# Property $(T)$ for non-unital $C^*$-algebras ☆

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## Abstract

Inspired by the recent work of Bekka, we study two reasonable analogues of property $(T)$ for not necessarily unital $C^*$-algebras. The stronger one of the two is called "property $(T)$" and the weaker one is called "property $(T_e)$." It is shown that all non-unital $C^*$-algebras do not have property $(T)$ (neither do their unitalizations). Moreover, all non-unital $\sigma$-unital $C^*$-algebras do not have property $(T_e)$.

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## 1. Introduction

Property $(T)$ for locally compact groups was first introduced and studied by Kazhdan in 1960's (see [6]). This notion was proved to be very useful in the study of topological groups. In 1980's, Connes defined the related concept of property $(T)$ for type II$_1$-factors (see [3]) which was also proved to be important. Recently, Bekka considered in [1] property $(T)$ for unital $C^*$-algebras and this was later on studied by Brown in [2].

In this short article, we will consider property $(T)$ for not necessarily unital $C^*$-algebras. Roughly speaking, a unital $C^*$-algebra $A$ is said to have *property $(T)$* if every Hilbert $A$-bimodule having an almost central unit vector for $A$ contains a central unit vector for $A$ (see Section 2). Notice that in the unital case, in order to check a $C^*$-algebra having property $(T)$, it suffices to consider only the class of essential Hilbert bimodules (see Proposition 2.2(b)). However, there is no guarantee that it is the case for non-unital $C^*$-algebras. For this reason, we introduce two notions of property $(T)$ for general $C^*$-algebras. We called the stronger one of the two "property $(T)$" and the weaker one

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"property $(T_e)$" (according to whether we consider the class of all Hilbert bimodules, or we restrict our attention to essential Hilbert bimodules).

The main results of this paper is the (somehow discouraging) fact that any non-unital $\sigma$-unital $C^*$-algebra does not even have property $(T_e)$. Other classes of infinite dimensional $C^*$-algebras definitely not having property $(T_e)$ include abelian $C^*$-algebras, group $C^*$-algebras of compact groups and AF-algebras. However the authors do not know whether all non-unital $C^*$-algebras do not have property $(T_e)$ (if it is the case, then property $(T)$ and property $(T_e)$ are actually equivalent).

On the other hand, the authors are grateful to the referee for showing us how to use a generalization of Brown's result in [2, Theorem 3.4] to give an elegant proof that non-unital $C^*$-algebras do not have property $(T)$ (see Theorem 2.9).

## 2. Main results

*Throughout this article, A will denote a $C^*$-algebra (not necessarily unital) and $\widehat{A}$ is the set of all unitarily equivalence classes of irreducible representations of A.*

A *Hilbert bimodule over A* is a Hilbert space $H$ together with two commuting $*$-homomorphisms $\rho_\ell : A \to \mathcal{L}(H)$ and $\rho_r : A^{op} \to \mathcal{L}(H)$ (where $A^{op}$ is the opposite algebra of $A$). For $x \in A$ and $\xi \in H$, we shall write $x \cdot \xi = \rho_\ell(x)(\xi)$ and $\xi \cdot x = \rho_r(x^{op})(\xi)$. A Hilbert bimodule $H$ over $A$ is said to be *essential* if the linear span of $\{x \cdot \xi \cdot y \mid x, y \in A$ and $\xi \in H\}$ is dense in $H$. On the other hand, a net $(\xi_i)$ of unit vectors in $H$ is called *an almost central unit vector* if $\|a \cdot \xi_i - \xi_i \cdot a\| \to 0$ for all $a \in A$. Moreover, an element $\xi \in H$ is said to be *central* if $a \cdot \xi = \xi \cdot a$ for all $a \in A$.

**Definition 2.1.** A $C^*$-algebra $A$ is said to have *property $(T)$* (respectively *property $(T_e)$*) if every Hilbert bimodule (respectively essential Hilbert bimodule) over $A$ having an almost central unit vector will contain a non-zero central vector.

It is clear that if $A$ has property $(T)$, then $A$ has property $(T_e)$. In contrary to the unital case in [1, Remark 17], a $C^*$-algebra without tracial state may not have property $(T_e)$ as can be seen in Theorem 2.7 below.

