Addendum

Cross characteristic representations of odd characteristic symplectic groups and unitary groups ✪

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Theorem 3.2 of [3], formulated below, classifies cross characteristic representations \( \Phi \) of finite symplectic and unitary groups \( G \) in which certain elements \( g \) of \( G \) satisfy the inequality \( d_\Phi(g) < o(g) \), where \( d_\Phi(g) \) is the degree of the minimal polynomial of \( \Phi(g) \) and \( o(g) \) is the order of \( g \) modulo \( Z(G) \).

**Theorem 1.** Let \( G = \text{Sp}_{2n}(q) \) with \( n > 1 \) and \( (n, q) \neq (2, 3) \), or \( G = \text{GU}_n(q) \) with \( n > 2 \). Let \( p \) be a prime not dividing \( q \) and let \( g \in G \) be a non-central element such that \( g \) belongs to a proper parabolic subgroup of \( G \) and \( o(g) \) is a power of \( p \). Let \( \Phi \) be an absolutely irreducible \( G \)-representation in characteristic coprime to \( q \) of degree \( > 1 \) such that \( d_\Phi(g) < o(g) \). Then \( \Phi \) is a Weil representation.

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The proof of this theorem relied on the main result of [1], which unfortunately overlooks one extra case. However, our Theorem 1 remains valid in this case.

For the reader’s convenience, we reproduce the correct version of the main result of [1] as follows. In this formulation, a finite classical group means any group lying between the isometry group $I(V)$ and its commutator subgroup $I(V)'$, where $V$ is endowed with either the zero or a nondegenerate orthogonal, symplectic, or Hermitian form; furthermore, $I(V)'$ is assumed to be quasisimple.

**Theorem 2.** [2] Let $G$ be a finite classical group in characteristic $r$. Let $p \neq r$ be a prime and $g \in G$ be a non-central element such that $g$ belongs to a proper parabolic subgroup of $G$ and $o(g)$ is a power of $p$. Let $\Phi$ be any absolutely irreducible representation of $G$ of degree $> 1$ over a field $\mathbb{F}$ of characteristic $\ell \neq r$. Then either $d_\Phi(g) = o(g)$, or $k := o(g)/(q + 1)$ is an integer and $d_\Phi(g) \geq o(g) - k$. Moreover, if $d_\Phi(g) < o(g)$, then for some $z \in Z(I(V))$, one of the following holds.

(i) $G = Sp_{2n}(r), r > 2, n \geq 2, o(g) = r + 1$, and $\text{rank}(g - z) = 2$.

(ii) $SU_n(r) \leq G \leq GU_n(r), r > 2, n > 2, o(g) = r + 1$, and $\text{rank}(g - z) = 1$.

(iii) $SU_n(q) \leq G \leq GU_n(q), r = 2, n > 2, o(g) = p = q + 1$, and $\text{rank}(g - z) = 1$.

(iv) $SU_n(8) \leq G \leq GU_n(8), n > 2, o(g) = 9$, and $\text{rank}(g - z) = 1$.

(v) $SU_n(2) \leq G \leq GU_n(2), n > 4, o(g) = 9$, and $\text{rank}(g - z) = 3$.

(vi) $SU_n(q) \leq G \leq GU_n(q), o(g) = k(q + 1), k > 1, n \equiv 1 \pmod{k}, \text{rank}(g^k - z) = 1$. Moreover, $n > k + 1$ if $p$ is odd.

The case (vi) of Theorem 2 is the extra case that has been overlooked in [1]. The aim of this note is to show that the conclusion of Theorem 1 holds in the extra case (vi) of Theorem 2. In fact, we will prove more.

**Theorem 3.** Under the notation and assumptions of Theorem 2, assume we are in the case (vi). Then $\Phi$ is a Weil representation of $G$, and $d_\Phi(g) = o(g) - k$.

The rest of the note is devoted to prove Theorem 3.

Write $q + 1 = p^c, k = p^b$, so that $o(g) = p^{b+c}$. By assumption, there is some orthonormal basis $(e_1, \ldots, e_n)$ of $V = \mathbb{F}_{q^2}$ such that $h := g^k = \text{diag}(\alpha, \alpha, \ldots, \alpha, \alpha\beta)$ for some $0 \neq \alpha, \beta \in \mathbb{F}_{q^2}$ and $\beta \neq 1$. Since $[g, h] = 1$, $g$ preserves $U := \langle e_1, \ldots, e_{n-1} \rangle \mathbb{F}_{q^2}$ and $\langle e_n \rangle \mathbb{F}_{q^2}$. In particular, $g = \text{diag}(g_2, \gamma)$ in the given basis, with $\gamma \in \mathbb{F}_{q^2}$ and $\gamma^k \neq 1$. Furthermore, $C_{GU_n(q)}(g)G = GU_n(q)$, so without loss we may assume $G = GU_n(q)$.

