Journal of Computational and Applied Mathematics 129 (2001) 195–208

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.nl/locate/cam

A smoothing Gauss–Newton method for the generalized HLCP ☆

Naihua Xiu[a], Jianzhong Zhang[b,*]

[a] *Department of Applied Mathematics, Northern Jiaotong University, Beijing 100044, China*
[b] *Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong, China*

Received 18 March 1999

Abstract

In this paper, we present a smoothing Gauss–Newton method for solving the generalized horizontal linear complementarity problem and prove that the method is both globally and locally quadratically convergent under reasonable assumptions. As a by-product of our analysis, we obtain a sufficient condition for the existence and boundedness of the solutions to the problem.

MSC: 90C33; 65H10

Keywords: Generalized horizontal linear complementarity problem; Smoothing Gauss–Newton method; Global quadratic convergence; Local quadratic convergence

1. Introduction

Let $\{A,B\}$ be a pair of matrices in $\mathbb{R}^{m\times n}$ and q be a vector in $\mathbb{R}^m$. The generalized horizontal linear complementarity problem is to find a vector $(x,y)\in\mathbb{R}^{2n}$ such that

$$\begin{aligned} &Ax-By=q,\\ &x^{\mathrm{T}}y=0,\quad x\geqslant 0,\ \ y\geqslant 0. \end{aligned}\tag{1}$$

We denote the problem by $\widehat{\mathrm{H}}\mathrm{LCP}(A,B,q)$. This problem is a special case of the XLCP by Mangasarian and Pang [15] or the general LCP by Ye [26]. It arises from economic equilibrium problems, noncooperative games, traffic assignment problems, and optimization problems. When

☆ This research is partially supported by the National Natural Science Foundation of China (No.19971002), and the Strategic Research Grant of the City University of Hong Kong.

* Corresponding author. Tel.: +852-2788-8662; fax: +852-2788-8463.
E-mail addresses: nhxiu@center.njtu.edu.cn (N. Xiu), mazhang@cityu.edu.hk (J. Zhang).

PII: S0377-0427(00)00550-1

$m = n$, it reduces to the well-known horizontal linear complementarity problem which is denoted by HLCP(A,B,q); when $m = n$ and $B = I$, it becomes the linear complementarity problem which is denoted by LCP(A,q).

In the last couple of years the HLCP attracted attention of many researchers. Zhang [27] used it as a unifying framework for the convergence analysis of a class of infeasible interior-point algorithms for solving linear programs and complementarity problems. Subsequent work in this area includes Billups and Ferris [1], Bonnans and Gonzaga [2], Güler [11], Monteiro and Tsuchiya [16], etc. Some basic properties in the context of HLCP have been studied in [8,21,22,24], etc.

In this paper, we address the $\hat{\text{H}}$LCP. Our contribution is three-fold: first, we describe and characterize P_0-property in $\hat{\text{H}}$LCP along the line of the classical LCP in Section 2; second, based on the smoothed Fischer–Burmeister function, we present a smoothing Gauss–Newton method for solving problem (1) with $m \geqslant n$ in Section 3; the third, we establish global and local quadratic convergence of this smoothing method under reasonable conditions, and get a sufficient condition for the existence and boundedness of solutions to the problem in Section 4. We conclude with some final remarks in Section 5.

Throughout the paper, all vectors are column vectors with the superscript T denoting a transpose. For simplicity, we use (x,y) for column vector $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}}$. The notations $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ are used for nonnegative and positive orthants, respectively, in $\mathbb{R}^n$. We denote by $x = \mathrm{Vec}\{x_i\}$ a vector x whose ith element is x_i. $A \geqslant 0$ (> 0) means the matrix A is positive semi-definite (definite). Finally, $\|\cdot\|$ denotes l_2 norm of a vector or a matrix.

2. P_0-property and its characterizations

It is well known that, in the standard LCP, there are various classes of matrices, which play an important role in developing theory and methods for LCP (see, e.g., [6]). In this section we shall establish the generalized P_0-matrices in the context of $\hat{\text{H}}$LCP.

Definition 1. Given two matrices $A, B \in \mathbb{R}^{m\times n}$, we say that $\{A,B\}$ has the column P_0-property if rank$[A, -B] = 2n$, or the condition

$$Au - Bv = 0,\ (u,v) \neq (0,0) \Rightarrow \exists u_{i_0} \neq 0, \quad u_{i_0}v_{i_0} \geqslant 0, \text{ or } \exists v_{j_0} \neq 0, \quad u_{j_0}v_{j_0} \geqslant 0 \tag{2}$$

holds; and that $\{A,B\}$ has the row P_0-property if it satisfies the condition

$$(A^{\mathrm{T}}u, B^{\mathrm{T}}u) \neq 0, \quad u \in \mathbb{R}^m \Rightarrow \exists (A^{\mathrm{T}}u)_{i_0} \neq 0,\ (A^{\mathrm{T}}u)_{i_0}(B^{\mathrm{T}}u)_{i_0} \geqslant 0, \tag{3}$$

or

$$\exists (B^{\mathrm{T}}u)_{j_0} \neq 0,\ (A^{\mathrm{T}}u)_{j_0}(B^{\mathrm{T}}u)_{j_0} \geqslant 0.$$

$\{A,B\}$ is said to have P_0-property if it has both column and row P_0-properties.

It is easily seen that, when $m=n$, the column P_0-property is the same as the P_0-property introduced in [24], where they show that their P_0-property is equivalent to the W_0-pair of Willson Jr. [25]. It is clear that the matrix $A + B$ is of full column rank and hence $m \geqslant n$ if $\{A,B\}$ has the column P_0-property. In fact, assuming that there is a nonzero vector $x \in \mathbb{R}^n$ satisfying $(A + B)x = 0$, then by this property, there exists an index i_0 such that $x_{i_0} \neq 0$ and $x_{i_0}(-x_{i_0}) \geqslant 0$, a contradiction. Note also that, in the definition of the row P_0-property, we do not require the assumption: for any nonzero vector $u \in \mathbb{R}^m \Rightarrow (A^T u, B^T u) \neq 0$. In other words, $rank[A, -B] = m$ is not needed. In addition, the row P_0-property is different from the row W_0-property of Sznajder and Gowda [21] for the extended vertical LCP (see Remark 2 of this section for the details).

Recalling the definitions of the column and row sufficient properties in [9], we immediately obtain the following result.

