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Unified BRST description of AdS gauge fields

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Abstract

A concise formulation for mixed-symmetry gauge fields on AdS space is proposed. It is explicitly local, gauge invariant, and has manifest AdS symmetry. Various other known formulations (including the original formulation of Metsaev and the unfolded formulation) can be derived through the appropriate reductions and gauge fixing. As a byproduct, we also identify some new useful formulations of the theory that can be interesting for further developments. The formulation is presented in the BRST terms and extensively uses Howe duality. In particular, the BRST operator is a sum of the term associated to the spacetime isometry algebra and the term associated to the Howe dual symplectic algebra.

1. Introduction

There have been numerous approaches to mixed-symmetry higher spin gauge fields on the AdS space. In contrast to the totally symmetric case where a simple Lagrangian formulation is available [1], describing mixed-symmetry AdS fields is not so straightforward. In particular, general AdS gauge fields have been described much later [2] and only at the level of equations of motion. Moreover, these equations are not truly gauge-invariant as the gauge parameters satisfy differential constraints. The true gauge fields were then identified in [3,4] within the unfolded approach [5,6]. The unfolded formulation of AdS gauge fields was recently proposed in [7,8]. However, beyond the totally symmetric field case [6] this formulation happens to be rather involved technically because the constraints imposed on the fields bring the respective projectors to the equations of motion. As far as particular cases of mixed-symmetry AdS fields are concerned

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there are other successful approaches available in the literature [9–21]. Light-cone formulation for mixed-symmetry fields of any spins was elaborated in [22–24].

Although all these formulations are believed (and partially proved) to describe the same physical degrees of freedom their explicit interrelations remain unclear. Moreover, further developments and especially a search for mixed-symmetry fields consistent interactions call for a simple and algebraically transparent formulation that is free of the above difficulties. This paper is devoted to constructing a candidate formulation that meets these criteria. This turns out to be a natural generalization of the recent [25] (see also [26] for the case of Fronsdal fields) formulation for Minkowski space mixed-symmetry fields. At the same time, it naturally generalizes the formulation [27] of totally symmetric AdS fields to the mixed-symmetry case. In particular, the equations of motion and gauge symmetries has one and the same structure for massless fields of arbitrary symmetry type in both Minkowski and AdS spaces.

An important technical ingredient used throughout the paper is the twisted version of the Howe dual [28] realisation of symplectic and orthogonal algebras (in the case of Fronsdal fields, *i.e.* for $sp(4)$ algebra, this realization was first used in [27]). Though equivalent to the usual one in the space of polynomials it turns out inequivalent in the space of formal power series because the equivalence transformation is not well-defined in this space. In the same way as usual Howe duality is useful in describing finite-dimensional irreducible modules (*e.g.* irreducibility conditions for one algebra are highest weight conditions for its Howe dual) the twisted realization also describes infinite-dimensional indecomposable representations. This is crucial because both type of modules are necessary to describe gauge fields. Namely, the generalized curvatures take values in the indecomposable module (known as Weyl module) while the generalized gauge potentials in the irreducible modules (known as gauge modules) of the AdS isometry algebra [29]. The twisted Howe duality allows to embed both type of modules in one and the same $o(d-1,2) - sp(2n)$ bimodule.

An attractive feature of the proposed construction is that the irreducibility constraints commute with the equations of motion. Strictly speaking, they are BRST invariant with respect to the BRST operator defining the equations of motion and gauge symmetries. This allows to simultaneously describe a collection of irreducible fields such that an individual field can be then singled out by the appropriate constraints. This feature is important from the string theory perspective, where the string spectrum contains a huge collection of mixed-symmetry fields. Although string theory leads to massive mixed-symmetry fields and is not well-defined on AdS space, in the appropriate limit it is expected to incorporate massless fields and to admit AdS background (see *e.g.* [30,31]). Motivated by this relationship we also propose other equivalent reformulations of the AdS mixed-symmetry fields including that defined in terms of the ambient space and based on the BRST operator,[1] analogous to the standard one associated to the bosonic string.

2. Preliminaries

2.1. Howe dual realizations

In this section we introduce main technical tools of our construction that make the whole consideration manifestly $o(d-1,2)$ covariant.

[1] It is also similar to the formulation of [15] for totally symmetric fields.

The anti-de Sitter spacetime AdS can be described as a hyperboloid $\mathcal{X}$ embedded in the ambient flat pseudo-Euclidean space $\mathbb{R}^{d+1}$. Labelling the coordinates in $\mathbb{R}^{d+1}$ as X^A, $A=0,\dots,d$, the embedding equation is

$$\eta_{AB}X^AX^B+1=0, \qquad \eta_{AB}=(-+\cdots+-). \tag{2.1}$$

Infinitesimal isometries of the hyperboloid form a pseudo-orthogonal algebra $o(d-1,2)$.

Let A^A_I, where $A=0,\dots,d$ and $I=0,\dots,n-1$ be commuting variables transforming as vectors of $o(d-1,2)$. The realization of $o(d-1,2)$ on the space of functions in A^A_I reads

$$J^{AB}=A^A_I\frac{\partial}{\partial A_{BI}}-A^B_I\frac{\partial}{\partial A_{AI}}. \tag{2.2}$$

The realization of $sp(2n)$ reads

$$T_{IJ}=A^A_IA_{JA}, \qquad T_I{}^J=\frac{1}{2}\left\{A^A_I,\frac{\partial}{\partial A^A_J}\right\}, \qquad T^{IJ}=\frac{\partial}{\partial A^A_I}\frac{\partial}{\partial A_{JA}}. \tag{2.3}$$

These two algebras form a Howe dual pair $o(d-1,2)-sp(2n)$ [28]. The diagonal elements $T_I{}^I$ form a basis in the Cartan subalgebra while T^{IJ} and $T_I{}^J$, $I<J$ are the basis elements of the appropriately chosen upper-triangular subalgebra. Let us note that $gl(n)$ algebra is realized by the generators $T_I{}^J$ as a subalgebra of $sp(2n)$ while its $sl(n)$ subalgebra is generated by $T_I{}^J$ with $I\neq J$.

In what follows we also need to pick up a distinguished direction in the space of oscillators A^A_I. Without loss of generality we take it along A^A_0 so that from now on we consider variables A^A_0 and A^A_i, $i=1,\dots,n-1$ separately. In particular, we identify $sp(2n-2)\subset sp(2n)$ subalgebra preserving the direction. We use the following notation for some of $sp(2n-2)$ generators

$$N_i{}^j\equiv T_i{}^j=A^A_i\frac{\partial}{\partial A^A_j}, \quad i\neq j, \qquad N_i=N_i{}^i\equiv T_i{}^i-\frac{d+1}{2}=A^A_i\frac{\partial}{\partial A^A_i}, \tag{2.4}$$

which form $gl(n-1)$ subalgebra, and

$$T_{ij}=A^A_iA_{jA}, \qquad T^{ij}=\frac{\partial}{\partial A^A_i}\frac{\partial}{\partial A_{jA}}, \tag{2.5}$$

that complete above set of elements to $sp(2n-2)$ algebra.

In what follows we use two different realizations of $sp(2n)$ generators involving A^A_0 and/or $\partial/\partial A^A_0$:

- realization on the space of polynomials in A^A_i with coefficients in functions on $\mathbb{R}^{d+1}$ with the origin excluded. In this case

$$A^A_0=X^A, \qquad \frac{\partial}{\partial A^A_0}=\frac{\partial}{\partial X^A}, \tag{2.6}$$

  where X^A are Cartesian coordinates in $\mathbb{R}^{d+1}$. We keep the previous notation (2.4), (2.5) for generators that do not involve X^A and/or $\partial/\partial X^A$ while those that do are denoted by

$$\mathcal{S}^\dagger_i=A^A_i\frac{\partial}{\partial X^A}, \qquad \bar{\mathcal{S}}^i=X^A\frac{\partial}{\partial A^A_i},$$
$$\mathcal{S}^i=\frac{\partial}{\partial A^A_i}\frac{\partial}{\partial X_A}, \qquad \Box_X=\frac{\partial}{\partial X^A}\frac{\partial}{\partial X_A}. \tag{2.7}$$

It is convenient to split the $o(d-1,2)$ generators J^{AB} in two pieces as $J^{AB} = L^{AB} + M^{AB}$, where an orbital part L^{AB} is given by

$$L^{AB} = X^A \frac{\partial}{\partial X_B} - X^B \frac{\partial}{\partial X_A}. \tag{2.8}$$

- realization on the space of polynomials in A_i^A with coefficients in formal power series in variables Y^A such that

$$A_0^A = Y'^A = Y^A + V^A, \qquad \frac{\partial}{\partial A_0^A} = \frac{\partial}{\partial Y^A}, \tag{2.9}$$

where V^A is some $o(d-1,2)$ vector normalized as $V^A V_A = -1$. Respective $sp(2n)$ generators are realized by inhomogeneous differential operators on the space of functions in A_i^A and Y^A. We use for them the following notation

$$\begin{aligned} S_i^\dagger &= A_i^A \frac{\partial}{\partial Y^A}, \qquad & \bar{S}^i &= \left(Y^A + V^A\right)\frac{\partial}{\partial A_i^A}, \\ S^i &= \frac{\partial}{\partial A_i^A}\frac{\partial}{\partial Y_A}, \qquad & \Box_Y &= \frac{\partial}{\partial Y^A}\frac{\partial}{\partial Y_A}. \end{aligned} \tag{2.10}$$

Note that this realization is the same as in [25] but with Y^A replaced by $Y^A + V^A$. Shifting by V^A is crucial because this realization is inequivalent with the usual one (*i.e.*, the one with $V^A = 0$). This happens because the change of variables $Y^A \to Y^A + V^A$ is ill-defined in the space of formal power series. In contrast to the usual realization where highest (lowest) weight conditions of $sp(2n-2)$ determine finite-dimensional irreducible $o(d-1,2)$-modules, the inhomogeneous counterpart of these conditions can determine both finite-dimensional irreducible or infinite-dimensional indecomposable $o(d-1,2)$-modules. In particular, it allows one to describe finite-dimensional gauge modules and infinite-dimensional Weyl modules associated with gauge fields in AdS at the equal footing. Note that the case $n = 1, 2$ has been originally described in [27]. Analogous representation has been also used in [32] to describe conformal fields.

