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Twisted actions and the obstruction to extending unitary representations of subgroups[☆]

Astrid an Huef[a,*], Iain Raeburn[b]

[a] School of Mathematics, University of New South Wales, Sydney NSW 2052, Australia

[b] School of Mathematical and Physical Sciences, University of Newcastle, NSW 2308, Australia

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Abstract

Suppose that G is a locally compact group and π is a (not necessarily irreducible) unitary representation of a closed normal subgroup N of G on a Hilbert space $\mathscr{H}$. We extend results of Clifford and Mackey to determine when π extends to a unitary representation of G on the same space $\mathscr{H}$ in terms of a cohomological obstruction.

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Let G be a group and $\pi : N \to U(\mathscr{H})$ a unitary representation of a normal subgroup N of G. When is π the restriction of a unitary representation of G?

If π does extend to a representation ρ of G, then $\rho(s)$ implements a unitary equivalence between π and $\pi^s : n \mapsto \pi(sns^{-1})$. So an obvious necessary condition is that π should be equivalent to π^s for each $s \in G$ (we say that π is *G-invariant*), and the problem is to decide when a G-invariant representation extends.

Clifford answered this extension problem in [2] when G is discrete, π is irreducible and $\mathscr{H}$ is finite-dimensional. In modern language, Clifford showed that if π is G-invariant, then there is an obstruction to extending the representation in the cohomology group $H^2(G/N, \mathbb{T})$, where $\mathbb{T}$ is the unit circle. Mackey extended Clifford's result

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* Corresponding author.

E-mail addresses: astrid@maths.unsw.edu.au (A. an Huef), iain@frey.newcastle.edu.au (I. Raeburn).

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to irreducible unitary representations of normal closed subgroups of locally compact groups [9, Theorem 8.2]. Mackey's solution involves Borel cocycles, so his obstruction lies in a cohomology theory where all the cochains are Borel. The resulting cohomology groups were subsequently analysed by Moore in [11–14]. Mackey's theorem was used in [3] to study the extension problem when the subgroup is a simply connected nilpotent real Lie group.

The extension problem has recently resurfaced in the context of compact Lie groups in [1], where it was tackled using the structure theory of Lie groups, and in [4], where it was studied in the context of nonabelian duality for locally compact groups and crossed products of C^*-algebras. Here we investigate a cohomological obstruction to the extension of an arbitrary G-invariant unitary representation π of N, and its relationship to the results in [1,4]. Our obstruction is a twisted action of G/N on the von Neumann algebra $\pi(N)'$ of operators which commute with every $\pi(n)$; the representation extends if and only if this twisted action is equivalent, in a natural sense, to an ordinary action. We then use a stabilisation trick to show that if π is G-invariant then infinite multiples $\pi \otimes 1$ of π always extend.

Preliminaries. Let G be a second-countable locally compact group with a closed normal subgroup N. We endow the group $U(\mathscr{H})$ of all unitary operators on a separable Hilbert space $\mathscr{H}$ with the strong operator topology, and note that $U(\mathscr{H})$ is a Polish group (in the sense that the topology is given by a complete metric). A unitary representation ρ of G is a continuous homomorphism $\rho : G \to U(\mathscr{H})$. A function $f : G \to \mathscr{H}$ is Borel if $f^{-1}(O)$ is a Borel set for each open set O of $\mathscr{H}$; equivalently, if $s \mapsto (f(s) \mid h) : G \to \mathbb{C}$ is a Borel function for each $h \in \mathscr{H}$. We use a left-invariant Haar measure on G.

Let $\mathscr{A}$ be a von Neumann algebra acting on a separable Hilbert space $\mathscr{H}$. The group $U(\mathscr{A})$ of unitary elements in $\mathscr{A}$ is a Polish group in the ultra-weak topology, and it is then a closed subgroup of $U(\mathscr{H})$. The group $\mathrm{Aut}(\mathscr{A})$ of automorphisms of $\mathscr{A}$ is Polish in the topology of pointwise ultra-weak convergence; this is called the u-topology in [6, Definition 3.4]. For $u \in U(\mathscr{A})$, we denote by $\mathrm{Ad}\, u$ the automorphism $a \mapsto uau^*$ of $\mathscr{A}$, and note that $\mathrm{Ad} : U(\mathscr{A}) \to \mathrm{Aut}(\mathscr{A})$ is a continuous homomorphism.