Proposition 2. *The column (row) sufficiency of $\{A, B\}$ implies the column (row) P_0-property.*

Now, we further describe the characterization of P_0-property, which can be regarded as an extension of the seminal work by Kojima, Megiddo, Noma and Yoshise [14].

Theorem 3. *Given two matrices $A, B \in \mathbb{R}^{m\times n}$ and define the matrix Q associated with $\{A, B\}$ by*

$$Q := \begin{bmatrix} A & -B \\ D & \bar{D} \end{bmatrix}_{(m+n)\times(2n)}, \tag{4}$$

where $D = \mathrm{diag}(d_j)$ and $\bar{D} = \mathrm{diag}(\bar{d}_j)$ are two positive semi-definite matrices. Then

(a) *the pair $\{A, B\}$ has the column P_0-property if and only if the matrix Q defined in (4) is of full column rank for any $D > 0$ and $\bar{D} > 0$;*

(b) *the pair $\{A, B\}$ has the row P_0-property and $rank[A, -B] = m$ if and only if the matrix Q defined in (4) is of full row rank for any $D > 0$ and $\bar{D} > 0$.*

Proof. (a) $\Rightarrow$: Assume that, for some $D > 0$ and $\bar{D} > 0$, the matrix Q defined in (4) is not of full column rank. Then there exists a nonzero vector $(u, v) \in \mathbb{R}^{2n}$, such that $Q(u, v) = 0$, i.e.,

$$Au - Bv = 0, \quad Du + \bar{D}v = 0. \tag{5}$$

By the properties of D and $\bar{D}$, we know from the second equation of (5) that $u_i = -d_i^{-1}\bar{d}_i v_i$ for $i = 1, 2, \dots, n$. This yields that

$$u_i v_i < 0, \quad \text{or} \quad u_i = v_i = 0, \quad i = 1, 2, \dots, n. \tag{6}$$

This contradicts the column P_0-property of $\{A, B\}$.

$\Leftarrow$: Assume $\{A, B\}$ does not have the column P_0-property. Then by $\mathrm{rank}[A, -B] < 2n$, there is a nonzero vector $(u^0, v^0) \in \mathbb{R}^{2n}$ satisfying $Au^0 - Bv^0 = 0$ and (6). Define the matrix Q in (4) by taking

$$D = \mathrm{diag}(d_j), \quad d_j = 1 \text{ for } v_j^0 = 0, \quad \text{else} |v_j^0|,$$

$$\bar{D} = \mathrm{diag}(\bar{d}_j), \quad \bar{d}_j = 1 \text{ for } u_j^0 = 0, \quad \text{else } |u_j^0|.$$

It is easy to verify that $D > 0$, $\bar{D} > 0$, and $Q(u^0, v^0) = 0$. Since Q is of full column rank by the assumption, we have $(u^0, v^0) = 0$, a contradiction.

(b) $\Rightarrow$: Assume that, for some $D > 0$ and $\bar{D} > 0$, Q defined in (4) is not of full row rank. Then there is a nonzero vector $(u, v) \in \mathbb{R}^{m+n}$ such that $Q^T(u, v) = 0$, i.e.,

$$A^T u + Dv = 0, \quad -B^T u + \bar{D}v = 0. \tag{7}$$

This implies that

$$(A^T u)_i (B^T u)_i = -(Dv)_i (\bar{D}v)_i = -d_i \bar{d}_i v_i^2, \quad i = 1, 2, \dots, n. \tag{8}$$

By $D>0$ and $\bar{D}>0$, from (7) and (8) we have

$$(A^{\mathrm{T}}u)_i(B^{\mathrm{T}}u)_i<0, \quad \text{or} \quad (A^{\mathrm{T}}u)_i=(B^{\mathrm{T}}u)_i=0, \quad i=1,2,\ldots,n. \tag{9}$$

From $(u,v)\neq 0$, $\operatorname{rank}[A,-B]=m$ and (7), we have $(A^{\mathrm{T}}u,B^{\mathrm{T}}u)\neq 0$. So, the row P_0-property implies that $(A^{\mathrm{T}}u)_{i_0}(B^{\mathrm{T}}u)_{i_0}\geqslant 0$ for some $(A^{\mathrm{T}}u)_{i_0}\neq 0$, or $(A^{\mathrm{T}}u)_{j_0}(B^{\mathrm{T}}u)_{j_0}\geqslant 0$ for some $(B^{\mathrm{T}}u)_{j_0}\neq 0$, a contradiction to (9).

$\Longleftarrow$: $\operatorname{rank}[A,-B]=m$ follows directly from $\operatorname{rank}(Q)=m+n$. Suppose $\{A,B\}$ does not have the row P_0-property. Then by the definition, there is a nonzero vector $u^0\in\mathbb{R}^m$ such that

$$(A^{\mathrm{T}}u^0)_i(B^{\mathrm{T}}u^0)_i<0, \quad \text{or} \quad (A^{\mathrm{T}}u^0)_i=(B^{\mathrm{T}}u^0)_i=0, \quad i=1,2,\ldots,n. \tag{10}$$

Define the matrix Q in (4) by

$$D=\operatorname{diag}(d_j), \quad d_j=1 \text{ for } (A^{\mathrm{T}}u^0)_j=0, \quad \text{else } |(A^{\mathrm{T}}u^0)_j|,$$

$$\bar{D}=\operatorname{diag}(\bar{d}_j), \quad \bar{d}_j=1 \text{ for} (B^{\mathrm{T}}u^0)_j=0, \quad \text{else } |(B^{\mathrm{T}}u^0)_j|.$$

Then by a simple operation and (10), we know that the vector $(u^0,-\operatorname{sign}(A^{\mathrm{T}}u^0))\neq 0$, where $\operatorname{sign}(A^{\mathrm{T}}u^0)=\operatorname{Vec}\{\operatorname{sign}(A^{\mathrm{T}}u^0)_i\}$, satisfies the equation $Q^{\mathrm{T}}(u^0,-\operatorname{sign}(A^{\mathrm{T}}u^0))=0$. This contradicts the condition that Q has full row rank. The proof is complete. □

Remark. (1) The special case of result (a) in the above theorem, for $m=n$, was obtained by Sznajder and Gowda [21] and Tütüncü and Todd [24], respectively. Furthermore, for the special case where $m=n$ and $B=I$, the result (a) was proved by Kojima et al. [14], who were interested in developing a unified interior point algorithm for P_0-matrices.

