On the thermal description of the BTZ black holes

Norman Cruz \textsuperscript{a}, Samuel Lepe \textsuperscript{b}

\textsuperscript{a} Departamento de Física, Facultad de Ciencia, Universidad de Santiago de Chile, Casilla 307, Correo 2, Santiago, Chile
\textsuperscript{b} Instituto de Física, Facultad de Ciencias Básicas y Matemáticas, Pontificia Universidad Católica de Valparaíso, Avenida Brasil 2950, Valparaíso, Chile

Received 5 April 2004; accepted 26 April 2004
Available online 1 June 2004
Editor: M. Čvetić

Abstract

We investigate the limitations on the thermal description of three-dimensional BTZ black holes. We derive on physical grounds three basic mass scales that are relevant to characterize these limitations. The Planck mass in 2+1 dimensions indicates the limits where the black hole can emit Hawking’s radiation. We show that the back reaction is meaningless for spinless BTZ black hole. For stationary BTZ black holes the nearly extreme case is analyzed showing that may occur a breakdown of its description as a thermal object.

1. Introduction

An important point in discussion is if the extreme black holes behave as thermal objects. More than a decade ago, Preskill et al. \cite{1} pointed out that the thermal description of a black hole becomes ill-defined as the black hole approaches the extreme limit. In its treatment the use of the criterion that the standard semiclassical description, which neglects the back reaction, is not self-consistent if the emission of a typical quantum radiation changes the temperature by an amount comparable to the value of the temperature. For a Schwarzschild black hole, the thermal description breaks down simultaneously as the classical description of spacetime does \cite{1}. At the final stages of the evaporation, the black hole field has a high curvature and the radius of the horizon is of the order of the Planck length, besides, as the temperature rises as the mass diminishes, the problem of the back reaction becomes unavoidable. Thus, the Planck mass is the only relevant scale of mass that it is needed to take account of in a full theory of quantum gravity. Nevertheless, for a Kerr–Newman black hole the breakdown occurs near the extreme case when the mass is much larger than the Planck mass and the corrections due to quantum gravity are expected to be negligible.

In (2+1)-dimensional gravity the BTZ black hole solution is a spacetime of constant negative curvature, but it differs from anti-de Sitter (AdS) space in its
global properties [2]. The related investigations of the thermodynamics properties of this solution have not included considerations about the nature of a possible breakdown of its thermal description. An obvious question is if this type of black hole experiment, as its four-dimensional counterpart, shows a breakdown of its thermal description and what the relevant mass scale involved is. It was found by Reznik [3] that the Planck mass in 2 + 1 dimensions has a physical significance related to the description of the thermodynamics of the Hawking emission process, for static BTZ black holes. This is a key point when we try to shed some light in the behavior of BTZ black holes in relation to the limits to its description as thermal objects. The purpose of this Letter is to investigate the limits in the thermal description, using the approach described in [1], for static and rotating BTZ black holes.

In Section 2 we expose briefly the principal parameters related with the thermodynamics of the BTZ black hole. In Section 3 we show the existence of three mass scales: \( m_p, m_x, \) and \( m_T \) and discuss its physical meaning. The mass scale \( m_T \) has not been previously discussed in the literature; it appears when the approach described in [1] is applied to the thermodynamics of BTZ black holes. We have restricted our discussion to the case when the semiclassical description of the BTZ spacetime is valid. In Section 4 we discuss the particular limitations that experiment imposes on the thermodynamical description of BTZ black holes. We show the role played by the above three masses in these limitations. In particular, we explicitly calculate the conditions imposed on the black hole parameters in order to have a good description of the thermodynamics of the Hawking emission process. The problem of the back reaction is considered for spinless BTZ black holes. The near extreme black hole is discussed and the breakdown of its thermal description, in the sense discussed in [1], is analyzed. Finally, in Section 5 some concluding remarks are presented.

