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Ore extensions of Baer and p.p.-rings

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Abstract

We investigate Ore extensions of Baer rings and p.p.-rings. Let α be an endomorphism and δ an α-derivation of a ring R. Assume that R is an α-rigid ring. Then (1) R is a Baer ring if and only if the Ore extension $R[x;\alpha,\delta]$ is a Baer ring if and only if the skew power series ring $R[[x;\alpha]]$ is a Baer ring, (2) R is a p.p.-ring if and only if the Ore extension $R[x;\alpha,\delta]$ is a p.p.-ring.

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Throughout this paper R denotes an associative ring with identity. In [13] Kaplansky introduced *Baer* rings as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. According to Clark [9], a ring R is called to be *quasi-Baer* if the right annihilator of each right ideal of R is generated (as a right ideal) by an idempotent. Further works on quasi-Baer rings appear in [3–5,15]. Recently, Birkenmeier et al. [6] called a ring R a *right* (resp. *left*) *principally quasi-Baer* (or simply *right* (resp. *left*) *p.q.-Baer*) ring if the right (resp. left) annihilator of a principal right (resp. left) ideal of R is generated by an idempotent. R is called a *p.q.-Baer* ring if it is both right and left p.q.-Baer.

Another generalization of Baer rings is a p.p.-ring. A ring R is called a *right* (resp. *left*) *p.p.*-ring if the right (resp. left) annihilator of an element of R is generated by an idempotent. R is called a *p.p.*-ring if it is both a right and left p.p.-ring.

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It is natural to ask if any or all of these properties can be extended from R to $R[x]$ and $R[[x]]$. The extensions of Baer, quasi-Baer, right p.q.-Baer and p.p.-rings have been investigated by many authors [1,6,11–13, etc.].

In this paper, we study Ore extensions of Baer rings and p.p.-rings. In particular, we show: Let α be an endomorphism and δ an α-derivation of a ring R. Suppose that R is an α-rigid ring. Then (1) R is a Baer ring if and only if the Ore extension $R[x;\alpha,\delta]$ is a Baer ring if and only if the skew power series ring $R[[x;\alpha]]$ is a Baer ring, (2) R is a p.p.-ring if and only if the Ore extension $R[x;\alpha,\delta]$ is a p.p.-ring. Thereby several known results are extended.

For a nonempty subset X of a ring R, we write $r_R(X)=\{c\in R\,|\,Xc=0\}$ and $\ell_R(X)=\{c\in R\,|\,cX=0\}$, which are called the right annihilator of X in R and the left annihilator of X in R, respectively.

We begin with the following lemma. Recall that a ring R is *reduced* if R has no nonzero nilpotent elements. Observe that reduced rings are abelian (i.e., all idempotent are central).

Lemma 1. *Let R be a reduced ring. Then the following statements are equivalent*:
(i) *R is a right p.p.-ring.*
(ii) *R is a p.p.-ring.*
(iii) *R is a right p.q.-Baer ring.*
(iv) *R is a p.q.-Baer ring.*
(v) *For any $a\in R$, $r_R(a^nR)=eR$ for some $e=e^2\in R$, where n is any positive integer.*

Proof. These follow from the fact $r_R(a)=\ell_R(a)=r_R(aR)=\ell_R(Ra)=r_R(a^nR)$ for any $a\in R$ and any positive integer n because R is reduced. □

However, the following example shows that there exists an abelian right p.q.-Baer ring which is neither right nor left p.p. (see also [8, Example 14.17]). Due to Chase [7], there is a left p.p.-ring which is not right p.q.-Baer.

Example 2. (1) Let $\mathbb{Z}$ be the ring of integers and $\mathrm{Mat}_2(\mathbb{Z})$ the 2×2 full matrix ring over $\mathbb{Z}$. We consider the ring

$$R=\left\{\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in \mathrm{Mat}_2(\mathbb{Z})\,\middle|\, a\equiv d,\ b\equiv 0 \text{ and } c\equiv 0\,(\mathrm{mod}\,2)\right\}.$$