(2) Result (b) in the above theorem shows that, when $m=n$ and $\operatorname{rank}[A,-B]=m$, the row P_0-property of $\{A,B\}$ is equivalent to the row W_0-property of $\{A^{\mathrm{T}},B^{\mathrm{T}}\}$ in the VLCP by Sznajder and Gowda [21, Theorem 6].

(3) This theorem exhibits an interesting fact that, for $m<n$ the column P_0-property does not hold; for $m>n$, when $\operatorname{rank}[A,-B]=m$, the row P_0-property does not hold. So, for the case $m>n$, $\{A,B\}$ may have both the column and row P_0-properties only if $\operatorname{rank}[A,-B]\neq m$. For example, let $A=[1,1]^{\mathrm{T}}$ and $B=[1,1]^{\mathrm{T}}$, and we see that $\{A,B\}$ has these two properties. For the case $m=n$, the column P_0-property implies the row P_0-property; however the row P_0-property together with $\operatorname{rank}[A,-B]=m$ imply the column P_0-property. This indicates that the $\hat{\mathrm{H}}$LCP is more complicated, and may lose many good properties which the HLCP possesses.

Following the above discussion, we can introduce the concepts of column and row P-properties provided that all "$\geqslant$" in Definition 1 are replaced by "$>$", and get similar characterization results for the two concepts.

Theorem 4. *Given two matrices $A,B\in\mathbb{R}^{m\times n}$. Then*

(a) *the pair $\{A,B\}$ has the column P-property if and only if the matrix Q defined in* (4) *is of full column rank for any $D\geqslant 0$, $\bar{D}\geqslant 0$ and $D+\bar{D}>0$;*

(b) *the pair $\{A,B\}$ has the row P-property and* $\operatorname{rank}[A,-B]=m$ *if and only if the matrix Q defined in* (4) *is of full row rank for any $D\geqslant 0$, $\bar{D}\geqslant 0$ and $D+\bar{D}>0$.*

We say that $\{A, B\}$ has weak P-property if it has both the column and row P-properties, i.e., the following equivalent conditions hold:

(P_1) $Au - Bv = 0, \quad u_i v_i \leqslant 0, \quad i = 1, 2, \ldots, n \Rightarrow (u, v) = 0;$
(P_2) $(A^{\mathrm{T}}u)_i (B^{\mathrm{T}}u)_i \leqslant 0, \quad i = 1, 2, \ldots, n \Rightarrow (A^{\mathrm{T}}u, B^{\mathrm{T}}u) = 0.$

The word "weak" is related to the P-property by Gowda [9], who defined the P-property by (P_1) and $(\mathrm{P}_2)'$, where

$(\mathrm{P}_2)'$ $(A^{\mathrm{T}}u)_i (B^{\mathrm{T}}u)_i \leqslant 0, \; i = 1, 2, \ldots, n \Rightarrow u = 0.$

Similar to the previous analysis, the weak P-property can be introduced for $m \geqslant n$; however the P-property in [9] requires the condition $m = n$.

As in the standard LCP and HLCP, the full-rank property of Q defined in (4) permits us to develop a class of smoothing methods for solving problem (1).

3. Smoothing Gauss–Newton method

In the remainder of this paper we only address problem (1) with $m \geqslant n$. Obviously, (1) is equivalent to a nonsmooth equation

$$F(x, y) := \begin{bmatrix} Ax - By - q \\ \Psi(x, y) \end{bmatrix} = 0, \tag{1$'$}$$

where $\Psi(x, y) = (\psi(x_1, y_1), \ldots, \psi(x_n, y_n))^{\mathrm{T}}$, $\psi(a, b) = a + b - \sqrt{a^2 + b^2}$ (Fischer–Burmeister function). By using the smoothed Fischer–Burmeister function

$$p(a, b, \mu) = a + b - \sqrt{a^2 + b^2 + 2\mu^2}, \tag{11}$$

where $\mu > 0$ is a smoothing parameter which is also viewed as a variable, we can construct a smooth approximation to (1$'$) with respect to variable $(w, \mu) \in \mathbb{R}^{2n} \times \mathbb{R}_{++}$:

$$H(w, \mu) := \begin{bmatrix} Ax - By - q \\ \Phi(x, y, \mu) \end{bmatrix} = 0 \tag{12}$$

with $w = (x, y)$ and

$$\Phi(x, y, \mu) := \begin{bmatrix} p(x_1, y_1, \mu) \\ \cdots \\ p(x_n, y_n, \mu) \end{bmatrix}.$$

It is easy to check that $H(w, \mu)$ is twice continuously differentiable on $(w, \mu) \in \mathbb{R}^{2n} \times \mathbb{R}_{++}$ and continuous at $(w, 0)$. Of course $H(w, 0) = F(w)$. The Jacobian matrix of (12) with respect to variable w is given by

$$\nabla_w H(w, \mu) = \begin{bmatrix} A & -B \\ \nabla_x \Phi(x, y, \mu) & \nabla_y \Phi(x, y, \mu) \end{bmatrix}, \tag{13}$$

where

$$\nabla_x \Phi(x,y,\mu) = \mathrm{diag}\left(1 - \frac{x_i}{\sqrt{x_i^2 + y_i^2 + 2\mu^2}}\right), \quad \nabla_y \Phi(x,y,\mu) = \mathrm{diag}\left(1 - \frac{y_i}{\sqrt{x_i^2 + y_i^2 + 2\mu^2}}\right).$$

By using Theorem 3 and noting that both $\nabla_x \Phi(x,y,\mu)$ and $\nabla_y \Phi(x,y,\mu)$ are in interval $(0,2)$ for any $\mu > 0$ (see Lemma 5(ii) in the next section), we know that, under the column P_0-property of $\{A,B\}$, $\nabla_w H(w,\mu)$ is of full column rank, and hence $G(w,\mu) := \nabla_w H(w,\mu)^{\mathrm{T}} \nabla_w H(w,\mu)$ is nonsingular for any $\mu > 0$.

