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Hyponormality and subnormality for powers of commuting pairs of subnormal operators ☆

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Abstract

Let $\mathfrak{H}_0$ (respectively $\mathfrak{H}_\infty$) denote the class of commuting pairs of subnormal operators on Hilbert space (respectively subnormal pairs), and for an integer $k \geqslant 1$ let $\mathfrak{H}_k$ denote the class of k-hyponormal pairs in $\mathfrak{H}_0$. We study the hyponormality and subnormality of powers of pairs in $\mathfrak{H}_k$. We first show that if $(T_1, T_2) \in \mathfrak{H}_1$, the pair (T_1^2, T_2) may fail to be in $\mathfrak{H}_1$. Conversely, we find a pair $(T_1, T_2) \in \mathfrak{H}_0$ such that $(T_1^2, T_2) \in \mathfrak{H}_1$ but $(T_1, T_2) \notin \mathfrak{H}_1$. Next, we show that there exists a pair $(T_1, T_2) \in \mathfrak{H}_1$ such that $T_1^m T_2^n$ is subnormal (for all $m, n \geqslant 1$), but (T_1, T_2) is not in $\mathfrak{H}_\infty$; this further stretches the gap between the classes $\mathfrak{H}_1$ and $\mathfrak{H}_\infty$. Finally, we prove that there exists a large class of 2-variable weighted shifts (T_1, T_2) (namely those pairs in $\mathfrak{H}_0$ whose cores are of tensor form (cf. Definition 3.4)), for which the subnormality of (T_1^2, T_2) and (T_1, T_2^2) does imply the subnormality of (T_1, T_2).

Keywords: Jointly hyponormal pairs; Subnormal pairs; 2-variable weighted shifts; Powers of commuting pairs of subnormal operators

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1. Introduction

The Lifting Problem for Commuting Subnormals (LPCS) asks for necessary and sufficient conditions for a pair of subnormal operators on Hilbert space to admit commuting normal extensions. It is well known that the commutativity of the pair is necessary but not sufficient [1,3, 19–21], and it has recently been shown that the joint hyponormality of the pair is necessary but not sufficient [10], thus disproving the conjecture in [13]. An abstract answer to the Lifting Problem was obtained in [14], by stating and proving a multivariable analogue of the Bram–Halmos criterion for subnormality, and then showing concretely that no matter how k-hyponormal a pair might be, it may still fail to be subnormal. While this provides new insights into the LPCS, it stops short of identifying other types of conditions that, together with joint hyponormality, may imply subnormality.

Our previous work [10–12,14,25,26] has revealed that the nontrivial aspects of the LPCS are best detected within the class $\mathfrak{H}_1$ of commuting hyponormal pairs of subnormal operators; we thus focus our attention on this class. More generally, we will denote the class of commuting pairs of subnormal operators on Hilbert space by $\mathfrak{H}_0$, the class of subnormal pairs by $\mathfrak{H}_\infty$, and for an integer $k \geqslant 1$ the class of k-hyponormal pairs in $\mathfrak{H}_0$ by $\mathfrak{H}_k$. Clearly, $\mathfrak{H}_\infty \subseteq \cdots \subseteq \mathfrak{H}_k \subseteq \cdots \subseteq \mathfrak{H}_2 \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_0$; the main results in [10] and [14] show that these inclusions are all proper. (The LPCS thus asks for necessary and sufficient conditions for a pair $\mathbf{T} \in \mathfrak{H}_0$ to be in $\mathfrak{H}_\infty$.)

In [15], E. Franks proved that if $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_0$ and $p(\mathbf{T})$ is subnormal for all polynomials $p \in \mathbb{C}[z]$ with $\deg p \leqslant 5$, then $\mathbf{T}$ is necessarily subnormal. Motivated in part by this result, and in part by J. Stampfli's work in [22] and [23], in this article we consider the role of the powers of a pair in ascertaining its subnormality. Clearly, if $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_\infty$, and if $m, n \geqslant 1$, then $\mathbf{T}^{(m,n)} := (T_1^m, T_2^n) \in \mathfrak{H}_\infty$, and therefore $T_1^m T_2^n$ is a subnormal operator. It is thus natural to ask whether the subnormality of both $\mathbf{T}^{(2,1)}$ and $\mathbf{T}^{(1,2)}$ can force the subnormality of $\mathbf{T}$.

Our first main result shows that the class $\mathfrak{H}_1$ is not invariant under squares, as follows: we construct a pair $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_1$ such that $\mathbf{T}^{(2,1)} = (T_1^2, T_2) \notin \mathfrak{H}_1$ (Theorem 2.7). Conversely, we find a pair $\mathbf{T} \in \mathfrak{H}_0$ such that $\mathbf{T}^{(2,1)} = (T_1^2, T_2) \in \mathfrak{H}_1$ but $\mathbf{T} \notin \mathfrak{H}_1$. We then show that for a large class of commuting pairs of subnormal operators, the subnormality of both $\mathbf{T}^{(2,1)}$ and $\mathbf{T}^{(1,2)}$ does force the subnormality of $\mathbf{T}$. Concretely, if $\mathbf{T} \in \mathcal{TC}$, the class of all 2-variable weighted shifts $\mathbf{T} \in \mathfrak{H}_0$ whose cores are of *tensor form* (see Definition 3.4), then $\mathbf{T}^{(1,2)} \in \mathfrak{H}_\infty \Leftrightarrow \mathbf{T}^{(2,1)} \in \mathfrak{H}_\infty \Leftrightarrow \mathbf{T} \in \mathfrak{H}_\infty$ (Theorem 3.9). Our results thus seem to indicate that the subnormality of $\mathbf{T}^{(2,1)}, \mathbf{T}^{(1,2)}$ may very well be essential in determining the subnormality of $\mathbf{T}$ within the class $\mathfrak{H}_0$ (Conjecture 3.11). Next, we prove that it is possible for a pair $\mathbf{T} \in \mathfrak{H}_1$ to have all powers $T_1^m T_2^n$ $(m, n \geqslant 1)$ subnormal, without being subnormal (Example 4.5). This provides further evidence that the gap between the classes $\mathfrak{H}_\infty$ and $\mathfrak{H}_1$ is fairly large.

To prove our results, we resort to tools introduced in previous work (e.g., the Six-point Test to check hyponormality (Lemma 2.1) and the Backward Extension Theorem for 2-variable weighted shifts (Lemma 3.3)), together with a new direct sum decomposition for powers of 2-variable weighted shifts which parallels the decomposition used in [9] to analyze k-hyponormality for powers of (one-variable) weighted shifts. Specifically, we split the ambient space $\ell^2(\mathbb{Z}_+^2)$ as an orthogonal direct sum $\mathcal{H}^0 \oplus \mathcal{H}^1$, where $\mathcal{H}^m := \bigvee_{k=0}^{\infty}\{e_{(j,2k+m)}\colon j = 0, 1, 2, \ldots\}$ $(m = 0, 1)$. Each of the subspaces $\mathcal{H}^0$ and $\mathcal{H}^1$ reduces T_1 and T_2, and $\mathbf{T}^{(1,2)}$ is subnormal if and only if each of $\mathbf{T}^{(1,2)}|_{\mathcal{H}^0}$ and $\mathbf{T}^{(1,2)}|_{\mathcal{H}^1}$ is subnormal (cf. Fig. 4).

We devote the rest of this section to establishing our basic terminology and notation. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. We say that $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal and

$N(\mathcal{H}) \subseteq \mathcal{H}$, and *hyponormal* if $T^*T \geqslant TT^*$. For $S, T \in \mathcal{B}(\mathcal{H})$ let $[S,T] := ST - TS$. We say that an n-tuple $\mathbf{T} \equiv (T_1, \ldots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive on the direct sum of n copies of $\mathcal{H}$ (cf. [2,8,13]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. Finally, we say that a pair $\mathbf{T} \equiv (T_1, T_2)$ is 2-hyponormal if $\mathbf{T}$ is commuting and $(T_1, T_2, T_1^2, T_1T_2, T_2^2)$ is hyponormal. Clearly,

$$\text{normal} \quad \Rightarrow \quad \text{subnormal} \quad \Rightarrow \quad \text{2-hyponormal} \quad \Rightarrow \quad \text{hyponormal}.$$

The Bram–Halmos criterion for subnormality states that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if

$$\sum_{i,j} \left(T^i x_j, T^j x_i\right) \geqslant 0$$

for all finite collections $x_0, x_1, \ldots, x_k \in \mathcal{H}$ [4,5]. Using Choleski's algorithm for operator matrices, it is easy to verify that this condition is equivalent to the assertion that the k-tuple $(T, T^2, \ldots, T^k)$ is hyponormal for all $k \geqslant 1$.

For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*) let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geqslant 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. For notational convenience, we will often write $\mathit{shift}(\alpha_0, \alpha_1, \ldots)$ to denote W_α. In particular, we shall let $U_+ := \mathit{shift}(1, 1, \ldots)$ (U_+ is the (unweighted) unilateral shift) and $S_a := \mathit{shift}(a, 1, 1, \ldots)$. For a weighted shift W_α, *the moments of* α are given by

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1 & \text{if } k = 0, \\ \alpha_0^2 \cdots \alpha_{k-1}^2 & \text{if } k > 0. \end{cases}$$

It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_0 \leqslant \alpha_1 \leqslant \cdots$. Similarly, consider double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$, and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. (Recall that $\ell^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $\ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$.) We define the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ by

$$\begin{cases} T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}, \\ T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2}, \end{cases}$$

where $\boldsymbol{\varepsilon}_1 := (1,0)$ and $\boldsymbol{\varepsilon}_2 := (0,1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \quad \Leftrightarrow \quad \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \quad \left(\text{for all } \mathbf{k} \in \mathbb{Z}_+^2\right). \tag{1.1}$$

In an entirely similar way one can define multivariable weighted shifts.

A 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ is called *horizontally flat* if $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ for all $k_1, k_2 \geqslant 1$; $\mathbf{T}$ is called *vertically flat* if $\beta_{(k_1,k_2)} = \beta_{(1,1)}$ for all $k_1, k_2 \geqslant 1$. If $\mathbf{T}$ is horizontally and vertically flat, then $\mathbf{T}$ is simply called *flat*.

For an arbitrary 2-variable weighted shift $\mathbf{T}$, we shall let $\mathcal{R}_{ij}(\mathbf{T})$ denote the restriction of $\mathbf{T}$ to $\mathcal{M}_i \cap \mathcal{N}_j$, where $\mathcal{M}_i$ (respectively $\mathcal{N}_j$) is the subspace of $\ell^2(\mathbb{Z}_+^2)$ spanned by the canonical orthonormal basis vectors associated to indices $\mathbf{k} = (k_1, k_2)$ with $k_1 \geqslant 0$ and $k_2 \geqslant i$ (respectively $k_1 \geqslant j$ and $k_2 \geqslant 0$).