The orbital part L^{AB} of the generators J^{AB} takes the form

$$L^{AB} = \left(Y^A + V^A\right)\frac{\partial}{\partial Y_B} - \left(Y^B + V^B\right)\frac{\partial}{\partial Y_A}. \tag{2.11}$$

This realization of the dual orthogonal and symplectic algebras will be refereed to as twisted Howe dual realization.

2.2. *Fields on the hyperboloid in terms of the ambient space*

We start with the description of the unitary irreducible $o(d-1,2)$-modules originally developed by Fronsdal [1] for totally symmetric fields and then extended to mixed-symmetry fields by Metsaev [2]. However, we need the description in terms of a slightly different basis for the irreducibility conditions and in terms of fields defined on $\mathbb{R}^{d+1}$ rather than on the hyperboloid. We show that irreducibility conditions imposed on fields on $\mathbb{R}^{d+1}$ within the Metsaev formulation can be seen as the highest weight conditions for an upper-triangular subalgebra of $sp(2n)$. This is natural as $o(d-1,2)$ and $sp(2n)$ are dual in this representation in the sense of Howe duality. To make this algebraic interpretation manifest we reformulate the Metsaev description using the basis elements (2.4), (2.7) and restoring the radial dependence of the fields on the hyperboloid.

For the moment, we restrict our consideration to unitary massless fields[2] in AdS space in the explicitly $o(d-1,2)$-invariant way. It is useful to define them as tensor fields on the ambient space $\mathbb{R}^{d+1}$ with the origin excluded and the radial dependence eliminated through the appropriate $o(d-1,2)$-invariant constraint. More technically, tensor fields are represented by functions on $\mathbb{R}^{d+1}/\{0\}$ taking values in the space of polynomials in variables A^A_i, $i=1,\dots,n-1$, $A=0,\dots,d$ introduced in Section 2. Such a filed can be viewed as a function $\phi=\phi(X,A)$. The radial coordinate dependence is effectively eliminated through the homogeneity condition

$$(N_X-k)\phi=0, \qquad N_X=X^A\frac{\partial}{\partial X^A}, \tag{2.12}$$

where k is a number whose explicit value will be fixed later. This allows to uniquely represent any field defined on hyperboloid in terms of the ambient space field satisfying (2.12). More explicitly, taking a new coordinate system (r,x^m) in $\mathbb{R}^{d+1}$, such that $r=\sqrt{-X^2}$ is a radius and x^m are dilation-invariant coordinates $N_X x^m=0$, one finds $\phi=\phi_0(x,A)r^k$.

In order to describe irreducible representation let us impose the following irreducibility conditions (in the sector of A^A_i-variables):

$$T^{ij}\phi=0, \qquad N_i{}^j\phi=0 \quad i<j, \qquad (N_i-s_i)\phi=0. \tag{2.13}$$

In addition, we also impose the transversality, divergencelessness, and the “mass-shell” conditions (in the sector of X^A-variables):

$$\bar{\mathcal{S}}^i\phi=0, \qquad \mathcal{S}^i\phi=0, \qquad \Box_X\phi=0. \tag{2.14}$$

In contrast to purely algebraic conditions (2.13) the latter ones explicitly involve space–time coordinates.

All together, constraints (2.12), (2.13), and (2.14) form the upper-triangular subalgebra of $sp(2n)$ algebra supplemented with Cartan elements. Because $sp(2n)$ and $o(d-1,2)$ commute these constraints single out an $o(d-1,2)$-module.[3] By solving the homogeneity condition (2.12) and transversality constraints $\bar{\mathcal{S}}^i\phi=0$ one finds the description in terms of $o(d-1,1)$-tensor fields defined on the hyperboloid as was originally observed in the case of totally symmetric fields [1].

For fields subjected to the irreducibility conditions (2.12) and (2.14), one derives the following wave equation [2]

$$\left(\Box_{AdS}+m^2\right)\phi=0, \qquad \Box_{AdS}\equiv\frac{1}{2}L_{AB}L^{AB}, \tag{2.15}$$

where L_{AB} is an orbital part of $o(d-1,2)$ generators (2.8). Evaluating $\frac{1}{2}L_{AB}L^{AB}=\Box_X-N_X(d-1+N_X)$ on the hyperboloid (2.1) and substituting the mass-shell condition (2.14) one finds an explicit value of the mass-like term

$$m^2=N_X(d-1+N_X). \tag{2.16}$$

Comparing with the original formula $m^2=E_0(d-1+E_0)$ derived in [2] we see that an eigenvalue of the $o(d-1,2)$ energy operator E_0 and an eigenvalue of $sp(2n)$ Cartan element N_X defined by (2.12) are linearly dependent.

[2] The case of non-unitary massless fields as well as partially-massless fields in AdS space is discussed in Section 3.4.

[3] Note that this module is not necessarily irreducible. In the space of polynomials in X^A these conditions are known to determine a finite-dimensional irreducible $o(d-1,2)$-module. This is not the case for functions on $\mathbb{R}^{d+1}/\{0\}$ though.

2.3. Gauge invariance

The theory determined by conditions (2.13) and (2.14) does not in general describe irreducible fields. More precisely, depending on the value of N_X and N_i the space of solutions may contain singular vectors. In this case we obtain a gauge theory.

It is useful to describe the gauge symmetry using the BRST formalism. To this end, we introduce Grassmann odd ghost variables b_α, $\alpha = 1, \dots, p \leqslant n-1$. The BRST description comes together with the ghost number grading $\mathrm{gh}(b_\alpha) = -1$ and $\mathrm{gh}(X^A) = \mathrm{gh}(A_i^A) = 0$. The gauge invariance is encoded in the BRST operator

$$\Omega_p = \mathcal{S}_\alpha^\dagger \frac{\partial}{\partial b_\alpha}, \tag{2.17}$$

acting in the space of functions $\Psi = \Psi(X, A|b)$ in X^A taking values in polynomials in A_i^A and ghost variables b_α.

Field $\phi = \phi(X, A)$ considered in the previous section should be identified as the physical field which is the ghost-number-zero component of $\Psi(X, A|b)$ while the ghost number -1 component is identified with gauge parameters. The gauge transformation is defined as

$$\delta\phi = \Omega_p \chi, \qquad \mathrm{gh}(\chi) = -1, \qquad \mathrm{gh}(\phi) = 0, \tag{2.18}$$

where $\chi = \chi^\alpha(X, A) b_\alpha$ is a gauge parameter.

In order to consistently impose conditions (2.12), (2.13), and (2.14) some of them are to be extended by ghost contributions to make Ω_p act in the subspace. More precisely, in the ghost extended space one imposes unchanged constraints

$$\bar{\mathcal{S}}^i \Psi = 0, \qquad \mathcal{S}^i \Psi = 0, \qquad \Box_X \Psi = 0, \qquad T^{ij} \Psi = 0, \tag{2.19}$$

and modified constraints

$$\widehat{\mathcal{N}}_i{}^j \Psi = 0 \quad i < j, \qquad (\widehat{\mathcal{N}}_i - s_i)\Psi = 0, \qquad \widehat{\mathcal{J}}_\alpha{}^\beta \Psi = 0, \tag{2.20}$$

where

$$\widehat{\mathcal{N}}_i{}^j = N_i{}^j + B_i{}^j, \qquad \widehat{\mathcal{N}}_i = N_i + B_i{}^i, \qquad \widehat{\mathcal{J}}_\alpha{}^\beta = \widehat{\mathcal{N}}_\alpha{}^\beta - \delta_\alpha^\beta (N_X - B + p + 1). \tag{2.21}$$

Here and in what follows we use the following useful notations:

$$B_i{}^j = \delta_i^\alpha \delta_\beta^j b_\alpha \frac{\partial}{\partial b_\beta}, \qquad B_\alpha = b_\alpha \frac{\partial}{\partial b_\alpha}, \qquad B = \sum_\alpha B_\alpha. \tag{2.22}$$

Note that for ghost independent elements constraints (2.20) impose additional restrictions compared to their counterparts in (2.12) and (2.13). Additional constraints $\widehat{\mathcal{J}}_\alpha{}^\beta$ appear as the consistency condition following from the commutators of the BRST operator Ω_p with constraints $\bar{\mathcal{S}}^\alpha$.

Requiring gauge invariance restricts possible values of weights s_i. Their admissible values are specified by consistency of the second and the third conditions in (2.20) which, in turn, originate from the consistency with the gauge transformation. Indeed, from the third condition it follows that $\mathcal{N}_\alpha{}^\beta \Psi = 0$ for $\alpha \neq \beta$. This implies $(\mathcal{N}_\alpha - \mathcal{N}_\beta)\Psi = 0$ and hence $s_\alpha = s_\beta$ for all α, β. In other words, Ψ has vanishing $sl(p)$ weights so that fields are $sl(p)$ singlets. At the same time, they cannot be $sl(p+k)$ singlets for $k > 0$, therefore $s_p > s_{p+k}$. For the later convenience, we introduce a notation $s_1 \equiv s$ and order the weights as follows

$$s \equiv s_1 = s_2 = \dots = s_p > s_{p+1} \geqslant s_{p+2} \geqslant \dots \geqslant s_{n-1}. \tag{2.23}$$

Consistency with the gauge transformation also fixes the value of constant term in the constraint $(N_X - k)\phi = 0$ (2.12) because it is now encoded in the constraint $\mathcal{J}_\alpha{}^\alpha$. More precisely, for a physical field ϕ one gets

$$N_X\phi = (s - p - 1)\phi. \tag{2.24}$$

By virtue of formula (2.16) one explicitly calculates a value of the mass-like term

$$m^2 = (s - p - 1)(s - p + d - 2), \tag{2.25}$$

thereby recovering the result of [2] for unitary massless AdS fields having the uppermost block of length s and height p.

From a more algebraic point of view, consistency with the gauge transformation (2.18) extends conditions (2.12), (2.13), and (2.14) that form upper-triangular subalgebra of $sp(2n)$ (including Cartan elements) to an extended set of conditions (2.19) and (2.20) whose ghost independent parts form the parabolic subalgebra of $sp(2n)$. The consistency can be immediately seen from the fact that together with $\mathcal{S}^\dagger_\alpha$ these conditions also form a parabolic subalgebra.

