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Relaxation procedures on graphs

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Abstract

The procedures studied in this paper originate from a problem posed at the International Mathematical Olympiad in 1986. We present several approaches to the IMO problem and its generalizations. In this context we introduce a "signed mean value procedure" and study "relaxation procedures on graphs". We prove that these processes are always finite, thus confirming a conjecture of Akiyama, Hosono and Urabe [J. Akiyama, K. Hosono, M. Urabe, Some combinatorial problems. Discrete Mathematics 116 (1993) 291–298]. Moreover, we indicate relations to sorting and to an iterative method used in circle packing.

Keywords: Graph algorithms; Reflection processes; Sorting; Electrical networks; Discrete harmonic functions; Signed mean values; Circle packing; IMO problems

1. The pentagon game

Our starting point is Problem 3 of the International Mathematical Olympiad (IMO) in 1986 which arguably belongs to the hardest challenges this contest has ever seen [7,14,15]. The problem was proposed by the first author of the present paper and originally emerged as a side product of investigations of an old geometric question concerning partial reflections of non-convex polygons (see [13] pp. 30–34). After the competition it turned out that the problem can be generalized in various directions and has interrelations with several other topics. In this paper we collect some known facts and present new perspectives of the problem.

The pentagon game: *Five integers with positive sum are assigned to the vertices of a pentagon. If there is at least one negative number, the player may pick one of them, say y, add it to its two neighbors x and z, and then reverse the sign of y. The game terminates when all the numbers are nonnegative. Prove that this game must always terminate.*

1.1. Decreasing quadratic functions

A standard approach for proving finiteness of a procedure is to construct a positive integer-valued function which decreases in every step.

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First solution. We denote the five numbers in consecutive order by x_1, x_2, x_3, x_4, x_5 and remark that the sum $s := x_1+x_2+x_3+x_4+x_5$ remains invariant. A simple calculation shows that the function f of $x = (x_1, x_2, x_3, x_4, x_5)$, given by

$$f(x) := (x_1 - x_3)^2 + (x_2 - x_4)^2 + (x_3 - x_5)^2 + (x_4 - x_1)^2 + (x_5 - x_2)^2, \tag{1}$$

is strictly decreasing in each step of the game. In fact the value of f changes from $f(x_{\text{old}})$ to

$$f(x_{\text{new}}) = f(x_{\text{old}}) + 2ys < f(x_{\text{old}}).$$

Since all values of f are nonnegative integers, the game must stop after at most $f(x) - 1$ steps. □

This argument was found by all but one of the eleven students who succeeded in solving the problem during contest, and it coincides with the solution suggested by the proposer, but there are other quadratic functions which work as well. One example proposed by Géza Kóz (see [18], p. 321) is the function

$$2(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1) + 3(x_1x_3 + x_2x_4 + x_3x_5 + x_4x_1 + x_5x_2),$$

which is strictly increasing and bounded from above by s^2.

Although the function f provides the simplest solution, it does not immediately generalize to the analogous game played on an arbitrary polygon. For a square, for instance, the required decreasing quadratic function is not the naively expected expression

$$(x_1 - x_3)^2 + (x_2 - x_4)^2,$$

as the counterexample $x_{\text{old}} = (-1, 3, -5, 4)$ with $x_{\text{new}} = (-1, -2, 5, -1)$ shows. A substitute which works is

$$3\{(x_1 - x_3)^2 + (x_2 - x_4)^2\} + \{(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + (x_4 - x_1)^2\}.$$

To handle the hexagon in a similar way one may use something like

$$\sum_{k=1}^{6}\{(x_k - x_{k+3})^2 + 2(x_k - x_{k+2})^2 + 2(x_k + x_{k+1} - 2x_{k+3})^2 + 6(x_k + 2x_{k+1} - 3x_{k+3})^2\}.$$

In the next subsection we describe a construction given by Alon, Krasikov, and Peres [2] which solves the problem for all n.

1.2. Sums of consecutive elements

During the IMO contest all but one participant who solved the problem used the function f defined in (1) (see [14], p. 20). The only alternative solution was found by Joseph G. Keane from the US team, whose idea had the rare distinction to be honored by a special prize. Instead of a sum of squares he considered a function involving absolute values.