I. Observe that $|\alpha| = |\beta| = q + 1$. For, $(\alpha\beta)^{(q+1)/p} = \gamma^{p^{b+c-1}} = (\gamma^{q+1})^{p^{b-1}} = 1$. Since $o(h) = q + 1$, we must have $|\alpha| = q + 1$. Now if $|\beta| < q + 1$ then $\beta^{(q+1)/p} = 1$ and so $\omega^{(q+1)/p} = 1$, a contradiction. Thus $g^k$ is a scalar multiple of the pseudoreflection $\alpha^{-1}h$ of order $q + 1$. Notice if $\Psi$ is a Weil representation of $G$ then $d_\Psi(\alpha^{-1}h) = q$ and so $d_\Psi(g) \leq qk = o(g) - k$, whence the equality holds by Theorem 2. Thus we are done if $\Phi$ is a Weil representation. From now one we will assume that $\Phi$ is not a Weil representation. Replacing $g$ by $\gamma^{-1}g$, we may assume that $g = \text{diag}(g_2, 1)$ fixes $e_n$, and so $o(g) = |g|$.
2. Here we assume \( p > 2 \). Then we claim that one can choose an orthonormal basis of \( U \) such that \( g_2 = \text{diag}(g_1, \ldots, g_1) \) and \( g_1 \) is an irreducible \( p \)-element of \( GU_k(q) \). Indeed, consider any eigenvalue \( \lambda \) of \( g_2 \). Then \( \lambda p^b = \alpha \) and so \( |\lambda| = p^{b+c} \). Since \( p > 2 \), it follows that \( \lambda \) is a primitive element of \( \mathbb{F}_{q^{2k}} \) over \( \mathbb{F}_{q^2} \) and so it has the minimal polynomial \( f(t) := t^k - \alpha \) over \( \mathbb{F}_{q^2} \). Hence \( g_2 \) has the minimal polynomial \( f(t) \) and the characteristic polynomial \( f(t)^{(n-1)/k} \). One can choose an irreducible \( p \)-element in \( GU_k(q) \) with the characteristic polynomial \( f(t) \). Up to conjugacy in \( GU_{n-1}(q) \), we may now assume that \( g_2 = \text{diag}(g_1, \ldots, g_1) \).

Next we claim that we may assume \( n = 2k + 1 \). Indeed, \( n \geq 2k + 1 \) by assumption. Assume \( n > 2k + 1 \). We can embed \( g \) in a standard subgroup \( H = H_1 \times H_2 \) of \( G \), with \( H_1 \simeq GU_{n-2k-1}(q) \) and \( H_2 \simeq GU_{2k+1}(q) \). Consider any irreducible constituent \( \Phi_1 \otimes \Phi_2 \) of \( \Phi|_H \) with \( \Phi_1 \in \text{IBr}(H_1) \) and \( \dim(\Phi_2) > 1 \). Such a constituent exists, as otherwise \( \Phi \) is trivial on the perfect group \( SU_{2k+1}(q) \) and so \( \dim(\Phi) = 1 \), a contradiction. Notice that \( g \) projects onto the element \( g' = \text{diag}(g_1, g_1, 1) \) and \( o(g') = o(g) \). Clearly, \( o(g) > d_\Phi(g) \geq d_\Phi(g') \). Assuming the theorem holds in the case \( n = 2k + 1 \), we conclude that \( \Phi_2 \) is a Weil representation. Thus any irreducible constituent of \( \Phi|_{H_2} \) is either of degree 1 or a Weil representation. By [3, Theorem 2.5], \( \Phi \) is also a Weil representation, contrary to our assumption.

In the case \( n = 2k + 1 \), it is not difficult to show that \( g \) stabilizes a \( k \)-dimensional totally singular subspace \( W \) of \( V \).

3. Consider the case \( p = 2 \). Let \( g_3 := g_2^{q^{b-1}} \) and let \( \mu \) be any eigenvalue of \( g_3 \). Then \( \mu^2 = \alpha \) and \( \mu^{q+1} = \alpha^{(q+1)/2} = -1 \). In particular, \( \mu^{-q} = -\mu \). It follows that \( g_3 \) has two eigenvalues \( \pm \mu \) on \( U \) and, moreover, the corresponding eigenspaces are totally singular. Hence \( n = 2m + 1 \), and \( g \) stabilizes the \( m \)-dimensional totally singular subspace \( W \), where \( W \) is the \( \mu \)-eigenspace for \( g_3 \).

4. We have shown in Parts 2, 3 that we may assume \( n = 2m + 1 \) and \( g \) belongs to the parabolic subgroup \( P := \text{Stab}_G(W) \), where \( W \) is an \( m \)-dimensional totally singular subspace in \( U \). Recall that for \( Q := O_r(P) \), \( Z(Q) \) is an elementary abelian \( r \)-subgroup of order \( q^{m^2} \).