To complete the description of unitary gauge fields in terms of the ambient space let us spell out in components the gauge transformation of the physical fields (2.18). It takes the following form

$$\delta\phi = \mathcal{S}^\dagger_1\chi^1 + \cdots + \mathcal{S}^\dagger_p\chi^p, \tag{2.26}$$

where gauge parameters $\chi^\alpha = \chi^\alpha(X, A)$ are components of ghost-number -1 element $\chi = \chi^\alpha b_\alpha$ satisfying constraints (2.19) and (2.20). The peculiar feature of the gauge transformation is that gauge parameters χ^α do not satisfy Young symmetry conditions and are linearly dependent. By virtue of constraints (2.20) one can show that gauge parameters χ^α at $\alpha < p$ are expressed through parameter χ^p satisfying Young symmetry conditions $N_i{}^j\chi^p = 0$, $i < j$ and weight conditions $N_\alpha\chi^p = (s - \delta_{\alpha p})\chi^p$. With the help of gauge parameter χ^p gauge variation (2.26) can be equivalently rewritten in the form

$$\delta\phi = \Pi\mathcal{S}^\dagger_p\chi^p \equiv \left(\mathcal{S}^\dagger_p - \mathcal{S}^\dagger_{p-1}N_p{}^{p-1} - \mathcal{S}^\dagger_{p-2}N_p{}^{p-2} - \cdots - \mathcal{S}^\dagger_1 N_p{}^1\right)\chi^p, \tag{2.27}$$

where Π involves appropriate Young symmetrizations needed to adjust symmetry properties of both sides [2].

3. Generating BRST formulation

The formulation of the unitary gauge fields developed in the previous section is not completely satisfactory. First of all, it is not a genuine local gauge field theory because gauge parameters are subjected to the differential constraints (*i.e.*, constraints involving derivatives with respect to X^A-coordinates). Furthermore, the way it is formulated is not explicitly local because fields are defined in terms of the ambient space. A natural question is to find a realization of the theory in terms of internal coordinates on the hyperboloid and gauge parameters not subjected to differential constraints. This can be done following the procedure used in [27] in the case of totally symmetric fields.

The idea suggested from [27][4] is to put the ambient space to the fiber of the vector bundle over AdS space and then eliminate additional degrees of freedom through auxiliary constraints. More

[4] In its turn it originates in (the generalization [33,34,26] to constrained systems of) the Fedosov quantization procedure [35] and Vasiliev unfolded formalism [36,37,39]. In the related context it was also used in [40,32].

technically, one replaces coordinates X^A with formal variables Y^A and then consider fields on the AdS space with values in the fiber that is in the space of "functions" in Y^A and A^A_i variables. In this procedure all the algebraic constraints stay the same while those involving X^A and $\frac{\partial}{\partial X^A}$ (in particular, those entering the BRST operator) are replaced with the respective constraints for Y^A variables and hence also become algebraic. The extra degrees of freedom are then eliminated by introducing additional constraints.

3.1. BRST operator and field equations

The well-known approach to describe AdS geometry structure on manifold $\mathcal{X}$ is to consider vector bundle $\mathcal{V}_0$ over $\mathcal{X}$ with the fiber being $(d-1,2)$-dimensional pseudo-Euclidean space. The AdS geometry structure is then encoded in the compatible flat $o(d-1,2)$-connection $\omega^{AB}(x)$ and a given section $V^A(x)$ of $\mathcal{V}_0$ satisfying $\eta_{AB}V^AV^B=-1$, where η_{AB} are coefficients of the fiber-wise pseudo-Euclidean bilinear form. If in addition ∇V^A seen as a map from the tangent bundle to $\mathcal{V}_0$ is of maximal rank then indeed $e^A=\nabla V^A$ can be identified with the vielbein and $\eta(e,e)$ with the AdS metric. Here ∇ denotes the covariant derivative determined by connection ω.

The space of polynomials in A^A_i with coefficients in formal power series in Y^A is equipped with the action of $sp(2n)$ and $o(d-1,2)$ defined by (2.4), (2.5), (2.10) and (2.11), respectively. Taking this space as a fiber gives a vector bundle $\mathcal{V}$ associated to $\mathcal{V}_0$. In what follows, the fibre is also assumed to contain ghost variables b_α on which $o(d-1,2)$ and $sp(2n)$ act trivially. The $o(d-1,2)$-connection ω^{AB} determines the following covariant derivative (also denoted by ∇) in the associated bundle $\mathcal{V}$

$$\nabla=\boldsymbol{d}+\frac{1}{2}\theta^m\omega_m^{AB}J_{AB}\equiv\theta^m\frac{\partial}{\partial x^m}-\theta^m\omega^A_{mB}\left(\left(Y^B+V^B\right)\frac{\partial}{\partial Y^A}+A^B_i\frac{\partial}{\partial A^A_i}\right),\tag{3.1}$$

where ω_m^{AB} and V^A are components of $\omega^{AB}(x)$ and $V^A(x)$ introduced using a suitable local frame and x^m are local coordinates on $\mathcal{X}$. Here the frame is chosen such that $V^A=const$; the expression for ∇ gets additional terms if a local frame where $V^A\neq const$ is used. We have replaced basis differential forms dx^m with extra Grassmann odd ghost variables θ^m, $m=0,\dots,d-1$ because ∇ will be interpreted later as a part of BRST operator.

Let us consider the following BRST operator

$$\widehat{\Omega}=\nabla+Q_p,\quad Q_p=S^\dagger_\alpha\frac{\partial}{\partial b_\alpha},\tag{3.2}$$

defined on the space of sections of the bundle above. We assign the following gradings to the ghost variables $\mathrm{gh}(\theta^m)=-\mathrm{gh}(b_\alpha)=1$ so that BRST operator $\widehat{\Omega}$ has a standard ghost-number $\mathrm{gh}(\widehat{\Omega})=1$. The BRST operator is nilpotent because of the following obvious relations[5]

$$\nabla^2=Q_p^2=0,\quad [Q_p,\nabla]=0.\tag{3.3}$$

The former relation holds in virtue of the zero-curvature condition for connection ω^{AB}. The latter one is true because ∇ and Q_p are build of generators of two commuting (Howe dual) algebras $o(d-1,2)$ and $sp(2n)$.

[5] Here and in what follows the commutator denotes the graded commutator, $[f,g]=fg-(-)^{|f||g|}gf$, where $|f|$ is the Grassmann parity of f.

That BRST operator is build out of the flat $o(d-1,2)$ connection and $sp(2n)$ generators implies the explicit $o(d-1,2)$-invariance of the theory described by $\widehat{\Omega}$. To see how $o(d-1,2)$-algebra acts on fields let us note that $o(d-1,2)$ naturally acts on the fibre at any point $x_0 \in \mathcal{X}$. This determines an action on fields by taking as parameter a covariantly constant section of the associated bundle with the fibre being $o(d-1,2)$ considered as the adjoint module.[6] In terms of components, let $\xi^0_{AB} = -\xi^0_{BA}$ represent an $o(d-1,2)$-element. It can be extended to a covariantly constant $\xi_{AB}(x)$ satisfying $\nabla \xi_{AB}(x) = 0$ and $\xi_{AB}(x_0) = \xi^0_{AB}$, where $x_0 \in \mathcal{X}$ is a given point of $\mathcal{X}$. If $\phi = \phi(x, Y, A, \text{ghosts})$ represents a field then the $o(d-1,2)$-action can be defined as

$$R\big(\xi^0\big)\phi = \frac{1}{2}\xi_{AB} J^{AB}\phi, \qquad \nabla \xi_{AB}(x) = 0, \qquad \xi_{AB}(x_0) = \xi^0_{AB}. \tag{3.4}$$

Note that the above expression is not unique since it is defined modulo a gauge transformation and terms proportional to the equations of motion. For instance, one can also represent the action such that coordinates x^m are affected (see [25] for a more extensive discussion).

Let us recall how the BRST operator and its representation space encode a gauge field theory. Physical fields are identified as elements $\Psi^{(0)}$ at ghost number 0, gauge parameters as elements $\chi^{(-1)}$ at ghost number -1. The equations of motion and the gauge transformations read as

$$\widehat{\Omega}\Psi^{(0)} = 0, \qquad \delta\Psi^{(0)} = \widehat{\Omega}\chi^{(-1)}, \tag{3.5}$$

where the gauge parameters have ghost number $\mathrm{gh}(\chi^{(-1)}) = -1$. Elements at other ghost numbers correspond to higher structures of the gauge algebra. For instance, order k, $k = 1, \dots, p-1$ reducibility parameters are described by ghost-number $-k$ elements. The respective reducibility identities read as $\delta\chi^{(-k)} = \widehat{\Omega}\chi^{(-k-1)}$.

Specializing to the case at hand: an element of vanishing ghost degree reads as

$$\Psi^{(0)} = \psi_0 + \psi_1 + \cdots + \psi_p, \quad \psi_k = \psi^{\alpha_1\dots\alpha_k}_{m_1\dots m_k}(x, Y, A) b_{\alpha_1} \cdots b_{\alpha_k} \theta^{m_1} \cdots \theta^{m_k}. \tag{3.6}$$

The expansion coefficients $\psi^{\alpha_1\dots\alpha_k}_{m_1\dots m_k}$ are identified as differential k-forms ($k \leqslant p$) on $\mathcal{X}$. The equations of motion take the form

$$\begin{aligned}
&\nabla\psi_0 + S^\dagger_\alpha \frac{\partial}{\partial b_\alpha}\psi_1 = 0,\\
&\nabla\psi_1 + S^\dagger_\alpha \frac{\partial}{\partial b_\alpha}\psi_2 = 0,\\
&\vdots\\
&\nabla\psi_p = 0.
\end{aligned} \tag{3.7}$$

First order gauge parameters can be represented as

$$\xi^{(-1)} = \xi_1 + \xi_2 + \cdots + \xi_p, \quad \xi_k = \xi^{\alpha_1\dots\alpha_k}_{i_1\dots i_{k-1}}(x, A, Y) b_{\alpha_1} \dots b_{\alpha_k} \theta^{i_1} \dots \theta^{i_{k-1}}. \tag{3.8}$$

For instance, gauge parameter $\xi_1 = \xi^\alpha b_\alpha$ is a 0-form. The gauge transformations have the form

$$\delta_\xi \psi_0 = S^\dagger_\alpha \frac{\partial}{\partial b_\alpha}\xi_1,$$

[6] See [25] for a discussion of a general symmetry algebra and the example of Poincaré algebra.

$$
\begin{aligned}
&\delta_\xi \psi_1 = \nabla \xi_1 + S^\dagger_\alpha \frac{\partial}{\partial b_\alpha} \xi_2,\\
&\vdots\\
&\delta_\xi \psi_p = \nabla \xi_p.
\end{aligned}
\tag{3.9}
$$

In the same way one can spell out the reducibility relations.