Second solution. Let the function g be defined by

$$g(x) := \sum_{j=1}^{5}(|x_j| + |x_j + x_{j+1}| + |x_j + x_{j+1} + x_{j+2}| + |x_j + x_{j+1} + x_{j+2} + x_{j+3}|),$$

where all indices are reduced modulo 5. In each step of the algorithm all but one of the summands remain invariant or switch places. Only the term $|s - y|$ is changed to $|s + y|$, where y denotes the negative number chosen by the player. Consequently g decreases by the positive integer $d := |s - y| - |s + y|$. □

The function g can easily be adapted to the corresponding game played on *polygons* with *real* numbers $x_1, \ldots, x_n$. In order to simplify notations we extend the sequence (x_j) periodically to all integers j and define

$$s_{ij} := x_i + x_{i+1} + \cdots + x_{i+j-1}, \quad i, j \geq 1. \tag{2}$$

Then the generalized function g is a sum of absolute values of the s_{ij},

$$g(x) := \sum_{i=1}^{n}\sum_{j=1}^{n-1}|s_{ij}|. \tag{3}$$

Again g decreases in each step by $d := |s-y| - |s+y|$, which shows that the algorithm stops if the x_j are integers. In order to prove that it also terminates if the x_j are real numbers, we denote by S the multiset of all numbers $|s_{ij}|$ with $1 \le i \le n$ and $1 \le j \le n-1$. As stated above, if the value of an element a in S is changed, then $a = |s-y|$, so $y < 0$ is equivalent to $a > s$. If $s < a \le 2s$ then a is replaced with the new number $|s+y| = 2s - a \le s$, which then must remain constant forever. If $a \ge 2s$, then $|s+y| = a - 2s$, i.e. a is reduced by $2s$. Since this can happen only a finite number of times, any element of S is eventually trapped in the interval $[0, s]$, and then the algorithm must stop.

This argument also shows that the number of steps needed to turn every number non-negative depends only on the initial configuration and *not* on the player's choice.

Indeed, if we denote by $\lfloor x \rfloor'$ that integer satisfying $x - 1 \le \lfloor x \rfloor' < x$, then any $y \in S$ may be reduced $\lfloor \frac{y+s}{2s} \rfloor'$ many times in the manner described above. Note that in each step of the algorithm exactly one element of S is diminished and that there must still be a negative number as long as there remain elements of S outside the interval $[0, s]$. Consequently, the formula

$$N = \sum_{y \in S} \left\lfloor \frac{y+s}{2s} \right\rfloor' \tag{4}$$

gives the total number of operations to be performed.

Alon, Krasikov, and Peres [2] derived a similar formula using the squares s_{ij}^2 instead of the absolute values $|s_{ij}|$. In order to show that (4) coincides with their result we denote by T the multiset of all numbers

$$s_{ij} = x_i + x_{i+1} + \cdots + x_{i+j-1}, \quad 1 \le i \le n, \ 1 \le j \le n-1.$$

Taking into account that $x \mapsto s - x$ maps T bijectively onto itself, we obtain

$$N = \sum_{t \in T, t>0} \left\lfloor \frac{s+t}{2s} \right\rfloor' + \sum_{t \in T, t \le 0} \left\lfloor \frac{s-t}{2s} \right\rfloor' = \sum_{t \in T, t>0} \left\lfloor \frac{s+t}{2s} \right\rfloor' + \sum_{t \in T, t \ge s} \left\lfloor \frac{t}{2s} \right\rfloor'.$$

As $\lfloor \frac{t}{2s} \rfloor'$ and $\lfloor \frac{t+s}{2s} \rfloor'$ give the number of even and odd integers in the interval $(0, \frac{t}{s})$, respectively, both sum up to $\lfloor \frac{t}{s} \rfloor'$. Denoting by $\lceil x \rceil$ the integer with $x \le \lceil x \rceil < x + 1$, we arrive at the formula given in [2],

$$N = \sum_{t \in T, t>s} \left\lfloor \frac{t}{s} \right\rfloor' = \sum_{t \in T, t<0} \left\lfloor \frac{s-t}{s} \right\rfloor' = \sum_{t \in T, t<0} \left\lceil \frac{|t|}{s} \right\rceil. \tag{5}$$

An unbeatably elegant application of sums of consecutive elements is due to Bernard Chazelle ([5], [7] p. 6).