Consider the least-squares problem associated with (12)

$$\min_w r(w,\mu) = \tfrac{1}{2} H(w,\mu)^{\mathrm{T}} H(w,\mu). \tag{14}$$

Then the gradient is given by $g(w,\mu) := \nabla_w r(w,\mu) = \nabla_w H(w,\mu)^{\mathrm{T}} H(w,\mu)$; the Hessian matrix is given by

$$\nabla_w^2 r(w,\mu) = G(w,\mu) + \sum_{j=1}^{n} \Phi_j(w,\mu) \nabla_w^2 \Phi_j(w,\mu).$$

We try to solve (14) inexactly using just one step of Gauss–Newton method for a fixed value of $\mu > 0$, and then reduce this value of μ by some rule. By repeating the above computation we will get a sequence of iterates. The details of our algorithm are described as follows. For simplicity, at the kth iteration we use $r^k = r(w^k, \mu_k)$, $g^k = g(w^k, \mu_k)$, $\nabla_w H^k = \nabla_w H(w^k, \mu_k)$, etc.

Algorithm (I)

Step 0: Given constants $\sigma \in (0,1)$, $\alpha \in (0,1)$, $\eta \in (0,1)$ and $\gamma > 0$.

Choose any initial point $(w^0, \mu_0) \in \mathbb{R}^{2n} \times \mathbb{R}_{++}$. Set $\sigma_0 = 1$ and $k = 0$.

Step 1: The search direction. If $\mu_k = 0$ then stop. If $g^k = 0$ then $w^{k+1} = w^k$, and go to the “If” part of Step 3. Otherwise, compute $d^k \in \mathbb{R}^{2n}$ as the solution of the linear system

$$(\nabla_w H^k)^{\mathrm{T}} (\nabla_w H^k) d = -g^k. \tag{15}$$

Step 2: The fast step. Let $\bar{w}^{k+1} = w^k + d^k$, $\bar{\mu}_{k+1} = \min\{\sigma_{k+1}, \eta\}\mu_k$ where $\sigma_{k+1} = \min\{\sigma_k, ||F(\bar{w}^{k+1})||^2\}$. If $\bar{r}^{k+1} \leqslant \eta r^k$ where $\bar{r}^{k+1} = r(\bar{w}^{k+1}, \bar{\mu}_{k+1})$, then $w^{k+1} = \bar{w}^{k+1}, \mu_{k+1} = \bar{\mu}_{k+1}$, and go to Step 4.

Step 3: The safe step. Set $w^{k+1} = w^k + \lambda_k d^k$, where $\lambda_k = \alpha^{m_k}$ and m_k is the smallest nonnegative integer m satisfying the Armijo condition

$$r(w^k + \alpha^m d^k, \mu_k) \leqslant r^k + \sigma \alpha^m \cdot (g^k)^{\mathrm{T}} d^k. \tag{16}$$

If $||g(w^{k+1}, \mu_k)|| \leqslant \gamma \mu_k$, then $\mu_{k+1} = \min\{\sigma_{k+1}, \eta \mu_k\}$ where $\sigma_{k+1} = \min\{\sigma_k, ||F(w^{k+1})||^2\}$; otherwise $\mu_{k+1} = \mu_k$, $\sigma_{k+1} = \sigma_k$.

Step 4: $k := k + 1$, and go to Step 1.

Remark. (1) In the above algorithm we use the Gauss–Newton equation (15) to get the search direction. So, our algorithm is different from the smoothing Newton methods (see, e.g., [18–20]) and the noninterior path-following methods (see, e.g., [3–5,23]), and is called the smoothing Gauss–Newton algorithm.

(2) If the case where $\mu_k = 0$ occurs in Step 1, then by the updating rule for μ_k in Step 2 and Step 3, we know that $F(w^k) = 0$ and hence w^k is a solution of (1). If Algorithm (I) produces an infinite sequence $\{\mu_k\}$, then from $\mu_{k+1} \leqslant \eta \mu_k$ or $\mu_{k+1} = \mu_k$, it must be a decreasing sequence.

(3) For global convergence, only the safe step is needed. The fast step is added to ensure local quadratic convergence. Design of the fast step is motivated by Qi [17], Facchinei and Soares [7] and Chen and Xiu [5], and the safe step by Kanzow [13].

4. Theoretical analysis

In this section, we shall consider the global convergence and local rate of convergence for Algorithm (I). To do so, we need to study further the properties of the smoothed Fischer–Burmeister function $p(a,b,\mu)$ and the target function $r(w,\mu)$.

Lemma 5. *For the smoothed Fischer–Burmeister function* $p(a,b,\mu)$ *defined in* (11), *it is twice continuously differentiable on* $\mathbb{R}\times\mathbb{R}\times\mathbb{R}_{++}$. *Moreover,*

(i) $0 < p(a,b,0) - p(a,b,\mu) < \sqrt{2}\mu, \quad \forall (a,b,\mu)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}_{++}$;
(ii) $0 < p_i'(a,b,\mu) < 2\ (i=1,2), \quad -\sqrt{2}\leqslant p_3'(a,b,\mu) < 0, \quad \forall (a,b,\mu)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}_{++}$;
(iii) [12, Lemma 2.1] $p(a,b,\mu)$ *is a strongly semismooth function on* $(a,b,\mu)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}_{+}$;
(iv) let $a'\to a, \quad b'\to b, \quad \mu\to 0$, then

$$\lim_{a'\to a,\, b'\to b,\, \mu\to 0} p_i'(a',b',\mu)p(a',b',\mu) = \partial_i p(a,b,0)p(a,b,0), \quad i=1,2,$$

where $\partial_i p(a,b,0)$ *stands for the generalized derivative of the function* p *on the ith variable at* $(a,b,0)$ *in the sense of Clarke.*

Proof. (i) Follows from a direct calculation. (ii) Comes from the fact that

$$\nabla p(a,b,\mu) = \begin{bmatrix} 1 - \dfrac{a}{\sqrt{a^2+b^2+2\mu^2}} \\ 1 - \dfrac{b}{\sqrt{a^2+b^2+2\mu^2}} \\ -\dfrac{2\mu}{\sqrt{a^2+b^2+2\mu^2}} \end{bmatrix}.$$

We now prove (iv). If $p(a,b,0)=0$, then by (ii) we have for $i=1,2$,

$$\begin{aligned} &|p_i'(a',b',\mu)p(a',b',\mu) - \partial_i p(a,b,0)p(a,b,0)| \\ &\quad = |p_i'(a',b',\mu)p(a',b',\mu) - 0| \\ &\quad = |p_i'(a',b',\mu)|\cdot|p(a',b',\mu) - p(a,b,0)| \\ &\quad \leqslant 2|p(a',b',\mu) - p(a,b,0)|. \end{aligned}$$

