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ScienceDirect
Applied Mathematics Letters 21 (2008) 1069–1073
Applied Mathematics Letters
www.elsevier.com/locate/aml

Solving the generalized Sylvester matrix equation $AV + BW = EVF$ via a Kronecker map

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Received 24 April 2006; received in revised form 14 November 2007; accepted 11 December 2007

Abstract

This note considers the solution to the generalized Sylvester matrix equation $AV + BW = EVF$ with F being an arbitrary matrix, where V and W are the matrices to be determined. With the help of the Kronecker map, some properties of the Sylvester sum are first proposed. By applying the Sylvester sum as tools, an explicit parametric solution to this matrix equation is established. The proposed solution is expressed by the Sylvester sum, and allows the matrix F to be undetermined.

Keywords: Generalized Sylvester matrix equation; Kronecker map; Sylvester sum; Parametric solutions

1. Introduction

When dealing with many problems for descriptor linear systems, such as, eigenstructure assignment [1,2], output regulation [3], observer design and fault detection [4], the following generalized Sylvester matrix equation is often encountered:

$$AV + BW = EVF, \tag{1}$$

where $A, E \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{p\times p}$ are known matrices, and $V \in \mathbb{R}^{n\times p}$ and $W \in \mathbb{R}^{r\times p}$ need to be determined. When $E = I$ in (1), it arises in solutions of related problems of eigenstructure assignment, eigenvalue assignment and observer design for conventional linear systems [5,6].

For the solution of (1), there exist several numerical solutions, such as the SVD-based block algorithm [7,8] and the large-scale algorithms [9,10]. It is well known that one can obtain only a special solution by applying a numerical method. When dealing with some problems related to optimization, for example, the robust pole assignment problem [11], it is better to request complete explicit solutions of the matrix equation (1). Efforts in this direction have been made. When F is in Jordan form, an analytical and restriction-free solution for (1) with $E = I$ is presented in [12].

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2007.12.004

Reference [13] proposes two solutions to the matrix equation (1) in an iterative form and an explicit parametric form, also for the case where the matrix F is in Jordan form. Besides, when the matrix F is an arbitrary matrix, a neat parametric solution is presented in [15] for the matrix equation (1) in terms of an R-controllability matrix, an observability matrix and a so-called generalized symmetric operator matrix. This solution has been used in [14] to design a kind of observer for descriptor linear systems.

In this note, the main objective is to give a closed-form solution for the linear matrix equation (1) with F being an arbitrary matrix with the aid of the Kronecker map. The concept of the so-called Kronecker map was first introduced in [15], based on the underlying idea of [16]. In [15], the Kronecker map is utilized to prove the completeness of the proposed solution. In contrast to the idea in [15], in this note the Kronecker map is first applied to obtain some good properties of the Sylvester sum. The solution of the linear matrix equation (1) is then established with the aid of the Sylvester sum.

Throughout this note, $\sigma(A)$ denotes the set of eigenvalues of matrix A. For a matrix $A \in \mathbb{C}^{m\times n}$, $\mathrm{vec}(A)$ is defined as

$$\mathrm{vec}(A) = \begin{bmatrix} a_1^{\mathrm{T}} & a_2^{\mathrm{T}} & \cdots & a_n^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

where a_i is the i-th column of the matrix A. The symbol "$\otimes$" denotes the Kronecker product of two matrices. For two matrices $A = [a_{ij}]_{m\times n}$ and B, the Kronecker product $A \otimes B$ is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdots & \cdots & & \cdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

For matrices M, X and N with appropriate dimensions, the following property of the Kronecker product is well known:

$$\mathrm{vec}(MXN) = (N^{\mathrm{T}} \otimes M)\mathrm{vec}(X). \tag{2}$$

2. Kronecker map

First, we introduce the concepts of the Sylvester sum and the Kronecker map which were first proposed in [15].

Definition 1. Let $T(s) = \sum_{i=0}^{t} T_i s^i \in \mathbb{R}^{m\times q}[s]$, $F \in \mathbb{R}^{p\times p}$ and $Z \in \mathbb{R}^{q\times p}$. The following matrix sum:

$$\mathrm{Syl}(T(s), F, Z) = \sum_{i=0}^{t} T_i Z F^i$$

is called the Sylvester sum associated with $T(s)$, F and Z.

Definition 2. Let $T(s) = \sum_{i=0}^{t} T_i s^i \in \mathbb{R}^{m\times q}[s]$, $F \in \mathbb{R}^{p\times p}$. The following map:

$$\mathcal{F}[T(s)] = \sum_{i=0}^{t} (F^{\mathrm{T}})^i \otimes T_i$$

is called the $\mathcal{F}$-Kronecker map.

