ON $\phi^0$ BEAM ELEMENTS WITH SHEAR AND THEIR CORRESPONDING PENALTY FUNCTION FORMULATION

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Abstract—On a model of clamped beam problem we study a finite element method with approximate constraints and the corresponding penalty function approach. By a small perturbation analysis we show that the rotation is independent from the penalization. This method allows us to use simple $\phi^0$ elements, hence we can compute approximate solutions to obstacle problems. The one dimensional nature of all the results is underlined and the extension of this FEM to flexible pipe lines is indicated.

1. INTRODUCTION

We consider a class of finite element discretization of the clamped beam problem:

$$\frac{d^4w}{dx^4} = f(x), \quad x \in [0, 1]$$

$$w(0) = w(1) = 0$$

$$\frac{dw}{dx} \bigg|_{0} = \frac{dw}{dx} \bigg|_{1} = 0.$$  (1)

There were two motivations for the study of this model one dimensional problem. First (and originally) we wanted to devise simple $\phi^0$ plate bending elements (two dimensional biharmonic problem) and the penalty function approach based on a Mindlin type plate model seemed to be a good way to do so. It was justified to look at the simpler one dimensional case. In the second place we wanted a simple finite element method for the study of (nearly) inextensible pipe lines, including unilateral (obstacle) problems.

Our present study relies on a method we exposed in a preceding paper [3]. It was motivated here by the corresponding pioneer work of Fried [12]. This problem has also been studied in the general setting of the “penalty with reduced integration” method by Malkus and Hughes [15]. Related theoretical results and convergence examples are also given by Arnold [1]. Like him we use the “penalty on consistent constraint” approach rather than the reduced integration method. We introduce here a small perturbation analysis which gives us new results and more insight on the influence of the penalty parameter.

Our approach, following [3], is based on the use of an abstract theorem due to Brezzi [8] (see also Babuska [2]). The corresponding “consistent constraint”, mixed finite element discretization as well as the perturbation analysis underline the fact that all results obtained rely heavily on the one dimensional character of the problem! (Hence extension of this method to plate elements should be made very cautiously!) The simplest element considered here ($P_1$ elements with constant constraint) can be used to study unilateral (contact) variational problems. We illustrate that by two numerical examples. From there it is natural to define a simple finite element discretization to study (nearly) inextensible pipe lines with obstacles. This class of problems were studied for instance by Bourgat, Dumay and Glowinski [7] using Hermite type elements which are more delicate to use on general obstacles.
2. MIXED VARIATIONAL FORMULATION

Since all quantities have been undimensioned for the sake of simplicity, we shall consider the following Sobolev spaces:

\[ H^0_0([0,1]); H^1_0([0,1]); H^{-1}([0,1]) \ldots \]

Since there is no confusion possible we shall write \( H^m_0 \) instead of \( H^m_0([0,1]) \), \( L^2 \) instead of \( L^2([0,1]) \) and so on . . . On \( L^2 \) we shall use the standard scalar product notation:

\[ (u, v) = \int_0^1 u(x)v(x) \, dx, \quad u, v \in L^2. \]

The variational problem corresponding to (1) is:

find \( w \in V = H^1_0 \), such that for \( f \in H^{-2} \):

\[ \left( \frac{d^2 w}{dx^2}, \frac{d^2 v}{dx^2} \right) = (f, v)_{H^{-2}, H^1_0} \quad \text{for all } v \in V. \]

We can view \( f \) as a vertical load acting on an horizontal beam of unit length. The beam vertical displacement is \( w(x) \), solution of (2). Discretization of (2) requires \( c^1 \) elements. This is simple here, but as we explained in the introduction we would rather use \( c^0 \) elements. For that purpose we first introduce a mixed variational formulation:

find \( (w, \theta) \in V = H^1_0 \times H^1_0 \) and \( \lambda \in W = L^2/R \) such that

\[ \left( \frac{d^2 w}{dx^2}, \frac{d^2 v}{dx^2} \right) - \left( \lambda, \frac{d v}{dx} - \varphi \right) = (f, v)_{H^{-1}, H^1_0} \]

\[ \left( \mu, \frac{d w}{dx} - \theta \right) = 0 \]

for all \( (v, \varphi) \in V \) and \( \mu \in W \).