Third solution. Let $\widetilde{S}$ be the *infinite multiset* of all sums s_{ij} defined in (2) with $i, j \in \mathbb{Z}$, $1 \le i \le n$ and $1 \le j$. Since the sum $s = x_1 + \cdots + x_n$ is positive, the number of negative elements in $\widetilde{S}$ is finite. In each step of the game all elements of $\widetilde{S}$, except one, remain invariant or switch places with others. Only the negative number y chosen by the player is changed to $-y$.

In order to verify this we arrange the elements of $\widetilde{S}$ in the following table

$$\begin{array}{cccccccc}
x_1 & x_1 + x_2 & x_1 + x_2 + x_3 & \dots & x_1 + \cdots + x_{n-1} & s & s + x_1 & \dots \\
x_2 & x_2 + x_3 & x_2 + x_3 + x_4 & \dots & x_2 + \cdots + x_n & s & s + x_2 & \dots \\
x_3 & x_3 + x_4 & x_3 + x_4 + x_5 & \dots & x_3 + \cdots + x_1 & s & s + x_3 & \dots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
x_n & x_n + x_1 & x_n + x_1 + x_2 & \dots & x_n + \cdots + x_{n-2} & s & s + x_n & \dots.
\end{array}$$

If, without loss of generality, the player chooses the number $y = x_1$ then the elements in every row, except in the first and the second, are preserved. Apart from the element $-x_1$, the new first row has the same elements as the old second row, and the new second row coincides with the old first row without x_1.

Hence, in every move exactly one negative element of $\widetilde{S}$ is changed to positive. Since the sum s is positive, the number N of negative elements in $\widetilde{S}$ is finite and the algorithm must terminate after at most N steps. In fact it cannot stop earlier, since then $\widetilde{S}$ would still have negative elements, which is impossible if none of the x_i is negative. As $\widetilde{S}$ may be constructed as the infinite multiset of all $t + z \cdot s$, where t runs through T and z through the non-negative integers, we again obtain formula (5). □

Alon, Krasikov, and Peres as well as Chazelle also proved that the final position is uniquely determined by the initial configuration.

A similar solution given by John M. Campbell ([6], see [12]) uses only one infinite two-sided sequence (v_i) of consecutive sums (here shown for $n = 5$)

$$\ldots, -x_4 - x_5, -x_5, 0, x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots, x_1 + \ldots + x_5, 2x_1 + x_2 + \cdots + x_5, \ldots,$$

which is constructed by adding (respectively subtracting) cyclically the numbers x_i.

Fourth solution Any move exchanges the elements v_{i+nj} and v_{i+1+nj} for some $i \in \{1, \ldots, n\}$ with $v_i > v_{i+1}$ and all $j \in \mathbb{Z}$. The procedure can be performed as long as there exists $i \in \{1, \ldots, n\}$ with $v_i > v_{i+1}$. We denote by N_i the (finite) number of elements v_j which stand right of v_i and are less than v_i. It is clear that every move reduces the sum $N = N_1 + \cdots + N_n$ by one, and a little thought then shows that the procedure stops after exactly N steps with a strictly increasing sequence (v_i). □

Campbell's solution reveals that the pentagon game can be reformulated as a sorting procedure, which clearly explains why the final position is uniquely determined. We explore this idea further in the next sections.

1.3. Breaking symmetry – a sorting procedure

The next approach is a finite version of Campbell's solution. The first author learned it in 1987 from Sergej Steinberg [19] during a personal communication in Pushchino. The idea is to represent the numbers x_i as differences.

Fifth solution. Let $y_1 = 0$ and define $y_2, \ldots, y_n$ by

$$y_i = x_1 + x_2 + \cdots + x_{i-1}.$$

Then $x_i = y_{i+1} - y_i$ for $i = 1, \ldots, n-1$ and $x_n = s - y_n = y_1 - y_n + s$. Rewriting the rules of the game for the numbers $y_1, \ldots, y_n$ we get the following two possible operations.