Trivially, a pair of unilateral weighted shifts W_α and W_β gives rise to a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$, if we let $\alpha_{(k_1,k_2)} := \alpha_{k_1}$ and $\beta_{(k_1,k_2)} := \beta_{k_2}$ (all $k_1, k_2 \in \mathbb{Z}_+^2$). In this case, $\mathbf{T}$ is subnormal (respectively hyponormal) if and only if so are T_1 and T_2; in fact, under the canonical identification of $\ell^2(\mathbb{Z}_+^2)$ and $\ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$, $T_1 \cong I \otimes W_\alpha$ and $T_2 \cong W_\beta \otimes I$, and $\mathbf{T}$ is also doubly commuting. For this reason, we do not focus attention on shifts of this type, and use them only when the above mentioned triviality is desirable or needed. Given $\mathbf{k} \in \mathbb{Z}_+^2$, the moment of (α, β) of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha, \beta) := \begin{cases} 1 & \text{if } \mathbf{k} = 0, \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geqslant 1 \text{ and } k_2 = 0, \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geqslant 1, \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geqslant 1 \text{ and } k_2 \geqslant 1. \end{cases}$$

(We remark that, due to the commutativity condition (1.1), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0, 0)$ to (k_1, k_2).) We now recall a well-known characterization of subnormality for multivariable weighted shifts [18], due to C. Berger (cf. [5, III.8.16]) and independently established by Gellar and Wallen [16]) in the single variable case: $\mathbf{T} \equiv (T_1, T_2)$ admits a commuting normal extension if and only if there is a probability measure μ (which we call the Berger measure of $\mathbf{T}$) defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$ (where $a_i := \|T_i\|^2$) such that $\gamma_{\mathbf{k}} = \int_R s^{k_1} t^{k_2}\, d\mu(s, t)$, for all $\mathbf{k} \in \mathbb{Z}_+^2$. In the single variable case, if W_α is subnormal with Berger measure ξ_α and $h \geqslant 1$, and if we let $\mathcal{L}_h := \bigvee\{e_n\colon n \geqslant h\}$ denote the invariant subspace obtained by removing the first h vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_\alpha|_{\mathcal{L}_h}$ is $\frac{s^h}{\gamma_h}\, d\xi_\alpha(s)$; alternatively, if $S : \ell^\infty(\mathbb{Z}_+) \to \ell^\infty(\mathbb{Z}_+)$ is defined by

$$S(\alpha)(n) := \alpha(n+1) \quad \bigl(\alpha \in \ell^\infty(\mathbb{Z}_+),\ n \geqslant 0\bigr), \tag{1.2}$$

then

$$d\xi_{S(\alpha)}(s) = \frac{s}{\alpha_0^2}\, d\xi(s). \tag{1.3}$$

2. The class $\mathfrak{H}_1$ is not invariant under squares

For a general operator T on Hilbert space, it is well known that the hyponormality of T does not imply the hyponormality of T^2 [17]. However, for a unilateral weighted shift W_α, the hyponormality of W_α (detected by the condition $\alpha_k \leqslant \alpha_{k+1}$ for all $k \geqslant 0$) clearly implies the hyponormality of every power W_α^m ($m \geqslant 1$). For 2-variable weighted shifts, one is thus tempted to expect that a similar result would hold, especially if we restrict attention to the class $\mathfrak{H}_1$ of commuting hyponormal pairs of subnormal operators. Somewhat surprisingly, it is actually possible to build a 2-variable weighted shift $\mathbf{T} \in \mathfrak{H}_1$ such that $\mathbf{T}^{(2,1)} \notin \mathfrak{H}_1$, and we do this in this section.

We begin with some basic results. First, we recall a hyponormality criterion for 2-variable weighted shifts.

Lemma 2.1. *(See Six-point Test [6].) Let $\mathbf{T} \equiv (T_1, T_2)$ be a 2-variable weighted shift, with weight sequences α and β. Then $\mathbf{T}$ is hyponormal if and only if*

$$H_{\mathbf{T}}(\mathbf{k}) := \begin{pmatrix} \alpha^2_{\mathbf{k}+\boldsymbol{\varepsilon}_1} - \alpha^2_{\mathbf{k}} & \alpha_{\mathbf{k}+\boldsymbol{\varepsilon}_2}\beta_{\mathbf{k}+\boldsymbol{\varepsilon}_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\boldsymbol{\varepsilon}_2}\beta_{\mathbf{k}+\boldsymbol{\varepsilon}_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta^2_{\mathbf{k}+\boldsymbol{\varepsilon}_2} - \beta^2_{\mathbf{k}} \end{pmatrix} \geqslant 0 \quad \bigl(\text{for all } \mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}^2_+\bigr).$$

Next, given integers i and ℓ ($\ell \geqslant 1, 0 \leqslant i \leqslant \ell - 1$), consider $\mathcal{H} \equiv \ell^2(\mathbb{Z}_+) = \bigvee_{j=0}^{\infty}\{e_j\}$. Define $\mathcal{H}_i := \bigvee_{j=0}^{\infty}\{e_{\ell j+i}\}$, so $\mathcal{H} = \bigoplus_{i=0}^{\ell-1} \mathcal{H}_i$. Following the notation in [9], for a weight sequence α let

$$P_{i\ell}(\alpha) \equiv \alpha(\ell : i) := \left\{\prod_{m=0}^{\ell-1} \alpha_{\ell j+i+m}\right\}_{j=0}^{\infty}; \tag{2.1}$$

that is, $\alpha(\ell : i)$ denotes the sequence of products of weights in adjacent packets of size ℓ, beginning with $\alpha_i \cdots \alpha_{i+\ell-1}$. For example, $\alpha(2:0) : \alpha_0\alpha_1, \alpha_2\alpha_3, \alpha_4\alpha_5, \ldots$, $\alpha(2:1) : \alpha_1\alpha_2, \alpha_3\alpha_4, \alpha_5\alpha_6, \ldots$ and $\alpha(3:2) : \alpha_2\alpha_3\alpha_4, \alpha_5\alpha_6\alpha_7, \alpha_8\alpha_9\alpha_{10}, \ldots$. Observe that, using the notation introduced in (1.2), $P_{i\ell} = P_{0\ell}S^i$. For a subnormal weighted shift W_α, it was proved in [9] that $W_{P_{i\ell}(\alpha)}$ is also subnormal (all $\ell \geqslant 1$, $0 \leqslant i \leqslant \ell - 1$). In fact, more is true.

Lemma 2.2. *(See [9].) For $\ell \geqslant 1$, and $0 \leqslant i \leqslant \ell - 1$, $W_{P_{i\ell}(\alpha)}$ is unitarily equivalent to $W_\alpha^\ell|_{\mathcal{H}_i}$. Therefore, W_α^ℓ is unitarily equivalent to $\bigoplus_{i=0}^{\ell-1} W_{P_{i\ell}(\alpha)}$. Consequently, W_α^ℓ is k-hyponormal if and only if $W_{P_{i\ell}(\alpha)}$ is k-hyponormal for each i such that $0 \leqslant i \leqslant \ell - 1$. Moreover, if W_α is subnormal with Berger measure ξ_α, then $W_{P_{i\ell}(\alpha)}$ is subnormal with Berger measure*

$$d\xi_{P_{i\ell}(\alpha)}(s) = d\xi_{P_{0\ell}S^i(\alpha}(s) = \frac{s^i}{\gamma_i(\alpha)}\, d\xi_{P_{0\ell}}(s) = \frac{s^{i/\ell}}{\gamma_i(\alpha)}\, d\xi_\alpha\bigl(s^{1/\ell}\bigr) \quad (0 \leqslant i \leqslant \ell - 1). \tag{2.2}$$

Example 2.3. Let $W_\alpha \equiv \mathit{shift}(\alpha_0, \alpha_1, \ldots)$ be a subnormal weighted shift, with Berger measure ξ_α. Then $\mathit{shift}(\alpha_2\alpha_3, \alpha_4\alpha_5, \ldots) \equiv W_{P_{22}(\alpha)}$ is also subnormal, with Berger measure $\frac{s}{\alpha_0^2\alpha_1^2}\, d\xi_\alpha(\sqrt{s}\,)$.

To produce an example of $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_1$ such that $\mathbf{T}^{(2,1)} \notin \mathfrak{H}_1$, we start with an example given in [14]. For $0 < \kappa \leqslant 1$, let $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ be defined by

$$\alpha_n := \begin{cases} \kappa\sqrt{\frac{3}{4}} & \text{if } n = 0, \\ \frac{\sqrt{(n+1)(n+3)}}{(n+2)} & \text{if } n \geqslant 1. \end{cases} \tag{2.3}$$

We know that W_α is subnormal, with Berger measure

$$d\xi_\alpha(s) := \bigl(1 - \kappa^2\bigr)\, d\delta_0(s) + \frac{\kappa^2}{2}\, ds + \frac{\kappa^2}{2}\, d\delta_1(s) \quad \text{[14, Proposition 4.2]},$$

where δ_p denotes the Dirac measure at p.

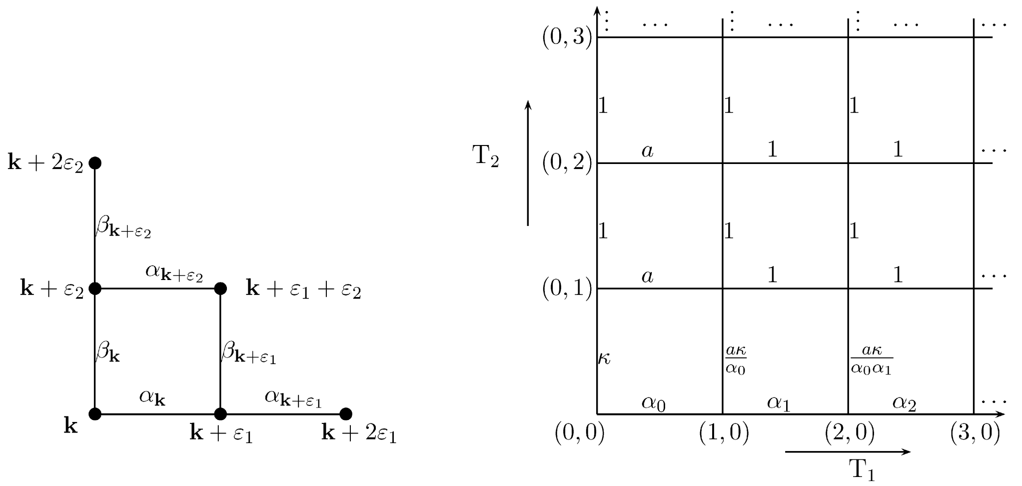

Fig. 1. Weight diagram used in the Six-point Test and weight diagram of the 2-variable weighted shift in Lemma 2.4.

For $0 < a < 1$, consider the 2-variable weighted shift given by Fig. 1, with $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ as above.

Lemma 2.4. *(See [14].) Let* $\mathbf{T} \equiv (T_1, T_2)$ *be the* 2*-variable weighted shift whose weight diagram is given by Fig.* 1*, with* $0 < a \leqslant \sqrt{\frac{1}{2}}$*. Then*

(i) *T_1 and T_2 are subnormal*;
(ii) $\mathbf{T} \in \mathfrak{H}_1$ *if and only if* $0 < \kappa \leqslant h_1(a) := \sqrt{\frac{32-48a^4}{59-72a^2}}$;
(iii) $\mathbf{T} \in \mathfrak{H}_2$ *if and only if* $0 < \kappa \leqslant h_2(a) := \sqrt{\frac{81-144a^2}{157-360a^2+144a^4}}$;
(iv) $\mathbf{T} \in \mathfrak{H}_\infty$ *if and only if* $0 < \kappa \leqslant h_\infty(a) := \frac{1}{\sqrt{2-a^2}}$.

