rightarrow_{nt}^2 S$ in table $P_1{}^i$ for $i = 2, 3, 5, 6$ and 7, we have by Lemma 3.4 that F_2, F_3, F_5, F_6 and F_7 are ET0L-complete. F_{10} is ET0L-complete by Corollary 5.4. Now $F_{10} \lhd F_8$ hence F_8 is ET0L-complete. Adding $N \to N$ to $P_1{}^9$ does not change $\mathscr{L}(F_9)$, this gives $F_8 \lhd F_9$, hence F_9 is ET0L-complete. Finally, from F_4 obtain $F'_4 \lhd F_4$, F'_4: $\{S \to a \mid S \mid SS; a \to NN; N \to N\}$, $\{S \to S; a \to S;$

$N \to N\}$, and now replacing $a \to NN$ by $a \to N$ in the first table does not change $\mathscr{L}(F'_4)$. Now $F'_4 = F_8$, hence F_4 is ET0L-complete.

THEOREM 5.6. *Each of the ET0L forms F_i, $1 \leqslant i \leqslant 9$ in Theorem 5.5 remains ET0L-complete even when $P_2{}^i = \{S \to S; a \to a\}$. Add $A \to A$ and/or $N \to N$ to $P_2{}^i$ when necessary.*

Proof. F_1 is ET0L-complete by Theorem 5.2. F_2, F_3, F_4, F_5, F_6, F_7 and F_9 are ET0L-complete by the arguments given in Theorem 5.5. F_8 is ET0L-complete since we can "partially" synchronize F_1 as in Corollary 5.4 to obtain $\bar{F}_1$ which is an interpretation of F_8.

Remark. The above two theorems seem to indicate that adding either $\{S \to S; a \to S\}$ or $\{S \to S; a \to a\}$ to an E0L-complete E0L form over the alphabet $\{S, a\}$ gives an ET0L-complete ET0L form. This seems reasonable since the second table enables two or more E0L-like tables to be simulated.

We now demonstrate a 2-ET0L form which is E0L-complete, neither of whose tables gives an E0L-complete E0L form when taken alone.

THEOREM 5.7. *$F = (\{S, a\}, \{a\}, P_0, P_1, S)$, where $P_0 = \{S \to S \mid SS; a \to a\}$ and $P_1 = \{S \to a; a \to a\}$ is E0L-complete.*

Proof. (1) $\text{E0L} \subseteq \mathscr{L}(F)$. Any E0L language L can be generated by some interpretation $\bar{F}'$ of $\bar{F} = (\{S, N, a\}, a, \{S \to a \mid S \mid SS; a \to N; N \to N\}, S)$; see [MSW]. Since $\bar{F}'$ is synchronized $L(\bar{F}')$ can be obtained from some interpretation, F' of F. Synchronization is carried out by the two tables in F'.

(2) $\mathscr{L}(F) \subseteq \text{E0L}$. Take any interpretation F' of F with terminal alphabet $\Sigma' = \{a_1, ..., a_t\}$. For any bit sequence x define a substitution N_x on Σ' as follows:

$$N_x(a) = \begin{cases} \{a\}, & \text{if } x = \epsilon, \\ \{b: a \to b \text{ is in } P_x\}, & \text{if } x = 0 \text{ or } 1, \\ N_{x'}(N_p(a)), & \text{if } x = px',\ p = 0 \text{ or } 1. \end{cases}$$

Intuitively $N_x(Y)$, where $Y \subseteq \Sigma'$, is the set of all terminals obtainable from terminals of Y by the table sequence x.

Since $|N_x(a)| \leqslant |\Sigma'|$ for every x there are only finitely many different substitutions of the form N_x; let these substitutions be $N_{x_1}, ..., N_{x_r}$.

Consider the interpretation F'' of F with exactly the same rules for the nonterminals, but with $a \to a$ as the only rule for the terminals.

Note that

$$L(F') = L(F'') \cup \bigcup_x N_x(L(F''))$$

$$= L(F'') \cup \bigcup_{i=1}^{r} N_{x_i}(L(F'')).$$

Since $L(F'')$ is in E0L, and E0L is closed under union and finite substitution $L(F')$ is in E0L as desired.

The normal form given by Theorem 5.7 is interesting since in [MSW] it is shown that propagating, binary E0L-complete E0L forms must have rules of the form $a \to A$, a in Σ, A in $V - \Sigma$. Clearly for E0L-complete ET0L forms (and also ET0L-complete forms, see the following theorem) this is just not true.

THEOREM 5.8. *There exists ET0L-complete ET0L forms whose only terminal rules are* $a \to a$.

Proof. This is a generalization of the result for E0L forms in Theorem 5.7.