Definition 1. A *twisted action* of G on a von Neumann algebra $\mathscr{A}$ is a pair (α, σ) of maps $\alpha : G \to \mathrm{Aut}(\mathscr{A})$ and $\sigma : G \times G \to U(\mathscr{A})$ such that

(1) α and σ are Borel,
(2) $\alpha_e = \mathrm{id}, \sigma(e,s) = \sigma(s,e) = 1$ for $s \in G$,
(3) $\alpha_s \circ \alpha_t = \mathrm{Ad}\sigma(s,t) \circ \alpha_{st}$ for $s,t \in G$, and
(4) $\alpha_r(\sigma(s,t))\sigma(r,st) = \sigma(r,s)\sigma(rs,t)$ for $r,s,t \in G$.

Two twisted actions (α, σ) and (β, ω) of G on $\mathscr{A}$ are *exterior equivalent* if there is a Borel map $v : G \to U(\mathscr{A})$ such that

(1) $\beta_s = \mathrm{Ad} v_s \circ \alpha_s$, and
(2) $\omega(s,t) = v_s \alpha_s(v_t)\sigma(s,t)v_{st}^*$.

These definitions are the von-Neumann algebraic analogues of [15, Definitions 2.1 and 3.1]. Our definition of twisted action is slightly different from the one used in [18, Definition 2.1], where the map $s \mapsto \alpha_s$ is required to be continuous.

Main results. In Theorem 2 we prove that the obstruction to extending a G-invariant unitary representation π of N is a twisted action of G/N on the von Neumann algebra $\pi(N)'$, and in Theorem 5 we discuss the extension problem in the context of non-abelian duality for amenable groups G. We reconcile the two approaches in Remark 6; to do so one needs to understand not only the statement of Theorem 2 but also its proof.

Theorem 2. *Let N be a closed normal subgroup of a second-countable locally compact group G. Suppose $\pi : N \to U(\mathscr{H})$ is a unitary representation of N which is G-invariant. Then there is a twisted action (α, σ) of G/N on the commutant $\pi(N)'$ of $\pi(N)$ such that π extends to a unitary representation ρ of G on $\mathscr{H}$ if and only if (α, σ) is exterior equivalent to an action.*

Proof. We start by constructing the twisted action (α, σ). Since π^s is unitarily equivalent to π for all $s \in G$, there exist unitary operators $W_s \in U(\mathscr{H})$ such that $W_s\pi(n)W_s^* = \pi(sns^{-1})$. We claim that we can choose W_s such that the map $s \mapsto W_s$ is Borel. To see this, let

$$H = \{(W,s) : W \in U(\mathscr{H}),\ s \in G \text{ and } \pi(n)W^* = \pi(sns^{-1}) \text{ for } n \in N\}.$$

Then H is a subgroup of $U(\mathscr{H}) \times G$; we claim that H is closed. So suppose that the net $(W_\beta, s_\beta) \in H$ converges to (W,s). Then $W_\beta\pi(n)$ converges strongly to $W\pi(n)$ for each $n \in N$, and since multiplication in $U(\mathscr{H})$ is jointly continuous, $\pi(s_\beta n s_\beta^{-1})W_\beta$ converges strongly to $\pi(sns^{-1})W$. Thus $W\pi(n) = \pi(sns^{-1})W$, and H is closed. Now H is Polish since it is a closed subgroup of a Polish group, and the quotient map $H \to G : (W,s) \mapsto s$ has a Borel section $s \mapsto (W_s, s)$ by [13, Proposition 4].

The quotient map $G \to G/N$ admits a Borel section $c : G/N \to G$ by [8, Lemma 1.1]. We set

$$V_s = W_{c(sN)}\pi(c(sN)^{-1}s). \tag{1}$$

Then $s \mapsto V_s$ is Borel and $V_s\pi(n)V_s^* = \pi(sns^{-1})$, $V_{sn} = V_s\pi(n)$ and $V_{ns} = \pi(n)V_s$ for $s \in G$ and $n \in N$. We define $\alpha_s = \operatorname{Ad} V_s$. For $T \in \pi(N)'$ and $s \in G$, we have

$$\begin{aligned}\alpha_s(T)\pi(n) &= V_sTV_s^*\pi(n) = V_sT(V_s^*\pi(n)V_s)V_s^* = V_sT\pi(s^{-1}ns)V_s^* \\ &= V_s\pi(s^{-1}ns)TV_s^* = V_s\pi(s^{-1}ns)V_s^*V_sTV_s^* = \pi(n)\alpha_s(T);\end{aligned}$$

thus $\alpha_s : \pi(N)' \to \pi(N)'$, and α_s is an automorphism of $\pi(N)'$ because V_s is unitary.