Remark 2.5. Close inspection of the proof of Lemma 2.4 reveals that the hyponormality of the 2-variable weighted shift $\mathbf{T}$ whose weight diagram is given by Fig. 1 extends beyond the range $0 < a \leqslant \sqrt{\frac{1}{2}}$. As a matter of fact, the hyponormality of $\mathbf{T}$ is controlled by the nonnegativity of the two expressions, $f(a) := 84 - 95a^2$ and $g(a,\kappa) := (72a^2 - 59)\kappa^2 + 32 - 48a^4$. Of course, the nonnegativity of f requires $a \leqslant \sqrt{\frac{84}{95}}$, while to analyze the second expression we need to consider three cases:

$$\text{(i)}\quad 72a^2 - 59 < 0; \qquad \text{(ii)}\quad 72a^2 - 59 = 0; \quad \text{and} \quad \text{(iii)}\quad 72a^2 - 59 > 0.$$

In case (i),

$$g(a,\kappa) \geqslant 0 \quad \Leftrightarrow \quad a^4 \leqslant \frac{2}{3} \quad \text{and} \quad \kappa^2 \leqslant \frac{32-48a^4}{59-72a^2};$$

in case (ii),

$$a^2 = \frac{59}{72} \quad \text{and} \quad g(a,\kappa) = 32 - 48\left(\frac{59}{72}\right)^2 < 0;$$

and in case (iii),

$$g(a,\kappa) \geqslant 0 \quad \Leftrightarrow \quad a^2 > \frac{59}{72} \quad \text{and} \quad \kappa^2 \geqslant \frac{32-48a^4}{59-72a^2}.$$

Now, it is easy to verify that on the interval $(\sqrt{\frac{59}{72}}, \sqrt{\frac{84}{95}}\,]$ the expression $\frac{32-48a^4}{59-72a^2}$ is always greater than 1, and since we must have $\kappa \leqslant 1$, case (iii) cannot really happen. If we now observe that $a \leqslant \sqrt{\frac{84}{95}}$ is implied by the condition $a^4 \leqslant \frac{2}{3}$, we conclude that $\mathbf{T}$ is hyponormal if and only if $a \leqslant \sqrt[4]{\frac{2}{3}}$ and $\kappa \leqslant \sqrt{\frac{32-48a^4}{59-72a^2}} = h_1(a)$.

Theorem 2.6. *Let* $\mathbf{T} \equiv (T_1, T_2)$ *be the* 2*-variable weighted shift whose weight diagram is given by Fig.* 1*. Then* $\mathbf{T}^{(2,1)} \equiv (T_1^2, T_2)$ *is hyponormal if and only if* $0 < \kappa \leqslant h_{21}(a) := 3\sqrt{\frac{3-5a^4}{47-60a^2}}$*, with* $0 < a \leqslant \sqrt[4]{\frac{3}{5}}$.

Proof. For $m = 0, 1$, let $\mathcal{H}_m := \bigvee_{j=0}^{\infty}\{e_{(2j+m,k)}\colon k = 0, 1, 2, \ldots\}$. Then $\ell^2(\mathbb{Z}_+^2) \equiv \mathcal{H}_0 \oplus \mathcal{H}_1$, and each of $\mathcal{H}_0$ and $\mathcal{H}_1$ reduces T_1^2 and T_2. We can thus write

$$\left(T_1^2, T_2\right) \cong \left(W_{\alpha(2:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_0}\right) \oplus \left(W_{\alpha(2:1)} \oplus (I \otimes U_+), T_2|_{\mathcal{H}_1}\right).$$

By [10, Theorem 5.2 and Remark 5.3], the second summand, $(W_{\alpha(2:1)} \oplus (I \otimes U_+), T_2|_{\mathcal{H}_1})$, is subnormal. Thus, the hyponormality of (T_1^2, T_2) is equivalent to the hyponormality of the first summand, $(W_{\alpha(2:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_0})$. Now, to check the hyponormality of the first summand, by Lemma 2.1 it suffices to apply the Six-point Test at $\mathbf{k} = (0, 0)$. We have

$$\begin{aligned}
H_{(W_{\alpha(2:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_0})}(\mathbf{0}) &\equiv \begin{pmatrix} \alpha_3^2\alpha_2^2 - \alpha_1^2\alpha_0^2 & \frac{a^2\kappa}{\alpha_0\alpha_1} - \kappa\alpha_0\alpha_1 \\ \frac{a^2\kappa}{\alpha_0\alpha_1} - \kappa\alpha_0\alpha_1 & 1-\kappa^2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{9}{10} - \frac{2}{3}\kappa^2 & \sqrt{6}(\frac{1}{2}a^2 - \frac{1}{3}\kappa^2) \\ \sqrt{6}(\frac{1}{2}a^2 - \frac{1}{3}\kappa^2) & 1-\kappa^2 \end{pmatrix} \geqslant 0 \\
\Leftrightarrow \quad & \left(1-\kappa^2\right)\left(\frac{9}{10} - \frac{2}{3}\kappa^2\right) \geqslant 6\left(\frac{a^2}{2} - \frac{\kappa^2}{3}\right)^2 \\
\Leftrightarrow \quad & \frac{9}{10} - \frac{47}{30}\kappa^2 - \frac{3}{2}a^4 + 2a^2\kappa^2 \geqslant 0 \\
\Leftrightarrow \quad & h(a,\kappa) := \left(60a^2 - 47\right)\kappa^2 + 27 - 45a^4 \geqslant 0.
\end{aligned}$$

As in Remark 2.5, three cases arise:

$$\text{(i)} \quad 60a^2 - 47 < 0; \qquad \text{(ii)} \quad 72a^2 - 59 = 0; \quad \text{and} \quad \text{(iii)} \quad 60a^2 - 47 > 0.$$

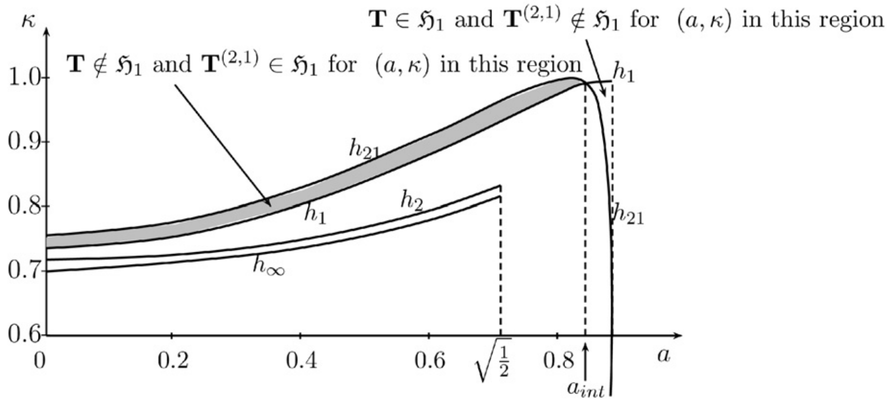

Fig. 2. Graphs of h_1, h_{21}, h_2 and h_∞ on the interval $[0, \sqrt[4]{\frac{3}{5}}\,]$.

In case (i),

$$h(a,\kappa) \geqslant 0 \quad \Leftrightarrow \quad a^4 \leqslant \frac{3}{5} \quad \text{and} \quad \kappa^2 \leqslant \frac{9(3-5a^4)}{47-60a^2};$$

in case (ii),

$$a^2 = \frac{47}{60} \quad \text{and} \quad h(a,\kappa) = 27 - 45\left(\frac{47}{60}\right)^2 < 0;$$

and in case (iii),

$$h(a,\kappa) \geqslant 0 \quad \Leftrightarrow \quad a^2 > \frac{47}{60} \quad \text{and} \quad \kappa^2 \geqslant \frac{27-45a^4}{47-60a^2}.$$

As before, it is easy to verify that on the interval $(\sqrt{\frac{47}{60}}, 1]$ the expression $\frac{27-45a^4}{47-60a^2}$ is always greater than 1, and since we must have $\kappa \leqslant 1$, case (iii) cannot really happen. We conclude that $\mathbf{T}$ is hyponormal if and only if $a \leqslant \sqrt[4]{\frac{3}{5}}$ and $\kappa \leqslant \sqrt{\frac{9(3-5a^4)}{47-60a^2}} \equiv h_{21}(a)$, as desired. □

We are now ready to formulate our first main result. Consider the two functions h_1 and h_{21} in Remark 2.5 and Theorem 2.6, respectively, restricted to the common portion of their domains, namely the interval $(0, \sqrt[4]{\frac{3}{5}}\,]$. A calculation shows that there exists a unique point $a_{\text{int}} \in (0, \sqrt[4]{\frac{3}{5}}\,]$ such that $h_1(a_{\text{int}}) = h_{21}(a_{\text{int}})$; in fact, $a_{\text{int}} \cong 0.8386$. Fig. 2 shows two regions in the (a,κ)-plane, one where $\mathbf{T}$ is hyponormal but $\mathbf{T}^{(2,1)}$ is not, and one where $\mathbf{T}^{(2,1)}$ is hyponormal but $\mathbf{T}$ is not. For added emphasis, we include the graphs of h_2 and h_∞ mentioned in Lemma 2.4, which are only defined on the interval $(0, \sqrt{\frac{1}{2}}\,]$. We thus have:

Theorem 2.7. *Let* $\mathbf{T}$ *be the* 2*-variable weighted shift whose weight diagram is given by Fig.* 1. *Then*

(i) $\mathbf{T} \in \mathfrak{H}_1$ *and* $\mathbf{T}^{(2,1)} \notin \mathfrak{H}_1 \Leftrightarrow a_{\mathrm{int}} < a \leqslant \sqrt[4]{\frac{3}{5}}$ *and* $h_{21}(a) < \kappa \leqslant h_1(a)$ (*see Fig.* 2).
(ii) $\mathbf{T} \notin \mathfrak{H}_1$ *and* $\mathbf{T}^{(2,1)} \in \mathfrak{H}_1 \Leftrightarrow 0 < a < a_{\mathrm{int}}$ *and* $h_1(a) < \kappa \leqslant h_{21}(a)$ (*see Fig.* 2).

3. A large class for which $(T_1^2, T_2) \in \mathfrak{H}_\infty \Leftrightarrow (T_1, T_2^2) \in \mathfrak{H}_\infty \Leftrightarrow (T_1, T_2) \in \mathfrak{H}_\infty$

It is well known that for a single operator T, the subnormality of all powers T^n $(n \geqslant 2)$ does not imply the hyponormality of T, even if T is a unilateral weighted shift [23]. In the multivariable case, the analogous result is nontrivial if one further assumes that each component is subnormal. To study this, we begin by recalling some useful notation and results. Given a weighted shift W_α, a (one-step) backward extension of W_α is the weighted shift $W_{\alpha(x)}$, where $\alpha(x): x, \alpha_0, \alpha_1, \alpha_2, \ldots.$

Lemma 3.1. *(Subnormal backward extension of a* 1*-variable weighted shift; cf. [7], [10, Proposition 1.5].) Let* W_α *be a weighted shift whose restriction* $W_\alpha|_{\mathcal{L}}$ *to* $\mathcal{L} := \bigvee\{e_1, e_2, \ldots\}$ *is subnormal, with Berger measure* $\mu_{\mathcal{L}}$. *Then* W_α *is subnormal* (*with Berger measure* μ) *if and only if*:

(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{L}})$, *and*
(ii) $\alpha_0^2 \leqslant (\|\frac{1}{t}\|_{L^1(\mu_{\mathcal{L}})})^{-1}$.