Remark 1

\( f \) is in \( H^{-1} \) and not in \( H^{-2} \) as for (2). Introducing a mixed variational formulation requires some additional regularity. But so does \( c^1 \) interpolation!

Remark 2

(3b) expressed that \( d w/dx = \theta \) (a.e.) thus \( \theta \) is the rotation of the beam. \( \lambda \) is the shear stress as we shall see.

THEOREM 1

For \( f \in H^{-1} \), (3a), (3b) has a unique solution \( [(w, \theta), \lambda] \), more over \( w \) is the solution of (2) and \( \lambda = d^3 w/dx^3 \).

Proof. Uniqueness is evident. Let \( w \) be the solution (2) since \( f \in H^{-1} \), it is clear that \( d^3 w/dx^3 \in L^2 \). Hence \( [(w, d^2 w/dx), d^3 w/dx^3] \) is the solution of (3a). (3b)—this direct proof does not extend to mixed finite element discretization nor to the penalty function approach. Thus we are going to give a second proof longer and based on the result of Brezzi[8], this will provide us with the background we need for our extensions.

First we have to show that the Brezzi–Babuska hypothesis holds, that is:

There exists \( k > 0 \), independent of \( \lambda \) such that:

\[ \sup_{(w, \theta) \in V} \left\| \left( \lambda, \frac{d w}{dx} - \theta \right) \right\| \leq k \| \lambda \|_{L^2/R}. \]

Since \( \lambda \in L^2/R \) we take as a representative the element \( \lambda \) such that \( \int_0^1 \lambda \, dx = 0 \). We have

\[ \| \lambda \|_{L^2/R} = \| \lambda \|_{L^2}. \]
There is a solution \( w \in H_0^1 \) to
\[
\frac{d w}{d w} = \lambda, \tag{5}
\]
and we have
\[
\|w\|_{H_0^1} = \|\lambda\|_{L^2}.
\]

Thus we see that (4) holds by taking \((w, 0) \in V\).

Next we must show that our bilinear form in (3a) is definite positive that is:
\[
\frac{d \theta}{d x}^2 + \left( \frac{d w}{d x} - \theta \right)^2 \geq c \left( \frac{d \theta}{d x}^2 + \frac{d w}{d x}^2 \right). \tag{6}
\]

We have:
\[
\frac{d \theta}{d x}^2 + \left( \frac{d w}{d x} - \theta \right)^2 = \frac{d \theta}{d x}^2 + \frac{d w}{d x}^2 + \|\theta\|_{L^2}^2 - 2 \left( \frac{d w}{d x}, \theta \right)
\]
\[
\geq \frac{1}{2} \left( \frac{d \theta}{d x}^2 + \left( \frac{2}{2} + 1 \right) \|\theta\|_{L^2}^2 + \frac{d w}{d x}^2 - \frac{1}{\alpha} \frac{d w}{d x}^2 - \|\theta\|_{L^2}^2 \right),
\]
where we used the fact that \( \|\theta\|_{L^2}^2 \leq \|d\theta/dx\|_{L^2}^2 \). Now choose \( \alpha \) such that \( 5/4 \leq \alpha \leq 3/2 \), then:
\[
\frac{d \theta}{d x}^2 + \left( \frac{d w}{d x} - \theta \right)^2 \geq \frac{1}{2} \frac{d \theta}{d x}^2 + \frac{1}{5} \frac{d w}{d x}^2.
\]

Following Brezzi[8] and using (6), (4) and Remark 2 we get our theorem.

**Remark 3**

Under Remark 2 and provided (6) holds (4) is a necessary and sufficient condition for the existence and uniticity of \([w, \theta, \lambda]\). It is clear that to prove (4) here we used the fact that we had a one dimensional problem. Another way to state (4) is to say that any function \( \lambda \in L^2 \), such that \( \int \lambda \ dx = 0 \) is the gradient of a function in \( H_0^1 \), obviously a statement that does not carry to 2 or 3 dimensions!

