Discrete Mathematics 125 (1994) 11–14
North-Holland

On the mean chromatic number

Martin Anthony

Department of Statistical and Mathematical Sciences, The London School of Economics, London WC2A 2AE, UK

Received 12 July 1991
Revised 19 July 1991

Abstract

The mean chromatic number of a graph is a measure of the expected performance of the greedy vertex-colouring algorithm when each ordering of the vertices is equally likely. Some results on the value of the mean chromatic number and its asymptotic behaviour are presented.

1. The mean chromatic number

Taking the colours to be the positive integers, the greedy vertex-colouring algorithm can be described as follows. The vertices of a graph G are ordered and the algorithm assigns colours to the vertices in that order, giving each vertex the first available colour (i.e., the first colour which has not already been assigned to a vertex adjacent to it).

There are some results which show that the greedy algorithm is reasonably good in the worst case for certain families of graphs, such as the interval graphs [7] and the complements of chordal graphs [5]. However, there are many graphs for which the algorithm is very bad in the worst case [2, 6]. We discuss here the expected performance of the greedy algorithm on particular graphs.

We may regard an ordering σ of the vertex set of a graph $G=(V,E)$ as a permutation of V. For any ordering σ of the vertex set V, denote by $\chi(G,\sigma)$ the number of colours used by the greedy algorithm to colour G when the vertices are presented in the order described by σ. It is easy to see, and well known, that if G has maximum degree $\Delta(G)$ then for any σ,

$$\chi(G)\leqslant\chi(G,\sigma)\leqslant\Delta(G)+1,$$

where $\chi(G)$ is the chromatic number of G. Further, there is an ordering σ^* of the vertices for which $\chi(G,\sigma^*)=\chi(G)$, so the lower bound is always attained.

Correspondence to: Martin Anthony, Department of Statistical and Mathematical Sciences, The London School of Economics, London WC2A 2AE, UK.

SSDI 0012-365X(92)00176-T

In order to quantify the expected behaviour of the greedy algorithm, we define the *mean chromatic number* of G to be

$$\bar{\chi}(G)=\frac{1}{n!}\sum \chi(G,\sigma),$$

where the sum is over all orderings σ of the vertex set. It is easily verified [2] that if $f_i(G)$ is the proportion of orderings σ for which $\chi(G,\sigma)\geqslant\chi(G)+i$ then

$$\bar{\chi}(G)=\chi(G)+f_1(G)+f_2(G)+\cdots$$

(This sum is finite since, for any σ, $\chi(G,\sigma)\leqslant\Delta(G)+1$.)

As an example, we first describe a result of Biggs [2]. The *cocktail-party graph* or *n-dimensional octahedron* $\mathrm{CP}(n)$ $(n\geqslant 3)$ is defined to be the complete bipartite graph $K_{n,n}$ with a one-factor removed. As σ runs through all orderings of the vertex set, $\chi(\mathrm{CP}(n),\sigma)$ takes on all values from 2 (the chromatic number) to n. Further, it is shown in [2] that $f_1(\mathrm{CP}(n))=1/n$ and, for $2\leqslant i\leqslant n-2$,

$$f_i(\mathrm{CP}(n))=\left(\frac{3}{2}\right)^{i-1}\frac{1}{(n-i+1)}\frac{1}{(2n-1)(2n-3)\cdots(2n-2i+3)},$$

for $n\geqslant 4$.

We remark that in any greedy colouring of the complete bipartite graph $K_{n,n}$ (and indeed, of any complete bipartite graph), exactly two colours are used. Hence, for any n, $\bar{\chi}(K_{n,n})=2$.

As a second example, we consider the even cycles. Let C_{2n} denote the even cycle graph on vertex set $\{1,2,\ldots,2n\}$ $(n\geqslant 2)$. For any n and any ordering σ of the vertex set of C_{2n}, we have $\chi(C_{2n},\sigma)=2$ or 3. Results from [3,4] yield the generating function for the mean chromatic numbers $\bar{\chi}(C_{2n})$. Specifically, these results show that $\sum_{n=0}^{\infty}(3-\bar{\chi}(C_{2n+4}))z^{2n+1}$ is the Taylor series (about the origin) for the function $f(z)=\sinh z/(\cosh z-z\sinh z)$. It follows that the generating function for the numbers $\bar{\chi}(C_{2n})$ $(n\geqslant 2)$ is

$$c(z)=\frac{3z^4}{1-z^2}+\frac{z^3\sinh z}{z\sinh z-\cosh z}.$$

For example $\bar{\chi}(C_{14})=88679/31185$, which is approximately 2.844.

2. Asymptotic behaviour

The above expressions for $f_i(\mathrm{CP}(n))$ give rise to an explicit formula for $\bar{\chi}(\mathrm{CP}(n))$. Using this, it is not difficult to prove the following result; we omit the details.

Result 2.1. *Let* CP(n) *denote the cocktail-party graph on* $2n$ *vertices. Then,*

$$2+\frac{1}{n}<\bar{\chi}(\mathrm{CP}(n))<2+\frac{3}{n}$$

for all $n\geqslant 3$.