To see that $s \mapsto \alpha_s : G \to \operatorname{Aut}(\pi(N)')$ is Borel, we will show that if V_β converges to V in the strong operator topology and $\operatorname{Ad} V_\beta$ and $\operatorname{Ad} V$ leave $\pi(N)'$ invariant, then $\operatorname{Ad} V_\beta$

converges to $\mathrm{Ad}V$ in $\mathrm{Aut}(\pi(N)')$. It then follows that $s \mapsto \alpha_s : G \to \mathrm{Aut}(\pi(N)')$ is Borel because $s \mapsto V_s$ is Borel. The topology on $\mathrm{Aut}(\pi(N)')$ is the topology generated by the seminorms $\alpha \mapsto \|f \circ \alpha\|$, where $f \in \pi(N)'_*$ and the pre-dual $\pi(N)'_*$ has been identified with the ultra-weakly continuous functionals on $\pi(N)'$. The ultra-weakly continuous functionals on $\pi(N)'$ have the form $f(T) = \sum_{n=1}^{\infty} (Th_n \mid k_n)$, where $h_n, k_n \in \mathscr{H}$ satisfy $\sum_{n=1}^{\infty} \|h_n\|^2, \sum_{n=1}^{\infty} \|k_n\|^2 < \infty$ (see, for example, [7, pp. 482–483]). Let $\varepsilon > 0$. If K is the maximum of $\left(\sum_{n=1}^{\infty} \|h_n\|^2\right)^{1/2}$ and $\left(\sum_{n=1}^{\infty} \|k_n\|^2\right)^{1/2}$, then

$$\begin{aligned}
\|f \circ \mathrm{Ad}V_\beta - f \circ \mathrm{Ad}V\| &= \sup\{|f(V_\beta T V_\beta^* - VTV^*)| : \|T\| = 1, T \in \pi(N)'\} \\
&= \sup\left\{\left|\sum_{n=1}^{\infty} (V_\beta^* h_n \mid T^* V_\beta^* k_n) - (TV^* h_n \mid V^* k_n)\right| : \|T\| = 1, T \in \pi(N)'\right\} \\
&\leqslant \sup\left\{\sum_{n=1}^{\infty} \|(V_\beta^* - V^*)h_n\| \, \|T^* V_\beta^* k_n\| + \|TV^* h_n\| \, \|(V_\beta^* - V^*)k_n\|\right\} \\
&\leqslant \sum_{n=1}^{\infty} \|(V_\beta^* - V^*)h_n\| \, \|k_n\| + \|h_n\| \, \|(V_\beta^* - V^*)k_n\| \\
&\leqslant K\left(\sum_{n=1}^{\infty} \|(V_\beta^* - V^*)h_n\|^2\right)^{1/2} + K\left(\sum_{n=1}^{\infty} \|(V_\beta^* - V^*)k_n\|^2\right)^{1/2} \qquad (2)
\end{aligned}$$

by Hölder's inequality. Since each V_β is a normal operator, we have $V_\beta^* \to V^*$ in the strong operator topology. Now choose $N > 0$ such that $\sum_{n=N}^{\infty} \|h_n\|^2 < \varepsilon^2(16K^2)^{-1}$ and $\sum_{n=N}^{\infty} \|k_n\|^2 < \varepsilon^2(16K^2)^{-1}$. Then, for each $n < N$, choose a strong-operator open neighbourhood O_n of V^* such that

$$\|(V_\beta^* - V^*)h_n\|^2 < \frac{\varepsilon^2}{8(N-1)K^2} \quad \text{and} \quad \|(V_\beta^* - V^*)k_n\|^2 < \frac{\varepsilon^2}{8(N-1)K^2}$$

whenever $V_\beta^* \in O_n$, and check that if $V_\beta^* \in \bigcap_{n=1}^{N-1} O_n$ then $(2) < \varepsilon$. This proves that $\mathrm{Ad}V_\beta$ converges to $\mathrm{Ad}V$, and it follows that $\alpha : G \to \mathrm{Aut}(\pi(N)') : s \mapsto \mathrm{Ad}V_s$ is Borel.