### 3. SMALL PERTURBATION AND PENALTY FUNCTION

Following [3] we introduce a small perturbation in (3a), (3b) and look for the solution of the problem:

\[
\text{find } (w_\epsilon, \theta_\epsilon) \in V \text{ and } \lambda_\epsilon \in W \text{ such that }\]
\[
\left( \frac{d \theta_\epsilon}{d x}, \frac{d w_\epsilon}{d x} \right) - \left( \lambda_\epsilon, \frac{d w_\epsilon}{d x} - \varphi \right) = (f, v)_{H^{-1}, H_0^1}, \tag{7a}
\]
\[
\epsilon (\lambda_\epsilon, \mu) + \left( \mu, \frac{d w_\epsilon}{d x} - \theta_\epsilon \right) = 0 \tag{7b}
\]

for all \((v, \varphi) \in V\) and all \( \mu \in W \).

Since (7b) is equivalent to
\[
\lambda_\epsilon = - \frac{1}{\epsilon} \left( \frac{d w_\epsilon}{d x} - \theta_\epsilon \right), \tag{8}
\]
We can replace (7a), (7b) by the following penalty function formulation:

\[
\begin{aligned}
\text{find } (w, \theta) \in V \text{ such that } \\
\left( \frac{d\theta}{dx}, \frac{d\varphi}{dx} \right) + \frac{1}{\epsilon} \left( \frac{d^2w}{dx^2} - \frac{d\varphi}{dx} \right) = \langle f, \psi \rangle_{H^{-1}, H_0},
\end{aligned}
\]

for all \((v, \varphi) \in V\).

We see that as \(\epsilon \to 0\) we have a penalization of the constraint \(dw/dx = \theta\). (9) can be viewed as a thick beam model where \(1/\sqrt{\epsilon}\) is the beam thickness [1, 12].

It is clear that a solution of (8), (9) is a solution of (7a), (7b) and conversely. Thus the mixed variational formulation shows us that the direct penalization of the constraint is actually a small regular perturbation. This can be seen by deriving (formally) the system of differential equations corresponding to (9). We get

\[
\begin{aligned}
&\frac{d^2\theta}{dx^2} - \frac{1}{\epsilon} \left( \frac{d^2w}{dx^2} - \frac{d\varphi}{dx} \right) = 0 \\
&\frac{1}{\epsilon} \left( \frac{d^2w}{dx^2} - \frac{d\varphi}{dx} \right) = f
\end{aligned}
\]

and of course the Dirichlet boundary conditions on \(w\) and \(\varphi\). Deriving once more (10a) and inserting (10b). We get

\[
\frac{d^2\varphi}{dx^2} = f.
\]

This last equality suggests that \(\theta\) is actually independent of \(\epsilon\) and it must be \(\theta = dw/dx\) as given by (3a), (3b). The vertical displacement \(w\) is of the form \(w + \epsilon \theta\) where \(w\) is the solution of (2). More exactly we have the

**Theorem 2**

(9) (or resp. (7a), (7b)) has unique solution in \(V\) for any \(f \in H^{-1}\). Moreover the rotation component \(\theta\) is independent of \(\epsilon\) and the vertical displacement \(w\) is given by

\[
w = w + \epsilon \theta,
\]

where \(w\) is the solution of (2) (or (3a), (3b)) and \((w)\) of the membrane problem

\[
w_1 \in H_0^1; \left( \frac{d^2w}{dx^2}, \frac{d\varphi}{dx} \right) = \langle f, \psi \rangle_{H^{-1}, H_0}, \text{ for all } \psi \in H_0^1.
\]