In this case,

$$d\mu(t) = \frac{\alpha_0^2}{t}\, d\mu_{\mathcal{L}}(t) + \left(1 - \alpha_0^2\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{L}})}\right) d\delta_0(t).$$

In particular, W_α *is never subnormal when* $\mu_{\mathcal{L}}(\{0\}) > 0$.

Corollary 3.2. *Let* W_α *be a subnormal weighted shift, let* $\mathcal{L}_2 := \bigvee\{e_2, e_3, \ldots\}$ *and let* $\mu_{\mathcal{L}_2}$ *denote the Berger measure of* $W_\alpha|_{\mathcal{L}_2}$. *Then* α_1 *is completely determined by* $\mu_{\mathcal{L}_2}$, *namely* $\alpha_1^2 = (\|1/t\|_{L^1(\mu_{\mathcal{L}_2})})^{-1}$. *More generally, for* $j \geqslant 3$ *let* $\mathcal{L}_j := \bigvee\{e_j, e_{j+1}, \ldots\}$, *and let* $\mu_{\mathcal{L}_j}$ *denote the Berger measure of* $W_\alpha|_{\mathcal{L}_j}$; *then* $\alpha_{j-1} = (\|1/t\|_{L^1(\mu_{\mathcal{L}_j})})^{-1}$.

Proof. Without loss of generality, we prove only the first assertion. Since $W_a|_{\mathcal{L}}$ is subnormal, Lemma 3.1 implies that $\alpha_1^2 \leqslant (\|1/t\|_{L^1(\mu_{\mathcal{L}_2})})^{-1}$. If strict inequality occurred, then the measure $\mu_{\mathcal{L}}$ would have an atom at 0, which would render the subnormality of W_α impossible. □

To state the 2-variable version of Lemma 3.1, we need to recall two notions from [10]:

(i) given a probability measure μ on $X \times Y \equiv \mathbb{R}_+ \times \mathbb{R}_+$, with $\frac{1}{t} \in L^1(\mu)$, the *extremal measure* μ_{ext} (which is also a probability measure) on $X \times Y$ is given by $d\mu_{\mathrm{ext}}(s,t) := (1 - \delta_0(t))\frac{1}{t\|1/t\|_{L^1(\mu)}}\, d\mu(s,t)$; and
(ii) given a measure μ on $X \times Y$, the *marginal measure* μ^X is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection onto X. Thus, $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$.

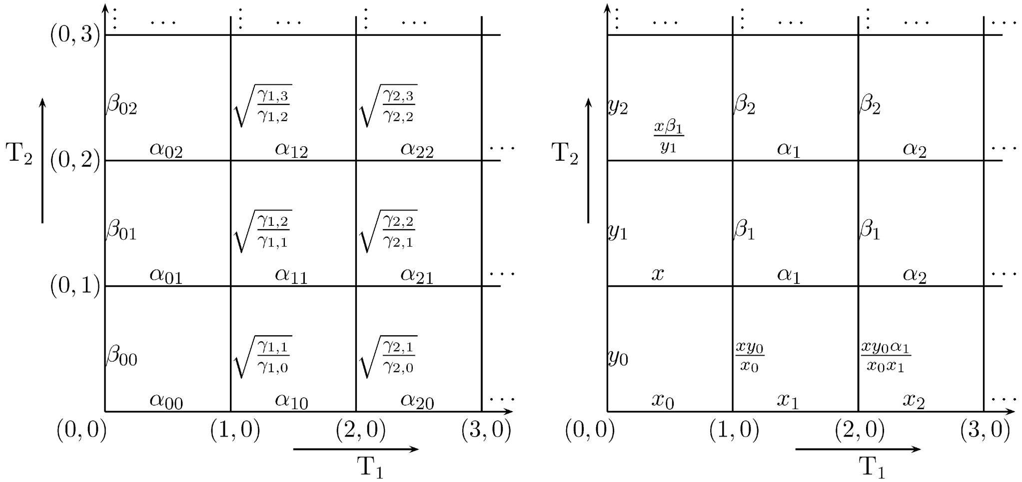

Fig. 3. Weight diagram of the 2-variable weighted shift in Lemma 3.3 and weight diagram of a 2-variable weighted shift with $\mathcal{R}_{11}(\mathbf{T}) \cong (I \otimes W_\alpha, W_\beta \otimes I)$, respectively.

Observe that if μ is a probability measure, then so is μ^X. For example,

$$d(\xi \times \eta)_{\mathrm{ext}}(s,t) = \bigl(1-\delta_0(t)\bigr)\frac{1}{t\|1/t\|_{L^1(\eta)}}\,d\xi(s)\,d\eta(t) \tag{3.1}$$

and $(\xi \times \eta)^X = \xi$.

Lemma 3.3. *(Subnormal backward extension of a 2-variable weighted shift; cf. [10, Proposition 3.10].) Consider the following 2-variable weighted shift (see Fig. 3), and let $\mathcal{M}$ be the subspace of $\ell^2(\mathbb{Z}_+^2)$ spanned by the canonical orthonormal basis vectors associated to indices $\mathbf{k} = (k_1, k_2)$ with $k_1 \geqslant 0$ and $k_2 \geqslant 1$. Assume that $\mathcal{R}_{10}(\mathbf{T}) \equiv \mathbf{T}|_{\mathcal{M}}$ is subnormal with Berger measure $\mu_{\mathcal{M}}$ and that $W_0 := \mathrm{shift}(\alpha_{00}, \alpha_{10}, \ldots)$ is subnormal with Berger measure ν. Then $\mathbf{T}$ is subnormal if and only if:*

(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;
(ii) $\beta_{00}^2 \leqslant (\|1/t\|_{L^1(\mu_{\mathcal{M}})})^{-1}$;
(iii) $\beta_{00}^2 \|1/t\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{\mathrm{ext}}^X \leqslant \nu$.

Moreover, if $\beta_{00}^2\|1/t\|_{L^1(\mu_{\mathcal{M}})} = 1$, then $(\mu_{\mathcal{M}})_{\mathrm{ext}}^X = \nu$. In the case when $\mathbf{T}$ is subnormal, the Berger measure μ of $\mathbf{T}$ is given by

$$d\mu(s,t) = \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu\mathcal{M})} d(\mu_{\mathcal{M}})_{\mathrm{ext}}(s,t) + \left(d\nu(s) - \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{\mathrm{ext}}^X(s)\right) d\delta_0(t).$$

Definition 3.4.

(i) The *core* of a 2-variable weighted shift $\mathbf{T}$ is the restriction of $\mathbf{T}$ to $\mathcal{M}_1 \cap \mathcal{N}_1$, in symbols, $c(\mathbf{T}) := \mathbf{T}|_{\mathcal{M}_1 \cap \mathcal{N}_1} \equiv \mathcal{R}_{11}(\mathbf{T})$.

(ii) A 2-variable weighted shift $\mathbf{T}$ is said to be of *tensor form* if $\mathbf{T} \cong (I \otimes W_\alpha, W_\beta \otimes I)$. When $\mathbf{T}$ is subnormal, this is equivalent to requiring that the Berger measure be a Cartesian product $\xi \times \eta$.

(iii) The class of all 2-variable weighted shifts $\mathbf{T} \in \mathfrak{H}_0$ whose cores are of tensor form will be denoted by $\mathcal{TC}$, that is, $\mathcal{TC} := \{\mathbf{T} \in \mathfrak{H}_0\colon c(\mathbf{T}) \text{ is of tensor form}\}$.

(iv) For each $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, let $A_\mathbf{k} := \{\mathbf{T} \in \mathfrak{H}_0\colon \mathcal{R}_{k_1 k_2}(\mathbf{T}) \in \mathcal{TC}\}$.

Observe that for $\mathbf{k}, \mathbf{m} \in \mathbb{Z}_+^2$ with $\mathbf{k} \leqslant \mathbf{m}$ (i.e., $\mathbf{m} - \mathbf{k} \in \mathbb{Z}_+^2$), we have $A_\mathbf{k} \subseteq A_\mathbf{m}$. Thus, the collection $\{A_\mathbf{k}\}$ forms an ascending chain with respect to set inclusion and the partial order induced by $\mathbb{Z}_+^2$. Moreover, $\mathcal{TC} = A_\mathbf{0} \subseteq A_\mathbf{k}$ for all $\mathbf{k} \in \mathbb{Z}_+^2$. All 2-variable weighted shifts considered in [10–12] and [14] are in $\mathcal{TC}$. Thus, $\mathcal{TC}$ is a rather large class; as a matter of fact, much more is true. The following theorem shows that an outer propagation phenomena occurs for $\mathcal{TC}$.

Theorem 3.5. *For all* $\mathbf{k} \in \mathbb{Z}_+^2$, $A_\mathbf{k} = \mathcal{TC}$.

Proof. Since we always have $\mathcal{TC} \subseteq A_\mathbf{k}$, we prove the reverse inclusion. Without loss of generality, it is enough to show that if $\mathbf{T} \in \mathfrak{H}_0$ and $\mathbf{T}|_{\mathcal{M}_2 \cap \mathcal{N}_2}$ is of tensor form, then $c(\mathbf{T})$ is of tensor form. If $\mathbf{T}|_{\mathcal{M}_2 \cap \mathcal{N}_2}$ is of tensor form, then $\mathit{shift}(\beta_{22}, \beta_{23}, \ldots) = \mathit{shift}(\beta_{k_1 2}, \beta_{k_1 3}, \ldots)$ for all $k_1 \geqslant 2$. The subnormality of T_2 then implies that $\mathit{shift}(\beta_{k_1 0}, \beta_{k_1 1}, \ldots)$ is subnormal for all $k_1 \geqslant 2$. By Corollary 3.2, we have $\beta_{k_1 1} = \sqrt{(\|1/t\|_{L^1(\xi_{k_1})})^{-1}}$ $(k_1 \geqslant 2)$, where ξ_{k_1} is the Berger measure of $\mathit{shift}(\beta_{k_1 2}, \beta_{k_1 3}, \ldots)$. Thus, $\mathit{shift}(\beta_{21}, \beta_{22}, \ldots) = \mathit{shift}(\beta_{k_1 1}, \beta_{k_1 2}, \ldots)$ for all $k_1 \geqslant 2$. Now, since $\beta_{21} = \beta_{k_1 1}$ (all $k_1 \geqslant 2$), the commutativity of T_1 and T_2 implies $\alpha_{k_1 2} = \alpha_{k_1 1}$ for all $k_1 \geqslant 2$. Thus, $\mathit{shift}(\alpha_{21}, \alpha_{31}, \ldots) = \mathit{shift}(\alpha_{2k_2}, \alpha_{3k_2}, \ldots)$ for all $k_2 \geqslant 1$. By the subnormality of T_1 and Lemma 3.1, we have $\mathit{shift}(\alpha_{11}, \alpha_{21}, \ldots) = \mathit{shift}(\alpha_{1k_2}, \alpha_{2k_2}, \ldots)$ for all $k_2 \geqslant 1$. Therefore $c(\mathbf{T})$ is of tensor form. $\square$

We now consider the 2-variable weighted shift given by Fig. 3, where $W_x := \mathit{shift}(x_0, x_1, \ldots)$ and $W_y := \mathit{shift}(y_0, y_1, \ldots)$ are subnormal with Berger measures μ_y and μ_x, respectively. Further, we let $W_\alpha := \mathit{shift}(\alpha_1, \alpha_2, \ldots)$ and $W_\beta := \mathit{shift}(\beta_1, \beta_2, \ldots)$ be subnormal with Berger measures ξ and η, respectively, and we let $r := \|1/s\|_{L^1(\xi)} \in (0, \infty]$ and $d\tilde{\xi}(s) := (d\xi(s))/s$. We then have:

Theorem 3.6. *Let* $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{TC}$. *Then* $\mathcal{R}_{10}(\mathbf{T}) \in \mathfrak{H}_\infty$ *if and only if* $x^2 r\eta \leqslant (\mu_y)_1$. *In this case, the Berger measure of* $\mathcal{R}_{10}(\mathbf{T})$ *is* $x^2\tilde{\xi} \times \eta + \delta_0 \times ((\eta_y)_1 - x^2 r\eta)$, *where* $(\eta_y)_1$ *is the Berger measure of the subnormal shift* $\mathit{shift}(y_1, y_2, \ldots)$.