If $y_{i+1} < y_i$ for some $i = 1, \ldots, n-1$ it is allowed to interchange y_{i+1} and y_i. If $y_n > y_1 + s$ it is allowed to replace the first number y_1 with $y_n - s$ and the last number y_n with $y_1 + s$.

It is convenient to think of the numbers y_i as written on cards which are arranged in a line and are numbered from left to right. Then the first operation exchanges two neighboring cards, the card carrying the greater number "moving right" and the card with the smaller number "moving left". The second operation is allowed if the difference $y_n - y_1$ is greater than s. Here the larger number y_n is diminished by s, the smaller number y_1 is increased by s, and the two cards change places.

Since the numbers y_i are only changed by multiples of s, all possible values belong to a discrete set R. This set is even finite, since the maximal number y_i never increases and the minimal number never decreases. Because R is finite, $\max y_i$ must remain constant after a number of steps. Let this maximum be m. Since no card with $y_i < m$ can ever reach the value m, at least one card must carry the number m forever. Now, analogously, each of the remaining $n - 1$ cards can change its value only a finite number of times. Going on by induction, we see that after some time the values of all cards remain unchanged. It is now clear that the remaining sorting process must stop, since it contains no cycle and the number of permutations is finite. □

As an alternative to the above reasoning one may consider the nonnegative function $f(y) = \sum y_j^2$. If the second operation is performed the value of f decreases by

$$d := 2s(y_n - y_1) - 2s^2 = 2s(y_n - y_1 - s) > 0.$$

Since y_n and y_1 belong to the finite set R, the number d is bounded from below by a positive constant. So the second operation can be applied only a finite number of times.

In order to determine the final configuration x_j^* we remark that the final values y_i^* of y_i must satisfy

$$y_1^* \le y_2^* \le \cdots \le y_n^* \le y_1^* + s.$$

Moreover, the final value on each card differs from its initial value by a multiple of s and the sums $\sum_{i=1}^n y_i$ and $\sum_{i=1}^n y_i^*$ must be equal. These observations allow to find y_i^* as follows:

Let $0 \le r_j < s$ be the remainders of y_j modulo s. Denote by r_j^* the rearrangement of the r_j such that $0 \le r_1^* \le \ldots \le r_n^* < s$. Then y_j^* is given by

$$y_i^* = r_{i+j}^* + ks \quad (i = 1, \ldots, n-j), \qquad y_i^* = r_{i+j-n}^* + (k+1)s \quad (i = n-j+1, \ldots, n),$$

where the integers j and k are chosen such that $0 \le j \le n-1$ and

$$\sum_{i=1}^{n} (y_i - r_i) = (kn + j)s.$$

For the final values x_i^* of x_i we then obtain

$$x_1^* = y_2^* - y_1^*, x_2^* = y_3^* - y_2^*, \ldots, x_{n-1}^* = y_n^* - y_{n-1}^*, x_n^* = y_1^* - y_n^* + s.$$

1.4. Keeping symmetry – threshold sorting

The solution via the above sorting procedure breaks the symmetry between the variables. In the sequel we develop a similar approach keeping symmetry. This will lead to a new kind of problems which are treated in more generality in the next section. Here we start with a simple situation.

The threshold sorting procedure. *Let d be a positive constant and let $y_1, \ldots, y_n$ be a finite sequence of real numbers. If there are $u = y_i$ and $v = y_j$ with $u > v + d$ then replace u by $v + d$ and v by $u - d$. Repeat this step as long as numbers u and v with $u > v + d$ exist. Determine whether this procedure always stops.*

First of all we observe that all numbers y_i are changed by multiples of d, their maximum is not increasing and their minimum is not decreasing. Consequently the set of all possible values is finite. Hence there exists a number c such that $u - v - d \ge c > 0$ for all u, v to which the operation might be applied during the whole process. It follows that the nonnegative function $f(y) = \sum y_i^2$ is decreasing in each step of the algorithm by

$$u^2 + v^2 - (u-d)^2 - (v+d)^2 = 2d(u - v - d) \ge 2cd.$$

Therefore the procedure always stops.