First we claim that R is right p.q.-Baer. Let

$$u=\begin{bmatrix} a & b\\ c & d\end{bmatrix}$$

be a nonzero element of R. Then we see that

$$\begin{bmatrix} 2a & 0\\ 2c & 0\end{bmatrix},\begin{bmatrix} 0 & 2a\\ 0 & 2c\end{bmatrix}\in uR$$

by multiplying

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

to the element u from the right-hand side, respectively. If

$$v = \begin{bmatrix} \alpha & \beta \\ \gamma & \varepsilon \end{bmatrix} \in r_R(uR),$$

then

$$\begin{bmatrix} 2a & 0 \\ 2c & 0 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \varepsilon \end{bmatrix} = \begin{bmatrix} 2a\alpha & 2a\beta \\ 2c\alpha & 2c\beta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So $\alpha = 0$ and $\beta = 0$ if $a \neq 0$ or $c \neq 0$. Also

$$\begin{bmatrix} 0 & 2a \\ 0 & 2c \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \varepsilon \end{bmatrix} = \begin{bmatrix} 2a\gamma & 2a\varepsilon \\ 2c\gamma & 2c\varepsilon \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So $\gamma = 0$ and $\varepsilon = 0$ if $a \neq 0$ or $c \neq 0$. Therefore

$$v = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

if $a \neq 0$ or $c \neq 0$. Suppose that $b \neq 0$ or $d \neq 0$. If we replace

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

by

$$\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix},$$

respectively, in the above, then by the same method we see that

$$v = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence $r_R(uR) = 0$ for any nonzero element $u \in R$. Therefore R is right p.q.-Baer.

Next we claim that R is neither right p.p. nor left p.p. For

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in R,$$

we have

$$r_R\left(\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} \mu & v \\ 0 & 0 \end{bmatrix} \mid \mu \equiv 0 \,(\mathrm{mod}\, 2) \text{ and } v \equiv 0 \,(\mathrm{mod}\, 2) \right\} \neq eR,$$

where $e = e^2 \in R$. Note that the only idempotents of R are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore R is not right p.p. Using the same method as the above, we can see that R is not left p.p.

(2) [8, Example 8.2] For a ring $\prod_{n=1}^{\infty} \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the ring of integers modulo 2, let

$$T = \{(a_n)_{n=1}^{\infty} \mid a_n \text{ is eventually constant}\}$$

and

$$I = \{(a_n)_{n=1}^{\infty} \mid a_n = 0 \text{ eventually}\}.$$

Then

$$R = \begin{pmatrix} T/I & T/I \\ 0 & T \end{pmatrix}$$

is a left p.p.-ring which is not right p.q.-Baer.

Recall that for a ring R with a ring endomorphism $\alpha : R \to R$ and an α-derivation $\delta : R \to R$, the *Ore extension* $R[x;\alpha,\delta]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication

$$xr = \alpha(r)x + \delta(r)$$

for all $r \in R$. If $\delta = 0$, we write $R[x;\alpha]$ for $R[x;\alpha,0]$ and is called an *Ore extension of endomorphism type* (also called a *skew polynomial ring*). While $R[[x;\alpha]]$ is called a *skew power series ring*.

Definition 3 (*Krempa* [14]). Let α be an endomorphism of R. α is called a *rigid endomorphism* if $r\alpha(r) = 0$ implies $r = 0$ for $r \in R$. A ring R is called to be *α-rigid* if there exists a rigid endomorphism α of R.

Clearly, any rigid endomorphism is a monomorphism. Note that α-rigid rings are reduced rings. In fact, if R is an α-rigid ring and $a^2=0$ for $a \in R$, then $a\alpha(a)\alpha(a\alpha(a)) = a\alpha(a^2)\alpha^2(a) = 0$. Thus $a\alpha(a) = 0$ and so $a = 0$. Therefore, R is reduced. But there exists an endomorphism of a reduced ring which is not a rigid endomorphism (see Example 9). However, if α is an inner automorphism (i.e., there exists an invertible element $u \in R$ such that $\alpha(r) = u^{-1}ru$ for any $r \in R$) of a reduced ring R, then R is α-rigid.

In this paper, we let α be an endomorphism of R and δ an α-derivation of R, unless especially noted.

Lemma 4. *Let R be an α-rigid ring and $a, b \in R$. Then we have the following*:

(i) *If $ab = 0$ then $a\alpha^n(b) = \alpha^n(a)b = 0$ for any positive integer n.*

(ii) *If $ab = 0$ then $a\delta^m(b) = \delta^m(a)b = 0$ for any positive integer m.*

(iii) *If $a\alpha^k(b) = 0 = \alpha^k(a)b$ for some positive integer k, then $ab = 0$.*

Proof. (i) It is enough to show that $a\alpha(b) = \alpha(a)b = 0$. If $ab = 0$, then $b\alpha(a)\alpha(b\alpha(a)) = b\alpha(ab)\alpha^2(a) = 0$. Since R is α-rigid, we have $b\alpha(a) = 0$. Since R is reduced, $(\alpha(a)b)^2 = 0$ implies $\alpha(a)b = 0$. Similarly, using $ba = 0$, we obtain $a\alpha(b) = 0$.