Let F be a synchronized, propagating and binary ET0L form which is ET0L-complete. Letting $F = (V, \Sigma, P_1, \ldots, P_n, S)$ we can also assume V contains a special symbol N whose only rule in each table is $N \to N$ (the blocking symbol), and each table contains $A \to N$ for all A in $V - \Sigma$.

Let $\bar{F} = (V, \Sigma, \bar{P}_1, \ldots, \bar{P}_n, P_1, \ldots, \bar{\bar{P}}_n, S)$ where for all i, $1 \leqslant i \leqslant n$,

(1) $\bar{P}_i = (P_i - \{A \to a, a \to A\colon A \text{ in } V - \Sigma,\ a \text{ in } \Sigma\}) \cup \{a \to a;\ a \text{ in } \Sigma\}$,

(2) $\bar{\bar{P}}_i = \{A \to a\colon A \to a \text{ is in } P_i\} \cup \{A \to N\colon A \to a \text{ is not in } P_i \text{ for any } a \text{ in } \Sigma\} \cup \{a \to a\colon a \text{ in } \Sigma\}$.

We show $\mathscr{L}(F) \subseteq \mathscr{L}(\bar{F})$ and since F is ET0L-complete this implies $\bar{F}$ is ET0L-complete.

Consider an $F' \lhd F(\mu)$, where $F' = (V', \Sigma', P'_1, \ldots, P'_n, S')$. F' must be binary, propagating and synchronized since F is. Add $A \to N'$, for some N' in $\mu(N)$, for all A in $\mu(V - \Sigma)$, to each table P'_i. Clearly this has no effect on $L(F')$.

Construct $\bar{F}' = (V', \Sigma', \bar{P}'_1, \ldots, \bar{P}'_n, \bar{\bar{P}}'_1, \ldots, \bar{\bar{P}}'_n, S')$ as before, except that in $\bar{\bar{P}}'_i$ add the rules $A \to N'$ for all N' in $\mu(N)$, rather than simply $A \to N$. Letting $\bar{\mu}(A) = \mu(A)$ for all A in V', it is clear that $\bar{F}' \lhd \bar{F}(\bar{\mu})$. Noticing that once a word x in $L(\bar{F}')$ is generated applying any table in $\bar{F}'$ leaves it unchanged, we have x in Σ'^* is in $L(F')$ iff

$$S' = x_0 \underset{P'_{i(0)}}{\Rightarrow} \quad \underset{P'_{i(1)}}{\Rightarrow} \cdots \underset{P'_{i(m-1)}}{\Rightarrow} x_m \underset{P'_{i(m)}}{\Rightarrow} x$$

is in F', where $m \geqslant 0$, x_i is in $(V' - \Sigma')^*$, $0 \leqslant i \leqslant m$, and $1 \leqslant i(j) \leqslant n$, $0 \leqslant j \leqslant m$, iff

$$S' = x_0 \underset{\bar{P}'_{i(0)}}{\Rightarrow} x_1 \underset{\bar{P}'_{i(1)}}{\Rightarrow} \cdots \underset{\bar{P}'_{i(m-1)}}{\Rightarrow} x_m \underset{\bar{\bar{P}}'_{i(m)}}{\Rightarrow} x$$

is in $\bar{F}'$ iff x is in $L(\bar{F}')$. Hence $\mathscr{L}(F) \subseteq \mathscr{L}(\bar{F})$ and since F is ET0L-complete, $\bar{F}$ is also ET0L-complete.

THEOREM 5.9. $F = (\{S, N, a\}, \{a\}, P_1, P_2, P_3, S)$ *where*

$$P_1 = \{S \to S \mid SS;\ a \to a\},$$
$$P_2 = \{S \to S;\ a \to a\},$$
$$P_3 = \{S \to a;\ a \to a\} \qquad \textit{is ET0L-complete.}$$

Proof.

$$P'_1\colon S \to a \mid S \mid SS;\ a \to N;\ N \to N,$$

$$P'_2\colon S \to S;\ a \to N;\ N \to N \text{ is ET0L-complete by Theorem 5.5.}$$

Now by the construction given in the proof of Theorem 5.8 we obtain:

$$\bar{P}_1\colon S \to S \mid SS;\ N \to N;\ a \to a, \qquad \bar{\bar{P}}_1\colon S \to a;\ N \to N;\ a \to a,$$

$$\bar{P}_2\colon S \to S;\ N \to N;\ a \to a, \qquad \bar{\bar{P}}_2\colon S \to N;\ N \to N;\ a \to a.$$

Notice that $\bar{\bar{P}}_2$ is superfluous since it blocks all nonterminals and carries out a finite alphabetic substitution on terminal words. This action of $\bar{\bar{P}}_2$ can be incorporated into either $\bar{P}_1$ or $\bar{P}_2$. Finally, since N is no longer reachable we reduce the ET0L form by removing N obtaining F.