Next we define $\sigma(s,t) = V_s V_t V_{st}^*$. Then

$$\begin{aligned}
\sigma(s,t)\pi(n) = V_s V_t V_{st}^* \pi(n) &= V_s V_t \pi((st)^{-1} nst) V_{st}^* \\
&= V_s V_t \pi((st)^{-1} nst) V_t^* V_t V_{st}^* \\
&= V_s \pi(s^{-1} ns) V_t V_{st}^* = \pi(n) V_s V_t V_{st}^*
\end{aligned}$$

for all $n \in N$, so $\sigma : G \times G \to U(\pi(N)')$. Note that σ is Borel because $s \mapsto V_s$ is Borel and both $V_s \mapsto V_s^*$ and $(s,t) \mapsto st$ are continuous. The equation $V_{sn} = V_s \pi(n)$ implies that $\sigma(s,n)=1=\sigma(n,s)$ for $s\in G$ and $n\in N$. We have

$$\sigma(r,s)\sigma(rs,t)=V_rV_sV_{rs}^*V_{rs}V_tV_{rst}^*=V_rV_sV_tV_{rst}^*$$
$$=V_rV_sV_t(V_{st}^*V_r^*V_rV_{st})V_{rst}^*=\alpha_r(\sigma(s,t))\sigma(r,st)$$

and, for $T\in\pi(N)'$,

$$\alpha_s(\alpha_t(T))=V_sV_tTV_t^*V_s^*=V_sV_tV_{st}^*V_{st}TV_{st}^*V_{st}V_t^*V_s^*=\sigma(s,t)\alpha_{st}(T)\sigma(s,t)^*.$$

Thus (α,σ) is a twisted action of G on $\pi(N)'$. But α_s depends only on sN since

$$\alpha_{sn}(T)=V_s\pi(n)T\pi(n)^*V_s^*=V_s\pi(n)\pi(n)^*TV_s^*=\alpha_s(T)$$

for all $n\in N$. We also have

$$\sigma(s,tn)=V_sV_{tn}V_{stn}^*=V_sV_t\pi(n)\pi(n)^*V_{st}^*=\sigma(s,t).$$

Since $V_{ms}=\pi(m)V_s$ for $m\in N$ we have $\sigma(ms,t)=\sigma(s,t)$, and hence $\sigma(sn,t)=\sigma(sns^{-1}s,t)=\sigma(s,t)$. So we can view (α,σ) as a twisted action of G/N on $\pi(N)'$.

Now suppose that π extends to a continuous representation ρ of G on $\mathscr{H}$. Let V_s be as in (1) and define $v:G/N\to U(\mathscr{H})$ by $v_{sN}=\rho(s)V_s^*$. Then v is Borel because $s\mapsto V_s^*$ is Borel and ρ is continuous, and

$$\omega(sN,tN):=v_{sN}\alpha_s(v_{tN})\sigma(sN,tN)v_{stN}^*=1.$$

If $\beta_{sN}:=\mathrm{Ad}v_{sN}\circ\alpha_{sN}$, then

$$\beta_{sN}(T)=\mathrm{Ad}v_{sN}(\alpha_{sN}(T)))=\rho(s)V_s^*(V_sTV_s^*)V_s\rho(s)^*=\mathrm{Ad}(\rho(s))(T)$$

for $T\in\pi(N)'$, and since ρ is a homomorphism so is β. Now $\beta:G/N\to\mathrm{Aut}(\pi(N)')$ is a Borel homomorphism between Polish groups and hence is continuous by [13, Proposition 5]. Thus $(\beta,1)$ is an ordinary action, and (α,σ) is exterior equivalent to an action.