(ii) It is enough to show that $a\delta(b)=\delta(a)b=0$. If $ab=0$, then $0=\delta(ab)=\alpha(a)\delta(b)+\delta(a)b$. So $\{\alpha(a)\delta(b)\}^2=-\delta(a)b\alpha(a)\delta(b)=0$ by (i). Hence $\alpha(a)\delta(b)=0$, since R is reduced. Then $\alpha(a\delta(b))=\alpha(a)\alpha(\delta(b))=0$ by (i). Since α is a monomorphism, we have $a\delta(b)=0$. Similarly, we obtain $\delta(a)b=0$.

(iii) Suppose that $a\alpha^k(b)=0$ for some positive integer k. Then, by (i) we obtain $\alpha^k(ab)=\alpha^k(a)\alpha^k(b)=0$. Since α is a monomorphism, we have $ab=0$. Similarly, $\alpha^k(a)b=0$ for some positive integer k implies $ab=0$. □

The following proposition extends [10, Lemma 3] and [14, Theorem 3.3].

Proposition 5. *A ring R is α-rigid if and only if the Ore extension $R[x;\alpha,\delta]$ is a reduced ring and α is a monomorphism of R. In this case, $\alpha(e)=e$, $\delta(e)=0$ for some $e=e^2\in R$.*

Proof. Suppose that R is α-rigid. Assume to the contrary that $R[x;\alpha,\delta]$ is not reduced. Then there exists $0\neq f\in R[x;\alpha,\delta]$ such that $f^2=0$. Since R is reduced, $f\notin R$. Thus we put $f=\sum_{i=0}^m a_ix^i$, where $a_i\in R$ for $0\le i\le m$ and $a_m\neq 0$. Since $f^2=0$, we have $a_m\alpha^m(a_m)=0$. By Lemma 4(iii) $a_m^2=0$ and so $a_m=0$, which is a contradiction. Therefore $R[x;\alpha,\delta]$ is reduced.

Conversely, suppose that $R[x;\alpha,\delta]$ is reduced. Clearly, R is reduced as a subring. If $a\in R$ and $a\alpha(a)=0$, then $(\alpha(a)xa)^2=0$ and so $\alpha(a)xa=0$. Thus $0=(\alpha(a)x)a=(\alpha(a))^2x+\alpha(a)\delta(a)$ and so $\alpha(a)=0$. Since α is a monomorphism, we have $a=0$. Therefore R is α-rigid.

Next, let e be an idempotent in R. Then e is central and so $ex=xe=\alpha(e)x+\delta(e)$. This implies that $\alpha(e)=e$ and $\delta(e)=0$. □

In this Proposition 5, if $R[x;\alpha,\delta]$ is reduced and $ab=0$ for $a,b\in R$, then we obtain $ax^mbx^n=0$ in $R[x;\alpha,\delta]$ for any nonnegative integers m and n.

Proposition 6. *Suppose that R is an α-rigid ring. Let $p=\sum_{i=0}^m a_ix^i$ and $q=\sum_{j=0}^n b_jx^j$ in $R[x;\alpha,\delta]$. Then $pq=0$ if and only if $a_ib_j=0$ for all $0\le i\le m$, $0\le j\le n$.*

Proof. Assume that $pq=0$. Then $(\sum_{i=0}^m a_ix^i)(\sum_{j=0}^n b_jx^j)=\sum_{k=0}^{m+n}(\sum_{i+j=k} a_ix^ib_jx^j)=c_{m+n}x^{m+n}+c_{m+n-1}x^{m+n-1}+\cdots+c_1x+c_0=0$. We claim that $a_sb_t=0$ for $s+t\ge 0$. We proceed by induction on $s+t$. It can be easily checked that $c_{m+n}=a_m\alpha^m(b_n)=0$. Then we obtain $a_mb_n=0$ by Lemma 4(iii). This proves for $s+t=m+n$. Now suppose that our claim is true for $s+t>k\ge 0$. Then by Proposition 5, we have $\sum_{i+j=l} a_ix^ib_jx^j=0$ for $l=m+n,m+n-1,\ldots,k+1$. However, using Lemma 4(i) and (ii) repeatedly, we see that for $i+j\ge k+1$, $a_i\alpha^{i_1}\delta^{j_1}\alpha^{i_2}\delta^{j_2}\cdots\alpha^{i_t}\delta^{j_t}(b_j)=0$ for each nonnegative integers $i_1,\ldots,i_t,j_1,\ldots,j_t$. Hence we obtain