**Proof.** Existence and uniqueness can be derived from (6). By our own regularization results [3], since (4) holds we know that \(\|w - w_1\|_{H_0^1}^2 + \|	heta - \theta_1\|_{H_0^1}^2 \leq c\epsilon\) where \(c\) is a constant independent of \(\epsilon\), \([w, \theta, \lambda]\) being the solution of (3a), (3b). Let us insert in (7a), (7b):

\[
\theta = \theta + \epsilon \theta; \quad w = w + \epsilon \theta_1; \quad \lambda = \lambda + \epsilon \lambda_1.
\]

We get:

\[
\left( \frac{d\theta_1}{dx}, \frac{d\varphi}{dx} \right) - \left( \frac{d\theta}{dx}, \frac{d\varphi}{dx} \right) = 0
\]

and

\[
\epsilon \lambda_1 \mu + \epsilon^2 (\lambda_1, \mu) + \epsilon \left( \mu, \frac{d\varphi}{dx} - \theta_1 \right) = 0.
\]
for all \( v, \varphi \in V \) and all \( \mu \in W \). From the first equation we get \( \lambda_1 = \text{constant, i.e. here } \lambda_1 = 0 \), thus \( \theta_1 = 0 \). From the second equation we get \( d\theta_1/dx = -\lambda_1 \), that is \(-d^2w_1/dx^2 = f\).

This regular perturbation result can hardly be expected in plate problems, except in very special situations. It so happens that most published examples are in those situations, but this will be the subject of another study.

Let us suppose now that we discretize (9) by taking \( V_h = V_h \times V_h \) where \( V_h \) is a \( C^0 \) subspace of \( H_0^1 \) such that

\[
\inf_{v_h \in V_h} \| u - v_h \|_{H_0^1} \leq ch^s.
\]

Arnold [1] has shown that:

\[
\| w_e - w_{e,h} \|_{H_0^1} \leq c \left( \min \left( h^s, \frac{h^{s+1}}{\sqrt{e}} \right) \right);
\]

\[
\| \theta_e - \theta_{e,h} \|_{H_0^1} \leq \begin{cases} c \max( h, \min( h^2/e, 1) ) & \text{if } s = 1 \\ c h & \text{if } s > 1 \end{cases}
\]

and

\[
\| w_e - w_{e,h} \|_{L^2} \leq \begin{cases} c \min( h^2/e, 1) & \text{if } s = 1 \\ c \min( h^{s+1}/\sqrt{e}, h^s) & \text{if } s > 1 \end{cases}
\]

(13a)

(13b)

\[
\| \theta_e - \theta_{e,h} \|_{L^2} \leq \begin{cases} c \min( h^2/e, 1) & \text{if } s = 1 \\ c \min( h^{s+1}/\sqrt{e}, h^s) & \text{if } s > 1 \end{cases}
\]

(13c)

(13d)

Now according to theorem 2, \( \theta = \theta_e \) so that the rotation is better approximated by any large \( e \) as for the displacement by Theorem 2 we have for instance (for \( s = 1 \)):

\[
\| w - w_{e,h} \|_{L^2} = c_1 e + c_2 \min \left( \frac{h^2}{e}, 1 \right)
\]

so that there is an optimal parameter \( e = ch \). We can also compute two results for \( e \) and \( e/2 \) and perform a Richardson extrapolation step taking

\[
w_{e,h} = 2w_{e,2h} - w_{e,h},
\]

the extrapolated result is now independent of \( c_1 e \) and the optimal result in \( w_{e,h} \) will be obtained for large \( e \).

Let us give a simple numerical illustration for \( s = 1 \) of these results. We take \( f \) such that \( w(x) = \cos 2\pi x - 1 \), we divide \([0, 1]\) into 30 equal elements and take the standard linear shape functions for \( w_{e,h} \) and \( \theta_{e,h} \). Figure 1 gives us the respective \( L^2 \) errors for \( w_{e,h}, \theta_{e,h} \) and \( w_{e,h} \). We see that the \( L^2 \) error for \( \theta_{e,h} \) is a linear function of \( 1/e \) as given in (13c) and so is the asymptotic behaviour of \( w_{e,h} \). As for \( w_{e,h} \) there is clearly an optimal \( e \).