Proof. This is a straightforward application of Lemma 3.3, if we think of $\mathcal{R}_{10}(\mathbf{T})$ as the backward extension of $c(\mathbf{T})$ (in the s direction). $\square$

Proposition 3.7. *Let* $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{TC}$. *Then* $\|\frac{1}{t}\|_{L^1((\eta_y)_1)} = y_1^2 \|\frac{1}{t}\|_{L^1((\eta_y)_2^2)}$, *where* $(\eta_y)_1$ (*respectively* $(\eta_y)_2^2$) *is the Berger measure of* $\mathit{shift}(y_1, y_2, \ldots)$ (*respectively* $\mathit{shift}(y_2 y_3, y_4 y_5, \ldots)$). *Moreover,* $\|\frac{1}{t}\|_{L^1(\eta)} = \beta_1^2 \|\frac{1}{t}\|_{L^1(\eta_1^2)}$, *where* η_1^2 *is the Berger measure of* $\mathit{shift}(\beta_2\beta_3, \beta_4\beta_5, \ldots)$.

Proof. Since *shift*$(y_0, y_1, \ldots)$ has Berger measure η_y, we have $(d\eta_y)_1 = \frac{t}{y_0^2}\, d\eta_y(t)$; moreover, the Berger measure of *shift*$(y_2, y_3, \ldots)$ is

$$(d\eta_y)_2(t) = \frac{t^2}{y_0^2 y_1^2}\, d\eta_y(t).$$

Thus by Lemma 2.2, *shift*$(y_2 y_3, y_4 y_5, \ldots)$ has Berger measure

$$(d\eta_y)_2^2 \equiv \frac{t}{y_0^2 y_1^2}\, d\eta_y(\sqrt{t}\,).$$

Observe that

$$\left\|\frac{1}{t}\right\|_{L^1((\eta_y)_1)} = \frac{1}{y_0^2} = \int_0^A \frac{1}{y_0^2}\, d\eta_y(t) = \frac{1}{y_0^2}\int_0^{A^2} d\eta_y(\sqrt{t}\,) = \frac{1}{y_0^2}\int_0^{A^2} \frac{y_0^2 y_1^2}{t}\, d(\eta_y)_2^2 = y_1^2 \left\|\frac{1}{t}\right\|_{L^1((\eta_y)_2^2)},$$

where $A := \|\textit{shift}(y_0, y_1, \ldots)\|^2$. Thus, we get

$$\left\|\frac{1}{t}\right\|_{L^1((\eta_y)_1)} = y_1^2 \left\|\frac{1}{t}\right\|_{L^1((\eta_y)_2^2)},$$

as desired.

Next, we observe that $d\eta_1(t) \equiv \frac{t}{\beta_1^2}\, d\eta(t)$ is the Berger measure of *shift*$(\beta_2, \beta_3, \ldots)$ and $d\eta_1^2(t) \equiv \frac{\sqrt{t}}{\beta_1^2} d\eta(\sqrt{t}\,)$ is the Berger measure of *shift*$(\beta_2\beta_3, \beta_4\beta_5, \ldots)$. Let $B := \|\textit{shift}(\beta_0, \beta_1, \ldots)\|^2$; we then have

$$\left\|\frac{1}{t}\right\|_{L^1(\eta)} = \int_0^B \frac{1}{t}\, d\eta(t) = \int_0^{B^2} \frac{1}{\sqrt{t}}\, d\eta(\sqrt{t}\,) = \beta_1^2 \int_0^{B^2} \frac{1}{t}\frac{\sqrt{t}}{\beta_1^2}\, d\eta(\sqrt{t}\,) = \beta_1^2 \left\|\frac{1}{t}\right\|_{L^1(\eta_1^2)},$$

as desired. $\square$

We next recall that (T_1, T_2^2) can be regarded as the orthogonal direct sum of two 2-variable weighted shifts. For $m = 0, 1$, let

$$\mathcal{H}^m := \bigvee_{k=0}^{\infty} \{e_{(j,2k+m)}\colon\ j = 0, 1, 2, \ldots\}.$$

Then $\ell^2(\mathbb{Z}_+^2) \equiv \mathcal{H}^0 \oplus \mathcal{H}^1$ and each of $\mathcal{H}^0$ and $\mathcal{H}^1$ reduces T_1 and T_2. Thus, (T_1, T_2^2) is subnormal if and only if each of $(T_1, T_2^2)|_{\mathcal{H}^0}$ and $(T_1, T_2^2)|_{\mathcal{H}^1}$ is subnormal. The weight diagrams of these 2-variable weighted shifts are shown in Fig. 4.

We first focus on $(T_1, T_2^2)|_{\mathcal{H}^1}$:

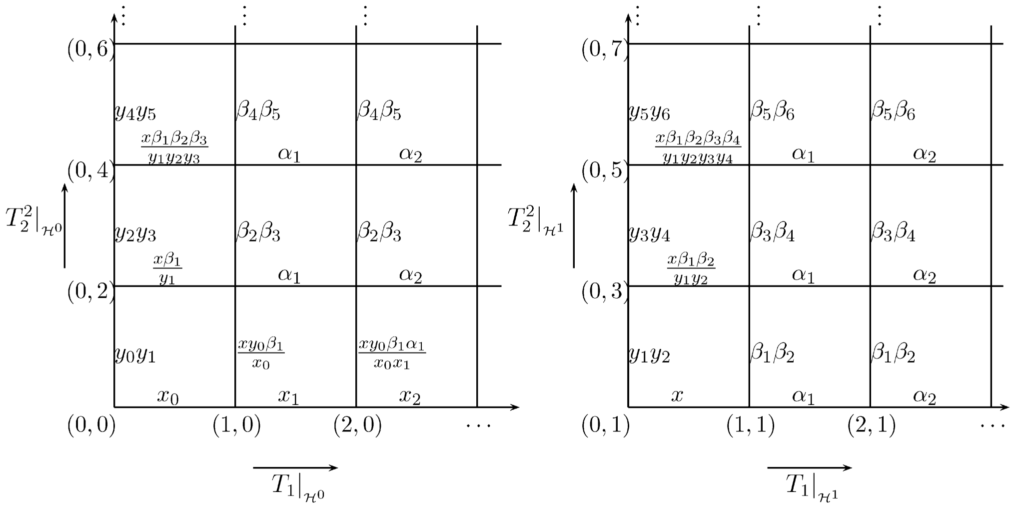

Fig. 4. Weight diagrams of $(T_1, T_2^2)|_{\mathcal{H}^0}$ and $(T_1, T_2^2)|_{\mathcal{H}^1}$ in the proof of Proposition 3.8 and Theorem 3.9.

Proposition 3.8. *Let* $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{TC}$. *Then* $(T_1, T_2^2)|_{\mathcal{H}_1}$ *is subnormal if and only if* $\mathcal{R}_{10}(\mathbf{T})$ *is subnormal.*

Proof. First, recall that *shift*$(y_0, y_1, y_2, \ldots)$ has Berger measure η_y, that $d(\eta_y)_1(t) = \frac{t}{y_0^2}\, d\eta_y(t)$ and that $d(\eta_y)_2(t) = \frac{t^2}{y_0^2 y_1^2}\, d\eta_y(t)$. Now, Theorem 3.6 states that

$$(T_1, T_2)|_{\mathcal{M}_1} \text{ is subnormal} \quad \Leftrightarrow \quad x^2 r\eta \leqslant (\eta_y)_1.$$

On the other hand, Theorem 3.6 (applied to $(T_1, T_2^2)|_{\mathcal{H}_1}$) says that

$$\left(T_1, T_2^2\right)\big|_{\mathcal{H}_1} \text{ is subnormal} \quad \Leftrightarrow \quad x^2 r\eta^2 \leqslant (\eta_y)_1^2,$$

and if $(T_1, T_2)|_{\mathcal{H}_1}$ is subnormal, its Berger measure is $x^2\tilde{\xi} \times \eta^2 + \delta_0 \times ((\eta_y)_1^2 - x^2 r\eta^2)$, where $(\eta_y)_1^2$ is the Berger measure of *shift*$(y_1y_2, y_3y_4, \ldots)$ and η^2 is the Berger measure of $W_\beta :=$ *shift*$(\beta_1\beta_2, \beta_3\beta_4, \ldots)$. By observing that

$$x^2 r\eta^2 \leqslant (\eta_y)_1^2 \quad \Leftrightarrow \quad x^2 r\, d\eta(\sqrt{t}\,) \leqslant d(\eta_y)_1(\sqrt{t}\,) \quad \Leftrightarrow \quad x^2 r\, d\eta(t) \leqslant d(\eta_y)_1(t),$$

we obtain the desired result. $\square$

Theorem 3.9. *Let* $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{TC}$. *Then*

$$\left(T_1, T_2^2\right) \in \mathfrak{H}_\infty \quad \Leftrightarrow \quad \left(T_1^2, T_2\right) \in \mathfrak{H}_\infty \quad \Leftrightarrow \quad (T_1, T_2) \in \mathfrak{H}_\infty.$$

Corollary 3.10. *Let* $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{TC}$. *If* $(T_1, T_2^2), (T_1^2, T_2) \in \mathfrak{H}_\infty$, *then* $(T_1, T_2) \in \mathfrak{H}_\infty$.

In view of Corollary 3.10, the following conjecture for 2-variable weighted shifts seems natural.

Conjecture 3.11. *If* $(T_1, T_2^2), (T_1^2, T_2) \in \mathfrak{H}_\infty$, *then* $(T_1, T_2) \in \mathfrak{H}_\infty$.