Here we consider the case \( n \geq 5 \). As shown in the proof of [3, Lemma 12.5], \( P \) has \( m \) orbits on \( \text{IBr}(Z(Q)) \) \( \setminus \{1 \cdot Z(Q)\} \), labeled as \( O_j \) with \( 1 \leq j \leq m \). Moreover, one can identify \( O_j \) with the set of Hermitian \( m \times m \)-matrices of rank \( j \) over \( \mathbb{F}_{q^2} \) (and the multiplication of characters corresponds to the matrix addition). By [3, Theorem 2.6], \( \Phi|_{Z(Q)} \) has to afford some \( P \)-orbit \( O_j \) with \( j > 1 \), since \( \Phi \) is not a Weil representation.

We claim that \( g \) has some orbit of length \( p^b \) (notice that \( g^{p^b} = h \) acts trivially on \( Z(Q) \)). Assume the contrary. Then \( t := g^{p^{b-1}} \) fixes every element of \( O_j \). It is easy to see that \( O_j \cap O_l \supseteq O_1 \). (Indeed, let \( E_{ij} \) be the \( m \times m \)-matrix with the entry 1 at the \((i,j)\)-position and 0, elsewhere. Then \( X_1 = \sum_{1 \leq i \leq j} E_{i,j+1-i} \) and \( X_1 = E_{11} \) are Hermitian of rank \( j \).) Hence \( t \) fixes every element of \( O_1 \). Arguing as in the proof of [3, Lemma 12.5], we see that \( t \) is represented by \( \text{diag}(D, 1, D^{-1}q, 1) \) with \( D \in GL_m(q^2) \) and \( D(q)XD = X \) for any Hermitian \( m \times m \)-matrix \( X \) of rank 1. One can choose \( X = E_{ii} \), or \( X = b^{q+1}E_{ii} + bE_{i1} + \)
b^iE_{ii} + E_{ll}$ with $i \neq l$ and $b \in \mathbb{F}_{q^2}$. Then the equality $t^D(q)XD = X$ implies $D$ is the scalar matrix $\delta I_m$ with $\delta^{q+1} = 1$. Thus $g^{p^{b+c-1}} = 1$, contrary to $o(g) = p^{b+c}$.

Now we can fix a character $\lambda$ such that the $g$-orbit of $\lambda$ has length $p^b$; in particular, $\lambda^h = \lambda$. Then $Q/\text{Ker}(\lambda) = R \times A$, where $R$ is an extraspecial $r$-group of order $rq^{2j}$ and $A$ is abelian of order $q^{2(m-j)}$. Furthermore, $h = g^{p^b}$ centralizes $Z(R)$ and normalizes both $R$ and $A$. Clearly, the restriction of the $\lambda$-isotypic component $\Psi$ of $\Phi$ to $\langle R, h \rangle$ contains an irreducible constituent $\Psi_0$ of degree $q^j$ that lies above $\lambda$. It is easy to see that $\Psi_0$ is faithful. Also, $h^{p^{b-1}}$ acts regularly on $R/Z(R)$, as $h$ acts on $R/Z(R) = \mathbb{F}_{q^2}$ via multiplication by $\alpha$.

Since $|h| = q + 1$ and $j > 1$, $h$ cannot act irreducibly on $R/Z(R)$. Hence, by a theorem of Hall-Higman and Shult, cf. [1, Theorem 2.6], $d_{\Psi_0}(h) = |h| = p^c$, and so $d_{\Phi}(h) = p^c$ as well. By [1, Proposition 2.15], $d_{\Phi}(g) \geq p^bd_{\Psi}(g^{p^b})$, whence $d_{\Phi}(g) = o(g) = p^{b+c}$.

5. Finally, consider the case $n = 3$. Since $n \geq 2k + 1 \geq 5$ for odd $p$, we have $p = 2$ and $q$ a Mersenne prime in this case. As shown in Part 3, one can find a hyperbolic basis $(u, v)$ of $U$ such that $g = \text{diag}(\lambda, \lambda^{-q}, 1)$ in the basis $(u, v, e_3)$ of $V$. Recall that $g \in P$ and $\Phi$ is not a Weil representation. By [3, Proposition 11.3], $\Phi|_Q$ contains a nontrivial linear character of $Q$. One can identify the set of linear characters of $Q$ with $\text{F}_{q^2}$, so that $g$ acts on it via multiplication by $\lambda$. Hence $g$ acts regularly on the set of nontrivial linear characters of $Q$. Consequently, $d_{\Phi}(g) = |g|$.

The proof of Theorem 3 is complete.

References