Let us stress that in this formulation the structure of the equations of motion and gauge symmetries is exactly the same as of the formulation [25] for the Minkowski space fields. The difference is in $o(d-1,2)$-module structure of the fiber replaced with the $iso(d-1,1)$ (*i.e.* Poincaré) one. Respectively, $o(d-1,2)$ covariant derivative of the present formulation is replaced with Poincaré one. However, the explicit structure of the fibre is quite different for AdS and Poincaré gauge fields. In particular, the algebraic constraints imposed to describe irreducible fields belong to different algebras.

3.2. *Algebraic constraints*

The system just constructed does not describe an irreducible representation. Moreover, it is an off-shell system in a sense that it does not impose true differential equations on fields. All equations are equivalent to constraints and can be solved in terms of some unconstrained fields (see [41,42,40] for more details on the off-shell form of HS dynamics). To make it dynamical one should impose the fiber version of the constraints (2.19) and (2.20). These read as

$$
\begin{aligned}
&T^{IJ}\Psi = 0, \qquad \bar{S}^i \Psi = 0, \qquad \widehat{\mathcal{N}}_i{}^j \Psi \equiv \big(N_i{}^j + B_i{}^j\big)\Psi = 0 \quad i<j,\\
&\widehat{\mathcal{J}}_\alpha{}^\beta \Psi \equiv \big(\widehat{\mathcal{N}}_\alpha{}^\beta - \delta^\beta_\alpha (N_{Y'} - B + p + 1)\big)\Psi = 0,
\end{aligned}
\tag{3.10}
$$

where we introduced Euler operator $N_{Y'} = (Y^A + V^A)\frac{\partial}{\partial Y^A}$ (cf. (4.5)). In addition, conditions

$$
\widehat{\mathcal{N}}_i \Psi \equiv (N_i + B_i)\Psi = s_i \Psi \tag{3.11}
$$

single out a particular spin field. As before, all the constraints together with $S^\dagger_\alpha$ imposed through the BRST operator form a parabolic subalgebra of $sp(2n)$ represented on the fiber. This ensures the consistency of the system. Note that among the constraints (3.10), (3.11) those involving $\frac{\partial}{\partial Y^A}$ (except for $\widehat{\mathcal{J}}_\alpha{}^\beta$) lead to differential equations of motion while the remaining ones give rise to algebraic constraints.

Applying the same reasoning as in Section 2.3 one concludes that spins are arranged according to (2.23). In particular, it follows that constraints $\widehat{\mathcal{J}}_\alpha{}^\beta$ split in two parts

$$
\widehat{\mathcal{N}}_\alpha{}^\beta \Psi = 0, \qquad h\Psi \equiv (N_{Y'} - B + p + 1)\Psi = s\Psi, \tag{3.12}
$$

for $\alpha \neq \beta$ and $\alpha = \beta$, respectively.

3.3. *Equivalence to Metsaev formulation*

Our next aim is to show that the theory determined by BRST operator (3.2) and the constraints (3.10) and (3.11) indeed describes unitary gauge fields. To this end let us note that by eliminating auxiliary fields and fixing the gauge, equations of motion (3.7) can be written as

$$
\nabla \psi_0 = 0, \qquad \nabla \psi_1 + \cdots = 0, \qquad \nabla \psi_2 + \cdots = 0, \qquad \ldots, \tag{3.13}
$$

where by slight abuse of notation we denote by ψ_k the field of the reduced theory. More precisely, ψ_k is a k-form obtained by eliminating auxiliary components and fixing algebraic gauge symmetries from the respective fields in (3.6) while dots in the equations for ψ_k denote the extra terms depending on ψ_l with $l < k$. This form of the equations of motion is known as unfolded form [5,6]. It can be obtained [26,27] from BRST formulation (3.7) by reducing to Q_p-cohomology. Here we do need an explicit form of the unfolded equations. We only note that analysing the gauge invariance of (3.13) one concludes that ψ_0 is invariant while higher components are determined in terms of ψ_0 modulo gauge transformations. It follows that physical degrees of freedom are carried by ψ_0 only.[7]

It is then enough to concentrate on equations for ψ_0 that decouple from others. Field ψ_0 can be shown to take values in Q_p-cohomology at vanishing ghost degree. Because the cocycle condition is trivial for ghost-number-zero elements the cohomology class can be identified with the equivalence class of ψ_0 from (3.7) modulo the equivalence relation $\psi_0 \sim \psi_0 + S^\dagger_\alpha \chi^\alpha$. Because ∇ is flat there exists local frame where connection coefficients ω vanish. In such frame equations for ψ_0 takes the form

$$\nabla\psi_0 \equiv \theta^m\left(\frac{\partial}{\partial x^m} - \frac{\partial V^A(x)}{\partial x^m}\frac{\partial}{\partial Y^A}\right)\psi_0 = 0, \tag{3.14}$$

where we have reintroduced the term proportional to dV^A that was missing in (3.1) (because V^A was assumed constant there). Moreover, in this frame the compensator components V^A satisfying $V^2 = -1$ can be identified with the Cartesian coordinates on the ambient space $\mathbb{R}^{d+1}$ expressed through the intrinsic coordinates on $\mathcal{X} \subset \mathbb{R}^{d+1}$. Note also that in this frame the interpretation of ∇ as $o(d-1,2)$-connection is not straightforward.

On the other hand, let $\phi = \phi(X, A)$ be a field on the ambient space $\mathbb{R}^{d+1}$ satisfying (2.12), (2.13), (2.14) and subjected to gauge equivalence (2.26) with the gauge parameters χ^α satisfying (2.19) and (2.20). Let us introduce formal variables Y^A and represent ϕ and χ^α by $\psi(X, Y, A)$ and $\lambda^\alpha(X, Y, A)$ satisfying

$$\left(\frac{\partial}{\partial X^A} - \frac{\partial}{\partial Y^A}\right)\psi = 0, \quad \psi|_{Y=0} = \phi, \qquad \left(\frac{\partial}{\partial X^A} - \frac{\partial}{\partial Y^A}\right)\lambda^\alpha = 0, \quad \lambda^\alpha|_{Y=0} = \chi^\alpha. \tag{3.15}$$

This representation is obviously one-to-one.[8] In view of (3.15) one observes that $T\psi$ is equivalent to $\mathfrak{T}\phi$, where T and $\mathfrak{T}$ are two realization of an element of $sp(2n)$. More precisely, T and $\mathfrak{T}$ are related by the change $X^A \leftrightarrow X^A + V^A$ and $\frac{\partial}{\partial X} \leftrightarrow \frac{\partial}{\partial Y^A}$ (see Section 2.1). For instance, $\Box_X\phi$ is equivalent to $\Box_Y\psi$ if (3.15) is imposed.

The condition $(\frac{\partial}{\partial X^A} - \frac{\partial}{\partial Y^A})\psi = 0$ can be interpreted as a covariant constancy condition $\nabla_0\psi = 0$ with respect to an appropriate[9] connection ∇_0 so that it is similar to (3.14). Indeed, ψ and ψ_0 take values in the same space of polynomials in A_i with coefficients in formal series in Y^A-variables. Although ψ and ψ_0 are defined on different spaces ($\mathbb{R}^{d+1}$ and $\mathcal{X}$, respectively) it turns out that $\nabla_0\psi = 0$ and $\nabla\psi_0 = 0$ have isomorphic spaces of (equivalence classes modulo gauge invariance) solutions. To see this let us first introduce a trivial vector bundle $\mathcal{V}(\mathbb{R}^{d+1})$ with

[7] This is a general feature of the unfolded form of equations of motion [29].

[8] This is true both in the space of smooth functions and formal power series in Y^A-variables. As before we assume formal series.

[9] This can be seen as a standard $iso(d-1,2)$-connection.

the fiber being the space of polynomials in A_i with coefficients in formal series in Y^A-variables. It is associated to the tangent bundle over the ambient space $\mathbb{R}^{d+1}$.

In terms of general coordinates $X^{\underline{A}}$ on $\mathbb{R}^{d+1}$ the covariant derivative ∇_0 takes the form [27] :

$$\nabla_0 = \Theta^{\underline{A}}\left(\frac{\partial}{\partial X^{\underline{A}}} - \frac{\partial X^A}{\partial X^{\underline{A}}}\frac{\partial}{\partial Y^A}\right), \tag{3.16}$$

where new ghost variables $\Theta^{\underline{A}}$ stand for the basis differentials $dX^{\underline{A}}$. Note that using a general orthogonal local frame of the tangent bundle over $\mathbb{R}^{d+1}$ would also bring the usual term $\Theta^{\underline{A}} W_{\underline{A}}{}^{AB}(Y_A \frac{\partial}{\partial Y^B} + A_{iA}\frac{\partial}{\partial A_i^B})$ with the connection coefficients $W_{\underline{A}}{}^{AB}$. Furthermore, vector bundle $\mathcal{V}(\mathcal{X})$ introduced in Section 3.1 can be identified as a pullback of the bundle $\mathcal{V}(\mathbb{R}^{d+1})$ to $\mathcal{X} \subset \mathbb{R}^{d+1}$. Moreover, flat connection ∇ in $\mathcal{V}(\mathcal{X})$ can be seen as a pullback of ∇_0 in $\mathcal{V}(\mathbb{R}^{d+1})$ to $\mathcal{V}(\mathcal{X})$. More explicitly, reducing to the hyperboloid amounts to choosing a new coordinate system (r, x^m) in $\mathbb{R}^{d+1}$, where $r = \sqrt{-X^2}$ is a radial coordinate and x^m are dilation invariant coordinates. Then ∇ can be seen as restriction of ∇_0 to the surface of fixed radius $r = 1$ and radial ghost component $\theta_{(r)} = 0$, and is given by (3.1) if one identifies dilation-invariant coordinates and the intrinsic coordinates on $\mathcal{X}$.