Let us first give the following simple result.

**Proposition 2.2.** *Let A be a $C^*$-algebra.*

(a) *A has property $(T)$ if and only if its unitalization $\widetilde{A}$ has property $(T)$.*
(b) *If A is unital and has property $(T_e)$, then A has property $(T)$.*

**Proof.** (a) This part is clear.

(b) Let $H$ be a Hilbert bimodule over $A$ having an almost central unit vector $(\xi_i)$. If $P = \rho_\ell(1_A)$ and $Q = \rho_r(1_A^{op})$, then

$$H = PHQ \oplus PH(1-Q) \oplus (1-P)HQ \oplus (1-P)H(1-Q).$$

It is obvious that $H$ must contain a non-zero central vector if $(1-P)H(1-Q) \neq (0)$, and so, one can assume that $(1-P)H(1-Q) = (0)$. For each $\xi_i$, we write $\xi_i = \alpha_i + \beta_i + \gamma_i$ where $\alpha_i \in PHQ$, $\beta_i \in PH(1-Q)$ and $\gamma_i \in (1-P)HQ$. Then

$$\|x \cdot \alpha_i - \alpha_i \cdot x\|^2 + \|x \cdot \beta_i\|^2 + \|\gamma_i \cdot x\|^2 = \|x \cdot \xi_i - \xi_i \cdot x\|^2 \to 0 \quad (x \in A)$$

will imply that $\beta_i \to 0$ and $\gamma_i \to 0$ by taking $x = 1_A$. Since $\|\xi_i\| = 1$, we have $\alpha_i \nrightarrow 0$ and $PHQ$ has an almost central unit vector. As $PHQ$ is an essential Hilbert bimodule over $A$ and $A$ has property $(T_e)$, we know that $H$ contains a non-zero invariant vector.  □

**Remark 2.3.** (a) It is very tempting to use the argument of Proposition 2.2(b) to show that property $(T_e)$ is equivalent to property $(T)$. However, the problem is that if $(a_i)$ is an approximate unit for $A$, and $P$ and $Q$ are the strong operator limits of $\rho_l(a_i)$ and $\rho_r(a_i^{op})$ respectively, it is possible that $PHQ = (0)$, e.g. $\rho_l : \mathcal{K}(\ell^2) \to \mathcal{L}(\ell^2)$ is the canonical embedding and $\rho_r = 0$ (note that the canonical basis $\{e_n\}$ is an almost central vector).

(b) We do not know if property $(T_e)$ is also preserved under unitalization. According to Proposition 2.2(b), this statement is equivalent to saying that property $(T)$ being the same as property $(T_e)$.

Part (a) of the following corollary follows from Proposition 2.2 and part (b) is easy to verify.

**Corollary 2.4.**

(a) *If $B$ and $C$ are $C^*$-algebras having property $(T)$, then so is their direct sum $B \oplus C$.*
(b) *If $A$ has property $(T)$ (respectively property $(T_e)$), then so is any quotient of $A$.*

Note that if $A$ has property $(T)$, then so is its multiplier algebra $M(A)$. However, the converse is not true because the $C^*$-algebra of compact operators $\mathcal{K}(\ell^2)$ does not even have property $(T_e)$ (see Proposition 2.5(a) below) while the $C^*$-algebra of all bounded linear operators $\mathcal{L}(\ell^2)$ has.

In the following, we give several cases when $A$ will definitely not have property $(T_e)$.