4. CONSISTENT APPROXIMATE CONSTRAINTS

We now want to implement a mixed finite element method such that (4) will be satisfied and such that (8) will hold in some sense at element level [3]. We limit ourselves to the case where \( w \) and \( \theta \) are both approximated by means of \( C^0 \) shape functions of polynomial degree \( s \geq 1 \). By analogy with (5) we introduce the discontinuous shape functions of polynomial degree \( s - 1 \) per element for the Lagrange multipliers \( \lambda_h \in W_h \).

By adding the additional condition

\[
\lambda_h \in W_h \Rightarrow \int_0^1 \lambda_h \, dx = 0.
\]
We see that $W_h \subset W$, $V_h = V_h \times V_h \subset V$ and that (4) is satisfied independently of $h$ on $V_h \times V_h$ and $W_h$. Following [3] we note that (3b) is now equivalent to the introduction on an approximate constraint operator, $\{(dw/dx) - \theta\}_h$ defined element by element by:

$$\left( \frac{dw}{dx} - \theta \right)_h \in P_{s-1}(K)$$

and

$$\int_K \mu_h \left( \frac{dw}{dx} - \theta \right)_h \, dx = \int_K \mu_h \left( \frac{dw}{dx} - \theta \right)_h \, dx, \quad \forall (w, \theta) \in V$$

Following [3] we note that (3b) is now equivalent to the introduction on an approximate constraint operator, $\left( (dw/dx) - \theta \right)_h$ defined element by element by:

$$\int_K \mu_h \left( \frac{dw}{dx} - \theta \right)_h \, dx = \int_K \mu_h \left( \frac{dw}{dx} - \theta \right)_h \, dx, \quad \forall (w, \theta) \in V$$

From (15) we get a new penalty function formulation:

$$\left( \frac{d\theta}{dx} \cdot \frac{d\varphi}{dx} + \frac{1}{\varepsilon} \left( \left( \frac{dw}{dx} - \theta \right)_h, \left( \frac{dv}{dx} - \varphi \right)_h \right) \right) = \langle f, v \rangle_{H^{-1}, H^1}.$$

**Remark 4**

Using a standard trick [4] we can define $\mu_h \in P_{s-1}(K)$ by using the shape functions based on the Gauss Legendre quadrature nodes of $s$ point rules. (This gives us a numerical quadrature rule exact for polynomials up to degree $2s - 1$ on each element.) By plugging this into (15) and

![Fig. 1.](image-url)
using the order of this quadrature rule we get that for \((w, \theta)\) and \((v, \varphi)\) \(\in V_h^e\):

\[
\int_K \left( \frac{d^2 w}{dx^2} - \theta \right)(\frac{d^2 v}{dx^2} - \varphi) \, dx = \sum_{i=1}^s \left( \frac{d^2 w}{dx^2} - \theta \right)_{|\alpha_i} \times \left( \frac{d^2 v}{dx^2} - \varphi \right)_{|\alpha_i} \times \omega_i
\]

where \(\alpha_i\) are the nodes and \(\omega_i\) the corresponding weights of the Gaussian rule. If we wanted to integrate \(\int_K \left( \frac{d w}{dx} - \theta \right)(\frac{d v}{dx} - \varphi) \, dx\) "exactly" we would have to use a \(s+1\) points rule in order to achieve exact integration for polynomials of order \(2s\). Hence (16) together with (17) is called the penalty with reduced integration method. We must underline however that the approximate constraint and the reduced integration method are not equivalent in 2 or 3D problems in general (for counter examples see Ref. [6]). This implies that plate elements using penalty with reduced integration should be used with great care!

**Theorem 3**

Let \(V_h \times V_h(= V_h^e)\) be defined by \(C^0\) polynomial shape functions of degree \(s\) per element. Let \(W_h\) be defined by local polynomial shape functions of degree \((s-1)\) and condition (14), let \([(d w/ dx) - \theta|_{h}]\) be the operator introduced in (15), then there is a unique solution \((w_{eh}, \theta_{eh})\) to the variational problem (16) on \(V_h \times V_h^e\). Moreover:

\[
\begin{align*}
\|w - w_{eh}\|_{H_0^1} & \leq c_1 \varepsilon + c_2 h^3 \\
\|\theta - \theta_{eh}\|_{H_0^2} & \leq c_3 h^4
\end{align*}
\]

where \((w, \theta)\) is the solution given by Theorem 1.