Conversely, if (α,σ) is exterior equivalent to an action, then there exists a Borel map $v:G/N\to U(\mathscr{H})$ such that $v_{sN}\alpha_{sN}(v_{tN})\sigma(sN,tN)v_{stN}^*=1$. Set $\rho(s)=v_{sN}V_s$. Then

$$\rho(s)\rho(t)=v_{sN}V_sv_{tN}V_t=v_{sN}\alpha_{sN}(v_{tN})V_sV_t$$
$$=v_{sN}\alpha_{sN}(v_{tN})\sigma(sN,tN)V_{st}=v_{stN}V_{st}=\rho(st).$$

Thus $\rho:G\to U(\mathscr{H})$ is a Borel homomorphism between Polish groups, and hence is continuous by [13, Proposition 5]; ρ is the required extension of π. □

Corollary 3. *If $\pi:N\to U(\mathscr{H})$ is a G-invariant unitary representation of N, then there is a unitary representation ρ of G on $\mathscr{H}\otimes L^2(G/N)$ such that $\rho|_N=\pi\otimes 1$.*

From Corollary 3 we immediately obtain:

Corollary 4. *Suppose that $\pi:N\to U(\mathscr{H})$ is a unitary representation of N which is unitarily equivalent to $\pi\otimes 1$ on $\mathscr{H}\otimes L^2(G/N)$. Then π extends to a representation of G if and only if π is G-invariant.*

Proof of Corollary 3. Let (α, σ) be the twisted action of G/N on $\pi(N)'$ constructed above. Then the twisted action $(\beta, \omega) := (\alpha \otimes \mathrm{id}, \sigma \otimes 1)$ of G/N on $\pi(N)' \otimes B(L^2(G/N)) = (\pi \otimes 1)(N)'$ is the obstruction to extending $\pi \otimes 1$. We will show that (β, ω) is exterior equivalent to an action. Similar "stabilisation tricks" have been used in [19, Proposition 2.1.3] and [15, Theorem 3.4], for example.

We begin by identifying $\mathscr{H} \otimes L^2(G/N)$ with the space $L^2(G/N, \mathscr{H})$ of Bochner square-integrable functions. Since $\mathscr{H}$ is separable, $\xi \in L^2(G/N, \mathscr{H})$ if and only if ξ is a Borel function from G/N to $\mathscr{H}$ and $\int_{G/N} \|\xi(sN)\|^2 \, d(sN) < \infty$. Define $v : G/N \to U(L^2(G/N, \mathscr{H}))$ by

$$(v_{tN}\xi)(rN) = \sigma(tN, t^{-1}r^{-1}N)^* \xi(rtN) \Delta(tN)^{1/2},$$

where Δ is the modular function of G/N and $\xi \in L^2(G/N, \mathscr{H})$. (The modular function is necessary to ensure that v_{tN} is unitary.) Then

$$(v_{tN}^*\xi)(rN) = \sigma(tN, r^{-1}N)\xi(rt^{-1}N)\Delta(tN)^{-1/2},$$

and hence

$$\begin{aligned}
(\beta_{sN}(v_{tN}^*)v_{sN}^* v_{stN}\xi)(rN) &= \alpha_{sN}(\sigma(tN, r^{-1}N))(v_{sN}^* v_{stN}\xi)(rt^{-1}N)\Delta(tN)^{-1/2} \\
&= \alpha_{sN}(\sigma(tN, r^{-1}N))\sigma(sN, tr^{-1}N)(v_{stN}\xi)(rt^{-1}s^{-1}N)\Delta(stN)^{-1/2} \\
&= \alpha_{sN}(\sigma(tN, r^{-1}N))\sigma(sN, tr^{-1}N)\sigma(stN, r^{-1}N)^*\xi(rN) \\
&= \sigma(sN, tN)\xi(rN) \\
&= (\sigma(sN, tN) \otimes 1)\xi(rN) \\
&= \omega(sN, tN)\xi(rN).
\end{aligned}$$

It follows that

$$v_{sN}\beta_{sN}(v_{tN})\omega(sN, tN)v_{stN}^* = 1. \tag{3}$$

If we now define $\gamma : G/N \to \mathrm{Aut}(\pi(N)')$ by $\gamma_{sN} = \mathrm{Ad}\, v_{sN} \circ \beta_{sN}$, then (3) implies that γ is a homomorphism. It remains to show that v is Borel, and it then follows from [13, Proposition 5] that $\gamma = \mathrm{Ad}\, v \circ \beta : G/N \to \mathrm{Aut}(\pi(N)')$ is continuous.