If $p(a,b,0)\neq 0$, then $a^2+b^2\neq 0$ and hence the function p is continuously differentiable at $(a,b,0)$ with $a^2+b^2\neq 0$. These two cases show that (iv) is true. □

Consider the case where $\mu=0$ in (14):

$$\min_w r(w,0) = \tfrac{1}{2}H(w,0)^{\mathrm{T}}H(w,0) = \tfrac{1}{2}F(w)^{\mathrm{T}}F(w). \tag{17}$$

Similar to the proof of Proposition 3.4 in [7], we can verify that the function $r(w,0)$ is continuously differentiable on w and its gradient is $\partial_w H(w,0)^{\mathrm{T}} H(w,0) := g(w,0)$, where $\partial_w H(w,0)$ is the generalized Jacobian matrix of $H(w,0)$ on variable w. Furthermore, we have the following result which was actually obtained in [12, Proposition 2.1], but here we give the proof for reference.

Lemma 6. *The target function* $r(w,\mu)$ *in* (14) *is continuously differentiable on* $(w,\mu) \in \mathbb{R}^{2n} \times \mathbb{R}_+$.

Proof. It suffices to prove that $\lim_{w' \to w, \mu \to 0} g(w',\mu) = g(w,0)$. In fact, from

$$\begin{aligned}
&\|g(w',\mu) - g(w,0)\| \\
&\quad = \|\nabla_w H(w',\mu)^{\mathrm{T}} H(w',\mu) - \partial_w H(w,0)^{\mathrm{T}} H(w,0)\| \\
&\quad \leqslant \left\| \begin{bmatrix} A^{\mathrm{T}} \\ -B^{\mathrm{T}} \end{bmatrix} \right\| \, ||(Ax' - By' - q) - (Ax - By - q)|| \\
&\qquad + \left\| \begin{bmatrix} \nabla_x \Phi(w',\mu) \\ \nabla_y \Phi(w',\mu) \end{bmatrix} \Phi(w',\mu) - \begin{bmatrix} \partial_x \Phi(w,0) \\ \partial_y \Phi(w,0) \end{bmatrix} \Phi(w,0) \right\|
\end{aligned}$$

and Lemma 5(iv), we derive the desired result. □

A point w^* is said to be a stationary point of (17) (or (1)) if $g(w^*,0) = 0$. Obviously, the global minimum point of (17) must be a stationary point. The following theorem, which is an extension of Theorem 4.1 in [7], states that the inverse is true under some conditions.

Theorem 7. *Suppose* $\{A,B\}$ *has the row* P_0*-property. Then every stationary point* $w^* = (x^*, y^*)$ *of* (17) *is a global minimum point, and satisfies*

$$w^* \in \arg\min \tfrac{1}{2}\|Ax - By - q\|^2, \text{ and } \Phi(w^*,0) = 0. \tag{18}$$

Moreover, if $\operatorname{rank}[A,-B] = \operatorname{rank}[A,-B,q]$, *then* w^* *is a solution to* (1).

Proof. We prove (18) mainly by following the pattern developed in [7, Theorem 4.1]. Assume $g(w^*,0) = 0$. This means that

$$\begin{aligned}
&A^{\mathrm{T}}(Ax^* - By^* - q) + \partial_x \Phi(w^*,0)\Phi(w^*,0) = 0, \\
&-B^{\mathrm{T}}(Ax^* - By^* - q) + \partial_y \Phi(w^*,0)\Phi(w^*,0) = 0.
\end{aligned} \tag{19}$$

So, if $\Phi_i(w^*,0) = 0$ for some index i, then

$$(A^{\mathrm{T}}(Ax^* - By^* - q))_i = (B^{\mathrm{T}}(Ax^* - By^* - q))_i = 0; \tag{20}$$

otherwise, it holds that $x_i^* \neq 0$ and $y_i^* \neq 0$, or $x_i^* = 0$ and $y_i^* < 0$, or $x_i^* < 0$ and $y_i^* = 0$. From $(x_i^*)^2 + (y_i^*)^2 \neq 0$, $\partial_j p(x_i^*, y_i^*, 0) = p_j'(x_i^*, y_i^*, 0) > 0$ $(j = 1,2)$ and (19), we have

$$(A^{\mathrm{T}}(Ax^* - By^* - q))_i (B^{\mathrm{T}}(Ax^* - By^* - q))_i < 0. \tag{21}$$

That is, (20) or (21) holds for $i=1,2,\dots,n$. By the row P_0-property of $\{A,B\}$, we deduce $\Phi(w^*,0)=0$. This implies from (19) that

$$\begin{bmatrix} A^{\mathrm{T}} \\ -B^{\mathrm{T}} \end{bmatrix} [A,-B] \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} A^{\mathrm{T}} \\ -B^{\mathrm{T}} \end{bmatrix} q, \tag{22}$$

which means that w^* is a stationary point, and hence a minimum point of the least-squares problem $\min \frac{1}{2}\|Ax - By - q\|^2$. Therefore, (18) holds.

It is clear that, when $\text{rank}[A, -B] = \text{rank}[A, -B, q]$, i.e., the equation $Ax - By = q$ is consistent, w^* satisfies $Ax^* - By^* = q$ by (18). The proof is complete. □

Now, we recall the definition of R_0-property of $\{A, B\}$, see, e.g., [10].

Definition 8. *Given two matrices $A, B \in \mathbb{R}^{m\times n}$, we say that $\{A, B\}$ has the R_0-property if the system*

$$Au - Bv = 0, \quad u^{\mathrm{T}}v = 0, \quad u \geqslant 0, \ v \geqslant 0 \tag{23}$$

has only zero solution.

Under R_0-property we can obtain the following result whose proof is similar to the one of Theorem 3.7 in [13], and is omitted.

Lemma 9. *Given two matrices $A, B \in \mathbb{R}^{m\times n}$. If $\{A, B\}$ has the R_0-property, then the level set*

$$L(\varepsilon, \mu_0) := \{w \in \mathbb{R}^{2n} \,|\, \|H(w, \mu)\| \leqslant \varepsilon, \ \mu \leqslant \mu_0\}$$

is bounded for any $\varepsilon > 0$ and $\mu_0 > 0$ (it may be empty).

We now show, using Theorem 3, Lemmas 5 and 9, that Algorithm (I) is globally convergent under suitable assumptions.