2. The BTZ black hole

In this Letter we adopt units such that \( c = 1 = k_B \), where \( k_B \) is the Boltzmann constant. We keep the \( G \) constant explicitly in the above expressions. \( MG \) is the geometrized mass, which has no dimensions in 2 + 1 gravity. In the following section we will discuss the mass scales relevant in the thermodynamics of the BTZ black hole and the role of \( G^{-1} \) as Planck mass. We also keep \( h \) in the following thermodynamics parameters, since there exists another scale which contains explicitly this constant.

The action considered in [2] is

\[
I = \frac{1}{2\pi G} \int \sqrt{-g} \left(R + 2\ell^{-2} \right) d^2 x \ dt + B, \tag{1}
\]

where \( B \) is a surface term, and the radius of curvature \( \ell = (-\Lambda)^{-1/2} \) provides the length scale necessary to have a horizon (\( \Lambda \) is the cosmological constant).

The axially symmetric BTZ black hole written in stationary coordinates is

\[
ds^2 = -N^2(r) \ dt^2 + f^{-2}(r) \ dr^2 + r^2 \left(N^\phi(r) \ dt + d\phi \right)^2, \tag{2}
\]

where the lapse \( N(r) \) and the angular shift \( N^\phi(r) \) are given by

\[
N^2(r) = f^2(r) = -MG + \frac{r^2}{\ell^2} + \frac{(JG)^2}{4r^2},
\]

and

\[
N^\phi(r) = -\frac{JG}{2r^2}, \tag{3}
\]

with \( -\infty < t < \infty, \ 0 < r < \infty, \) and \( 0 \leq \phi < 2\pi \). The constant of integration \( M \) is the conserved charge associated with asymptotic invariance under time displacements and \( J \) (angular momentum) is that associated with rotational invariance. The lapse function vanishes for two values of \( r \) given by

\[
r_{\pm}(x) = \ell(MG)^{1/2} \left[\frac{1}{2} (1 \pm \sqrt{x}) \right]^{1/2}, \tag{4}
\]

where \( x \) is defined by the relation

\[
x \equiv 1 - \frac{J^2}{(\ell M)^2}. \tag{5}
\]

The black hole horizon is given by \( r_{+}(x), \) which exists only for \( M > 0 \) and \( |J| \leq \ell M \). We will denote the horizon of a spinless black hole by \( r_{+}(1) \equiv r_{+}(x = 1) = \ell(MG)^{1/2}. \) The entropy of this black hole is given by

\[
S = \frac{1}{\ell MG} 4\pi r_{+}(x). \tag{6}
\]

From Eq. (6) it is possible to write the entropy, \( S, \) as a function of \( M \) and \( J \) and obtain the first law of
thermodynamics \[4\]

\[
dM = T \, dS + \Omega \, dJ,
\]

where \(T\) is the Hawking’s temperature and \(\Omega\) is the angular velocity of the black hole horizon. Both parameters can be expressed in terms of the variable \(x\), yielding for the temperature

\[
T(x) = T(1) \sqrt{\frac{2x}{1 + \sqrt{x}}},
\]

(8)

where \(T(1)\) is the temperature of a spinless BTZ black hole given by

\[
T(1) = \frac{h \, G \, r_+(1)}{2\pi \, \ell^2}.
\]

(9)

The corresponding expression for \(\Omega\) is

\[
\Omega = \frac{1}{\ell} \frac{\sqrt{1 - x}}{\sqrt{1 + \sqrt{x}}},
\]

(10)

Important aspects of the thermodynamics systems need to know the heat capacities \(C_z\), where \(z\) denotes the set of parameters held constant. A useful expression, that we will use later on, for the heat capacity at constant angular momentum \(J\) is given by

\[
C_J = T \left( \frac{\partial S}{\partial T} \right)_J = C_0 \left( \frac{1}{2} \frac{\sqrt{(1 + \sqrt{x})x}}{\sqrt{x}} - \frac{1}{2} \right)^{1/2},
\]

(11)

where \(C_0 = (hG)^{-1}4\pi r_+\) is the heat capacity at \(J = 0\). The expression for \(C\Omega\) is given by

\[
C\Omega = T \left( \frac{\partial S}{\partial \Omega} \right)_\Omega = \frac{4\pi \ell}{hG} \left( \frac{MG}{2} [1 + \sqrt{x}] \right)^{1/2}.
\]