Proof of Theorem 3.9. Clearly, it is enough to show that $(T_1, T_2^2) \in \mathfrak{H}_\infty \Rightarrow (T_1, T_2) \in \mathfrak{H}_\infty$. Since $(T_1, T_2^2) \in \mathfrak{H}_\infty \Rightarrow (T_1, T_2^2)|_{\mathcal{H}^0} \in \mathfrak{H}_\infty$, our strategy consists of first characterizing the subnormality of $\mathbf{T}$ and of $(T_1, T_2^2)|_{\mathcal{H}^0}$ in terms of the given parameters (y_0, ν, etc.), and then establishing the desired implication at the parameter level. That is, we will show that $(T_1, T_2^2)|_{\mathcal{H}^0} \in \mathfrak{H}_\infty \Rightarrow \mathbf{T} \in \mathfrak{H}_\infty$ using their parametric characterizations. Proposition 3.8 will help us characterize the subnormality of $\mathbf{T}$. Recall that $(T_1, T_2^2)|_{\mathcal{H}_1}$ is subnormal if and only if $(T_1, T_2)|_{\mathcal{M}_1}$ is subnormal, and in that case the Berger measure of $(T_1, T_2)|_{\mathcal{M}_1}$ is

$$\mu_{\mathcal{M}} = x^2 \tilde{\xi} \times \eta + \delta_0 \times \big((\eta_y)_1 - x^2 r \eta\big).$$

We then have

$$\begin{aligned} \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} &= \int \frac{1}{t}\, d\mu_{\mathcal{M}}(s,t) = x^2 r \left\| \frac{1}{t} \right\|_{L^1(\eta)} + \int \frac{1}{t}\, d(\eta_y)_1(t) - x^2 r \left\| \frac{1}{t} \right\|_{L^1(\eta)} \\ &= \left\| \frac{1}{t} \right\|_{L^1((\eta_y)_1)}. \end{aligned} \tag{3.2}$$

Thus, we get

$$\begin{aligned} d(\mu_{\mathcal{M}})_{\mathrm{ext}}(s,t) &= d\big\{x^2 \tilde{\xi} \times \eta + \delta_0 \times \big((\eta_y)_1 - x^2 r \eta\big)\big\}_{\mathrm{ext}}(s,t) \\ &= \frac{1}{t\|1/t\|_{L^1(\mu_{\mathcal{M}})}} \big\{x^2 d\tilde{\xi}(s)\, d\eta(t) + d\delta_0(s)\big(d(\eta_y)_1(t) - x^2 r\, d\eta(t)\big)\big\} \\ &= \frac{1}{\|1/t\|_{L^1(\mu_{\mathcal{M}})}} \left\{x^2 d\tilde{\xi}(s) \frac{d\eta(t)}{t} + d\delta_0(s)\left(\frac{d(\eta_y)_1(t)}{t} - x^2 r \frac{d\eta(t)}{t}\right)\right\}. \end{aligned}$$

From (3.2), it follows that

$$(\mu_{\mathcal{M}})_{\mathrm{ext}}^X = \left(\frac{x^2 \|1/t\|_{L^1(\eta)}}{\|1/t\|_{L^1((\eta_y)_1)}}\right)\tilde{\xi} + \left(1 - \frac{x^2 r \|1/t\|_{L^1(\eta)}}{\|1/t\|_{L^1((\eta_y)_1)}}\right)\delta_0. \tag{3.3}$$

If we let φ denote the right-hand side in (3.3), it follows that

$$\begin{aligned} (T_1, T_2) \text{ is subnormal} \quad &\Leftrightarrow \quad y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{\mathrm{ext}}^X \leqslant \nu \quad \text{(by Lemma 3.3)} \\ &\Leftrightarrow \quad y_0^2 \left\| \frac{1}{t} \right\|_{L^1((\eta_y)_1)} \varphi \leqslant \mu_x \quad \big(\text{using (3.2)}\big). \end{aligned} \tag{3.4}$$

We have thus characterized the subnormality of $\mathbf{T}$.

We now consider the 2-variable weighted shift $(T_1, T_2^2)|_{\mathcal{H}^0}$ and the associated subspace $\mathcal{HM} := \bigvee\{e_{\mathbf{k}} \in \mathcal{H}^0\colon k_2 \geqslant 1\}$. Observe that $(T_1, T_2^2)|_{\mathcal{H}^0}$ can be regarded as a backward extension of $(T_1, T_2^2)|_{\mathcal{HM}}$, and that the latter is subnormal with Berger measure

$$\theta := \frac{x^2\beta_1^2}{y_1^2}\tilde{\xi} \times \eta_1^2 + \delta_0 \times \left((\eta_y)_2^2 - \frac{x^2 r\beta_1^2}{y_1^2}\eta_1^2\right),$$

where η_1^2 (respectively $(\eta_y)_2^2$) is the Berger measure of *shift*$(\beta_2\beta_3, \beta_4\beta_5, \ldots)$ (respectively *shift*$(y_2y_3, y_4y_5, \ldots)$). We then have

$$\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{HM}})} = \left\|\frac{1}{t}\right\|_{L^1((\eta_y)_2^2)}, \tag{3.5}$$

and

$$d(\mu_{\mathcal{HM}})_{\mathrm{ext}}(s,t) = d\theta_{\mathrm{ext}}(s,t) = \frac{1}{t\|1/t\|_{L^1(\mu_{\mathcal{HM}})}}\, d\theta(s,t).$$

From (3.5), we have

$$(\mu_{\mathcal{HM}})_{\mathrm{ext}}^X = \frac{1}{\|1/t\|_{L^1((\eta_y)_2^2)}}\left\{\frac{x^2\beta_1^2}{y_1^2}\left\|\frac{1}{t}\right\|_{L^1(\eta_1^2)}\tilde{\xi} + \left(\left\|\frac{1}{t}\right\|_{L^1((\eta_y)_2^2)} - \frac{x^2 r\beta_1^2}{y_1^2}\left\|\frac{1}{t}\right\|_{L^1(\eta_1^2)}\right)\delta_0\right\}. \tag{3.6}$$

If we now let ψ denote the expression in braces in the right-hand side of (3.6), Lemma 3.3 combined with (3.5) imply that

$$\begin{aligned}
&\big(T_1, T_2^2\big)\big|_{\mathcal{H}^0} \text{ is subnormal}\\
&\Leftrightarrow\quad y_0^2y_1^2\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{HM}})}(\mu_{\mathcal{HM}})_{\mathrm{ext}}^X \leqslant \nu \quad\Leftrightarrow\quad y_0^2y_1^2\psi \leqslant \mu_x\\
&\Leftrightarrow\quad y_0^2\left\{x^2\beta_1^2\left\|\frac{1}{t}\right\|_{L^1(\eta_1^2)}\tilde{\xi} + \left(y_1^2\left\|\frac{1}{t}\right\|_{L^1((\eta_y)_2^2)} - x^2 r\beta_1^2\left\|\frac{1}{t}\right\|_{L^1(\eta_1^2)}\right)\delta_0\right\} \leqslant \mu_x.
\end{aligned} \tag{3.7}$$

Observe that

$$\begin{aligned}
&y_0^2\left\|\frac{1}{t}\right\|_{L^1((\eta_y)_1)}\varphi \leqslant \mu_x\\
&\Leftrightarrow\quad y_0^2y_1^2\left\|\frac{1}{t}\right\|_{L^1((\eta_y)_2^2)}\varphi \leqslant \mu_x\\
&\Leftrightarrow\quad y_0^2\left\{x^2\left\|\frac{1}{t}\right\|_{L^1(\eta)}\tilde{\xi} + \left(y_1^2\left\|\frac{1}{t}\right\|_{L^1((\eta_y)_2^2)} - x^2 r\left\|\frac{1}{t}\right\|_{L^1(\eta)}\right)\delta_0\right\} \leqslant \mu_x\\
&\Leftrightarrow\quad y_0^2\left\{x^2\beta_1^2\left\|\frac{1}{t}\right\|_{L^1(\eta_1^2)}\tilde{\xi} + \left(y_1^2\left\|\frac{1}{t}\right\|_{L^1((\eta_y)_2^2)} - x^2 r\beta_1^2\left\|\frac{1}{t}\right\|_{L^1(\eta_1^2)}\right)\delta_0\right\} \leqslant \mu_x.
\end{aligned} \tag{3.8}$$

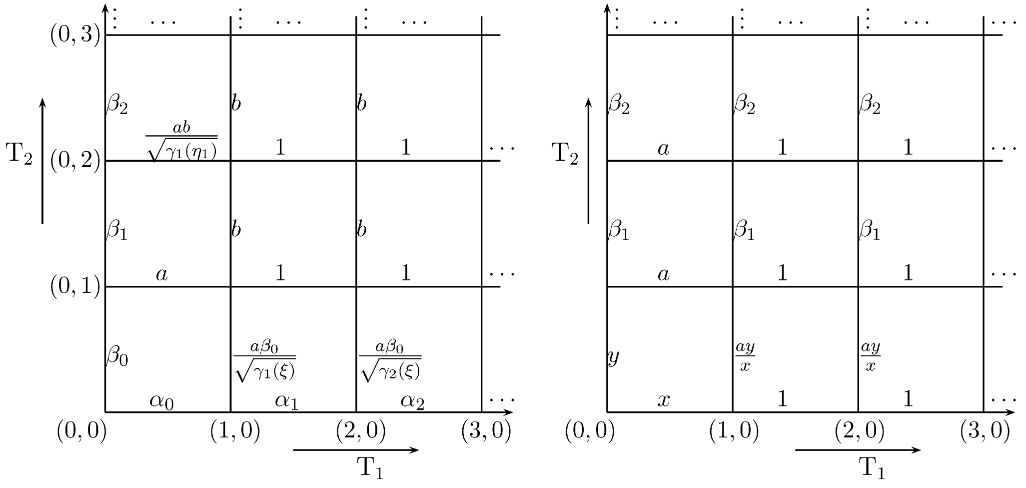

Fig. 5. Weight diagrams of the 2-variable weighted shifts in Theorem 3.13 and Lemma 4.3, respectively.

By combining (3.7) and (3.8), we easily see that

$$\big(T_1, T_2^2\big)\big|_{\mathcal{H}^0} \text{ is subnormal} \quad \Leftrightarrow \quad y_0^2\left\|\frac{1}{t}\right\|_{L^1((\eta_y)_1)}\varphi \leqslant \mu_x. \tag{3.9}$$

We thus have a characterization of the subnormality of $(T_1, T_2^2)|_{\mathcal{H}^0}$. From (3.4) and (3.9) it now follows that the subnormality of (T_1, T_2^2) implies the subnormality of (T_1, T_2). $\square$

It is straightforward from Definition 3.4 that a flat 2-variable weighted shift $\mathbf{T} \in \mathfrak{H}_0$ necessarily belongs to $\mathcal{TC}$. Thus, the following result is an easy consequence of Theorem 3.9.