Theorem 10. *Given two matrices $A, B \in \mathbb{R}^{m\times n}$ in (1) with $m \geqslant n$. Assume that $\{A, B\}$ has the column P_0- and R_0-properties. Let $\{w^k, \mu_k\}$ be an infinite sequence generated by Algorithm (I). Then (a) $\mu_k \downarrow 0$; (b) $\{w^k\}$ is bounded; (c) $\liminf_{k\to\infty} \|g(w^{k+1}, \mu_k)\| = 0$, or $\{r^k\} \downarrow 0$.*

Proof. (a) Assume that the result (a) is false. Then from $\mu_{k+1} \leqslant \eta\mu_k$ or $\mu_{k+1} = \mu_k$ for any k (see Steps 2 and 3), there exists an iteration index $\hat{k}$ such that $\mu_{\hat{k}+l} = \mu_{\hat{k}} > 0$ for all $l = 1, 2, \ldots.$ This means that Algorithm (I) eventually reduces to a damped Gauss–Newton method for the least-squares problem $\min\{r(w, \mu_{\hat{k}}) \,|\, w \in \mathbb{R}^{2n}\}$. In view of Lemma 9 and Theorem 3, we know that the level set

$$L(w^{\hat{k}}, \mu_{\hat{k}}) := \{w \in \mathbb{R}^{2n} \,|\, r(w, \mu_{\hat{k}}) \leqslant r(w^{\hat{k}}, \mu_{\hat{k}})\}$$

is bounded and $G(w, \mu_{\hat{k}})$ is nonsingular on this set. So, $\lim_{l\to\infty} g(w^{\hat{k}+l}, \mu_{\hat{k}}) = 0$. In particular, there is an index l_0 such that

$$\|g(w^{\hat{k}+l_0+1}, \mu_{\hat{k}+l_0})\| = \|g(w^{\hat{k}+l_0+1}, \mu_{\hat{k}})\| \leqslant \gamma\mu_{\hat{k}} = \gamma\mu_{\hat{k}+l_0}.$$

By the updating rule for μ_k in Step 3, $\mu_{\hat{k}+l_0+1} \leqslant \eta\mu_{\hat{k}+l_0} < \mu_{\hat{k}+l_0}$, a contradiction. So, result (a) is true.

(b) According to the updating rule for μ_k, we partition set $K := \{0, 1, 2, \ldots\}$ into three parts:

$$K_1 := \{k \in K \,|\, \mu_{k+1} = \min\{\sigma_{k+1}, \eta\}\mu_k\},$$

$$K_2 := \{k \in K \,|\, \mu_{k+1} = \mu_k\},$$

$$K_3 := \{k \in K \,|\, \mu_{k+1} = \min\{\sigma_{k+1}, \eta\mu_k\}\}.$$

For $k \in K_1$, $\|H^{k+1}\| \leqslant \sqrt{\eta}\|H^k\|$ by Step 2. For $k \in K_2$, $\|H^{k+1}\| = \|H(w^{k+1}, \mu_k)\| < \|H^k\|$ by (16). For $k \in K_3$, from Lemma 5(ii) and $0 \leqslant \mu_{k+1} \leqslant \eta\mu_k$ we have for $i = 1, 2, \ldots, n$,

$$\begin{aligned}\Phi_i(w^{k+1}, \mu_{k+1}) &\leqslant \Phi_i(w^{k+1}, \mu_k) + |\Phi_i(w^{k+1}, \mu_k) - \Phi_i(w^{k+1}, \mu_{k+1})|\\ &\leqslant \Phi_i(w^{k+1}, \mu_k) + \sqrt{2}|\mu_{k+1} - \mu_k|\\ &\leqslant \Phi_i(w^{k+1}, \mu_k) + \sqrt{2}\mu_k.\end{aligned}$$

Thus, for $k \in K_3$,

$$\begin{aligned}\|H(w^{k+1}, \mu_{k+1})\| &\leqslant \|H(w^{k+1}, \mu_k)\| + \sqrt{2n}\mu_k\\ &< \|H^k\| + \sqrt{2n}\mu_k. \qquad (24)\end{aligned}$$

This means that, for each $k \in K$,

$$\begin{aligned}\|H^{k+1}\| &\leqslant \|H^k\| + \sqrt{2n}\mu_k'\\ &\leqslant \|H^0\| + \sqrt{2n}(\eta^l + \eta^{l-1} + \cdots + 1)\mu_0\\ &\leqslant \|H^0\| + \sqrt{2n}\mu_0/(1-\eta), \qquad (25)\end{aligned}$$

where $\mu_k' := \mu_k$ for $k \in K_3$, or 0 for $k \in K_1 \cup K_2$; and $l := \max\{j \in \{0, 1, \ldots, k\} \mid j \in K_3\}$. So, result (b) follows immediately from Lemma 9.

(c) If K_3 is an infinite subset, then by result (a),

$$\lim_{k\in K_3, k\to\infty} \|g(w^{k+1}, \mu_k)\| \leqslant \gamma \lim_{k\in K_3, k\to\infty} \mu_k = 0. \qquad (26)$$

Otherwise, we may assume that $K = K_1 \cup K_2$. It follows from $\mu_k \downarrow 0$ that K_1 must be an infinite subset. From $r^{k+1} \leqslant \eta r^k$ for $k \in K_1$ and $r^{k+1} < r^k$ for $k \in K_2$, we have

$$\lim_{k\in K=K_1\cup K_2, k\to\infty} r^k = 0. \qquad (27)$$

So, result (c) follows from (26) and (27). The proof is complete. □

By using Theorems 7, 10 and Lemma 6, we obtain the following conclusion.

Theorem 11. *Given two matrices $A, B \in \mathbb{R}^{m\times n}$ in* (1) *with $m \geqslant n$. Assume $\{A, B\}$ has the column P_0- and R_0-properties. Let $\{w^k, \mu_k\}$ be an infinite sequence generated by Algorithm* (I). *Then,*

(a) *there is one limit point w^* in $\{w^k\}$ such that w^* is a stationary point of* (17);

(b) *if* $\operatorname{rank}[A, -B] = \operatorname{rank}[A, -B, q]$ *and $\{A, B\}$ has the row P_0-property, then $\{\|H^k\|\}$ converges to zero, and any limit point of $\{w^k\}$ is a solution to* (1).