(12)

One can verify directly that \(C_J\) and \(C\Omega\) are always positive, independently of the values of the parameters \(M\) and \(J\), for the BTZ black holes. These type of black holes never experience a phase transition. In the case of two independent thermodynamics variables \(Z_1, Z_2\), stability requires that \(\partial S, \partial Z_1, S \leq 0, \partial Z_2, S \leq 0, (\partial Z_1, \partial Z_2)^2 S - (\partial Z_1, \partial Z_2) S (\partial Z_2, \partial Z_2) S \leq 0 [5]\). It is straightforward to prove that the above conditions are satisfied taking \(Z_1 = M, Z_2 = J\), for static and rotating BTZ black holes, which ensure the thermal stability of this type of black holes. These conditions are always satisfied for any \(M\) and \(J\).

3. Mass scales

We will show in this section that the thermal description of a spinless BTZ black hole is ill defined when the black hole mass approaches a new mass scale, not discussed previously in the literature. For this reason we first discuss the relevant mass scales in the theory of black holes in \(2 + 1\) dimensions.

For a Schwarzschild black hole, at the Planck scale, the fluctuations of the geometry become important and this occurs when its radius becomes comparable to the Compton wavelength, i.e., \(r_{\text{horizon}} \sim \lambda_{\text{Compton}}\). From this relationship we obtain the expression for the Planck mass, \(m_P\). Nevertheless, we can also obtain the Planck mass imposing that \(r_{\text{horizon}} \sim \ell_P\), where \(\ell_P\) is the Planck length. In addition, we can define \(m_P\) as \(h/\ell_P\) from a straightforward dimensional analysis. In four dimensions, these criteria lead to a unique mass scale, the Planck mass, given by \(m_{P,3+1} = (h/G)^{1/2}\).

The situation is quite different in three-dimensional gravity. In general, for any dimension, the Planck mass, \(m_P\), is defined from the relation \(m_P \ell_P \sim h\), where \(\ell_P\) is fundamental length scale which is obtained imposing that the action in \(D\) dimensions

\[
I \sim \frac{1}{G} \int d^Dx \sqrt{-g} R,
\]

be of the order of \(h\). For \(2+1\) dimensions the fundamental unit of length is given by

\[
\ell_P = h/G,
\]

(14)

which leads to the corresponding Planck mass in \(2+1\) dimensions

\[
m_P = \frac{1}{G}.
\]

(15)

We restrict our discussion to the case when the semi-classical description of the BTZ spacetime is valid, i.e., when the condition \(S_{2+1}/h > 1\) holds or, in other words, when \(\ell > \ell_P\).

Since the Planck mass in \(2+1\) dimensions is a classical mass unit (contains only the gravitational constant), it is not surprising that it appears in classical phenomena associated with \((2+1)\)-dimensional fluids in hydrostatic equilibrium. If the cosmological constant is not included, classical results show that there exists a universal mass, in the sense that all rotationally invariant structures in hydrostatic equilibrium have a mass that is proportional to \(m_P\) [6]. In
this case there are no black hole solutions and the possibility of collapse is clearly forbidden. Nevertheless, the study of the structures, with a mass $M$ and a radius $R$, in hydrostatic equilibrium in AdS gravity leads to an upper bound on the ratio $M/R$ similar to the four-dimensional case. This result shows that the possibility of collapse exists for matter distributions that have a ratio $M/R$ over the above upper bound. It is possible that black holes with large masses exist (large with respect to $m_P$), even more, any fluid distribution in hydrostatic equilibrium has necessarily a mass larger than $m_P$. With the assumption that the BTZ black hole is the end of a collapse, the classical mass $m_P$ represents the lowest mass of any $(2+1)$-dimensional black hole [7].

For the Planck mass, $m_P$, $r_+ \sim \ell$, i.e., the size of the horizon is comparable with the associated length of the spacetime curvature $l$. Since $\ell > \ell_P$, the fluctuations of the black hole geometry are not important at this mass scale.