$$c_k=\sum_{i+j=k} a_i\alpha^i(b_j)=0. \tag{1}$$

By induction hypothesis, we have $a_s b_t = 0$ and so $a_s \alpha^s(b_t) = 0$ for $s + t > k$ by Lemma 4(i). Since R is reduced, we obtain $\alpha^s(b_t)a_s = 0$. Hence, multiplying a_k to Eq. (1) from the right-hand side, we obtain

$$\left\{ \sum_{i+j=k} a_i \alpha^i(b_j) \right\} a_k = a_k \alpha^k(b_0) a_k = 0.$$

Then $\{a_k \alpha^k(b_0)\}^2 = 0$. Since R is reduced, we obtain $a_k \alpha^k(b_0) = 0$ and hence $a_k b_0 = 0$ by Lemma 4(iii). Now Eq. (1) becomes

$$\sum_{\substack{i+j=k \\ 0 \le i \le k-1}} a_i \alpha^i(b_j) = 0. \tag{2}$$

Multiplying a_{k-1} to Eq. (2) from the right-hand side, we obtain $a_{k-1}\alpha^{k-1}(b_1)a_{k-1} = 0$. So by the same way as the above we obtain $a_{k-1}\alpha^{k-1}(b_1) = 0$ and so $a_{k-1}b_1 = 0$. Continuing this process, we can prove $a_i b_j = 0$ for all i, j with $i + j = k$. Therefore $a_i b_j = 0$ for all $0 \le i \le m$, $0 \le j \le n$.

The converse follows from Lemma 4. □

Corollary 7. *Let R be an α-rigid ring. If $e^2 = e \in R[x; \alpha, \delta]$, where $e = e_0 + e_1 x + \cdots + e_n x^n$, then $e = e_0$.*

Proof. Since $1 - e = (1 - e_0) - \sum_{i=1}^n e_i x^i$, we get $e_0(1 - e_0) = 0$ and $e_i^2 = 0$ for all $1 \le i \le n$ by Proposition 6. Thus $e_i = 0$ for all $1 \le i \le n$ and so $e = e_0 = e_0^2 \in R$. □

The Baerness and quasi-Baerness of a ring R do not inherit the Ore extension of R, respectively. The following example shows that there exists a Baer ring R but the Ore extension $R[x; \alpha, \delta]$ is not right p.q.-Baer.

Example 8. Let F be a field and consider the polynomial ring $R = F[y]$ over F. Then R is a commutative domain and so R is Baer. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha(f(y)) = f(0)$. Then the skew polynomial ring $R[x; \alpha]$ is not reduced. In fact, for $0 \ne yx \in R[x; \alpha]$, we have $yxyx = y\alpha(y)x^2 = 0$. So $R[x; \alpha]$ is not reduced.

Let $e = a_0(y) + a_1(y)x + \cdots + a_n(y)x^n \in R[x; \alpha]$ be a nonzero idempotent. Then $(a_0(y) + a_1(y)x + \cdots + a_n(y)x^n)(a_0(y) + a_1(y)x + \cdots + a_n(y)x^n) = a_0(y) + a_1(y)x + \cdots + a_n(y)x^n$. So $a_0(y)^2 = a_0(y)$. Since R is a domain, $a_0(y) = 0$ or $a_0(y) = 1$. Assume that $a_0(y) = 1$. Note that $(a_0(y)a_1(y) + a_1(y)\alpha(a_0(y)))x = a_1(y)x$. So

$$a_0(y)a_1(y) + a_1(y)\alpha(a_0(y)) = a_1(y). \tag{3}$$

Since $a_0(y) = 1$, we have $a_1(y) + a_1(y) = a_1(y)$. Hence $a_1(y) = 0$. Similarly, we obtain $(a_0(y)a_2(y) + a_2(y)\alpha^2(a_0(y)))x^2 = a_2(y)x^2$. So

$$a_0(y)a_2(y) + a_2(y)\alpha^2(a_0(y)) = a_2(y). \tag{4}$$

Then $a_2(y) + a_2(y) = a_2(y)$. Hence $a_2(y) = 0$. Continuing this process, we have $e = 1$.

Assume that $a_0(y)=0$. Then $\alpha(a_0(y))=0$. From Eqs. (3) and (4), we obtain $a_1(y)=0$ and $a_2(y)=0$. Continuing this process, we have $e=0$. Therefore the only idempotents of $R[x;\alpha]$ are 0 and 1.