Restriction to $\mathcal{X}$ clearly sends covariantly constant sections of $\mathcal{V}(\mathbb{R}^{d+1})$ to those of $\mathcal{V}(\mathcal{X})$. Moreover, this map is an isomorphism. To see this, let us note that this would be a trivial statement if the fiber were finite-dimensional. Indeed, a covariantly constant (with respect to ∇) section defined at $r = 1$ can be extended to a unique covariantly constant (with respect to ∇_0) section defined in the vicinity of $r = 1$. In the case at hand, however, the fiber is infinite-dimensional and solving for r-dependence could result in a nonconvergent series. This does note happen because the r-dependence of ϕ is fixed by constraint $X^A\frac{\partial}{\partial X^A}\phi = k\phi$ (2.12) which in turn originates from fiber constraint $(Y^A + V^A)\frac{\partial}{\partial Y^A}\psi = k\psi$. This shows that restriction to $\mathcal{X}$ is an isomorphism. In its turn, it determines an isomorphism between fields $\phi(X, A)$ satisfying (2.12), (2.13), and (2.14) and the covariantly constant sections $\psi_0(x, A)$ of $\mathcal{V}(\mathcal{X})$ satisfying (3.14).

This isomorphism is compatible with the $sp(2n)$ actions defined in Section 2.1. In particular, this guarantees that this map is compatible with the gauge transformation so that spaces of respective equivalence classes are also isomorphic. In addition, it is also compatible with the $o(d-1, 2)$ action. This implies that the value of the energy evaluated in Section 2.2 remains the same. Moreover, the computation of energy in Section 2.2 is only based on the relations of $o(d-1, 2)$ and $sp(2n)$ realized on the space of functions in X^A, A_i^A. Because the relations are the same for realizations of the same algebras on the fiber one immediately finds the same value for a fiber at a given point of $\mathcal{X}$. As the equations of motion have the form of a covariant constancy conditions one finds that this value is the same everywhere for a given field configuration.

3.4. Beyond the unitary case

We have by now constructed a compact gauge-invariant description of unitary gauge fields on AdS. It turns out that it can be generalized to a more general class of fields. To demonstrate the idea of such a generalization let us first show how the unitary fields can be seen as a subsector of a wider theory.

Let us consider BRST operator that can be obtained from (3.2) by taking $p = n - 1$

$$\Omega = \nabla + Q, \quad Q = S_i^\dagger \frac{\partial}{\partial b_i}, \quad i = 1, \ldots, n-1, \tag{3.17}$$

acting on the subspace of a space of functions $\Psi = \Psi(x, Y, A|\theta, b)$ singled out by constraints

$$T^{IJ}\Psi = 0, \qquad \big(N_i{}^j + B_i{}^j\big)\Psi = 0, \quad i < j, \qquad (N_i + B_i)\Psi = s_i\Psi. \tag{3.18}$$

Note that these form a subset of constraints (3.10) and (3.11). As we are going to see this theory describes a reducible system so that one or another irreducible field (not necessarily unitary) can be singled out by imposing further constraints.

For instance, suppose that in addition to (3.18) one impose the following constraints

$$\frac{\partial}{\partial b_i}\Psi = 0, \quad i = p+1, \dots, n-1. \tag{3.19}$$

One then observes that in this subspace Ω coincides with (3.2) while constraints (3.18) coincide with their counterparts from (3.10) and (3.11). Imposing the remaining constraints from (3.10) and (3.11) one indeed recovers the description of unitary gauge fields presented in Sections 3.1 and 3.2. This shows that unitary fields can indeed be singled out from a big theory (3.17) and (3.18) through farther algebraic conditions.

To give an example of non-unitary fields let us take $n = 2$ (totally symmetric fields) so that $Q = S^\dagger \frac{\partial}{\partial b}$ and impose the following constraints

$$\big(\bar{S}^\dagger\big)^t\Psi = 0, \qquad (N_{Y'} - B + t + 1)\Psi = s\Psi, \tag{3.20}$$

in addition to (3.18). One can check that Q indeed acts in the subspace.

To see which theory this defines let us evaluate Q-cohomology. In the minimal ghost number the coboundary condition is trivial so that the cohomology is defined by the cocycle condition $S^\dagger\Psi_1 = 0$. This, in particular, implies that Ψ_1 is a polynomial in Y^A and it is legitimate to re-express it in terms of change $Y'^A = Y^A + V^A$. The full list of conditions determining the cohomology at ghost degree -1 reads as

$$\begin{aligned} &N\Psi_1 = (s-1)\Psi_1, \qquad N_{Y'}\Psi_1 = (s-t)\Psi_1, \\ &(\bar{S})^t\Psi_1 = 0, \qquad S^\dagger\Psi_1 = 0. \end{aligned} \tag{3.21}$$

Being written in terms of variables A^A and Y'^A these give the description of cohomology classes in terms of two-row $o(d-1,2)$ Young diagrams with the first row of length $s-1$ and the second row of length $s-t$. These cohomology classes determine gauge fields that are 1-form connections with values in the respective $o(d-1,2)$ module originally considered in two-row Young diagrams [6,43]. These are known to describe partially-massless dynamics of spin s and depth t field [44–46,43].

The cohomology classes at vanishing ghost degree can be represented by elements satisfying

$$\bar{\mathsf{S}}\Psi_0 = 0, \quad \bar{\mathsf{S}} = Y^A\frac{\partial}{\partial A^A}. \tag{3.22}$$

Comparing with constraints (3.20) gives the following generalized V^A-transversality condition

$$V^{A_1}\cdots V^{A_m}\frac{\partial}{\partial A^{A_1}}\cdots\frac{\partial}{\partial A^{A_t}}\Psi_0 = 0, \tag{3.23}$$

and

$$\big(Y^A + V^A\big)\frac{\partial}{\partial Y^A}\Psi_0 = (s-t-1)\Psi_0. \tag{3.24}$$

Along with $\bar{\mathsf{S}}\Psi_0 = 0$ this gives the description of the respective Weyl module. Using the representation [27] of the Weyl module for massless spin-s fields as a subspace of totally traceless elements satisfying

$$Y^A \frac{\partial}{\partial A^A}\phi = 0, \qquad V^A \frac{\partial}{\partial A^A}\phi = 0, \qquad \left(Y^A + V^A\right)\frac{\partial}{\partial Y^A}\phi = (s-2)\phi, \tag{3.25}$$

one finds that the partially-massless Weyl module (3.23) decomposes into a collection of Weyl modules of massless Fronsdal fields of spins $s-t+1, \ldots, s-1, s$.

In this way we have extended the construction of previous sections to partially-massless fields originally described in [44–46,43]. In the similar manner, one can describe other irreducible AdS fields. Indeed, by Howe duality basis elements of $o(d-1,2)$ commutes with Q and therefore AdS algebra acts in the Q-cohomology. The Q-cohomology in non-zero negative ghost numbers $0 < p \leqslant n-1$ has been explicitly calculated in [25] and is represented by finite-dimensional irreps of $o(d-1,2)$ algebra. These give rise to p-form fields with values in the respective $o(d-1,2)$ irreps and coincide with those identified in [4]. According to [4] these fields correspond to all possible massless (unitary and non-unitary) fields and partially-massless fields of any symmetry types.

Let us finally comment on the relation of the theory determined by (3.17) and (3.18) to massless fields on $\mathbb{R}^{d+1}$. It turns out that this theory can be identified as a pull-back to $\mathcal{X}$ of a theory defined on $\mathbb{R}^{d+1}$. This can be constructed by considering fields on $\mathbb{R}^{d+1}$ and replacing ∇ with ∇_0 given by (3.16). The resulting theory describes massless mixed-symmetry fields[10] propagating in $\mathbb{R}^{d+1}$ spacetime [25,26]. Indeed, the BRST operator and the constraints simply coincide with those from [25]. Under the reduction to $\mathcal{X}$ the massless fields on $\mathbb{R}^{d+1}$ decomposes into a collection of gauge fields propagating on $\mathcal{X}$. As we have seen on examples one or another irreducible subsystem can be then singled out by auxiliary constraints compatible with (3.17) and (3.18). Let us note that this ideology is to some extent analogous to that of [7,8] where the unfolded form of the equations of motion for mixed-symmetry massless fields on AdS has been constructed starting from massless fields on the ambient space.

4. Parent form and other formulations

4.1. Parent form

Although the formulation constructed in Section 3.1 is rather compact and transparent other formulations can also be useful. An efficient way to handle various forms of the theory is to start with a sufficiently wide formulation such that other ones can be seen as one or another particular reductions. Such a formulation is refereed to as a parent form of the theory and is known for the case of totally symmetric [26] and mixed-symmetry [25] fields on Minkowski space as well as for totally symmetric AdS fields [27].