A remarkable result, as we shall see below, is that without the length scale provided by the cosmological constant it is not possible to build a unit mass containing $h$. Quantum phenomena are present in 2+1 dimensions with a length scale that can provide horizons and the basic mass units related contain both the Planck and the cosmological constant.

The fluctuations of the black hole geometry become important when $r_+ \sim \lambda_{\text{Compton}}$, i.e., when the radius of the black hole becomes comparable to the Compton wavelength. This yields the mass scale, $m_\lambda$, given by

$$m_\lambda = \left( \frac{\hbar^2}{\ell^2 G} \right)^{1/3}. \tag{16}$$

For this reason Reznik [3] identifies the Planck mass in 2+1 dimensions with $m_\lambda$.

We can obtain another mass scale when the limitations of the thermal description of a black hole are studied. In the following analysis we will consider a classical background geometry, ignoring the back reaction. As it was pointed out in [1] the semiclassical description of the black hole evaporation is not self-consistent if the emission of a typical quantum radiation changes the temperature by an amount comparable to the value of the temperature. If $T$ is the energy of the quantum and if $\Delta T$ is the change of temperature that the black hole experiences after the emission, then from the relation $C_J \Delta T = T$, where $C_J$ is specific heat at $J = \text{const}$, the condition for the thermal description to be self-consistent is

$$\left| T \frac{\partial T}{\partial (MG)} \right|_J \ll |T|, \tag{17}$$

which is equivalent to imposing

$$\frac{\partial T}{\partial (MG)} = C_J^{-1}$$

$$= C_0^{-1} (2 - \sqrt{x}) \left[ \frac{2}{(1 + \sqrt{x})^x} \right]^{1/2} \ll 1. \tag{18}$$

For the particular case of a static BTZ black hole, $x = 1$, the heat capacity is given by

$$C_0 = 4\pi \sqrt{\frac{m}{m_T}}. \tag{19}$$

where mass scale, $m_T$, has the following expression

$$m_T = \frac{\hbar^2 G}{\ell^2}. \tag{20}$$

The breakdown occurs when the black hole mass $M$ satisfies $M \sim m_T$. At this scale $r_{\text{horizon}} \sim \ell_P$ and corrections due to quantum gravity are expected to be very important. Note that this mass satisfies (up to a numerical factor), $T(M = m_T) \sim m_T$. For Schwarzschild black holes the corresponding relation is $T(M = m_{P+1}) \sim m_{P+1} / r_+$; i.e., black holes with masses of the order of the Planck mass radiate at Planck temperature.

4. Limitations on the thermal description

4.1. The static black hole

A crucial point of the characteristic behavior of thermodynamics of the spinless BTZ black hole was pointed out by Reznik [3]. He indicated that the physical significance of the mass unit $m_P$ is that for $M > m_P$ ($M < m_P$) the wavelength $\lambda$ of the Hawking radiation satisfies $\lambda < r_+$ ($\lambda > r_+$).

Since the process of energy emission can be thermodynamically well described when a typical wavelength of the Hawking radiation satisfies $\lambda \lesssim r_+$, we calculate for rotating BTZ black hole the conditions
on the parameters $M$ and $J$ in order to satisfy this restriction. We have the following relation for a typical wavelength, $\lambda$, related to Hawking’s temperature

$$T(x)\ell_p^{-1} \sim \lambda^{-1}. \quad (21)$$

The requirement for the wavelength is

$$\lambda \lesssim r_{+}(x). \quad (22)$$

From the relations (21) and (22) we obtain that

$$\left(\frac{r_{+}(1)}{\ell}\right)^2 \gtrsim \frac{1}{\sqrt{\lambda}}. \quad (23)$$

From this inequality we obtain, in terms of the dimensionless parameters $j \equiv J G/\ell$ and $m \equiv M/m_p$, that

$$j \lesssim m \left(1 - \frac{1}{m^4}\right)^{1/2}. \quad (24)$$

Notice that for a static BTZ black hole $x = 1$ and the inequality (23) yields $M \gtrsim m_p$, since $r_{+}(1)/\ell = M/m_p$. This was the result obtained in [3].