The function f gives a rather bad estimate of the number of steps. A better one can be obtained using the function g given by $g(y) := \sum |y_i - y_j|$.

In order to prove that g is decreasing, we remark that the new numbers $u - d$ and $v + d$ lie in the interval with endpoints u and v. It follows that $|(u-d) - (v+d)|$ is less than $|u - v|$ by at least $2 \min(c, d)$, and it is easy to see that for any w the sum $|(u-d) - w| + |w - (v+d)|$ is less than or equal to $|u - w| + |w - v|$.

The last result can be used to symmetrize the fourth solution of the Pentagon game.

Sixth solution. There exist uniquely determined values $y_1 := 0, y_2, \ldots, y_n,\ y_{n+1} := y_1$, such that the x_i have the symmetric representation

$$x_i = y_{i+1} - y_i + \frac{s}{n}, \quad i = 1, \ldots, n.$$

Reformulating the original algorithm for $x_1, \ldots, x_n$ in terms of $y_1, \ldots, y_n$ we get the following operation:

If there are two neighbors $u = y_i$ and $v = y_{i+1}$ such that $u > v + s/n$ then u is replaced with $v + s/n$ and v is replaced with $u - s/n$.

Clearly this rule is more restrictive than that of the Threshold Sorting Procedure, and so the algorithm must stop. □

It is interesting to see how the function g looks in terms of the x_i. In fact it is quite similar to the one considered earlier in (3), namely

$$g(x) = \sum_{i=1}^{n} \sum_{j=i}^{n-1} \left| \frac{(j-i+1)s}{n} - \sum_{k=i}^{j} x_k \right|.$$

2. The signed mean value procedure

It turns out that threshold sorting is just a special case of a more general algorithm which we introduce and investigate in this section. We start with fixing the rules of the game, which is now played on an arbitrary finite collection of real numbers.

The signed mean value procedure. *Fix a positive constant d and let $y_1, \ldots, y_n$ be real numbers. If there are numbers (signs) $\eta_1, \eta_2, \ldots, \eta_n \in \{-1, 0, 1\}$ such that*

$$s := \eta_1 y_1 + \eta_2 y_2 + \cdots + \eta_n y_n > d, \tag{6}$$

then set $m := \eta_1^2 + \eta_2^2 + \ldots + \eta_n^2$ and substitute

$$y_j \mapsto y_j - 2\eta_j \left(\frac{s-d}{m}\right), \quad j = 1, 2, \ldots, n. \tag{7}$$

Repeat this as long as numbers $\eta_1, \eta_2, \ldots, \eta_n \in \{-1, 0, 1\}$ with (6) exist.

Note that m always satisfies $1 \le m \le n$ and can vary from step to step.

The Threshold Sorting Procedure corresponds to the more restrictive rule where all the η_j are zero, except two which are 1 and -1, respectively.

Theorem 1. *The Signed Mean Value Procedure always stops.*

Proof. In what follows we assume that there is a procedure which does not stop.

1. The function $f(y) = y_1^2 + y_2^2 + \cdots + y_n^2$ is strictly decreasing. In fact

$$f(y_{\text{old}}) - f(y_{\text{new}}) = \sum_{j=1}^{n} y_j^2 - \sum_{j=1}^{n} \left(y_j - 2\eta_j \frac{s-d}{m}\right)^2 = 4\frac{d(s-d)}{m} > 0. \tag{8}$$

Let f_k denote the value of f at step k. Since the sequence (f_k) is monotone and bounded it converges to a limit f^*.

2. Let s_k, m_k, $\eta_{j,k}$ and $y_{j,k}$ denote the values of s, m, η_j and y_j in the k-th step, respectively. Since $1 \le m_k \le n$ it follows from (8) that

$$0 < s_k - d \le \frac{n}{4d}(f_k - f_{k+1}). \tag{9}$$

We write $f_1 - f^*$ as a (absolutely) convergent telescopic series,

$$f_1 - f^* = \sum_{k=1}^{\infty} (f_k - f_{k+1}).$$

Together with (9) this shows the (absolute) convergence of $\sum_{k=1}^{\infty}(s_k - d)$, and in particular we have $s_k \to d$. Further, by (7),

$$|y_{j,k+1} - y_{j,k}| \le 2(s_k - d), \quad j = 1, \ldots, n.$$

Consequently, the convergent series

$$y_{j,1} + 2\sum_{k=1}^{\infty}(s_k - d)$$

serves as a majorant for the representation

$$y_{j,k+1} = y_{j,1} + \sum_{i=1}^{k} |y_{j,i+1} - y_{j,i}|.$$

This implies that all sequences $(y_{j,k})$ converge to certain limits Y_j as k tends to infinity.