**Proof.** From the discrete analog of (4), and from Theorem 2 the result is a direct consequence of Theorem 5.1 in [3].

Equation (18) shows us that we have a kind of superconvergence for the rotation \((\theta)\). Actually as can be seen in [3] \(\|w - w_{eh}\|_{H_0^1}\) is bounded by \(c_3 h^4\) independently from \(\varepsilon\). Thus by Theorem 2 a one step Richardson extrapolation should give us the optimal solution \(\hat{w}_{eh} = 2w_{eh} - w_{2h}\).

**Numerical example**

As before we take \(P_1\) shape functions for \(w, \theta\) and we take \(\lambda\) constant per element. This is equivalent to the minimization of the following functional:

\[
\left\| \frac{d \theta}{dx} \right\|_{L^2}^2 + \frac{1}{\varepsilon} \left( \int_1^0 \frac{d \theta}{dx} - w \, dx \right)^2 - 2(f, w).
\]

The error estimate is given by (18) with \(s = 1\).

Taking the same analytical example as before and \(h = 1/30\) we get the results in Table 1. We see that the error is going down by half for each division of \(\varepsilon\) by 2 for large \(\varepsilon\).

<table>
<thead>
<tr>
<th>(c = 0.1^{(1/2)n})</th>
<th>n=0</th>
<th>n=1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L_2) error in (v_{eh})</td>
<td>3.959</td>
<td>1.977</td>
<td>.9869</td>
<td>.4916</td>
<td>.2440</td>
<td>.1202</td>
<td>.05824</td>
<td>.02729</td>
<td>.011381</td>
<td>.00407</td>
</tr>
<tr>
<td>(L_2) error on (\theta_{eh})</td>
<td>8.0043 (\times 10^{-6})</td>
<td>1d</td>
<td>1d</td>
<td>1d</td>
<td>1d</td>
<td>1d</td>
<td>1d</td>
<td>1d</td>
<td>1d</td>
<td>1d</td>
</tr>
</tbody>
</table>
For smaller $\epsilon$ the dominating term is clearly the error in $h$. A one step Richardson extrapolation gives us an error $\|w - w_{eh}\|_{L^2}$ of $3.666 \times 10^{-3}$ for all $\epsilon$ as expected. Moreover it is clear that $\theta_{eh}$ is actually independent of $\epsilon$ (and there is even a kind of superconvergence that needs further investigation!). This $P_1$ approximation for 4th order problem allows us to discretize unilateral variational inequalities since when a function $w$ is in a convex $C$ defined by an obstacle then its $P_1$ interpolate will be in the corresponding discretized convex $C_h$ provided that $C_h \subset C$.

5. OBSTACLE PROBLEMS FOR THE BEAM

4th order variational inequalities are studied in [11]. Here we shall consider only a simple case directly on the mixed variational formulation. We shall consider a convex $C \subset H^1_0$ such that

\[ \sup_{w \in C} \left( \frac{d w}{d x}, \lambda \right) \leq k \| \lambda \| \]  

and $0 \in C$.

Let us give an example of such a convex, let $\psi(s)$, $s \in [\alpha, \beta] \subset 0, 1]$, be a function such that $\inf \psi(s) > \gamma > 0$. Then:

\[ C = \{ w \in H^1_0 | w(s) \leq \psi(s), s \in [\alpha, \beta] \} \]

satisfies (19) and $0 \in C$. This is the kind of convex one encounters in the study of flexible pipe lines at sea.