A parent formulation for mixed-symmetry AdS fields can be constructed as follows: introduce Grassmann odd ghost variables c_i and c_0 associated to the constraints $\Box_Y$ and S^i. The total BRST operator reads then as

$$\Omega^{parent} = \nabla + \bar{\Omega}, \tag{4.1}$$

[10] Strictly speaking one also needs to take $\mathbb{R}^{d,1}$ rather then $\mathbb{R}^{d-1,2}$ in order to have a usual interpretation in terms of representations of Poincaré group.

where ∇ is given by (3.1) and $\bar{\Omega}$ is given by

$$\bar{\Omega} = Q_p + \text{“more”} = S^\dagger_\alpha \frac{\partial}{\partial b_\alpha} + c_i S^i + c_0 \Box_Y - c_\alpha \frac{\partial}{\partial b_\alpha}\frac{\partial}{\partial c_0}. \tag{4.2}$$

The representation space is that of formulation of Section 3.1 extended by polynomials in new ghost variables c_i, c_0 and satisfying the following constraints

$$\bar{\mathbb{S}}^i\Psi = 0, \qquad \mathcal{T}^{ij}\Psi = 0, \qquad \mathcal{N}_i{}^j\Psi = 0, \quad i<j, \qquad \mathcal{J}_\alpha{}^\beta\Psi = 0, \qquad \mathcal{N}_i\Psi = s_i\Psi, \tag{4.3}$$

where

$$\bar{\mathbb{S}}^i = \bar{S}^i + 2C_0{}^i, \qquad \mathcal{T}^{ij} = T^{ij} + G^{ij}, \qquad \mathcal{N}_i{}^j = \widehat{\mathcal{N}}_i{}^j + C_i{}^j, \tag{4.4}$$

and

$$\mathcal{J}_\alpha{}^\beta = \widehat{\mathcal{J}}_\alpha{}^\beta + \delta^\beta_\alpha(2C_0 - C). \tag{4.5}$$

Here in addition to B and B_i^j introduced above we have also used the following useful notation for operators involving new ghost variables

$$C_I{}^J = c_I \frac{\partial}{\partial c_J}, \qquad G^{ij} = \delta^i_\alpha \frac{\partial}{\partial c_j}\frac{\partial}{\partial b_\alpha} + \delta^j_\alpha \frac{\partial}{\partial c_i}\frac{\partial}{\partial b_\alpha}, \tag{4.6}$$

along with the respective Euler operators

$$C_I = c_I \frac{\partial}{\partial c_I} \quad \text{(no summation)}, \qquad C = \sum_i C_i. \tag{4.7}$$

To see that this formulation is equivalent to the one of Section 3.1 one introduces additional degree such that $\deg c_o = \deg c_i = -1$ and reduces the theory to the cohomology of the term $\Omega^{parent}_{-1} = c_0\Box_Y + c_i S^i$ from (4.2) which carries lowest degree. In its turn, Ω^{parent}_{-1}-cohomology is concentrated in vanishing degree, and hence the reduced theory coincides with the one of Section 3.1.[11]

The constraints (4.3) still contain those involving Y^A and $\frac{\partial}{\partial Y^A}$. These are ghost modified $\bar{S}^i$ and h (recall that $\mathcal{J}_\alpha{}^\beta$ can be split into $\mathcal{N}_\alpha{}^\beta$ and h, cf. (3.12)). It can be useful to implement these constraints through the BRST operator with their own ghost variables so that only purely algebraic constraints

$$\mathcal{T}^{ij}\Psi = 0, \qquad \mathcal{N}_i{}^j\Psi = 0, \quad i<j, \qquad \mathcal{N}_\alpha{}^\beta\Psi = 0, \quad \alpha \neq \beta, \qquad (\mathcal{N}_i - s_i)\Psi = 0 \tag{4.8}$$

are directly imposed in the representation space. To show that such formulation is equivalent to (4.1) one introduces a degree such that the term involving $\bar{S}^i$ and h is of degree -1 and then reduces to its cohomology. This gives back the theory (4.1).[12]

[11] See [26,27] for more details on the equivalent reductions in cohomological terms.

[12] This reduction is a straightforward generalization of that from [27] to which we refer for more details.

4.2. Ambient space parent theory

The parent theory constructed in the previous section can be seen as a reduction to the hyperboloid $\mathcal{X} \subset \mathbb{R}^{d+1}$ of the related theory defined on the ambient space $\mathbb{R}^{d+1}/\{0\}$. Indeed, the arguments analogous to those of Section 3.3 show that the theory determined by

$$\Omega^{parent\ amb.} = \nabla_0 + \bar{\Omega} \tag{4.9}$$

defined on $\mathbb{R}^{d+1}$ can be reduced to that determined by (4.1). Here, ∇_0 is the covariant derivative defined in (3.16). In addition, as this theory is defined on the entire $\mathbb{R}^{d+1}/\{0\}$ one also needs to replace components of V^A with Cartesian coordinates X^A on $\mathbb{R}^{d+1}/\{0\}$ in the expression of constraints.

Because all the constraints involving Y^A and $\frac{\partial}{\partial Y^A}$ can be assumed to be imposed through the BRST operator one can consistently eliminate variables Y^A and Θ^A. Indeed, using Cartesian coordinates on $\mathbb{R}^{d+1}$ and an appropriate degree one identifies $\Theta^A \frac{\partial}{\partial Y^A}$ as a lowest degree term in the total BRST operator. Because Y^A and Θ^A are unconstrained variables the cohomology can be identified with Y^A, Θ^A-independent elements (see [26], where the analogous reduction was discussed in more details). Under this reduction all the remaining operators are changed according to $Y^A + X^A \to X^A$ and $\frac{\partial}{\partial Y^A} \to \frac{\partial}{\partial X^A}$ so that the reduced BRST operator reads as

$$\Omega^{ambient} = c_0 \Box_X + c_i \mathcal{S}^i + \mathcal{S}^{\dagger}_{\alpha} \frac{\partial}{\partial b_\alpha} - c_\alpha \frac{\partial}{\partial b_\alpha} \frac{\partial}{\partial c_0}. \tag{4.10}$$

Here we assumed that BRST invariant extensions of constraints $\bar{\mathcal{S}}^i$ and h_X are imposed directly. All the algebraic constraints (4.8) stay the same.

Let us analyze the resulting ambient space theory in some more details. Fields $\Psi = \Psi(X, A|b, c)$ are convenient to represent in the form of the decomposition $\Psi = \Psi_1 + c_0 \Psi_2$. For the ghost-number-zero component $\Psi^{(0)}$ fields $\Psi_1^{(0)} \equiv \Phi$ and $\Psi_2^{(0)} \equiv C$ are the following decompositions with respect to the ghost variables

$$\begin{aligned} \Phi &= \sum_{k=0}^{p} c_{i_1} \cdots c_{i_k} b_{\alpha_1} \cdots b_{\alpha_k} \Phi^{i_1 \ldots i_k|\alpha_1 \ldots \alpha_k}, \\ C &= \sum_{k=0}^{p-1} c_{i_1} \cdots c_{i_k} b_{\alpha_1} \cdots b_{\alpha_{k+1}} C^{i_1 \ldots i_k|\alpha_1 \ldots \alpha_{k+1}}. \end{aligned} \tag{4.11}$$

The expansion coefficients in (4.11) are antisymmetric in each group of indices and the slash | implies that no symmetry properties between two groups are assumed. In other words, the expansion components take values in tensor products of $gl(n-1)$ and $gl(p)$ antisymmetric irreps. Note that these component fields can be seen as an AdS version of the generalized triplets discussed in [47–49,25].

Decomposing the BRST operator with respect to the homogeneity degree in c_0 as $\Omega^{ambient} = \Omega_1 + \Omega_0 + \Omega_{-1}$ one can reduce the original theory to the cohomology of $\Omega_{-1} = c_\alpha \frac{\partial}{\partial b_\alpha} \frac{\partial}{\partial c_0}$ (see [26,25] for details). One then concludes that fields C are auxiliary while some components of Φ are Stueckelberg. After the reduction one is left with the fields annihilated by operator $Z_+ = c_\alpha \frac{\partial}{\partial b_\alpha}$ which is naturally interpreted as a generator of $sl(2)$ realized on ghosts (see [25] for an explicit discussion of this issue in the similar context). This reduction provides a relationship between the AdS version of generalized triplet formulation and the ambient space metric-like formulation. In particular, one can show that subjecting the dynamical fields of the reduced theory to the BRST extended trace conditions yields the generalized double-tracelessness conditions introduced in [3].

5. Q_p-cohomology and BMV conjecture

For the sake of completeness, we show here that the constructed generating formulation reproduces infinite-dimensional Weyl module and finite-dimensional module of gauge fields of the unfolded formulation for AdS mixed-symmetry massless fields [3,7,8]. More precisely, the Q_p-cohomology in the zeroth ghost degree is identified as Weyl module, while Q_p-cohomology in the minimal ghost number $-p$ is identified as the gauge module. In all other ghost degrees the cohomology is empty. Representation of the Weyl module as Q_p-cohomology allows to describe it in terms of Lorentz irreducible fields that become Minkowski space gauge fields in the flat limit. In particular, this gives a proof of the Brink–Metsaev–Vasiliev (BMV) conjecture, put forward in [9], and partially proved in [7,8].

5.1. Elimination of $(d+1)$-th direction

So far we used manifestly $o(d-1,2)$ covariant language. In this section it is convenient to analyse the problem in terms of Lorentz (*i.e.* $o(d-1,1)$) tensor fields. To this end, we choose the local frame where $V^A = \delta^A_d$. Set $Y^a = y^a$ and $Y^d = z$. Analogously, $A^a_i = a^a_i$ and $A^d_i = w_i$. In what follows, we always assume that all elements $\Psi = \Psi(Y, A|b)$ are totally traceless, $T^{IJ}\Psi = 0$. The following statement shows how constraints $\bar{S}^i$ and h from (3.10), (3.12) eliminate the dependence on $(d+1)$-th variables z and w_i.

Proposition 5.1. *The space of all totally traceless elements $\Psi = \Psi(Y, A|b)$ satisfying*

$$\bar{S}^i\Psi = 0, \qquad (h+m)\Psi = 0, \tag{5.1}$$

is isomorphic to the space of all z, w_i-independent totally traceless elements. Here m denotes any integer. The isomorphism sends Ψ to the traceless component of $\Psi|_{z=w_i=0}$.

The dependence of elements on ghost variables b_α is inessential here and is introduced for future convenience.

Proof. The proof is a straightforward generalization of that from [27]. The idea is to introduce auxiliary differential

$$\delta = \gamma_i \bar{S}^i + \alpha(h+m) - \alpha\gamma_i\frac{\partial}{\partial\gamma_i}, \qquad \delta^2 = 0,$$

where γ_i, α are auxiliary Grassmann odd ghost variables, $\mathrm{gh}(\gamma_i) = \mathrm{gh}(\alpha) = 1$. For a ghost-number-zero element Ψ, equation $\delta\Psi = 0$ is equivalent to Eqs. (5.1). More formally, such elements can be identified with δ-cohomology at vanishing ghost number.