Now we claim that $R[x;\alpha]$ is not right p.q.-Baer. Note that $r_{R[x;\alpha]}(xR[x;\alpha]) \neq R[x;\alpha]$. Moreover $r_{R[x;\alpha]}(xR[x;\alpha]) \neq 0$. For, if $a_0(y)+a_1(y)x+\cdots+a_n(y)x^n \in R[x;\alpha]$, then $x(a_0(y)+a_1(y)x+\cdots+a_n(y)x^n)y = x(a_0(y)y+a_1(y)\alpha(y)x+\cdots+a_n(y)\alpha^n(y)x^n) = x(a_0(y)y) = \alpha(a_0(y)y)x = 0$, and so $y \in r_{R[x;\alpha]}(xR[x;\alpha])$. But the only idempotents of $R[x;\alpha]$ are 0 and 1. Therefore $R[x;\alpha]$ is not right p.q.-Baer.

The following example shows that there exists $R[x;\alpha,\delta]$ which is quasi-Baer, but R is not quasi-Baer.

Example 9. Let $\mathbb{Z}$ be the ring of integers and consider the ring $\mathbb{Z}\oplus\mathbb{Z}$ with the usual addition and multiplication. Then the subring

$$R=\{(a,b)\in\mathbb{Z}\oplus\mathbb{Z} \mid a\equiv b \pmod 2\}$$

of $\mathbb{Z}\oplus\mathbb{Z}$ is a commutative reduced ring. Note that the only idempotents of R are $(0,0)$ and $(1,1)$. In fact, if $(a,b)^2=(a,b)$, then $(a^2,b^2)=(a,b)$ and so $a^2=a$ and $b^2=b$. Since $a\equiv b \pmod 2$, $(a,b)=(0,0)$ or $(a,b)=(1,1)$. Now we claim that R is not quasi-Baer. For $(2,0)\in R$, we note that $r_R((2,0))=\{(0,2n) \mid n\in\mathbb{Z}\}$. So we can see that $r_R((2,0))$ does not contain a nonzero idempotent of R. Hence R is not quasi-Baer.

Now let $\alpha: R\to R$ be defined by $\alpha((a,b))=(b,a)$. Then α is an automorphism of R. Note that R is not α-rigid. We claim that $R[x;\alpha]$ is quasi-Baer. Let I be a nonzero right ideal of $R[x;\alpha]$ and $p\in I$. Put $p=(a_i,b_i)x^i+\cdots+(a_m,b_m)x^m$, where i is the smallest integer such that $(a_i,b_i)\neq(0,0)$, for all $0\le i\le m$. Then for some positive integer $2k-i$, $p(1,1)x^{2k-i}=(a_i,b_i)x^{2k}+\cdots+(a_m,b_m)x^{2k+m-i}\in I$ and $p(1,1)x^{2k+1-i}=(a_i,b_i)x^{2k+1}+\cdots+(a_m,b_m)x^{2k+1+m-i}\in I$. Suppose that $0\neq q\in r_{R[x;\alpha]}(I)$ and put $q=(u_j,v_j)x^j+\cdots+(u_n,v_n)x^n$, where j is the smallest integer such that $(u_j,v_j)\neq(0,0)$, for all $0\le j\le n$. Then $p(1,1)x^{2k-i}q=0$ and $p(1,1)x^{2k+1-i}q=0$. So we have

$$(a_i,b_i)x^{2k}(u_j,v_j)x^j+\cdots=(a_i,b_i)(u_j,v_j)x^{2k+j}+\cdots$$

and

$$(a_i,b_i)x^{2k+1}(u_j,v_j)x^j+\cdots=(a_i,b_i)(v_j,u_j)x^{2k+1+j}+\cdots.$$

Hence $(a_iu_j,b_iv_j)=(0,0)$ and $(a_iv_j,b_iu_j)=(0,0)$. This implies that $a_iu_j=b_iv_j=0$ and $a_iv_j=b_iu_j=0$. Since $(a_i,b_i)\neq(0,0)$, a_i or b_i is nonzero. Then we have $(u_j,v_j)=(0,0)$, which is a contradiction. So $r_{R[x;\alpha]}(I)=(0,0)$ and hence $R[x;\alpha]$ is quasi-Baer. Note that the only idempotents of $R[x;\alpha]$ are $(0,0)$ and $(1,1)$. But $r_{R[x;\alpha]}((2,0))$ cannot be generated by an idempotent since $\{(0,2n) \mid n\in\mathbb{Z}\}\subseteq r_{R[x;\alpha]}((2,0))\neq R[x;\alpha]$. Hence $R[x;\alpha]$ is not Baer.