**Proposition 2.5.** *Suppose that $A$ is a $C^*$-algebra and $M(A)$ is its multiplier algebra.*

(a) *If $\Omega$ is a locally compact Hausdorff space and $\varphi$ is a non-degenerate $*$-homomorphism from $C_0(\Omega)$ to $M(A)$ such that $1 \notin \varphi(C_0(\Omega))$, then $A$ does not have property $(T_e)$.*
(b) *If there exists an infinite directed set $I$ and a net of increasing projections $\{p_i\}_{i \in I}$ in $M(A)$ such that $p_i \to 1$ strictly and $p_i \lneq p_j \lneq 1$ for any $i, j \in I$ with $i \lneq j$, then $A$ does not have property $(T_e)$.*

**Proof.** (a) Let $I := \{K \subseteq \Omega : K \text{ is compact}\}$. For any $K \in I$, fix $f_K \in C_c(\Omega)$ with

$$\chi_K \leqslant f_K \leqslant 1_\Omega,$$

where $\chi_K$ is the characteristic function of $K$. Then by the non-degeneracy of $\varphi$, one has $\|a - a\varphi(f_K)\| + \|a - \varphi(f_K)a\| \to 0$ (along $K \in I$) for any $a \in A$. If $\pi : A \to \mathcal{L}(H)$ is any non-degenerate $*$-representation, it induces a unital $*$-representation $\varphi_\pi$ of $B(\Omega)$ on $H$ (where $B(\Omega)$ is the $C^*$-algebra of all bounded Borel measurable functions on $\Omega$). Suppose that there exists $K \in I$ such that $\varphi_\pi(\chi_K) = 1$ (and hence $\pi(\varphi(f_K)) = 1$) for every $\pi \in \widehat{A}$. Then the injectivity of $\bigoplus_{\pi \in \widehat{A}} \pi$ will imply that $\varphi(f_K) = 1$ which contradicts the hypothesis. This shows that for any $K \in I$, there exists $\pi \in \widehat{A}$ such that $\varphi_\pi(\chi_K) \neq 1$.

**Case 1.** There is $\pi \in \widehat{A}$ such that $\varphi_\pi(\chi_K) \neq 1$ for any $K \in I$.

Let $H_\pi$ be the underlying Hilbert space for $\pi$. Assume that $H_\pi$ is finite dimensional. Then $\pi(\varphi(C_0(\Omega)))$ is unital. It is not hard to check that $\|f - f\chi_K\|_{B(\Omega)} \to 0$ for any $f \in C_0(\Omega)$. Thus, $\varphi_\pi(\chi_K)$ converges to 1 in norm and there exists $K_0 \in I$ with $\varphi_\pi(\chi_{K_0}) = 1$ which contradicts the assumption of Case 1. Thus $H_\pi$ is infinite dimensional. It is easy to check that $HS(H_\pi) \cong H_\pi \otimes \overline{H}_\pi$ is an essential Hilbert bimodule over $A$ with multiplications:

$$a \cdot (\xi \otimes \bar{\eta}) \cdot b = \pi(a)(\xi) \otimes \overline{\pi(b^*)(\eta)} \quad (\xi, \eta \in H_\pi).$$

If $\Theta \in HS(H_\pi)$ is a central vector, then $\pi(a)\Theta = \Theta\pi(a)$ for all $a \in A$. This implies that $\Theta \in \mathbb{C}1$ (as $\pi$ is irreducible) and so $\Theta = 0$ (because $H_\pi$ is infinite dimensional). Thus, there is no non-zero central vector in $HS(H_\pi)$. For any $K \in I$, there exists $\xi_K \in H_\pi$ such that $\varphi_\pi(\chi_K)\xi_K = 0$ and $\|\xi_K\| = 1$. Define $\zeta_K := \xi_K \otimes \overline{\xi_K}$. Then for any $f \in C_0(\Omega)$ with supp $f \subseteq K$, we have $\pi(\varphi(f))\xi_K = 0$ and so

$$\|a \cdot \zeta_K - \zeta_K \cdot a\| = \left\| \pi(a)(\xi_K) \otimes \overline{\xi_K} - \pi(a\varphi(f))(\xi_K) \otimes \overline{\xi_K} - \xi_K \otimes \overline{\pi(a^*)(\xi_K)} + \xi_K \otimes \overline{\pi(a^*\varphi(f^*))(\xi_K)} \right\|$$
$$\leqslant \|a - a\varphi(f)\| + \|a - \varphi(f)a\|.$$

Now for any $\epsilon > 0$, there exists $K_0 \in I$ with $\|a - a\varphi(f_{K_0})\| + \|a - \varphi(f_{K_0})a\| < \epsilon$. For any $K \in I$ with supp $f_{K_0} \subseteq K$,

$$\|a \cdot \zeta_K - \zeta_K \cdot a\| \leqslant \|a - a\varphi(f_{K_0})\| + \|a - \varphi(f_{K_0})a\| < \epsilon.$$

Consequently, $\{\zeta_K\}_{K \in I}$ is an almost central vector for $A$ and $A$ does not have property $(T_e)$.