On the basis of the definition of the Kronecker map, by applying the property (2) of the Kronecker product we have the following relation:

$$\mathrm{vec}\,(\mathrm{Syl}(T(s), F, Z)) = \mathcal{F}[T(s)]\mathrm{vec}(Z).$$

The following lemmas give some important properties of the Kronecker map, which can be found in [15].

Lemma 1. *Let $X(s) \in \mathbb{R}^{q\times r}[s]$, $Y(s) \in \mathbb{R}^{r\times m}[s]$ and F be a square real matrix. Then*

$$\mathcal{F}[X(s)Y(s)] = \mathcal{F}[X(s)]\mathcal{F}[Y(s)].$$

Lemma 2. *For any unimodular matrix* $U(s) \in \mathbb{R}^{q\times q}[s]$ *and any square real matrix* $F \in \mathbb{R}$, *it holds that*

$$(\mathcal{F}[U(s)])^{-1} = \mathcal{F}\left[U^{-1}(s)\right].$$

On the basis of the above properties of the Kronecker map, the following conclusion is obtained. This conclusion can be found in [15]. For completeness, we give the proof of this result.

Theorem 1. *Let* $D(s) \in \mathbb{R}^{(n+r)\times r}[s]$, $F \in \mathbb{R}^{p\times p}$. *Then* $\operatorname{rank} \mathcal{F}[D(s)] = rp$ *if and only if* $\operatorname{rank} D(s) = r$ *for any* $s \in \sigma(F)$.

Proof. Let $P(s) \in \mathbb{R}^{(n+r)\times(n+r)}[s]$ and $Q(s) \in \mathbb{R}^{r\times r}[s]$ be two unimodular matrices that transform $D(s)$ into the Smith normal form, say

$$D(s) = P(s)\begin{bmatrix}\Sigma(s)\\ 0\end{bmatrix}Q(s), \tag{3}$$

with $\Sigma(s) = \operatorname{diag}(d_1(s),\ d_2(s), \ldots,\ d_r(s))$. Applying the result of Lemmas 1 and 2, we have

$$\operatorname{rank} \mathcal{F}[D(s)] = \operatorname{rank} \mathcal{F}[\Sigma(s)] = \sum_{i=1}^{r} \operatorname{rank} d_i(F^{\mathrm{T}}).$$

It is obvious that $\operatorname{rank} \mathcal{F}[D(s)] = rp$ if and only if

$$\operatorname{rank} d_i(F^{\mathrm{T}}) = p, \quad i = 1,\ 2, \ldots, r.$$

These relations hold if and only if

$$\det \Sigma(s) \neq 0, \quad \forall s \in \sigma(F).$$

Combining this with (3) shows that the conclusion is true. ■

Like in the proof of the above theorem, the following conclusion is obvious.

Corollary 1. *Let* $T(s) \in \mathbb{R}^{n\times(n+r)}[s]$, $F \in \mathbb{R}^{p\times p}$. *Then* $\operatorname{rank} \mathcal{F}[T(s)] = np$ *if and only if* $\operatorname{rank} T(s) = n$ *for any* $s \in \sigma(F)$.

As the end of this section, we propose some good properties of the Sylvester sum with the aid of the Kronecker map.

Theorem 2. *Let* $X(s) \in \mathbb{R}^{q\times r}[s]$, $Y(s) \in \mathbb{R}^{r\times m}[s]$, $Z \in \mathbb{R}^{m\times p}$ *and* $F \in \mathbb{R}^{p\times p}$. *Then*

$$\operatorname{Syl}(X(s), F, \operatorname{Syl}(Y(s), F, Z)) = \operatorname{Syl}(X(s)Y(s), F, Z).$$

Proof. According to the definition of the Kronecker map, by applying Lemma 1 and properties of Kronecker products we have

$$\begin{aligned}\operatorname{vec}(\operatorname{Syl}(X(s), F, \operatorname{Syl}(Y(s), F, Z))) &= \mathcal{F}[X(s)]\operatorname{vec}(\operatorname{Syl}(Y(s), F, Z))\\ &= \mathcal{F}[X(s)]\,\mathcal{F}[Y(s)]\operatorname{vec}(Z)\\ &= \mathcal{F}[X(s)Y(s)]\operatorname{vec}(Z)\\ &= \operatorname{vec}(\operatorname{Syl}(X(s)Y(s), F, Z)).\end{aligned}$$

This implies that the conclusion is true. ■

It follows from Theorem 1 and Corollary 1 that we have immediately the following conclusions on the Sylvester sum.