We now provide examples which show that the Baerness of R and $R[x;\alpha,\delta]$ does not depend on each other.

Example 10. (1) Let $R=\mathbb{Z}_2[y]/(y^2)$, where (y^2) is a principal ideal generated by y^2 of the polynomial ring $\mathbb{Z}_2[y]$. Note that the only idempotents of R are $0+(y^2)$ and $1+(y^2)$. Since $r_R(y+(y^2))=(y+(y^2))R$ cannot be generated by an idempotent, R is not right quasi-Baer and so it is not Baer. Now, let α be the identity map on R and we define an α-derivation δ on R by $\delta(y+(y^2))=1+(y^2)$. Then R is not α-rigid since R is not reduced. However, by [2, Example 11]

$$R[x;\alpha,\delta]=R[x;\delta]\cong \mathrm{Mat}_2(\mathbb{Z}_2[y^2])\cong \mathrm{Mat}_2(\mathbb{Z}_2[t]).$$

Since $\mathbb{Z}_2[t]$ is a principal integral domain, $\mathbb{Z}_2[t]$ is a Prüfer domain (i.e., all finitely generated ideals are invertible). So by [13, Exercise 3, p. 17], $\mathrm{Mat}_2(\mathbb{Z}_2[t])$ is Baer. Therefore $R[x;\alpha,\delta]=R[x;\delta]$ is Baer.

(2) Let $R=\mathrm{Mat}_2(\mathbb{Z})$. Then R is a Baer ring and so R is right p.p.. But $R[x]$ is not a right p.p.-ring (see [1] or [11]). Also $R[x;\alpha]$ is not Baer, in case α is the identity map of R.

Theorem 11. *Let R be an α-rigid ring. Then R is a Baer ring if and only if $R[x;\alpha,\delta]$ is a Baer ring.*

Proof. Assume that R is Baer. Let A be a nonempty subset of $R[x;\alpha,\delta]$ and A^* be the set of all coefficients of elements of A. Then A^* is a nonempty subset of R and so $r_R(A^*)=eR$ for some idempotent $e\in R$. Since $e\in r_{R[x;\alpha,\delta]}(A)$, we get $eR[x;\alpha,\delta]\subseteq r_{R[x;\alpha,\delta]}(A)$. Now, we let $0\neq g=b_0+b_1x+\cdots+b_tx^t\in r_{R[x;\alpha,\delta]}(A)$. Then $Ag=0$ and hence $fg=0$ for any $f\in A$. Thus $b_0,b_1,\ldots,b_t\in r_R(A^*)=eR$ by Proposition 6. Hence there exist $c_0,c_1,\ldots,c_t\in R$ such that $g=ec_0+ec_1x+\cdots+ec_tx^t=e(c_0+c_1x+\cdots+c_tx^t)\in eR[x;\alpha,\delta]$. Consequently, $eR[x;\alpha,\delta]=r_{R[x;\alpha,\delta]}(A)$. Therefore $R[x;\alpha,\delta]$ is Baer.

Conversely, assume that $R[x;\alpha,\delta]$ is Baer. Let B be a nonempty subset of R. Then $r_{R[x;\alpha,\delta]}(B)=eR[x;\alpha,\delta]$ for some idempotent $e\in R$ by Corollary 7. Thus $r_R(B)=r_{R[x;\alpha,\delta]}(B)\cap R=eR[x;\alpha,\delta]\cap R=eR$. Therefore R is Baer. □

Corollary 12. *Let R be an α-rigid ring. Then R is a quasi-Baer ring if and only if $R[x;\alpha,\delta]$ is a quasi-Baer ring.*

Proof. It follows from [3, Lemma 1], Proposition 5 and Theorem 11. □

Corollary 13 (Armendariz [1, Theorem B]). *Let R be a reduced ring. Then R is a Baer ring if and only if $R[x]$ is a Baer ring.*

From Example 10(2), we can see that there exists a right p.p.-ring R such that $R[x;\alpha,\delta]$ is not a right p.p.-ring. However we have the following.