Corollary 3.12. *Let $\mathbf{T} \equiv (T_1, T_2)$ be a flat 2-variable weighted shifts, that is, a 2-variable weighted shift $\mathbf{T} \in \mathfrak{H}_0$ given by Fig. 5. Then we have*

$$\big(T_1, T_2^2\big) \in \mathfrak{H}_\infty \quad \text{if and only if} \quad \big(T_1^2, T_2\big) \in \mathfrak{H}_\infty \quad \text{if and only if} \quad \big(T_1, T_2\big) \in \mathfrak{H}_\infty.$$

For a flat, contractive 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$, we can give a concrete condition for the subnormality of $\mathbf{T}$. To do this, let *shift*$(\alpha_0, \alpha_1, \ldots)$ and *shift*$(\beta_0, \beta_1, \ldots)$ have Berger measures ξ and η, respectively. Also, recall that for $0 < \alpha < \beta$, *shift*$(\alpha, \beta, \beta, \ldots)$ is subnormal with Berger measure

$$\left(1 - \frac{\alpha^2}{\beta^2}\right)\delta_0 + \frac{\alpha^2}{\beta^2}\delta_{\beta^2}.$$

To avoid trivial cases, and to ensure that each of T_1 and T_2 is a contraction, we need to assume that $ab^n < \prod_{j=1}^n \beta_j$, and we shall see in Theorem 3.13 that we also need $a^2/b^2 < \|1/t\|_{L^1(\eta_1)}$, where η_1 is the Berger measure of *shift*$(\beta_1, \beta_2, \beta_3, \ldots)$. Finally, we know from [11, Theorem 3.3] and [12, Section 5] that if $\mathbf{T} \equiv (T_1, T_2)$ is subnormal, then ξ and η are of the form

$$\begin{aligned} \xi &= p\delta_0 + q\delta_1 + \bigl[1-(p+q)\bigr]\rho, \\ \eta &= u\delta_0 + v\delta_{b^2} + \bigl[1-(u+v)\bigr]\sigma, \end{aligned} \tag{3.10}$$

where $0 < p, q, u, v < 1$, $p+q \leqslant 1$, $u+v \leqslant 1$, and ρ, σ are probability measures with $\rho(\{0\} \cup \{1\}) = 0$, $\sigma(\{0\} \cup \{b^2\}) = 0$. We then have:

Theorem 3.13. *Let* $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_0$ *be a contractive* 2*-variable weighted shift whose weight diagram is given by Fig.* 5*, let* $v := \eta(\{b^2\})$ *and* $\xi \equiv p\delta_0 + q\delta_1 + [1-(p+q)]\rho$*, with* $p, q > 0$*,* $p+q \leqslant 1$ *(cf.* (3.10)*), and let* η_1 *denote the Berger measure of shift*$(\beta_1, \beta_2, \ldots)$*. Then* $(T_1, T_2) \in \mathfrak{H}_\infty$ *if and only if*

$$\beta_0 \leqslant \min\left\{\frac{b}{a}\sqrt{v}, \sqrt{\frac{p}{\|1/t\|_{L^1(\eta_1)} - a^2/b^2}}, \frac{b}{a}\sqrt{q}, \sqrt{\frac{1}{\|1/t\|_{L^1(\eta_1)}}}\right\}.$$

Proof. We first observe that

$$\mu_{\mathcal{M}} = a^2\delta_1 \times \delta_{b^2} + \delta_0 \times \bigl(\eta_1 - a^2\delta_{b^2}\bigr). \tag{3.11}$$

Using (3.10) and (3.11), a calculation shows that $(T_1, T_2)|_{\mathcal{M}_1} \in \mathfrak{H}_\infty$ if and only if $\beta_0 \leqslant \frac{b}{a}\sqrt{v}$. Observe that

$$(\mu_{\mathcal{M}})^X_{\text{ext}} = \frac{1}{\|1/t\|_{L^1(\eta_1)}}\left\{\left(\left\|\frac{1}{t}\right\|_{L^1(\eta_1)} - \frac{a^2}{b^2}\right)\delta_0 + \frac{a^2}{b^2}\delta_1\right\}.$$

By [10, Theorem 5.2], $(T_1, T_2)|_{\mathcal{N}_1} \in \mathfrak{H}_\infty$. Therefore

$$\begin{aligned} &(T_1, T_2) \in \mathfrak{H}_\infty \\ &\Leftrightarrow \quad y_0^2\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}(\mu_{\mathcal{M}})^X_{\text{ext}} \leqslant \nu \quad \text{(by Lemma 3.3)} \quad \text{and} \quad \beta_0 \leqslant \frac{b}{a}\sqrt{v} \\ &\Leftrightarrow \quad \beta_0^2\left\{\left(\left\|\frac{1}{t}\right\|_{L^1(\eta_1)} - \frac{a^2}{b^2}\right)\delta_0 + \frac{a^2}{b^2}\delta_1\right\} \leqslant \xi \quad \text{and} \quad \beta_0 \leqslant \frac{b}{a}\sqrt{v} \\ &\Leftrightarrow \quad \beta_0 \leqslant \min\left\{\frac{b}{a}\sqrt{v}, \sqrt{\frac{p}{\|1/t\|_{L^1(\eta_1)} - a^2/b^2}}, \frac{b}{a}\sqrt{q}, \sqrt{\frac{1}{\|1/t\|_{L^1(\eta_1)}}}\right\}. \qquad \square \end{aligned} \tag{3.12}$$

4. Subnormality for powers of hyponormal pairs

In this section we study the connection between the joint subnormality of pairs $(T_1, T_2) \in \mathfrak{H}_1$ and the subnormality of the associated monomials $T_1^m T_2^n$ $(m, n \geqslant 1)$. Our results will further exhibit the large gap between the classes $\mathfrak{H}_\infty$ (subnormal pairs) and $\mathfrak{H}_0$ (commuting pairs of subnormal operators). We begin with the following proposition, which is a direct consequence of a well-known result of J. Stampfli's [22,23]: if T is hyponormal and T^n is normal for some $n \geqslant 1$, then T is necessarily normal.

Proposition 4.1. *Let* $\mathbf{T} \equiv (T_1, T_2)$ *be hyponormal, and assume that* (T_1^m, T_2^n) *is normal for some* $m \geqslant 1$ *and* $n \geqslant 1$. *Then* (T_1, T_2) *is normal.*

In view of Proposition 4.1, one might conjecture that if (T_1, T_2) is hyponormal and $T_1^m T_2^n$ is normal for some $m \geqslant 1$ and $n \geqslant 1$, then (T_1, T_2) is normal (cf. [23]). But this is not true even if we assume that (T_1, T_2) is subnormal and $T_1^m T_2^n$ is normal for all $m \geqslant 1$ and $n \geqslant 1$, as the following example shows.

Example 4.2. Let $T_1 := U_+ \oplus 0_\infty$ and $T_2 := 0_\infty \oplus U_+$, then (T_1, T_2) is subnormal and $T_1^m T_2^n$ is normal for all $m \geqslant 1$ and $n \geqslant 1$. However, (T_1, T_2) is not normal.

Whether Proposition 4.1 holds with “normal” replaced by “subnormal” is not at all obvious. Our main result of this section states that it is indeed possible to have a pair $(T_1, T_2) \in \mathfrak{H}_1$ with $T_1^m T_2^n$ subnormal for all $m \geqslant 1$ and $n \geqslant 1$, but such that $(T_1, T_2) \notin \mathfrak{H}_\infty$. (Observe, however, that the subnormality of the monomials $T_1^m T_2^n$ is a condition weaker than the subnormality of the pairs (T_1^m, T_2^n).) To do so, consider a subnormal weighted shift *shift*$(\beta_1, \beta_2, \ldots)$ with Berger measure η. For $0 < a < x < 1$ and $y > 0$, let

$$\alpha(\mathbf{k}) := \begin{cases} x & \text{if } k_1 = 0 \text{ and } k_2 = 0, \\ a & \text{if } k_1 = 0 \text{ and } k_2 \geqslant 1, \\ 1 & \text{otherwise} \end{cases}$$

and

$$\beta(\mathbf{k}) := \begin{cases} \beta_{k_2} & \text{if } k_2 \geqslant 1, \\ y & \text{if } k_1 = 0 \text{ and } k_2 = 0, \\ ay/x & \text{if } k_1 \geqslant 1 \text{ and } k_2 = 0, \end{cases}$$

($\mathbf{k} = (k_1, k_2) \in \mathbb{Z}_+^2$). We now let $\mathbf{T} := (T_1, T_2)$ denote the pair of 2-variable weighted shift on $\ell^2(\mathbb{Z}_+^2)$ defined by $\alpha(\mathbf{k})$ and $\beta(\mathbf{k})$. We then have:

Lemma 4.3. *Let* $\mathbf{T} \equiv (T_1, T_2)$ *be the* 2*-variable weighted shift associated with* α *and* β *above* (*see Fig.* 5). *Then*

(i) $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_1$ *if and only if*

$$y \leqslant \min\left\{ \frac{\beta_1 x \sqrt{1 - x^2}}{\sqrt{x^2 + a^4 - 2a^2 x^2}}, \sqrt{\|1/t\|_{L^1(\eta)}^{-1}} \right\}.$$

(ii) $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_\infty$ *if and only if*

$$y \leqslant \sqrt{\|1/t\|_{L^1(\eta)}^{-1}} \cdot \sqrt{\frac{1 - x^2}{1 - a^2}}.$$

Proof. First observe that if $shift(y, \beta_1, \beta_2, \ldots)$ is subnormal then T_2 is subnormal. To guarantee this, by Lemma 3.3 we must have $y \leqslant \sqrt{\|1/t\|^{-1}_{L^1(\eta)}}$. For the hyponormality of (T_1, T_2), it suffices to apply the Six-point Test to $\mathbf{k} = (0, 0)$, since

$$\mathcal{R}_{10}(\mathbf{T}) \equiv (T_1, T_2)|_{\mathcal{M}_1} \cong \left(I \otimes U_+, shift\left(\frac{ay}{x}, \beta_1, \beta_2, \ldots\right) \otimes I\right) \in \mathfrak{H}_\infty$$

and

$$\mathcal{R}_{01}(\mathbf{T}) \equiv (T_1, T_2)|_{\mathcal{N}_1} \cong \big(I \otimes S_a, shift(\beta_1, \beta_2, \beta_3, \ldots) \otimes I\big) \in \mathfrak{H}_\infty.$$

Thus,

$$\begin{aligned} &\begin{pmatrix} 1 - x^2 & \frac{a^2 y}{x} - xy \\ \frac{a^2 y}{x} - xy & \beta_1^2 - y^2 \end{pmatrix} \geqslant 0 \quad \text{(by Lemma 2.1)} \\ &\Leftrightarrow \quad y^2\left(1 + \frac{a^4}{x^2} - 2a^2\right) \leqslant \beta_1^2\big(1 - x^2\big) \\ &\Leftrightarrow \quad y \leqslant \frac{\beta_1 x\sqrt{1 - x^2}}{\sqrt{x^2 + a^4 - 2a^2x^2}}. \end{aligned}$$

Therefore, $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_1$ if and only if

$$y \leqslant \min\left\{\frac{\beta_1 x\sqrt{1 - x^2}}{\sqrt{x^2 + a^4 - 2a^2x^2}}, \sqrt{\|1/t\|^{-1}_{L^1(\eta)}}\right\}.$$