We close this section by examining necessary and sufficient conditions for ET0L-completeness.

As in [MSW] we have:

THEOREM 5.10. *If* $F = (V, \Sigma, P_1, \ldots, P_n, S)$ *is an ET0L-complete form then*

(1) $L(F)$ *must contain a word of length* m *for any* $m \geqslant 1$,

(2) *at least one* P_i *must contain a rule of the form* $A \to x$ *with* A *in* $V - \Sigma$ *and* x *in* Σ^*,

(3) F *is expansive,*

(4) F *is looping.*

Proof. (1) and (2) clear. (3) and (4) are proved by methods similar to those used for E0L forms (see [MSW].

Sufficient conditions follows from the ET0L-completeness of $P_1\colon \{S \to a \mid S \mid SS;\ a \to S\}$, $P_2\colon \{S \to S;\ a \to S\}$ and $\bar{P}_1\colon \{S \to S \mid SS;\ a \to a\}$, $\bar{P}_2\colon \{S \to S;\ a \to a\}$; $\bar{P}_3\colon \{S \to a;\ a \to a\}$. Any ET0L form which simulates either ET0L-complete form is ET0L-complete, giving:

THEOREM 5.11. *If* $F = (V, \Sigma, P_1, P_2, S)$ *is an ET0L form and all the following conditions* (1)–(8) *hold, then* F *is ET0L-complete. In* P_1:

(1) $S \overset{s_1}{\underset{nt}{\Rightarrow}} S$,

(2) $S \overset{s_2}{\underset{nt}{\Rightarrow}} a$,

(3) $S \overset{s_3}{\underset{nt}{\Rightarrow}} SS$,

(4) $a \overset{s_4}{\underset{nt}{\Rightarrow}} xSy$, x, y in V^*,

(5) *there exists an integer* $p \geqslant 1$ *and integers* $i_1, i_2, i_3, i_4 \geqslant 0$ *such that* $i_1 s_1 = s_2 + i_2 s_1 = s_3 + i_3 s_1 = s_4 + i_4 s_1 = p$.

In P_2:

(6) $S \overset{t_1}{\underset{nt}{\Rightarrow}} S$,

(7) $a \overset{t_2}{\underset{nt}{\Rightarrow}} uSv$, u, v in V^*, and

(8) *there exists an integer* $q \geqslant 1$ *and integers* $j_1, j_2 \geqslant 0$ *such that* $j_1 t_1 = t_2 + j_2 t_1 = q$.

Proof. This follows that given in [MSW] for Theorem 5.3. The proof technique is to reduce an arbitrary form F fulfilling these conditions to an intermediate form which simulates the ET0L-complete form $(\{S, a, N\}, \{a\}, \{S \to a \mid S \mid SS; a \to N; N \to N\}, \{S \to S; a \to N; N \to N\}, S)$.

Secondly, we obtain

THEOREM 5.12. *If* $F = (V, \Sigma, P_1, P_2, P_3, S)$ *is an ET0L form and all the following conditions* (1)–(11) *hold, then F is ET0L-complete.*

In P_1:

(1) $S \overset{r_1}{\underset{nt}{\Rightarrow}} S$,

(2) $S \overset{r_2}{\underset{nt}{\Rightarrow}} SS$,

(3) $a \overset{r_3}{\underset{nt}{\Rightarrow}} a$,

(4) *there exist an integer* $l \geqslant 1$ *and integers* i_1, *and* $i_2 \geqslant 0$ *such that* $i_1 r_1 = r_2 + i_2 r_1 = l$.

In P_2:

(5) $S \overset{s_1}{\underset{nt}{\Rightarrow}} S$,

(6) $a \overset{s_2}{\underset{nt}{\Rightarrow}} a$,

(7) *there exists an integer* $p \geqslant 1$ *and integers* j_1 *and* $j_2 \geqslant 0$ *such that* $j_1 s_1 = s_2 + j_2 s_2 = p$.

In P_3:

(8) $S \overset{t_1}{\underset{nt}{\Rightarrow}} a$,

(9) $S \overset{t_2}{\underset{nt}{\Rightarrow}} a$,

(10) $a \overset{t_3}{\underset{nt}{\Rightarrow}} a$,

(11) *there exists an integer* $q \geqslant 1$ *and integers* k_1 *and* $k_2 \geqslant 0$ *such that* $k_1 t_1 = t_2 + k_2 t_1 = q$.

Proof. This follows from Theorem 5.9 in a manner similar to the proof of Theorem 5.11.

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