**Case 2.** For any $\pi \in \widehat{A}$ there exists $K_\pi \in I$ such that $\varphi_\pi(\chi_{K_\pi}) = 1$.

Consider $H_0 := \bigoplus_{\pi \neq \sigma, \, \pi, \sigma \in \widehat{A}} H_\pi \otimes \overline{H_\sigma}$ as an essential Hilbert bimodule over $A$ with multiplications: $a \cdot (\xi \otimes \bar\eta) \cdot b = \pi(a)(\xi) \otimes \overline{\sigma(b^*)(\eta)}$ for any $\pi, \sigma \in \widehat{A}$, $\xi \in H_\pi$ and $\eta \in H_\sigma$. Let $\Theta \in H_0$ be a central vector. Then $\Theta = (\Theta_{\pi,\sigma})$ where $\Theta_{\pi,\sigma} \in H_\pi \otimes \overline{H_\sigma}$. By considering $\Theta_{\pi,\sigma}$ as an element in $HS(H_\sigma; H_\pi)$, the relation $\pi(a)\Theta_{\pi,\sigma} = \Theta_{\pi,\sigma}\sigma(a)$ $(a \in A)$, and the Schur's lemma tells us that $\Theta_{\pi,\sigma} = 0$ (as $\pi \neq \sigma$). This shows that $H_0$ has no non-zero central vector. We claim that for any $K \in I$, there exist at least two elements $\pi, \sigma \in \widehat{A}$ such that

$$\varphi_\pi(\chi_K) \neq 1 \quad \text{and} \quad \varphi_\sigma(\chi_K) \neq 1.$$

Indeed, as noted above, there exists $\pi \in \widehat{A}$ such that $\varphi_\pi(\chi_K) \neq 1$. Suppose on the contrary that $\varphi_\sigma(\chi_K) = 1$ for any $\sigma \in \widehat{A} \setminus \{\pi\}$. Then $\varphi_\sigma(\chi_L) = 1$ for any $L \in I$ with $K \subseteq L$ and any $\sigma \in \widehat{A} \setminus \{\pi\}$. If $K_\pi$ is as in the assumption of Case 2 and if $L \in I$ with $K \subseteq L$ and $K_\pi \subseteq L$, then $\bigoplus_{\sigma \in \widehat{A}} \sigma(\varphi(f_L)) = 1$ which contradicts the hypothesis that $1 \notin \varphi(C_0(\Omega))$. Now for any $K \in I$, we take two different elements $\pi, \sigma \in \widehat{A}$ with $\varphi_\pi(\chi_K) \neq 1$ and $\varphi_\sigma(\chi_K) \neq 1$, and we choose $\xi_K \in H_\pi$ and $\eta_K \in H_\sigma$ such that $\varphi_\pi(\chi_K)(\xi_K) = 0$, $\varphi_\sigma(\chi_K)(\eta_K) = 0$ and $\|\xi_K\| = 1 = \|\eta_K\|$. Define $\zeta_K := \xi_K \otimes \overline{\eta_K}$. For any $f \in C_0(\Omega)$ with $\text{supp}\, f \subseteq K$, we have

$$\|a \cdot \zeta_K - \zeta_K \cdot a\| = \|\pi(a)(\xi_K) \otimes \overline{\eta_K} - \pi(a\varphi(f))(\xi_K) \otimes \overline{\eta_K} - \xi_K \otimes \overline{\sigma(a^*)(\eta_K)} + \xi_K \otimes \overline{\sigma(a^*\varphi(f^*))(\eta_K)}\|$$
$$\leqslant \|a - a\varphi(f)\| + \|a - \varphi(f)a\|.$$

Now a similar argument as that of Case 1 will show that $A$ does not have property $(T_e)$.