It follows, in particular, that $\bar{\chi}(\mathrm{CP}(n))$ tends to the limit 2 as n tends to infinity. So the sequence $\{\mathrm{CP}(n)\}$ of cocktail-party graphs is a sequence of graphs for which the worst-case behaviour of the greedy algorithm is bad (since there is an ordering forcing the algorithm to use n colours to colour CP(n)), but for which the expected behaviour is very good; asymptotically the algorithm will give an optimal colouring.

Consider again the even cycles. Intuitively, it seems that $\bar{\chi}(C_{2n})$ should tend to 3 as n tends to infinity. This is indeed the case, and the generating functions of the previous section enable us to find an asymptotic expression for $\bar{\chi}(C_{2n})$.

Result 2.2. *Let* C_{2n} *denote the even cycle on* $2n$ *vertices. Then, as* $n\to\infty$,

$$3-\bar{\chi}(C_{2n})\sim\frac{2}{\alpha^{2n}},$$

where $\alpha>1$ *is the least positive solution of the equation* $\cosh x=x\sinh x$.

Proof. As this result has now appeared in [1], we shall omit some of the details. Let $\gamma_{2n}=3-\bar{\chi}(C_{2n})$, and consider the series $\sum_{n=2}^{\infty}\gamma_{2n}z^{2n}$. The results from [3, 4] mentioned in the previous section show that this series is the Taylor series (about the origin) of the function $h(z)=z^3\sin z/(\cosh z-z\sinh z)$. It is easy to show [1] that the radius of convergence of the series is α, the smallest positive solution of $\cosh x=x\sinh x$, and, further, that the only zeroes of $\cosh z-z\sinh z$ on the circle of convergence are $\pm\alpha$. Extending a technique from [8], let $R>\alpha$ be such that $h(z)$ has no poles other than $\pm\alpha$ in the closed disk $\bar{D}(0,R)$. Such an R exists by the above comments, and because the poles of h are discrete. Let $r<\alpha$ and let $C=C(0,r)$, $\bar{C}=C(0,R)$. By Cauchy's residue theorem [9], we have, for $n\geqslant 2$,

$$\gamma_{2n}=\frac{1}{2\pi\mathrm{i}}\int_C\frac{h(z)}{z^{2n+1}}\mathrm{d}z=\frac{1}{2\pi\mathrm{i}}\int_{\bar{C}}\frac{h(z)}{z^{2n+1}}\mathrm{d}z-\mathrm{res}(\alpha)-\mathrm{res}(-\alpha),$$

where

$$\mathrm{res}(\pm\alpha)=\mathrm{res}\left(\frac{h(z)}{z^{2n+1}},\pm\alpha\right)$$

are residues. Now, the poles at $\pm\alpha$ are simple, since the derivative of $\phi(z)=\cosh z-z\sinh z$ is nonzero at these points. Thus,

$$\mathrm{res}(\pm\alpha)=-\frac{(\pm\alpha)^3\sinh(\pm\alpha)}{(\pm\alpha)^{2n+1}(\pm\alpha)\cosh(\pm\alpha)}=-\frac{\sinh\alpha}{\alpha^{2n-1}\cosh\alpha}=-\frac{1}{\alpha^{2n}}.$$

So,

$$\left|\gamma_{2n}-\frac{2}{\alpha^{2n}}\right|=\left|\frac{1}{2\pi i}\int_{\bar{C}}\frac{h(z)}{z^{2n+1}}\,dz\right|\leqslant\frac{M}{R^{2n}},$$

where M is an upper bound for the absolute value of the continuous function h on the compact set $\bar{C}$. Hence,

$$\gamma_{2n}=\frac{2}{\alpha^{2n}}(1+k(n)),$$

where

$$|k(n)|\leqslant\frac{M}{2}\left(\frac{\alpha}{R}\right)^{2n}.$$

Now, $k(n)\to 0$ as $n\to\infty$, and so the required asymptotic expression for γ_{2n} follows. $\square$

In fact, α is about 1.19968 and, in particular, this result implies that $\bar{\chi}(C_{2n})$ tends 'quickly' to 3 as n tends to infinity.

References

[1] M. Anthony and N. Biggs, The mean chromatic number of paths and cycles, Discrete Math. 120 (1993) 227–231.
[2] N.L. Biggs, Some heuristics for graph colouring, in: R. Nelson and R.J. Wilson, eds., Graph Colourings (Longmans, New York, 1990) 87–96.
[3] T. Bouwer and Z. Star, A question of protocol, Amer. Math. Monthly 95 (1988) 118–121.
[4] I. Gessel, A colouring problem, Amer. Math. Monthly 98 (1991) 530–533.
[5] A. Gyarfas and J. Lehel, On-line and first-fit colourings of graphs, J. Graph Theory 12 (1988) 217–227.
[6] D.S. Johnson, Worst-case behaviour of graph colouring algorithms, Proc. 5th South-Eastern Conf. on Combinatorics, Graph Theory and Computation (Utilitas Mathematica Publ., Winnipeg, Canada, 1974) 513–528.
[7] H.A. Kierstead, The linearity of first-fit colouring of interval graphs, SIAM J. Discrete Math. 1 (1988) 526–530.
[8] J.K. Percus, Combinatorial Methods (Springer, New York, 1971).
[9] H.A. Priestley, Introduction to Complex Analysis (Oxford Univ. Press, Oxford, 1985).