3. We now consider the difference

$$(\eta_{1,k} y_{1,k} + \cdots + \eta_{n,k} y_{n,k}) - (\eta_{1,k} Y_1 + \cdots + \eta_{n,k} Y_n),$$

which converges to zero, since $y_{j,k} \to Y_j$ as $k \to \infty$. The first term in parentheses is s_k, which has been shown to converge to d. Consequently also

$$\eta_{1,k} Y_1 + \eta_{2,k} Y_2 + \cdots + \eta_{n,k} Y_n \to d. \tag{10}$$

However, the set

$$\{\eta_1 Y_1 + \eta_2 Y_2 + \cdots + \eta_n Y_n : \eta_j \in \{-1, 0, +1\}\}$$

contains only a finite number of elements, which together with (10) then implies that

$$\eta_{1,k} Y_1 + \eta_{2,k} Y_2 + \cdots + \eta_{n,k} Y_n = d \tag{11}$$

for all sufficiently large k, say for all $k \geq K$.

4. Finally, we observe that the values

$$I_k := \sum_{j=1}^{n} (y_{j,k} - Y_j)^2$$

all converge to zero, since $y_{j,k} \to Y_j$ as $k \to \infty$. In fact all I_k are equal for $k \geq K$, namely,

$$\begin{aligned}
I_{k+1} - I_k &= \sum_{j=1}^{n} \left(y_{j,k} - 2\eta_{j,k} \frac{s_k - d}{m_k} - Y_j \right)^2 - \sum_{j=1}^{n} \left(y_{j,k} - Y_j \right)^2 \\
&= 4 \sum_{j=1}^{n} \eta_{j,k}^2 \left(\frac{s_k - d}{m_k} \right)^2 - 4 \sum_{j=1}^{n} \eta_{j,k} \frac{s_k - d}{m_k} (y_{j,k} - Y_j) \\
&= 4 \frac{s_k - d}{m_k} \left[\sum_{j=1}^{n} \eta_{j,k}^2 \frac{s_k - d}{m_k} - \sum_{j=1}^{n} \eta_{j,k} y_{j,k} + \sum_{j=1}^{n} \eta_{j,k} Y_j \right] \\
&= 4 \frac{s_k - d}{m_k} \left[s_k - d - \sum_{j=1}^{n} \eta_{j,k} y_{j,k} + \sum_{j=1}^{n} \eta_{j,k} Y_j \right] = 0,
\end{aligned}$$

where we used (6) and (11) in the last step. Since $I_k \to 0$ it follows that $I_k = 0$ for all $k \geq K$, which implies $y_{j,k} = Y_j$. But then all variables y_j would be constant after the K-th step, a contradiction. □

3. Relaxation procedures

In this section we return to the IMO Pentagon Game and formulate a natural generalization to graphs.

A relaxation procedure on graphs: *Let G be a connected graph with at least two vertices. To each vertex v_j of G a real number x_j, called a label, is assigned. Assume that $s := \sum x_j > 0$. If the label x associated with a vertex v is negative then it is allowed to add $2x/m$ to each of the m labels at the vertices adjacent to v, and then to replace x by $-x$. This step is performed repeatedly as long as negative labels exist.*

In their paper [1], Akiyama, Hosono and Urabe asked if this procedure necessarily terminates for *regular graphs*. We shall prove that this indeed always happens for *arbitrary* graphs. Note that connectedness can be omitted if we assume that $s > 0$ holds in every component of the graph.