Proof. We only prove (b) because the proof of (a) can be observed from the one of (b). Based on the proof of Theorem 10(c), if K_3 is an infinite subset then by the boundedness of $\{w^k\}$, there is a subset $K_4 \subseteq K_3$ such that $\lim_{k\in K_4, k\to\infty} w^{k+1} = w^*$. This, together with (26) and Lemma 6, imply that

$$g(w^*, 0) = \lim_{k\in K_4, k\to\infty} g(w^{k+1}, \mu_k) = 0.$$

That is, w^* is a stationary point of (17) (from which and (27), we can derive result (a)). Moreover, from the assumptions in (b) and Theorem 7, w^* is a solution to (1). By (24), $\mu_k \downarrow 0$ and the continuity of $H(w, \mu)$, we have

$$\begin{aligned}\lim_{k \in K_4, k \to \infty} \|H(w^{k+1}, \mu_{k+1})\| &\leqslant \lim_{k \in K_4, k \to \infty} (\|H(w^{k+1}, \mu_k)\| + \sqrt{2n}\mu_k) \\ &= \|H(w^*, 0)\| + 0 \\ &= 0,\end{aligned}$$

which together with (27) show that there exists a subsequence of $\{\|H^k\|\}$ which converges to zero. This, plus the first inequality of (25) and $\sum_{k=0}^{\infty} \mu_k' < +\infty$, imply that $\{\|H^k\|\}$ converges to zero. Hence, any limit point of $\{w^k\}$ is a solution to (1). □

Theorem 11 tells us a sufficient condition for the existence and boundedness of solutions to (1), and can be viewed as an extension of the result which states that a standard LCP with P_0- and R_0-properties has a nonempty and bounded solution set (see, e.g., [6]).

At the end of this section, we discuss the local quadratic convergence of Algorithm (I).

Theorem 12. *Given two matrices $A, B \in \mathbb{R}^{m \times n}$ in* (1) *with $m \geqslant n$. Assume that* $\operatorname{rank}[A, -B] = \operatorname{rank}[A, -B, q]$, *and $\{A, B\}$ has the column P- and row P_0-properties. Let $\{w^k, \mu_k\}$ be an infinite sequence generated by Algorithm* (I). *Then $\{w^k\}$ converges to a unique solution, say w^*, to problem* (1). *Moreover,*

(a) *$\{w^k\}$ converges Q-quadratically to w^*;*
(b) *$\{\mu_k\}$ converges Q-superlinearly to zero.*

Proof. By Theorem 11 and the uniqueness of solution to (1) (see [9, Theorem 14]), we know that $\{w^k\}$ converges to a unique solution, say w^*, to (1).

We now observe the local rate of convergence. Since $\mu_k \downarrow 0$ and $K_1 \cup K_3$ is an infinite subset, we have, for some sufficiently large $(k-1) \in K_1 \cup K_3$,

$$\mu_k \leqslant \max\{1, \mu_0\}\|H(w^k, 0)\|^2 = \mathrm{O}(\|H(w^k, 0) - H(w^*, 0)\|^2) = \mathrm{O}(\|w^k - w^*\|^2), \tag{28}$$

where the local Lipschitz continuity of $H(w, 0)$ at w^* is used. This means that for some sufficiently large $(k-1) \in K_1 \cup K_3$,

$$\begin{aligned}&\|w^k + d^k - w^*\| \\ &\quad = \|w^k - w^* - (G^k)^{-1} g^k\| \\ &\quad \leqslant \|(G^k)^{-1}\| \cdot \|(\nabla_w H^k)^{\mathrm{T}}\| \cdot \|H(w^k, \mu_k) - (\nabla_w H^k)(w^k - w^*)\| \\ &\quad = \mathrm{O}(\|H(w^k, \mu_k) - H(w^*, 0) - \nabla_w H(w^k, \mu_k)(w^k - w^*)\|) \\ &\quad = \mathrm{O}(\|(w^k - w^*)\|^2) + \mathrm{O}(\mu_k) \\ &\quad = \mathrm{O}(\|(w^k - w^*)\|^2),\end{aligned} \tag{29}$$

where the second equality is due to the nonsingularity of $G^* := \partial_w H(w^*, 0)^{\mathrm{T}} \partial_w H(w^*, 0)$ (by Theorem 4) and the boundedness of $\{\|\nabla_w H^k\|\}$ (by Lemma 5(ii)); the third equality comes from the strong semismoothness of $H(w,\mu)$ at $(w^*,0)$ (by Lemma 5(iii)) and the boundedness of $\{\|\nabla_\mu\Phi(w^k,\mu_k)\|\}$ (by Lemma 5(ii)). Thus,

$$\begin{aligned}\|H(\bar{w}^{k+1},\bar{\mu}_{k+1})\| &= \|H(\bar{w}^{k+1},\bar{\mu}_{k+1})-H(w^*,0)\|\\ &= \mathrm{O}(\|(\bar{w}^{k+1}-w^*)\|)+\mathrm{O}(\bar{\mu}_{k+1}) \quad \text{(by Lemma 5(ii))}\\ &= \mathrm{O}(\|(w^k-w^*)\|^2)+\mathrm{O}(\mu_k) \quad (\text{by } \bar{\mu}_{k+1}\leqslant\eta\mu_k)\\ &= \mathrm{O}(\|(w^k-w^*)\|^2) \quad \text{(by (28))}\\ &= \mathrm{O}(\|H(w^k,\mu_k)\|^2), \end{aligned} \tag{30}$$

where the last equality is based on the relation

$$\|w^k-w^*\| = \mathrm{O}(\|H^k\|). \tag{31}$$

In fact, by (29), (15) and the boundedness of $\{\|(G^k)^{-1}(\nabla_w H^k)^{\mathrm{T}}\|\}$, we have for some sufficiently large $(k-1)\in K_1\cup K_3$,

$$\begin{aligned}\|w^k-w^*\| &\leqslant \|d^k\|+\|w^k+d^k-w^*\|\\ &= \|d^k\|+\mathrm{O}(\|w^k-w^*\|^2)\\ &= \|H^k\|+\mathrm{O}(\|w^k-w^*\|^2).\end{aligned}$$

That is, (31) holds.