Since $U(L^2(G/N, \mathscr{H}))$ has the strong operator topology, v is Borel if and only if $sN \mapsto v_{sN}\xi$ is Borel for every $\xi \in L^2(G/N, \mathscr{H})$, and hence if and only if $sN \mapsto (v_{sN}\xi \mid \eta)$ is Borel for every $\xi, \eta \in L^2(G/N, \mathscr{H})$. Since $(U, h) \mapsto Uh$ is continuous, the map $(sN, tN, rN) \mapsto (\sigma(sN, tN), \xi(rN)) \mapsto \sigma(sN, tN)\xi(rN)$ is Borel, and hence so is

$$(tN, rN) \mapsto |(\sigma(tN, t^{-1}r^{-1}N)^* \xi(rtN) \mid \eta(rN))|. \tag{4}$$

Since (4) is dominated by $\|\xi(rtN)\| \, \|\eta(rN)\|$, and an application of Tonelli's Theorem shows that this is integrable over $G/N \times G/N$, it follows from Fubini's Theorem that

$$tN \mapsto \int_{G/N} (\sigma(tN, t^{-1}r^{-1}N)^* \xi(rtN) \mid \eta(rN)) \, d(rN)$$

defines, almost everywhere, an integrable (and therefore Borel) function. Multiplying by $\Delta(tN)^{1/2}$ shows that $tN \mapsto (v_{tN}\xi \mid \eta)$ is Borel. Thus v is Borel and γ is continuous.

Thus v implements an exterior equivalence between (β,ω) and the ordinary action $(\gamma,1)$. It now follows from Theorem 2 that there is a representation ρ of G with $\rho|_N = \pi \otimes 1$. □

The irreducible case. When the representation π of N is irreducible, the commutant $\pi(N)'$ is $\mathbb{C}1$, the action α is trivial, and the obstruction σ to extending π is a Borel cocycle in the Moore cohomology group $H^2(G/N,\mathbb{T})$. Thus we recover Mackey's [9, Theorem 8.2] as it applies to ordinary (that is, non-projective) irreducible representations.

When the obstruction σ is non-trivial, we can recover Corollary 3 from another important part of the Mackey machine [9, Theorem 8.3]: π extends to a projective representation U of G with cocycle $\sigma \circ (q \times q)$, and tensoring with an irreducible $\bar{\sigma}$-representation W of G/N gives an irreducible representation $U \otimes (W \circ q)$ of G whose restriction to N is a multiple $\pi \otimes 1$ of π.

Applications to compact Lie groups. When Γ is a compact connected Lie group, Moore computed $H^2(\Gamma,\mathbb{T})$ as follows. Let $\widetilde{\Gamma}$ be the simply connected covering group of Γ; then the fundamental group $\pi_1(\Gamma)$ is isomorphic to a central subgroup of $\widetilde{\Gamma}$ and $\Gamma \cong \widetilde{\Gamma}/\pi_1(\Gamma)$. An inflation and restriction sequence identifies $H^2(\Gamma,\mathbb{T})$ with the quotient of the dual group $\pi_1(\Gamma)^\wedge = \mathrm{Hom}(\pi_1(\Gamma),\mathbb{T})$ by the image of the restriction map $\mathrm{Res} : (\widetilde{\Gamma})^\wedge \to \pi_1(\Gamma)^\wedge$ [11, pp. 55].

When $\Gamma = \mathbb{T}^n$, we have $\pi_1(\Gamma) = \mathbb{Z}^n$ and $\widetilde{\Gamma} = \mathbb{R}^n$, and the restriction map $\mathbb{R}^n = (\mathbb{R}^n)^\wedge \mapsto \mathbb{T}^n = (\mathbb{Z}^n)^\wedge$ is onto by duality. Thus $H^2(\mathbb{T}^n,\mathbb{T}) = 0$. Theorem 2 thus implies that if $G/N \cong \mathbb{T}^n$, then every G-invariant irreducible unitary representation of N extends to G. Because representations of compact groups are direct sums of irreducible representations, this observation includes [1, Corollary 3.5], and hence also [1, Theorem 1.1].

For non-compact groups G, one might want to prove Corollary 3 by reducing to the irreducible case using a direct-integral decomposition. There can be substantial technical difficulties; see, for example [5], where a direct-integral decomposition is used to find sufficient conditions for a unitary representation of a closed normal subgroup of a separable locally compact group to extend.