Theorem 14. *Let R be an α-rigid ring. Then R is a p.p.-ring if and only if $R[x;\alpha,\delta]$ is a p.p.-ring.*

Proof. Assume that R is a p.p.-ring. Let $p=a_0+a_1x+\cdots+a_mx^m\in R[x;\alpha,\delta]$. There exists an idempotent $e_i\in R$ such that $r_R(a_i)=e_iR$ for $i=0,1,\dots,m$. Let $e=e_0e_1\cdots e_m$. Then $e^2=e\in R$ and $eR=\bigcap_{i=0}^m r_R(a_i)$. So by Proposition 5, $pe=a_0e+a_1\alpha(e)x+\cdots+a_m\alpha^m(e)x^m=a_0e+a_1ex+\cdots+a_mex^m=0$. Hence $eR[x;\alpha,\delta]\subseteq r_{R[x;\alpha,\delta]}(p)$. Let $q=b_0+b_1x+\cdots+b_nx^n\in r_{R[x;\alpha,\delta]}(p)$. Since $pq=0$, by Proposition 6 we obtain $a_ib_j=0$ for all $0\le i\le m$, $0\le j\le n$. Then $b_j\in e_0e_1\cdots e_mR=eR$ for all $j=0,1,\dots,n$ and so $q\in eR[x;\alpha,\delta]$. Consequently $eR[x;\alpha,\delta]=r_{R[x;\alpha,\delta]}(p)$. Thus $R[x;\alpha,\delta]$ is a p.p.-ring.

Conversely, assume that $R[x;\alpha,\delta]$ is a p.p.-ring. Let $a\in R$. By Corollary 7, there exists an idempotent $e\in R$ such that $r_{R[x;\alpha,\delta]}(a)=eR[x;\alpha,\delta]$. Hence $r_R(a)=eR$. Therefore R is a p.p.-ring. □

Corollary 15. *Let R be an α-rigid ring. Then R is a p.q.-Baer ring if and only if $R[x;\alpha,\delta]$ is a p.q.-Baer ring.*

Proof. It follows from Proposition 5, Lemma 1 and Theorem 14. □

Corollary 16 (Armendariz [1, Theorem A]). *Let R be a reduced ring. Then R is a p.p.-ring if and only if $R[x]$ is a p.p.-ring.*

From Example 10(1), we can see that the condition "R is α-rigid" in Theorem 11 is not superfluous. On the other hand, Example 9 shows that the condition "R is α-rigid" in Corollary 12, Theorem 14 and Corollary 15 is not superfluous.

Now we turn our attention to the relationship between the Baerness of a ring R and the Baerness of the skew power series ring $R[[x;\alpha]]$.

Proposition 17. *Suppose that R is an α-rigid ring. Let $p=\sum_{i=0}^\infty a_ix^i$ and $q=\sum_{j=0}^\infty b_jx^j$ in $R[[x;\alpha]]$. Then $pq=0$ if and only if $a_ib_j=0$ for all $i\ge 0$ and $j\ge 0$.*

Proof. Assume that $pq=0$. Then

$$\sum_{k=0}^{\infty}\left(\sum_{i+j=k}a_ix^ib_jx^j\right)=\sum_{k=0}^{\infty}\left(\sum_{i+j=k}a_i\alpha^i(b_j)x^{i+j}\right)=0. \tag{5}$$

We claim that $a_ib_j=0$ for all i,j. We proceed by induction on $i+j$. Then we obtain $a_0b_0=0$. This proves for $i+j=0$. Now suppose that our claim is true for $i+j\le n-1$. From Eq. (5), we have

$$\sum_{i+j=n}a_i\alpha^i(b_j)=0. \tag{6}$$

Multiplying a_0 to Eq. (6) from the right-hand side, by Lemma 4(iii) we obtain $a_0b_na_0=0$. Since R is reduced, $a_0b_n=0$. Now Eq. (6) becomes

$$\sum_{\substack{i+j=n\\ 1\le i\le n}}a_i\alpha^i(b_j)=0. \tag{7}$$

Multiplying a_1 to Eq. (7) from the right-hand side, we obtain $a_1\alpha(b_{n-1})a_1=0$ and so $a_1b_{n-1}=0$. Continuing this process, we can prove $a_ib_j=0$ for all i,j with $i+j=n$. Therefore $a_ib_j=0$ for all i and j.

The converse follows from Lemma 4. □

Corollary 18. *A ring R is α-rigid if and only if $R[[x;\alpha]]$ is a reduced ring and α is a monomorphism.*

Proof. Suppose that R is α-rigid. Assume to the contrary that $R[[x;\alpha]]$ is not reduced. Then there exists $0\neq f\in R[[x;\alpha]]$ such that $f^2=0$. Since R is reduced, $f\notin R$. Thus we put $f=\sum_{i=s}^{\infty}a_ix^i$ with $a_i\in R$ for all i and $a_s\neq 0$. Then $f^2=0$ implies $a_s^2=0$ by Proposition 17 and Lemma 4(iii). Thus $a_s=0$, which is a contradiction. Therefore $R[[x;\alpha]]$ is reduced.