The name "Relaxation Procedure" is motivated by the following interpretation. If we consider the x_j as charges sitting at the vertices v_j, their distribution induces a "tension" of the graph G. We do not describe precisely what this means but vaguely speaking, the more an edge contributes to the tension, the greater the differences of charges at its incident vertices is. So "tension" is a measure for non-uniformity of a charge distribution. The rules of a "relaxation procedure" are such that the charges are allowed to be shifted along the edges so that the tension is reduced.

Quite recently we learned that procedures of this kind are basic for iterative methods in circle packing, as described in Kenneth Stephenson's beautiful and inspiring book [20]. Here the vertices of the graph correspond to the circles involved in the packing and the edges are defined by the prescribed tangency structure of the packing. Each circle (vertex) carries two labels, one is the "radius", the other one is the "angle sum" (which measures the "local curvature"). The angle sum at a circle C is expressed by the radii of C and all circles adjacent to C.

The goal is to find appropriate radii which "flatten" the packing, which happens if all interior angle sums are 2π (or a multiple thereof for branched packings). This is achieved by an iterative procedure, where in each step one circle is chosen and its radius is adjusted such that the local curvature becomes zero (or "small"). This changes the curvature labels of C and of the adjacent circles. In effect the "curvature overhead" of C is distributed among its neighbors according to a rule which resembles the setting of the above relaxation procedure. The iteration is stopped if the absolute values of all local curvatures are less then a positive threshold. For details we refer to [20], especially pages 243–244.

Does a relaxation procedure necessarily terminate? If yes, how many steps can be (or must be) performed and what are the possible final configurations? For the procedure defined above the following theorem gives an affirmative answer to the first question.

Theorem 2. *For each graph the above relaxation procedure stops.*

Proof. 1. We reduce the problem to a special case of the Signed Mean Value Procedure. In order to do so, G is first converted to a digraph by choosing arbitrary directions of its edges. Further, we double the number n of vertices of G by associating with any vertex v_j a new vertex v'_j which is adjacent (exactly) to v_j by an edge e_j directed from v'_j to v_j. The resulting digraph is denoted by G'. To each vertex v'_j of G' which does not belong to G we assign the label $-d$, where $d := s/n$ and n denotes the number of vertices of G. Then the total sum of all vertex labels of G' is zero.

2. With any directed edge e_i of G' we associate a label ("the conductance") y_i, such that the vertex labels x_j are equal to the sum of the edge labels at the incident incoming edges minus the sum of the edge labels at the incident outgoing edges.

The existence of such labels follows from Kirchhoff's law, using the fact that the sum of all vertex labels is zero. More directly, to find appropriate edge labels one can select a spanning tree T of G' and choose arbitrary labels at those edges of G' which do not belong to T. The remaining labels at the edges of T are then uniquely determined and can be found by successively deleting monovalent vertices of T together with the corresponding edges. Note that all edges from v'_j to v_j belong to any spanning tree and carry the label d.

3. We investigate the edge labels during a step of the relaxation procedure. Each vertex label x has the representation

$$x = d - \sum \eta_i y_i$$

where y_i are the labels of the incident edges belonging to G, with $\eta_i = -1$ for incoming, and $\eta_i = +1$ for outgoing edges.

Let x be the (negative) label of a vertex v with valency m selected in a step of the procedure. Then $x < 0$ is equivalent to $s' := \sum \eta_i y_i > d$. If the labels y_i of the incident edges belonging to G are replaced by $y_i - 2\eta_i(s'-d)/m$ and the label d of the edge between v and v' remains unchanged, these new values are compatible with the new vertex labels. In fact,

$$x_{\text{new}} = d - \sum_{i=1}^{m} \left(\eta_i y_i - 2\frac{s'-d}{m} \right) = d - s' + 2s' - 2d = -x_{\text{old}},$$

and changing the labels y_i of all edges incident with v according to the rule

$$y_i \mapsto y_i - 2\eta_i \frac{s'-d}{m} \equiv y_i + 2\eta_i \frac{x}{m}$$

has the same effect like adding $2x/m$ to the labels at all vertices adjacent to v.