Equality (30) shows that for some sufficiently large $(k-1)\in K_1\cup K_3$, $k\in K_1$. Repeating the above proof, we know that $k+1\in K_1$, $k+2\in K_1,\ldots$, i.e., the fast step is always taken after a finite number of iterations. Therefore, result (a) follows from (29), and result (b) is based on

$$\mu_{k+1}\leqslant\sigma_{k+1}\mu_k = \mathrm{o}(\mu_k).$$

This completes the proof. □

5. Final remarks

In this paper, we have studied the P_0-property which allows us to propose a Gauss–Newton algorithm for the solution of problem (1) with $m\geqslant n$. The new algorithm needs to solve only a linear equation per iteration and hence has a low cost. Its global convergence, and hence the existence and boundedness of solutions to (1), have been shown under the P_0- and R_0-properties. Its local quadratic convergence is established without the strict complementarity assumption.

Although we do not make numerical experiments, from theoretical results the new algorithm should be effective for obtaining the solution of problem (1) with $m>n$. For problem (1) with $m=n$, we suggest the readers to use the interior point methods or the noninterior path-following methods. For problem (1) with $m<n$, since the column P_0-property (hence the column sufficiency and column P-property) does not hold, we suggest the readers to use the finite SLP algorithm of Mangasarian and Pang [15], or the polynomial-time potential reduction algorithm of Ye [26], etc.

However, we should point out that the further research on the $\hat{\text{H}}$LCP (or XLCP) is necessary. For example, is it possible to establish a global error bound for this problem with some monotonicity assumption? Would the smoothing method solve the problem with such assumption?

Acknowledgements

The authors are very grateful to Dr. Defeng Sun for his suggestions and pointing out the relevant references so that Theorem 12 of this paper has been improved.

References

[1] S.C. Billups, M.C. Ferris, Convergence of an infeasible interior point algorithm from arbitrary positive starting points, SIAM J. Optim. 6 (1996) 316–325.
[2] J.F. Bonnans, C.C. Gonzaga, Convergence of interior point algorithms for the monotone linear complementarity problem, Math. Oper. Res. 21 (1996) 1–25.
[3] J. Burke, S. Xu, The global linear convergence of a non-interior path-following algorithm for linear complementarity problem, Math. Oper. Res. 23 (1998) 719–734.
[4] J. Burke, S. Xu, A non-interior predictor–corrector path-following algorithm for the monotone linear complementarity problem, Math. Programming 87 (2000) 113–130.
[5] B. Chen, N. Xiu, A global linear and local quadratic non-interior continuation method for nonlinear complementarity problems based on Chen–Mangasarian smoothing functions, SIAM J. Optim. 9 (1999) 605–623.
[6] R.W. Cottle, J.S. Pang, R.E. Stone, The Linear Complementarity Problem, Academic Press, Boston, MA, 1992.
[7] F. Facchinei, J. Soares, A new merit function for nonlinear complementarity problems and a related algorithm, SIAM J. Optim. 7 (1997) 225–247.
[8] M.S. Gowda, On reducing a monotone horizontal LCP to an LCP, Appl. Math. Lett. 8 (1994) 97–100.
[9] M.S. Gowda, On the extended linear complementarity problem, Math. Programming 72 (1996) 33–50.
[10] M.S. Gowda, An analysis of zero set and global error bound properties of a piecewise affine function via its recession function, SIAM J. Matrix Anal. Appl. 17 (1996) 594–609.
[11] O. Güler, Generalized linear complementarity problems, Math. Oper. Res. 20 (1995) 441–448.
[12] H. Jiang, Smoothed Fischer–Burmeister equation methods for the complementarity problem, Report, Department of Mathematics, University of Melbourne, Parkville, Australia, June, 1997.
[13] C. Kanzow, Some noninterior continuation methods for linear complementarity problems, SIAM J. Matrix Anal. Appl. 17 (1996) 851–868.
[14] M. Kojima, N. Megiddo, T. Noma, A. Yoshise, A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems, Lecture Notes in Computer Science, Vol. 538, Springer, Berlin, 1991.
[15] O.L. Mangasarian, J.S. Pang, The extended linear complementarity problem, SIAM J. Matrix Anal. Appl. 16 (1995) 359–368.
[16] R.D.C. Monteiro, T. Tsuchiya, Limiting behavior of the derivatives of certain trajectories associated with a monotone horizontal linear complementarity, Math. Oper. Res. 21 (1996) 793–814.
[17] L. Qi, Convergence analysis of some algorithms for solving nonsmooth equations, Math. Oper. Res. 18 (1993) 227–244.
[18] L. Qi, D. Sun, Smoothing functions and a smoothing Newton method for complementarity and variational inequality problems, Preprint, The University of New South Wales, Sydney 2052, Australia, October, 1998.
[19] L. Qi, D. Sun, Nonsmooth equations and smoothing methods, in: A. Eberhard, B. Glover, R. Hill, D. Ralph (Eds.), Progress in Optimization: Contributions from Australasia, Kluwer Academic Publisher, Nowell, MA, USA, 1999.
[20] L. Qi, D. Sun, G. Zhou, A new look at smoothing Newton methods for nonlinear complementarity problems and box constrained variational inequalities, Math. Programming 87 (2000) 1–35.
[21] R. Sznajder, M.S. Gowda, Generalization of P_0- and P-properties; extended vertical and horizontal LCP_s, Linear Algebra Appl. 223/224 (1995) 695–715.

[22] R. Sznajder, M.S. Gowda, On the Lipschitz continuity of the solution map in some generalized linear complementarity problems, in: T. Parthasarathy, et al., (Eds.), Game Theoretical Applications to Economics and Operations Research, Kluwer Academic Publishers, Boston, 1997.
[23] P. Tseng, Analysis of a non-interior continuation method based on Chen-Mangasarian smoothing functions for complementarity problems, in: M. Fukushima, L. Qi (Eds.), Reformulation-Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods, Kluwer Academic Publisher, Nowell, MD, 1998, pp. 383–404.
[24] R.H. Tütüncü, M.J. Todd, Reducing horizontal linear complementarity problem, Linear Algebra Appl. 223/224 (1995) 716–720.
[25] A.N. Willson Jr., A useful generalization of the P_0-matrix concept, Numer. Math. 17 (1971) 62–70.
[26] Y. Ye, A fully polynomial-time approximation algorithm for computing a stationary point of the general linear complementarity problem, Math. Oper. Res. 18 (1993) 334–345.
[27] Y. Zhang, On the convergence of a class of infeasible interior-point methods for the horizontal linear complementarity problem, SIAM J. Optim. 4 (1994) 208–227.