The statement amounts to showing that any traceless z, w_i-independent $\Psi(y, a|b)$ can be uniquely completed to a totally traceless element annihilated by δ. If one takes homogeneity in z, w_i as a degree such a completion can be constructed order by order using the homological perturbation theory. More precisely, decomposing δ according to the degree

$$\delta = \delta_{-1} + \delta_0, \quad \delta_{-1} = \alpha\frac{\partial}{\partial z} + \gamma^i\frac{\partial}{\partial w^i}, \tag{5.2}$$

one observes that such a completion exists and is unique provided δ_{-1}-cohomology is trivial (any z, w_i-independent elements). This is obviously the case in the space of all (not necessarily traceless) elements. That this is also the case in the traceless subspace is a straightforward generalization of the respective statement proved in [27]. □

Both the space of z, w_i-independent traceless elements and its isomorphic space are $sl(n-1)$-modules (in fact, also $gl(n-1)$-modules), with $sl(n-1)$ algebra generated by operators $N_i{}^j = A_i^A \frac{\partial}{\partial A_j^A}$, $i \neq j$ and $n_i{}^j = a_i^a \frac{\partial}{\partial a_j^a}$, $i \neq j$, respectively. The isomorphism above is also an isomorphism of $sl(n-1)$-modules. Indeed,

$$\mathcal{P}\big((N_i{}^j \Psi)|_{z=w=0}\big) = n_i{}^j\big(\mathcal{P}(\Psi|_{z=w=0})\big), \tag{5.3}$$

where $\mathcal{P}$ denotes the standard projector to a totally traceless component. That the spaces above are isomorphic as $sl(n-1)$-modules implies, in particular, that if $m = s$ then the subspace of (5.1) satisfying in addition irreducibility conditions (3.10) is isomorphic to a subspace of traceless z, w_i-independent elements satisfying the respective constraints in terms of $n_i{}^j$. One may formulate the above statement as follows: when reducing to Lorentz all constraints remain intact while $\bar{S}^i$ and h are relaxed. In particular, all the weights s_i remain the same.

Furthermore, the action of the BRST operator can be represented in terms of z, w_i-independent elements using the isomorphism of Proposition 5.1. It is easy to check that

$$\mathcal{P}\big((Q_p\Psi)|_{z=w=0}\big) = q_p\mathcal{P}(\Psi|_{z=w=0}), \quad \text{where } q_p = s_\alpha^\dagger \frac{\partial}{\partial b_\alpha} \equiv a_\alpha^a \frac{\partial}{\partial y^a}\frac{\partial}{\partial b_\alpha}. \tag{5.4}$$

This implies that the field theory determined by $\Omega = \nabla_0 + Q_p$ can be completely reformulated in terms of z, w_i-independent fields. In these terms the respective BRST operator reads as

$$\widetilde{\Omega} = \widetilde{\nabla} + q_p, \tag{5.5}$$

where $\widetilde{\nabla}$ represents the action of ∇ in terms of z, w_i-independent fields. It acts in the space of totally traceless functions $\phi = \phi(x, y, a|b, \theta)$ subjected to the following conditions

$$(n_i + B_i)\phi = s_i\phi, \qquad \big(n_i{}^j + B_i{}^j\big)\phi = 0, \quad i < j, \qquad \big(n_\alpha{}^\beta + B_\alpha{}^\beta\big)\phi = 0 \tag{5.6}$$

where spins are arranged as in (2.23). Although this form of the theory is not very useful because the explicit expression of $\widetilde{\nabla}$ and hence the form of the equations of motion is rather involved in terms of $o(d-1,1)$-tensor fields we are going to use it for the analysis of the spectra of unfolded fields. These can be found as Q_p-cohomology classes.

In the flat limit $\widetilde{\nabla}$ becomes $\widetilde{\nabla}|_{\Lambda=0} = \theta^a(\frac{\partial}{\partial x^a} - \frac{\partial}{\partial y^a})$, where we made use of standard flat coordinates x^a and the associated local frame. Remarkably, in this limit the theory describes a dynamics of a particular collection of Minkowski mixed-symmetry fields. Indeed, (5.5) coincides with the BRST operator from [25] describing mixed-symmetry Minkowski fields provided one replaces $\widetilde{\nabla}$ with a usual flat Poincaré covariant derivative. Moreover, for rectangular fields ($p = n-1$) conditions (5.6) explicitly coincides with their counterpart from [25] so that in this case the flat limit is simply identical with the respective Minkowski field. More generally, if $p < n-1$ the flat limit of the theory (5.5) has less gauge invariance (only $s_i^\dagger$ with $i \leqslant p$ determine gauge symmetry) then its Minkowski space counterpart and hence carries more degrees of freedom. The fact that in the flat limit an irreducible AdS gauge field decomposes into a collection of Minkowski fields is known as BMV conjecture [9]. In Section 5.3 we give a general proof of the conjecture for fields of any symmetry type. Note that for fields with at most four rows a correctness of the BMV conjecture has been recently established in [8,7].

5.2. AdS Weyl module

If one reduces the theory to Q_p-cohomology, elements of vanishing ghost number give rise to gauge invariant fields that are zero forms. In the literature the module where these fields take values is known as Weyl module. In the present context we have the following:

Definition 5.2. An AdS Weyl module $\widetilde{\mathcal{M}}_0$ of spins $s_1, \ldots, s_{n-1}$ is a ghost number zero Q_p-cohomology evaluated in the subspace of elements $\phi(Y, A|b)$ satisfying (3.10) and (3.11).

At ghost number zero the cocycle condition is trivial, while the coboundary condition says that any element of the form $S^\dagger_\alpha \chi^\alpha$ is trivial. As was explained above the Q_p-cohomology can be computed as cohomology of $q_p = s^\dagger_\alpha \frac{\partial}{\partial b_\alpha}$ in the subspace of z, w_i-independent traceless elements satisfying (5.6).

Before the actual analysis of the q_p-cohomology let us first introduce some useful notation and definitions. As only generators of $sl(n)$ algebra are involved in the constraints and the BRST operator, it is enough to compute cohomology in the subspace $\mathcal{K}^{(k)}$ of traceless homogeneity-k polynomials in a^a_i, y^a tensored with ghost variables, *i.e.* the respective eigenspace of the Euler operator $n = n_y + \sum_i n_i$, $n_y = y^a \frac{\partial}{\partial y^a}$. Indeed, all $sl(n)$ generators do not change the homogeneity degree.

In its turn, $\mathcal{K}^{(k)}$ decomposes into a collection of finite-dimensional irreducible $sl(n)$-modules. Obviously, the following sets

$$n_- = \{ n_i{}^j \; i < j \} \quad \text{and} \quad n_+ = \{ n_i{}^j \; i > j \}, \tag{5.7}$$

generate $sl(n-1) \subset gl(n)$ subalgebra and can be identified as the upper-triangular and the lower-triangular subalgebras of $sl(n-1)$.

In order to realize the AdS Weyl module in terms of representatives of the equivalence classes it is useful to restrict the analysis to a finite-dimensional irreducible $sl(n)$-module $V \subset \mathcal{K}^{(k)}$. In particular, module V is completely specified by the eigenvalues $m_y(\psi_0)$, $m_i(\psi_0)$ of its highest weight (HW) vector ψ_0 with respect to the Euler operators n_y, n_i.

Conditions $n_i{}^j \phi = 0$ for $i < j$ imposed on ϕ are in fact the HW conditions with respect to $sl(n-1)$ subalgebra. The space of n_--invariant elements can be then seen as a subspace $V_0 \subset V$ of $sl(n-1)$ HW vectors. Decomposing V into the irreducible $sl(n-1)$-submodules as

$$V = \bigoplus_i V_i, \tag{5.8}$$

and using the natural projection to the $sl(n-1)$ HW subspace of any irreducible $sl(n)$-module, one defines the projector $\Pi : V \to V$ such that $\Pi^2 = \Pi$ and $\operatorname{Im} \Pi$ is the n_--invariant subspace.

We have the following two lemmas. Integers $m_i(\phi)$ below are eigenvalues of the Euler operators n_i acting on ϕ.

Lemma 5.3. *Let ϕ be an $sl(n-1)$ HW vector from $V_i \subset V$ then ϕ can be represented as*

$$\phi = \Pi s^\dagger_{i_1} \cdots s^\dagger_{i_l} \Lambda^{i_1 \ldots i_l} \psi_0, \tag{5.9}$$

where ψ_0 is a HW vector of irreducible $sl(n)$-module V and $\Lambda^{i_1 \ldots i_l}$ are some coefficients.

Lemma 5.4. *Let ϕ be an $sl(n-1)$ HW vector from $V_i \subset V$ then the conditions $m_\alpha(\phi) = m_\alpha(\psi_0)$ and $\phi \neq s^\dagger_\alpha \chi^\alpha$ are equivalent.*

Both lemmas follow from basic properties of finite-dimensional irreducible $sl(k)$-modules (see Appendix A). Lemma 5.4 gives a description of Q_p-cohomology at zeroth ghost numbers in terms of HW vectors of irreducible $sl(n)$-modules.

5.3. AdS Weyl module in terms of Poincaré ones: BMV conjecture

Let us recall that a Poincaré Weyl module of spin $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_{n-1}$ [50] can be defined as a subspace of $sl(n)$ HW vectors in $\mathcal{K}$ satisfying the respective weight conditions (see [25] for more details). It turns out that the AdS Weyl $\widetilde{\mathcal{M}}_0$ can be decomposed into the direct sum of some Poincaré Weyl modules.