Let us discuss first the case of a spinless BTZ black hole. Since we only consider the regime where $\ell > \ell_p$ the mass scales obtained on physical grounds satisfy the relation $m_T < m_\lambda < m_p$. We have argued that for this black hole the Hawking’s radiation cannot be emitted if $M < m_p$. Notice that this is an effect due to the finite size of our system, which has no equivalent in the case of Schwarzschild or Kerr black holes. The other mass scales found allow to characterize further the limitations in the description of the thermodynamical behavior of the BTZ black hole.

At the scale of $m_p$ and unlike the Schwarzschild black hole, where the curvature of the spacetime is associated with the size of the black hole horizon, the BTZ black hole is an AdS space of constant curvature, which no changes through all the evaporation process, neglecting the back reaction, with a radius of curvature given by $\ell$. On the other hand, $r_{+}(1) > \lambda_{\text{Compton}}$ ($m_\lambda < m_p$) which means that the fluctuations of the black hole geometry never become important for the spinless BTZ black hole.

Contrary to the Schwarzschild black hole, at the scale of $m_p$ the temperature obeys the relation $T(M = m_p) < m_p$. At this scale the energy emitted by the black hole is not important with respect to the energy of the black hole itself. This is consistent with the fact that the thermal description is well defined only for black hole masses larger than $m_p$ ($m_T < m_p$). Or in other words, there is no breakdown of the thermal description, in the sense discussed in [1], for the spinless BTZ black hole.

Notice that the fluctuations, both in temperature and entropy, are important at mass scale $m_T$. The equilibrium thermodynamics fluctuation of BTZ black holes in the microcanonical ensemble, canonical ensemble and grand canonical ensemble was studied in [8]. The fluctuations in the temperature (microcanonical ensemble) are given by

$$\frac{\langle \delta T(1)\delta T(1) \rangle}{T^2(1)} = C_0^{-1} = \frac{1}{4\pi} \sqrt{\frac{m_T}{M}}, \quad (25)$$

and in the entropy (canonical ensemble) by

$$\langle \delta S \delta S \rangle = C_0 = 4\pi \sqrt{\frac{M}{m_T}} \quad (26)$$

The above equations allow us to say that at the Planck scale, the fluctuations of the temperature are not important and that the system has many states sampled spontaneously.

These results have important consequences for the back reaction problem. Martínez and Zanelli [9] have evaluated the back reaction of a massless conformal scalar field on the geometry of the spinless BTZ black hole. The authors calculate the $O(\hbar r_{+}^{-1})$ corrections to the metric, which do not change the value of the ADM mass. The corrections to the temperature and entropy are linear in the function $F(M) \sim e^{-\pi M/m_p}$, which implies that the back reaction becomes large only for small masses compared to the Planck mass. These results are consistent with the fact that at the mass scales lower than the Planck mass, for example, $m_T$, the fluctuations of the temperature are important. Nevertheless, since it only has physical sense to consider black holes above the Planck mass, calculations relative to the back reaction are meaningless.

For $\ell = \ell_p$, there is a unique mass scale, since $m_p \sim m_\lambda \sim m_T$. In this case the semiclassical description of the AdS spacetime is no longer valid due to the fluctuations of the geometry becoming important, independent of the black hole radius. It has no physical sense to consider the limitations on the thermal description.
4.2. The rotating black hole

One can directly see that the rotating BTZ black hole has a very different behavior with respect to its four-dimensional counterparts. Eq. (23) rewritten in terms of the black hole mass, we obtain

\[ M \gtrsim m_P x^{-1/4}. \]  

(27)

For a rotating black hole \( x < 1 \), which implies that the allowed minimum mass is always larger than the Planck mass. On the other hand, if, for example, \( M \approx m_p \), Eq. (24) imposes an upper bound on the angular momentum of the black hole, which in this case indicates that \( j \ll m \). In simple words, for black hole with a mass near to the Planck mass the upper bound on the angular momentum is very low.