Conversely, let $R[[x;\alpha]]$ be reduced. Clearly $R[x;\alpha]$ is reduced as a subring. If $a\alpha(a)=0$ for $a\in R$, then $(ax)^2=a\alpha(a)x^2=0$. Thus $ax=0$ and so $a=0$. Therefore R is α-rigid. □

Corollary 19. *Let R be an α-rigid ring. If $e^2=e\in R[[x;\alpha]]$, where $e=e_0+e_1x+\cdots+e_nx^n+\cdots$, then $e=e_0$.*

Proof. Since $1-e=(1-e_0)-\sum_{i=1}^{\infty}e_ix^i$, we get $e_0(1-e_0)=0$ and $e_i^2=0$ for all $i\geq 1$ by Proposition 17 and Corollary 18. Thus $e_i=0$ for all $i\geq 1$ and so $e=e_0=e_0^2\in R$. □

The following example shows that there exists a Baer ring R but the formal power series ring $R[[x]]$ is not Baer.

Example 20. Let $R=\mathrm{Mat}_2(\mathbb{Z})$. Then R is a Baer ring. Note that $R[[x]]\cong\mathrm{Mat}_2(\mathbb{Z}[[x]])$ and $\mathbb{Z}[[x]]$ is a commutative domain. Let $S=\mathrm{Mat}_2(\mathbb{Z}[[x]])$. If

$$\begin{bmatrix}2 & x\\ 0 & 0\end{bmatrix}\begin{bmatrix}f & g\\ h & k\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},$$

where $f,g,h,k\in\mathbb{Z}[[x]]$, then $2f+xh=0$ and $2g+xk=0$. So

$$r_S\left(\begin{bmatrix}2 & x\\ 0 & 0\end{bmatrix}\right)=\left\{\begin{bmatrix}-xu & -xv\\ 2u & 2v\end{bmatrix}\,|u,v\in\mathbb{Z}[[x]]\right\}.$$

Now if

$$\begin{bmatrix}-xu & -xv\\ 2u & 2v\end{bmatrix}\begin{bmatrix}-xu & -xv\\ 2u & 2v\end{bmatrix}=\begin{bmatrix}-xu & -xv\\ 2u & 2v\end{bmatrix},$$

then $x(xu-2v+1)u=0$ and $x(xu-2v+1)v=0$. But $x(xu-2v+1)\neq 0$ and so $u=0$ and $v=0$. Hence

$$\begin{bmatrix}-xu & -xv\\ 2u & 2v\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

Since

$$r_S\left(\begin{bmatrix} 2 & x \\ 0 & 0 \end{bmatrix}\right) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

we see that

$$r_S\left(\begin{bmatrix} 2 & x \\ 0 & 0 \end{bmatrix}\right)$$

cannot be generated by an idempotent. Thus S is not Baer and therefore $R[[x]]$ is not Baer.

Moreover, this example shows that the condition "R is α-rigid" in the following Theorem 21 and Corollary 22 is not superfluous.

Theorem 21. *Let R be an α-rigid ring. Then R is a Baer ring if and only if $R[[x;\alpha]]$ is a Baer ring.*

Proof. It is proved by the similar method in the proof of Theorem 11. □

Corollary 22. *Let R be an α-rigid ring. Then R is a quasi-Baer ring if and only if $R[[x;\alpha]]$ is a quasi-Baer ring.*

From [6, Example 2.5], we can see that there is a reduced right p.q.-Baer ring R such that $R[[x;\alpha]]$ is not a right p.q.-Baer ring. For a given field F, let

$$R = \left\{ (a_n) \in \prod_{n=1}^{\infty} F_n | a_n \text{ is eventually constant} \right\},$$

which is the subring of $\prod_{n=1}^{\infty} F_n$, where $F_n = F$ for $n = 1, 2, \ldots$. Then R is a commutative von Neumann regular ring and hence it is right p.q.-Bear. Let α be the identity map on R. Then R is an α-rigid ring since R is reduced. But $R[[x;\alpha]]$ is not right p.q.-Bear. Furthermore, $R[[x;\alpha]]$ is neither right p.p. nor left p.p. by Corollary 18 and Lemma 1.

Corollary 23. *Let R be a reduced ring. Then R is a Baer ring if and only if $R[[x]]$ is a Baer ring.*

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