So the relaxation procedure for the vertex labels induces a special Signed Mean Value Procedure (with preselected η_i) for the edge labels. By Theorem 1 the latter must terminate. □

In contrast to the problem for polygons neither the final configuration nor the number of steps is uniquely determined. For instance, if the labels $-1, -2, 3, 4$ are attached to the vertices of a complete graph of order four we get the following results (scaled by a common factor of 27), depending on whether one starts with -1 or -2 in the initial step:

$$(-27, -54, 81, 108) \to (27, -72, 63, 90) \to (-21, 72, 15, 42) \to (21, 58, 1, 28)$$

$$(-27, -54, 81, 108) \to (-63, 54, 45, 72) \to (63, 12, 3, 30).$$

Open problem: *Find a characterization of all graphs where the final configuration and/or the number of steps are unique.*

There are many possibilities to change the rules of the game. One option is to admit *weighted shifts* of the charges, which leads to *relaxation procedures on weighted digraphs*.

To define these procedures we assume that every edge e_{ij} of a digraph G with n vertices v_i carries a non-negative label c_{ij}, its "edge conductance", such that for all $i = 1, \ldots, n$

$$c_i := \sum_{j \neq i} c_{ij} > 0. \tag{12}$$

To simplify notations we assume that G is completely bi-oriented, which can be achieved by adding virtual edges with conductance zero, and define the weights w_{ij} for $i, j = 1, \ldots, n$ by

$$w_{ij} := \begin{cases} -1 & \text{if } i = j \\ c_{ij}/c_i & \text{if } i \neq j. \end{cases} \tag{13}$$

A relaxation procedure on weighted digraphs. *Let G be a digraph endowed with edge conductances c_{ij}, let the weights w_{ij} be defined by (12) and (13), and fix a "relaxation parameter" $\lambda \in \mathbb{R}_+$. Assume further that any vertex v_i of G carries a label x_i, its "charge", so that the total charge $x_1 + \cdots + x_n$ is positive.*

If there is at least one negative charge, say x_i, it is allowed to replace all charges according to the rule

$$x_j \mapsto x_j + 2\lambda w_{ij} x_i, \quad j = 1, \ldots, n. \tag{14}$$

Repeat this step as long as there are negative charges.

Remark. Condition (12) guarantees that any negative charge has the option to be distributed among their neighboring vertices. If one admits that c_i vanishes for some indices i, it is natural to set all corresponding weights w_{ij}, including w_{ii}, to zero and to modify the procedure by not allowing the substitution (14) for those values of i.

Open problem: *Assume that the edge conductances of G are given. Describe the set of all relaxation parameters λ such that, for any initial distribution of the charges, the relaxation procedure terminates.*

The rules of the relaxation procedure are designed so that the total charge remains invariant, which was motivated by our physical interpretation. One should not think that this condition is natural to get a "reasonable" procedure – on the contrary.

In 1987 Shahar Mozes invented what is now called **Mozes' Numbers Game** (see [16]), which corresponds to the substitution rule

$$x_j \mapsto \begin{cases} -x_j & \text{if } i = j \\ x_j + w_{ij} x_i & \text{if } i \neq j \end{cases} \tag{15}$$

with *integer* weights w_{ij}. Using Weyl groups and Kac-Moody algebras, Mozes gave an algebraic characterization of the initial positions giving rise to finite games and proved that for those the number of steps and the finial configuration do not depend on the moves of the player. For $w_{ij} \in \{0, 1\}$ (which includes the original Pentagon problem) Anders Björner [3] (see also [4], Section 4.3, and [8]) gave an elementary proof (without characterizing the initial configurations for which the game terminates). Kimmo Eriksson [9] showed that (for a subclass of problems) one can decide which positions are reachable from a given initial configuration, and his subsequent investigations [10, 11] revealed deep connections to Coxeter groups and greedoids. Proctor [17] discovered that Mozes' Numbers Game is related to Bruhat lattices.

Further modifications of the problem arise if one admits *simultaneous* substitutions of (some or all) negative labels. Another direction is to allow substitutions *without sign restrictions*, and to ask for the set of all reachable configurations. Does this set contain "minimal" configurations? Of, course one can also replace the real numbers by other (partially ordered) algebraic structures ...

Isn't it fascinating to find so much mathematics hidden in a simple game?

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