As the black hole mass increases the upper bound on the angular momentum allowed also increases. For a massive rotating black hole, i.e., \( m \gg 1 \), or equivalently, \( M \gg m_P \), we obtain that \( j \lesssim m \). This is the only type of rotating BTZ black hole that can approach the extreme case. To investigate the breakdown of the thermal description, in the sense discussed in [1], we approach the extreme case, \( |J| \lesssim \ell M \), taking \( x \simeq 0 \) in Eq. (18), which yields

\[ \frac{\partial T}{\partial (MG)} \simeq \frac{1}{2\pi} \left( \frac{2m_T}{M x} \right)^{1/2} \ll 1. \]  

(28)

Notice that for near extreme BTZ black holes, the thermal description breaks down when \( M x \sim m_T \), which implies that this breakdown may occur within the regime \( M \gg m_T \). In this case it is straightforward to see that with an adequate ratio \( \ell / \ell_P \) it is possible that a massive rotating black hole experiences a breakdown of its thermal description. The temperature fluctuations (microcanonical ensemble) can be evaluated from the relation

\[ \frac{\langle \delta T(x) \delta T(x) \rangle}{T^2(x)} = C_J^{-1}, \]  

(29)

which according to the rhs of Eq. (28) implies that as the near extreme case is approached, the temperature fluctuations become larger to the temperature itself. The entropy fluctuations (canonical ensemble) are given by

\[ \langle \delta S \delta S \rangle = C_J \simeq 2\pi \left( \frac{M x}{2m_T} \right)^{1/2}. \]  

(30)

So near the extreme case few states are sampled spontaneously. Notice the above arguments are valid if \( M x \to 0 \) as the black hole mass increases, which is correct since Eq. (24) implies that \( x \sim m^{-4} \) for massive black holes.

In this scenario, as we explained above, the constant curvature of the AdS spacetime at the horizon is small in Planck units, and corrections due to quantum gravity are expected to be negligible. For extreme Kerr–Newman black holes it was found in [1] that thermal description breaks down when \( M \gg m_P \), so the situation is equivalent in the sense that we are in a scenario in which it is not necessary to include quantum gravity.

5. Conclusions

We have shown the existence of three mass scales: \( m_P, m_3, m_T \), which have a clear physical meaning. We have restricted our discussion to the case when the semiclassical description of the BTZ spacetime is valid, i.e., when the condition \( S_{2+1}/\hbar > 1 \) holds, or, in other words, when \( \ell > \ell_P \). In this case the masses satisfy the relation \( m_T < m_3 < m_P \).

We have argued that the process of energy emission can be thermodynamically well described when the typical wavelength of the Hawking radiation satisfies \( \lambda \lesssim r_+ \).

We have shown that the radiation process takes place only for static black holes with masses above the Planck scale.

At this stage the size of the horizon is comparable with the associated length of the spacetime curvature and the fluctuations of the black hole geometry are not important. This means that in the evaporation of the BTZ black hole the horizon never loses its classical meaning. On the other hand, we have proved that the breakdown of the thermal description, in the sense discussed by Preskill et al. [1], never occurs. This is consistent with the result found in [9], in which the back reaction becomes large for small masses compared to the Planck mass. In this sense, the BTZ black hole never experience a back reaction that is comparatively important with respect to the background geometry.

We have shown that for rotating BTZ black hole the allowed minimum mass is always larger than the
Planck mass. Besides, there exists an upper bound on the angular momentum of the black hole which depends on the black hole mass. Only for massive rotating black holes \( j \lesssim m \) is possible, and the near extreme case can be reached. In this case the description of black holes as thermal objects may experience a breakdown. This also occurs for Kerr–Newman black holes in 3 + 1 dimensions [1].

Acknowledgements

Useful discussions with members of the GACG are gratefully acknowledged. S. Lepe acknowledges Departamento de Física, Universidad de Santiago de Chile, for hospitality. The authors also acknowledge the Universidad de Concepción for their kind hospitality. This work was supported by CONICYT through Grant FONDECYT No. 1040229, USACH-DICYT No. 04-0031CM (N.C.) and PUCV-DI No. 123.769/03 (S.L.). The authors were partially supported by the Ministerio de Educación through a MÉCESUP Grant, USA 0108. This work was partially realized during one of the Dichato Cosmological Meetings, Concepción, Chile (2004).

References