Corollary 2. *Let* $D(s) \in \mathbb{R}^{(n+r)\times r}[s]$, $F \in \mathbb{R}^{p\times p}$. *Then the mapping* $Z \to \operatorname{Syl}(D(s), F, Z)$ *is injective if and only if* $\operatorname{rank} D(s) = r$ *for any* $s \in \sigma(F)$.

Corollary 3. *Let* $T(s) \in \mathbb{R}^{n\times(n+r)}[s]$, $F \in \mathbb{R}^{p\times p}$. *Then the mapping* $Z \to \operatorname{Syl}(D(s), F, Z)$ *is surjective if and only if* $\operatorname{rank} T(s) = n$ *for any* $s \in \sigma(F)$.

3. The main result

In this section, we discuss the solution to the matrix equation (1) with the help of the Kronecker map, the degrees of freedom of the matrix equation (1) are given in the next lemma.

Lemma 3 ([15]). *Let $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $F \in \mathbb{R}^{p\times p}$. Then the matrix equation (1) has rp degrees of freedom if and only if*

$$\operatorname{rank}\begin{bmatrix} A - sE & B \end{bmatrix} = n, \quad \forall s \in \sigma(F). \tag{4}$$

Considering the solution of the matrix equation (1) leads to the following theorem.

Theorem 3. *Let $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $F \in \mathbb{R}^{p\times p}$ satisfy*

$$\operatorname{rank}\begin{bmatrix} A - sE & B \end{bmatrix} = n, \quad \text{for any } s \in \sigma(F).$$

Further, let $N \in \mathbb{R}^{(n+r)\times r}[s]$ be a polynomial matrix satisfying

$$\begin{bmatrix} A - sE & B \end{bmatrix} N(s) = 0_{n\times r}.$$

Then:

(1) *The matrices $V \in \mathbb{R}^{n\times p}$ and $W \in \mathbb{R}^{r\times p}$ given by*

$$\begin{bmatrix} V \\ W \end{bmatrix} = \operatorname{Syl}(N(s), F, Z) \tag{5}$$

satisfy the matrix equation (1) for any matrix $Z \in \mathbb{R}^{r\times p}$.

(2) *When* rank $N(s) = r$ *for any $s \in \sigma(F)$, all the matrices V and W satisfying the matrix equation (1) can be explicitly expressed by (5).*

Proof. Let $T(s)$ and X be defined as

$$T(s) = \begin{bmatrix} A - sE & B \end{bmatrix}, \qquad X = \begin{bmatrix} V \\ W \end{bmatrix}.$$

Then the matrix equation (1) can be rewritten as $\operatorname{Syl}(T(s), F, X) = 0$. By applying Theorem 2, we have

$$\begin{aligned} \operatorname{Syl}(T(s), F, \operatorname{Syl}(N(s), F, Z)) &= \operatorname{Syl}(T(s)N(s), F, Z) \\ &= \operatorname{Syl}(0, F, Z) \\ &= 0. \end{aligned}$$

This implies that the matrices V and W given by (5) satisfy the matrix equation (1).

On the other hand, it follows from Corollary 2 that the mapping $Z \to \operatorname{Syl}(N(s), F, Z)$ is injective when rank $N(s) = r$ for any $s \in \sigma(F)$. Combining this fact and $Z \in \mathbb{R}^{r\times p}$ with Lemma 3 gives the conclusion in Item 2. ∎

According to the above theorem, we have the following two remarks on the solution of the matrix equation (1).

Remark 1. In this section, we provide a general complete parametric solution for the matrix equation (1). The presented solution is in an explicit form with respect to the matrix F. Therefore, this matrix F, together with the parameter matrix Z, can be further utilized to achieve some system performances in some applications. This will give some convenience and advantages for practical applications.

Remark 2. From the result in this note, we have shown that the Kronecker map and the Sylvester sum are crucial and suitable for serving as the theoretical basis of the matrix equations considered. It is expected that further insight into the intrinsic properties of the matrix equation mentioned in this note and the Sylvester sum can be developed with the aid of the Kronecker map.

4. Conclusions

With the help of the Kronecker map, some good properties of the Sylvester sum are proposed. By applying the properties of the Sylvester sum, an explicit solution is established for the generalized Sylvester matrix equation $AV + BW = EVF$, where the matrix F can be an arbitrary matrix. This solution can offer all the degrees of freedom of the matrix equation, which is represented by the free parameter matrix Z. It is recommended that one can apply the concept of the Kronecker map to exploit further properties of the aforementioned matrix equation.

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