More precisely, given AdS Weyl module of spin (2.23) a Poincaré Weyl module is called admissible associated module if $l_i = s_i - \nu_i$, where nonnegative integers $\nu_i = 0$, $i \leqslant p$ and $\nu_i \neq 0$, $i > p$, are chosen in a way compatible with the Young symmetry. We have:

Proposition 5.5. *AdS Weyl module $\widetilde{\mathcal{M}}_0$ of a given spin is isomorphic to a direct sum of the admissible associated Poincaré Weyl modules.*

Proof. We prove the statement by constructing the isomorphism explicitly. Let us first restrict to $\mathcal{K}^{(k)}$. As usual, we decompose $\mathcal{K}^{(k)}$ into the direct sum of irreducible $sl(n)$-modules. Let V be a given irreducible component. Its highest weight vector ψ_0 by definition belongs to some Poincaré Weyl module. Two things can happen: either ψ_0 is admissible or not. If not then in V there are no elements from $\widetilde{\mathcal{M}}_0$. If ψ_0 is admissible then there are nonnegative integers ν_i such that $s_i = l_i + \nu_i$ and $\nu_i = 0$ for $i \leqslant p$. It then follows from Lemma 5.4 that

$$\phi = \mathcal{I}_V(\psi_0) = \Pi\big[\big(s^\dagger_{p+1}\big)^{\nu_{p+1}} \cdots \big(s^\dagger_{n-1}\big)^{\nu_{n-1}} \psi_0\big] \tag{5.10}$$

belongs to $\mathcal{M}_0$. Note that ϕ is the only element in V that belongs to $\widetilde{\mathcal{M}}_0$. Defining the map $\mathcal{I}_V$ for each irreducible V (if V is not admissible $\mathcal{I}_V$ is trivial) one determines $\mathcal{I}$ for any element of the Poincaré module. By construction, $\mathcal{I}_V$ is an isomorphism. □

5.4. AdS gauge module

To complete the description of the spectrum of unfolded fields let us identify the cohomology at negative ghost degrees. The respective fields take values in the so-called gauge module. At ghost degree $-p$ the fields are identified as differential p-forms taking values in the respective $o(d-1,2)$ modules [3]. Namely, the coboundary condition is trivial while the cocycle implies that $S^\dagger_\alpha \phi = 0$, where $\phi = \phi_{m_1 \ldots m_p} \theta^{m_1} \cdots \theta^{m_p}$ takes values in a subspace singled out by constraints (3.10), (3.11). In particular, the field ϕ fulfills the following conditions: $\bar{S}^i \phi = 0$ for all i and $(N_\alpha - s + 1)\phi = 0$ and $(N_i - s_i)\phi = 0$ for $i > p$. In view of these conditions representatives can be chosen polynomials in Y^A. In terms of $Y^{A\prime} = Y^A + V^A$ all the conditions give an explicit characterization of gauge modules in terms of $o(d-1,2)$ Young tableaux having the uppermost block of length $s-1$ and height $p+1$ [3]. It turns out that Q_p-cohomology at ghost numbers other than 0, $-p$ vanish. To see this we again use the representation in terms of z, w_i-independent elements.

First of all we note that constraints $\widehat{N}_\alpha{}^\beta \psi = 0$ (3.12) for $\alpha \neq \beta$ imply that element $\psi = \psi^{\alpha_1 \ldots \alpha_k} b_{\alpha_1} \ldots b_{\alpha_k}$ with fixed weights contains just one independent component $\phi_{(k)} \equiv$

$\psi^{p-k+1\dots p-1p}$ satisfying $sl(n-1)$ HW conditions $N_i{}^j\phi_{(k)} = 0$, $i < j$. The Young tableau associated to $\phi_{(k)}$ includes the uppermost block of size $[s, p-k]$, the neighboring block of size $[s-1, k]$, while the rest of the diagram has rows of lengths s_i.

Operator q_p obviously acts in the space of $sl(n-1)$ HW elements of definite weights. More precisely, $q_p : \phi_{(k+1)} \mapsto \phi_{(k)} = \Pi s^\dagger_{p-k}\phi_{(k+1)}$, where Π is a projector on $sl(n-1)$ HW elements (see Section 5.2).

For the q_p-cohomology at ghost number $-k$ we have the following cocycle and the coboundary conditions:

$$\Pi s^\dagger_{p-k+1}\phi_{(k)} = 0, \qquad \phi_{(k)} \sim \phi_{(k)} + \Pi s^\dagger_{p-k}\chi_{(k+1)}, \tag{5.11}$$

where $\chi_{(k+1)}$ are some $sl(n-1)$ HW elements of definite weights. Note that for $k = 0$ the cocycle condition is trivial as was already discussed in Section 5.2. For $k = p$ the coboundary condition is missing so we are left with the cocycle condition only. For intermediate values of the ghost number $0 < k < p$ we have the following lemmas describing solutions to (5.11).

Lemma 5.6. *Let $\phi_{(k)}$ be an $sl(n-1)$ HW vector from $V_i \subset V$ then conditions $m_\alpha(\phi_{(k)}) = m_\alpha(\psi_0)$ at $1 \leqslant \alpha \leqslant p-k$, and $\phi_{(k)} \neq \Pi s^\dagger_{p-k}\chi_{(k+1)}$ are equivalent.*

Lemma 5.7. *Let $\phi_{(k)}$ be an $sl(n-1)$ HW vector from $V_i \subset V$ then $\Pi s^\dagger_{p-k+1}\phi_{(k)} = 0$ iff $m_\alpha(\phi_{(k)}) > m_\alpha(\psi_0)$ for some α such that $1 \leqslant \alpha \leqslant p-k$.*

Here ψ_0 denote respective $sl(n)$ HW vectors from Lemma 5.3. Both lemmas result from comparing admissible weights of $sl(n-1)$ HW elements $\phi_{(k)}$ and their associated $sl(n)$ HW elements ψ_0 (see Appendix A). Since there are no $sl(n-1)$ HW elements that simultaneously satisfy both the cocycle and the coboundary conditions, one concludes that the cohomology is empty for $k \neq 0, p$.

6. Conclusions

In this paper, we have proposed the unified formulation for unitary dynamics of free bosonic HS fields of any symmetry type in the AdS space. We have also observed and discussed how to generalize the theory to include non-unitary fields. In particular, we have explicitly described such a generalization for totally symmetric partially-massless fields. The theory is formulated on the level of equations of motion using the usual BRST first quantized language. This makes the formulation somewhat analogous to the usual string-inspired BRST approach to higher spin fields. In particular, this can make the proposed formulation useful in describing relation to (a tensionless limit of) the bosonic string theory on the AdS background.

Another motivation and possible application of these results have to do with studying consistent interactions for mixed-symmetry AdS fields. While in the case of totally symmetric fields consistent interactions are known to cubic order in the Lagrangian formulation [51,6,52–57] and to all orders at the level of equations of motion [38,39], interactions of mixed-symmetry AdS gauge fields are not known so far. We hope that the transparent algebraic structure and a due control of the gauge invariance through the BRST technique make the present formulation useful in searching for nonlinear theory. Moreover, a possible nonlinear deformation is necessarily related to the appropriate algebraic structure – higher spin algebra. In the case of totally symmetric fields the respective algebra [39,58] can be identified with higher symmetries [59] of the scalar singleton, the corresponding algebra in the mixed-symmetry case is expected to be related to singletons of nonvanishing spins. The respective candidate higher spin algebras have been recently identified in [32] using a framework closely related to the present one (see also a discussion of singleton composites in [8]).

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Appendix A. Proofs of lemmas of Section 5

Proof of Lemma 5.3. Any element from $V \subset \mathcal{K}^{(k)}$ can be represented as a linear combination of elements obtained by acting on ψ_0 with n_+ and $s_i^\dagger$. Representing ϕ in this way, moving n_+ to the left by using the algebra commutation relations, and applying Π one finds that $\phi = \Pi\phi = \Pi s_{i_1}^\dagger \dots s_{i_l}^\dagger \Lambda^{i_1\dots i_l}\psi_0$ because all the terms involving n_+ cannot contribute. Indeed, $\Pi n_+ \chi = 0$ for any χ because n_+ cannot map to HW subspace. □

Proof of Lemma 5.4. Let us first show that $m_\alpha(\phi) \neq m_\alpha(\psi_0)$ iff ϕ is trivial in the sense that $\phi = s_\alpha^\dagger \chi^\alpha$ for some χ^α. To this end introduce the following notation: n_{-1} denotes an element from the subalgebra n_+ of the form N^α_{p+i}, where $p+i$ denote indices running $p+1, \dots, n-1$, n_0 either N_α^β or N^{p+j}_{p+i} from the subalgebra n_+; s_0 denotes $s^\dagger_{p+i}$ and s_1 denotes $s^\dagger_\alpha$. Note that commutation relations have the structure

$$
\begin{aligned}
&[\mathsf{n}_{-1}, \mathsf{n}_{-1}] = 0, \qquad [\mathsf{n}_{-1}, \mathsf{n}_0] = \mathsf{n}_{-1}, \qquad [\mathsf{n}_0, \mathsf{n}_0] = \mathsf{n}_0,\\
&[\mathsf{n}_0, \mathsf{s}_0] = \mathsf{s}_0, \qquad [\mathsf{n}_0, \mathsf{s}_1] = \mathsf{s}_1, \qquad [\mathsf{n}_{-1}, \mathsf{s}_1] = \mathsf{s}_0, \quad [\mathsf{n}_{-1}, \mathsf{s}_0] = 0.
\end{aligned}
\tag{A.1}
$$

According to Lemma 5.3 a given HW vector can be represented as $\phi = \Pi(\mathsf{s}_1)^l(\mathsf{s}_0)^m\psi_0$ for some nonnegative integers l, m. The terms originating from the projector have the following structure

$$
(\mathsf{n}_0)^i(\mathsf{n}_{-1})^j(\mathsf{s}_1)^{l+j}(\mathsf{s}_0)^{m-j}\psi_0
\tag{A.2}
$$

where the weights m_y and m_i of ϕ have been taken into account. Then using the commutation relations above one moves all s_1 to the left. This results in the expression of the form $\mathsf{s}_1(\dots)$ iff $l > 0$. Indeed, the terms without s_1 can arise in this process only if $l = 0$ (indeed only commuting n_{-1} with s_1 one can get rid of s_1; but the power of s_1 is higher than that of n_{-1} unless $l = 0$). If $l = 0$ then analogous arguments show that ϕ is nontrivial $\phi \neq \mathsf{s}_1(\dots)$ and other way around. □

Proof of Lemma 5.6. It is analogous to that of Lemma 5.4.

In summary, both Lemma 5.4 and its generalization Lemma 5.6 mean that nontrivial $sl(n-1)$ HW elements representing the equivalence relation cannot be generated from the respective $sl(n)$ HW elements by the first $p-k$ generators $s_\alpha^\dagger$ (for $k = 0$ we recover Lemma 5.4). □

Proof of Lemma 5.7. The proof reduces to the following two observations. Firstly, one observes that acting by $s_i^\dagger$ increases a value of weight s_i by one and recalls that $sl(n-1)$ HW elements with weights $s_j < s_{j+1}$ vanish identically. Secondly, given $sl(n-1)$ HW element ϕ it is easily seen that the relation $\Pi s_i^\dagger s_{i+1}^\dagger \phi = 0$ holds provided that at least two subsequent weights are equal, *i.e.*, $m_i(\phi) = m_{i+1}(\phi)$. $\square$

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