(b) The proof is similar to that of part (a) (but we need to replace $\{\chi_K\}$ with $\{p_i\}$). $\quad\square$

Although property $(T)$ is preserved under a finite direct sum (see Corollary 2.4), it does not hold for an infinite $c_0$-direct sum. More precisely, we have the following direct application of Proposition 2.5(b).

**Corollary 2.6.** *If $(A_\lambda)_{\lambda \in \Lambda}$ is any infinite family of nonzero $C^*$-algebras, then the $c_0$-direct sum $\bigoplus_{\lambda \in \Lambda} A_\lambda := \{(x_\lambda)_{\lambda \in \Lambda} \in \Pi_{\lambda \in \Lambda} A_\lambda : (\|x_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)\}$ does not have property $(T_e)$.*

Suppose that a non-unital $C^*$-algebra $A$ contains a strictly positive element $h$ (see [8, 3.10.6]). The smallest $C^*$-subalgebra $B \subseteq A$ generated by $h$ is isomorphic to $C_0(\Omega)$ for some non-compact locally compact space. Since $\{h^{1/n}\}$ is an approximate identity for $A$, Proposition 2.5(a) gives the following result (which implies that property $(T_e)$ is equivalent to property $(T)$ for any $\sigma$-unital $C^*$-algebra).

**Theorem 2.7.** *Every non-unital $\sigma$-unital $C^*$-algebra* (*in particular, any separable non-unital $C^*$-algebra*) *does not have property $(T_e)$.*

Proposition 2.5 also gives the following corollary. Part (a) of it follows from Proposition 2.5(a) and [1, Proposition 15] while part (b) follows from [7, Theorem 28.40] and Corollary 2.6. To show part (c), one needs (on top of Theorem 2.7) [2, Proposition 5.1] as well as the fact that any unital AF-algebra has a tracial state (see [5]).

**Corollary 2.8.** *Let $A$ be a $C^*$-algebra. If $A$ is in one of the following three classes of $C^*$-algebras, then $A$ having property $(T_e)$ will imply that $A$ is finite dimensional*: (a) *$A$ is commutative*; (b) *$A = C^*(G)$ for a compact group $G$*; (c) *$A$ is an AF-algebra.*

It is believed that one can remove the $\sigma$-unital assumption in Theorem 2.7 (note that the two cases considered in Proposition 2.5 do not have such assumption) although we still do not have a proof. However, if only property $(T)$, instead of property $(T_e)$, is concerned, the referee has kindly informed us that this is true (see the following theorem). As an application, we see that if $A$ is a non-unital $C^*$-algebra, then $\widetilde{A}$ will never have property $(T)$.

**Theorem 2.9.** *All non-unital $C^*$-algebras do not have property $(T)$.*

**Proof** *(Provided by the referee).* In [2, Theorem 3.4], N.P. Brown showed that if $B$ is a separable unital $C^*$-algebra with property $(T)$ and $\pi : B \to M_n(\mathbb{C})$ is any irreducible representation, then the central cover $c(\pi)$ (i.e., a central projection in $B^{**}$ defined by $B^{**}(1 - c(\pi)) = \ker \pi^{**}$) of $\pi$ must belong to $B$. Indeed, the separability assumption can be removed by replacing Voiculescu's theorem with Glimm's lemma [4, Lemma II.5.1] in the proof of [2, Theorem 3.4].

Let $A$ be a non-unital $C^*$-algebra. Suppose on the contrary that $A$ had property $(T)$. Then its unitization $\widetilde{A}$ also has property $(T)$. Let $\pi : \widetilde{A} \to \mathbb{C}$ be the canonical map. By the extension of [2, Theorem 3.4] as stated above, the central cover $c(\pi)$ of $\pi$ is contained in $\widetilde{A}$. This yields the following $C^*$-algebras decompositions:

$$\widetilde{A} = \big(1 - c(\pi)\big)\widetilde{A} \oplus c(\pi)\widetilde{A} = \ker \pi \oplus \mathbb{C} = A \oplus \mathbb{C},$$

which implies the contradiction that $A$ is unital.  □

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