We are now looking for a solution $[(w, \theta, \lambda)]$ in $C \times V \times W$ of

\begin{align*}
\left( \frac{d \theta}{d x}, \frac{d \varphi}{d x} \right) - \left( \lambda, \frac{d v}{d x} - \frac{d w}{d x} - \varphi \right) &\geq (f, v - w) \\
\left( \mu, \frac{d w}{d x} - \theta \right) &\geq 0
\end{align*}

(20a) (20b)

for all $v \in C$, $\varphi \in V$ and $\mu \in W$.

THEOREM 4

Under hypothesis (19) there exists a unique solution to (20a), (20b).

The proof of this theorem follows closely another result on variational inequalities[5] and we shall not repeat it here. (Note that we have unicity for the shear stress $\lambda$). It is based on the fact that there is also a unique solution $(w_e, \theta_e)$ to the penalty formulation

\[ \left( \frac{d \theta_e}{d x}, \frac{d \varphi}{d x} \right) + \frac{1}{\epsilon} \left( \frac{d w_e}{d x} - \theta_e \frac{d (v - w_e)}{d x} - \varphi \right) \geq (f, w_e - v) \]

and we have of course:

\[ \|w - w_e\|_{H^1_0} + \|\theta - \theta_e\|_{H^1_0} + \left\| \left( \lambda - \frac{1}{\epsilon} \frac{d w_e}{d x} - \theta_e \right) \right\|_{L^2} \leq c \sqrt{\epsilon}. \]  

(21) (22)

Numerical examples

Computations were done by using the $P_1 - P_0$ elements of the preceding paragraph using a duality (Uzawa type) method. In the two examples we do not know if condition (19) is satisfied. First we defined a step like obstacle $\psi(s) = 0, 0.375 \lesssim s \lesssim 0.625$. Then using a uniform load $f$ we get a one point contact only and convergence is fast (7 iterations). Figure 2 gives the results, they can be compared to the analytical solution of Hobbs[13].
For the second example we used a circle tangent to the beam axis of $x = 0.5$ and uniform load of $f = 200$. Convergence was quite slow (1200 iterations!) and the beam is still slightly below the circle (i.e. outside the convex) in Fig. 3 (by an amount of order $10^{-3}$). Since then we have used penalty and duality methods that are much faster. Error estimates on $w - w_{ss}$ and $\theta - \theta_m$ can easily be obtained when $C_h \subseteq C$ and for simple convexes satisfying (19) and a discrete analog of it.

**Pipe line model**

The interest of our beam finite element model lies in the possibility to define a simple finite element for nearly inextensible pipe lines [7]. We now have to "minimize" the following energy:

$$
\frac{EI}{2} \int_0^L \left( \left( \frac{dx}{ds} \right)^2 + \left( \frac{dy}{ds} \right)^2 \right) ds + \frac{N}{2} \int_0^L \left( \left( \frac{dx}{ds} \right)^2 + \left( \frac{dy}{ds} \right)^2 - 1 \right) ds + \int_0^L f_x ds + \int_0^L f_y ds \tag{23}
$$

the terms corresponding respectively to a bending energy ($EI$: flexural stiffness of the pipe) an extension energy ($N$: modules of extension) and load work. This can be done over a "convex" defining an obstacle (bottom of the sea for instance). When $N$ is large we have a nearly inextensible pipe line. This can be used to approximate the inextensibility constraint,

$$
\left( \frac{dx}{ds} \right)^2 + \left( \frac{dy}{ds} \right)^2 = 1,
$$

which is introduced in Elastica type models. Discretization of (23) is done by means of linear element for $x(s), y(s)$ and linear rotations $\theta_1(s), \theta_2(s)$, with penalization as in Section 4. A detailed study of the finite element discretization of (23) will be given elsewhere, together with 3D computations.

**CONCLUSION**

The model problem we studied cannot give much insight for biharmonic or plate approximations. One should rather use the work of Destuynder [10], see also Ciarlet-Destuynder [9] as a basis for thick plate elements. Error estimates were also obtained by Johnson and Pitkaranta [14] in two dimensional problems with penalty type bending elements. Nevertheless, it provides an easy and elegant discretization method for complex unilateral Elastica type problems.

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