We now study the subnormality of $\mathbf{T}$. Since $\mu_{\mathcal{M}}(s, t) = [(1 - a^2)\delta_0(s) + a^2\delta_1(s)] \cdot \eta(t)$ is the Berger measure of $(I \otimes S_a, shift(\beta_1, \beta_2, \beta_3, \ldots) \otimes I)$, Lemma 3.3 implies that

$$\begin{aligned} &\mathbf{T} \text{ is subnormal} \\ &\Leftrightarrow \quad y^2\|1/t\|_{L^1(\mu_{\mathcal{M}})}\mu_{\mathcal{M}}(s, t)^X_{\text{ext}} \leqslant \big(1 - x^2\big)\delta_0(s) + x^2\delta_1(s) \quad \text{and} \quad y \leqslant \sqrt{\|1/t\|^{-1}_{L^1(\eta)}} \\ &\Leftrightarrow \quad y^2\|1/t\|_{L^1(\eta)}\big[\big(1 - a^2\big)\delta_0(s) + a^2\delta_1(s)\big] \leqslant \big(1 - x^2\big)\delta_0(s) + x^2\delta_1(s) \\ &\qquad \text{and} \quad y \leqslant \sqrt{\|1/t\|^{-1}_{L^1(\eta)}} \\ &\Leftrightarrow \quad y \leqslant \min\left\{\sqrt{\|1/t\|^{-1}_{L^1(\eta)}} \cdot \sqrt{\frac{1 - x^2}{1 - a^2}}, \sqrt{\|1/t\|^{-1}_{L^1(\eta)}} \cdot \frac{x}{a}, \sqrt{\|1/t\|^{-1}_{L^1(\eta)}}\right\} \\ &\Leftrightarrow \quad y \leqslant \sqrt{\|1/t\|^{-1}_{L^1(\eta)}} \cdot \sqrt{\frac{1 - x^2}{1 - a^2}} \\ &\qquad \left(\text{because } x > a \text{ implies } \sqrt{\frac{1 - x^2}{1 - a^2}} < \frac{x}{a} \text{ and } \sqrt{\frac{1 - x^2}{1 - a^2}} < 1\right). \qquad \square \end{aligned}$$

We now detect the hyponormality and subnormality of the powers of (T_1, T_2) in Lemma 4.3. Let

$$\mathcal{H}_{(m,i)} := \bigvee_{j=0}^{\infty} \left\{ e_{(mj+i,k)}\colon\ m \geqslant 1,\ 0 \leqslant i \leqslant m-1 \text{ and } k = 0, 1, 2, \ldots \right\}.$$

Then $\ell^2(\mathbb{Z}_+^2) \equiv \bigoplus_{i=0}^{m-1} \mathcal{H}_{(m,i)}$. Under this decomposition, we have

$$T_1^m \cong T_1 \oplus (I \otimes U_+) \oplus \cdots \oplus (I \otimes U_+)$$

and

$$T_2 \cong T_2 \oplus \left(\mathit{shift}\left(\frac{ay}{x}, \beta_1, \beta_2, \ldots \right) \otimes I \right) \oplus \cdots \oplus \left(\mathit{shift}\left(\frac{ay}{x}, \beta_1, \beta_2, \ldots \right) \otimes I \right).$$

Thus, for all $m \geqslant 1$ and $n \geqslant 1$,

$$\left(T_1^m, T_2^n \right) \cong \left(T_1, T_2^n \right) \oplus \bigoplus_{i=1}^{m-1} (C, D),$$

where $C := I \otimes U_+$ and $D := (\mathit{shift}((ay)/x, \beta_1, \beta_2, \ldots))^n \otimes I$. But, since (C, D) is subnormal, the hyponormality (or subnormality) of (T_1^m, T_2^n) is equivalent to the hyponormality (or subnormality) of (T_1, T_2^n). Therefore, (T_1, T_2^n) is hyponormal (or subnormal) if and only if (T_1^m, T_2^n) is hyponormal (or subnormal) for all $m \geqslant 1$.

Theorem 4.4. *For the* 2*-variable weighted shift* $\mathbf{T} \equiv (T_1, T_2)$ *in Lemma* 4.3*, the following are equivalent.*

(i) $T_1^m T_2^n$ *is subnormal for all* $m \geqslant 1$ *and* $n \geqslant 1$;
(ii) $T_1 T_2^n$ *is subnormal for all* $n \geqslant 1$;
(iii) *The shift*$\left(\frac{ay \cdot \prod_{j=1}^{n-1} \beta_j}{x}, \prod_{j=n}^{2n-1} \beta_j, \prod_{j=2n}^{3n-1} \beta_j, \ldots\right)$ *is subnormal for all* $n \geqslant 1$;
(iv) $y \leqslant \frac{x}{a} \cdot \frac{1}{\prod_{j=1}^{n-1} \beta_j} \sqrt{\|1/t\|_{L^1(\eta^{(n)})}^{-1}}$ *for all* $n \geqslant 1$*, where* $d\eta^{(n)}(t) := \frac{t^{1-1/n}}{\beta_1^2 \cdots \beta_{n-1}^2} d\eta(t^{1/n})$.

Proof. (i) $\Leftrightarrow$ (ii). From the above observations, we can see that $T_1^m T_2^n$ is subnormal for all $m \geqslant 1$ and $n \geqslant 1$ if and only if $T_1 T_2^n$ and CD are subnormal for all $n \geqslant 1$. But observe that CD is always subnormal if $\mathit{shift}(ay/x, \beta_1, \beta_2, \ldots)$ is subnormal.

(ii) $\Leftrightarrow$ (iii). Let $\mathcal{M}_{(i,j)} := \bigvee \{e_{i+k,j+k}\colon\ k = 0, 1, 2, \ldots\}$ for $i, j \geqslant 0$ with $ij = 0$. Then $\ell^2(\mathbb{Z}_+^2) \equiv \bigoplus_{i,j=0}^{\infty} \mathcal{M}_{(i,j)}$. Under this decomposition, we have

$$T_1 T_2^n \cong \cdots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \cdots,$$

where

$$W_{-1} := shift\left(a\prod_{j=n}^{2n-1}\beta_j, \prod_{j=2n}^{3n-1}\beta_j, \prod_{j=3n}^{4n-1}\beta_j, \ldots\right) : \mathcal{M}_{(0,1)} \to \mathcal{M}_{(0,1)},$$

$$W_0 := shift\left(ay \cdot \prod_{j=1}^{n-1}\beta_j, \prod_{j=n}^{2n-1}\beta_j, \prod_{j=2n}^{3n-1}\beta_j, \ldots\right) : \mathcal{M}_{(0,0)} \to \mathcal{M}_{(0,0)}, \quad \text{and}$$

$$W_1 := shift\left(\frac{ay}{x} \cdot \prod_{j=1}^{n-1}\beta_j, \prod_{j=n}^{2n-1}\beta_j, \prod_{j=2n}^{3n-1}\beta_j, \ldots\right) : \mathcal{M}_{(1,0)} \to \mathcal{M}_{(1,0)}.$$

Since W_{-1} is subnormal, the result follows from the fact that if W_1 is subnormal then W_0 is also subnormal.

(iii) $\Leftrightarrow$ (iv). Since $shift(\beta_1, \beta_2, \beta_3, \ldots)$ has Berger measure η, we can use mathematical induction to show that $shift(\beta_n, \beta_{n+1}, \beta_{n+2}, \ldots)$ has Berger measure $\frac{t^{n-1}}{\beta_1^2 \cdots \beta_{n-1}^2}\, d\eta(t)$ for each $n \geqslant 1$. Thus by Lemma 2.2, $shift(\prod_{j=n}^{2n-1}\beta_j, \prod_{j=2n}^{3n-1}\beta_j, \prod_{j=3n}^{4n-1}\beta_j, \ldots)$ has Berger measure $d\eta^{(n)}(t) \equiv \frac{t^{1-1/n}}{\beta_1^2 \cdots \beta_{n-1}^2}\, d\eta(t^{1/n})$ for each $n \geqslant 1$. Therefore, by Lemma 3.1 we see that $shift(\frac{ay \cdot \prod_{j=1}^{n-1}\beta_j}{x}, \prod_{j=n}^{2n-1}\beta_j, \prod_{j=2n}^{3n-1}\beta_j, \ldots)$ is subnormal if and only if

$$y \leqslant \frac{x}{a} \cdot \frac{1}{\prod_{j=1}^{n-1}\beta_j}\sqrt{\|1/t\|_{L^1(\eta^{(n)})}^{-1}}. \qquad \square$$

For a concrete example, let

$$d\eta(t) := dt \quad \text{on } [1/2, 3/2],$$

so that

$$\beta_1 = 1 \quad \text{and} \quad \|1/t\|_{L^1(\eta)} = \ln 3.$$

Since

$$\gamma_{n-1} = \beta_1^2\beta_2^2 \cdots \beta_{n-1}^2 = \int_{1/2}^{3/2} t^{n-1}\, d\eta(t) = \frac{1}{n}\left(\frac{3^n - 1}{2^n}\right) \quad \text{and} \quad \gamma_{2n-1} = \frac{1}{2n}\left(\frac{3^{2n} - 1}{2^{2n}}\right),$$

it follows that $shift(\beta_n, \beta_{n+1}, \ldots)$ has Berger measure $\frac{n \cdot 2^n \cdot t^{n-1}}{3^n - 1} dt$ for each $n \geqslant 1$ on $[\frac{1}{2}, \frac{3}{2}]$ and $shift(\prod_{j=n}^{2n-1}\beta_j, \prod_{j=2n}^{3n-1}\beta_j, \ldots)$ has Berger measure

$$d\eta^{(n)}(t) = \frac{2^n}{3^n - 1}\, dt \quad \text{on } \left[(1/2)^n, (3/2)^n\right] \quad (\text{for all } n \geqslant 1).$$

Moreover,

$$\sqrt{\|1/t\|_{L^1(\eta^{(n)})}^{-1}} = \sqrt{\frac{3^n - 1}{n 2^n \ln 3}}.$$

Thus, Lemma 4.3 implies that:

(i) T_1 is subnormal if $0 < a < x < 1$;
(ii) T_2 is subnormal $\Leftrightarrow y \leqslant \sqrt{\frac{1}{\ln 3}}$;
(iii) $(T_1, T_2) \in \mathfrak{H}_1 \Leftrightarrow y \leqslant m := \min\{\frac{x\sqrt{1-x^2}}{\sqrt{x^2+a^4-2a^2x^2}}, \sqrt{\frac{1}{\ln 3}}\}$;
(iv) $(T_1, T_2) \in \mathfrak{H}_\infty \Leftrightarrow y \leqslant s := \sqrt{\frac{1}{\ln 3}\frac{1-x^2}{1-a^2}}$.

Therefore, we have the following result.

Example 4.5. For $s < y \leqslant m$ and $0 < a < x < 1$, we have

(i) $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_1$;
(ii) $\mathbf{T} \equiv (T_1, T_2) \notin \mathfrak{H}_\infty$;
(iii) $T_1^m T_2^n$ is subnormal for all $m \geqslant 1, n \geqslant 1$.

For, observe that if $0 < a < x < 1$, then

$$s \equiv \sqrt{\frac{1}{\ln 3}\frac{1-x^2}{1-a^2}} < \frac{x\sqrt{1-x^2}}{\sqrt{x^2+a^4-2a^2x^2}} \quad \text{and} \quad s < \sqrt{\frac{1}{\ln 3}};$$

thus, $s < m$, and it is then possible to choose values of y between these two quantities. From Theorem 4.4, we can see that $T_1^m T_2^n$ is subnormal for all $m \geqslant 1, n \geqslant 1$ if and only if

$$y \leqslant \frac{x}{a} \cdot \frac{1}{\prod_{j=1}^{n-1}\beta_j}\sqrt{\|1/t\|^{-1}_{L^1(\mu_\eta)}} = \frac{x}{a}\sqrt{\frac{1}{\ln 3}}.$$

But since

$$y \leqslant \sqrt{\frac{1}{\ln 3}} < \frac{x}{a}\sqrt{\frac{1}{\ln 3}},$$

it follows that $T_1^m T_2^n$ is subnormal for all $m \geqslant 1, n \geqslant 1$.

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