The nonabelian duality approach. If $\alpha : G \to \mathrm{Aut}(A)$ is a strongly continuous action of a locally compact group G on a C^*-algebra A, a *covariant representation* of (A,G,α) consists of a representation μ of A and a unitary representation U of G such that

$$\mu(\alpha_t(a)) = U_t \mu(a) U_t^* \quad \text{for } a \in A \quad \text{and } t \in G;$$

covariant representations can take values either in abstract C^*-algebras or in the concrete C^*-algebra $B(\mathscr{H})$. The *crossed product* $A \times_\alpha G$ is the C^*-algebra generated by a universal covariant representation in the multiplier algebra $M(A \times_\alpha G)$ (see [16] for details of what this means). The covariant representations (μ,U) of (A,G,α) therefore give representations $\mu \times U$ of $A \times_\alpha G$, and all representations of $A \times_\alpha G$ have this form. We shall be particularly interested in the actions $\mathrm{lt} : G \to \mathrm{Aut}(C_0(G/N))$ and

$\mathrm{rt} : G/N \to \mathrm{Aut}(C_0(G/N))$ defined by

$$\mathrm{lt}_s(f)(uN) = f(s^{-1}uN) \quad \text{and} \quad \mathrm{rt}_{tN}(f)(uN) = f(utN).$$

The automorphisms rt_{tN} commute with the automorphisms lt_s, and hence induce an action β of G/N on the crossed product $C_0(G/N) \times_{\mathrm{lt}} G$.

If π is a unitary representation of N, then the induced representation $\mathrm{Ind}\,\pi$ of G acts in the completion $\mathscr{H}(\mathrm{Ind}\,\pi)$ of

$$\{\xi \in C_b(G, \mathscr{H}) : \xi(tn) = \pi(n)^{-1}(\xi(t)) \text{ and } (tN \mapsto \|\xi(t)\|) \in C_c(G/N)\}$$

with respect to the inner product $(\xi \mid \eta) = \int_{G/N} (\xi(t) \mid \eta(t))\, d(tN)$, according to the formula $(\mathrm{Ind}\,\pi)_t(\xi)(r) = \xi(t^{-1}r)$. (See, for example, [17, pp. 296]; because N is normal there is a G-invariant measure on G/N, and we can take the rho-function in the usual formula to be 1.)

Let M be the representation of $C_0(G/N)$ by multiplication operators on $\mathscr{H}(\mathrm{Ind}\,\pi)$, and note that $(M, \mathrm{Ind}\,\pi)$ is a covariant representation of $(C_0(G/N), G, \mathrm{lt})$. The nonabelian duality approach to the extension problem yields the following theorem.

Theorem 5. *Suppose that N is a closed normal subgroup of an amenable and second-countable locally compact group G, and suppose that $\pi : N \to U(\mathscr{H})$ is a unitary representation. Then π extends to a unitary representation of G if and only if there exists a unitary representation $Q : G/N \to U(\mathscr{H}(\mathrm{Ind}\,\pi))$ such that $(M \times \mathrm{Ind}\,\pi, Q)$ is a covariant representation of $(C_0(G/N) \times_{\mathrm{lt}} G, G/N, \beta)$.*

Proof. The induction-restriction theory of [4] says that π is the restriction of a representation of G if and only if $M \times \mathrm{Ind}\,\pi$ is induced, in a dual sense, from a representation of the group C^*-algebra $C^*(G) = \mathbb{C} \times G$. To deduce this from [4, Theorem 5.16], we need to recall some ideas of nonabelian duality. The group C^*-algebra $C^*(G)$ is generated by a universal unitary representation $\imath : G \to UM(C^*(G))$. The comultiplication $\delta : C^*(G) \to M(C^*(G) \otimes C^*(G))$ is the representation corresponding to the unitary representation $\imath \otimes \imath$; it has a restriction $\delta|$ which is a coaction of G/N on $C^*(G)$. Since G is amenable, $C^*(G)$ coincides with the reduced group C^*-algebra $C^*_r(G)$, and hence we can apply results from [4,10] concerning reduced crossed products. In particular, we can induce representations from $C^*(G)$ to the coaction crossed product $C^*(G) \times_{\delta|} G/N$ by tensoring with a $(C^*(G) \times_{\delta|} G/N)$–$C^*(G)$ bimodule Y constructed by Mansfield [10]; the resulting map on representations is denoted by Y-Ind.

We recall from [17, Theorem C.23] that there is a Morita equivalence between $C_0(G/N) \times_{\mathrm{lt}} G$ and $C^*(N)$ which is implemented by an imprimitivity bimodule X; we denote by X-Ind the corresponding map on representations. The algebras $C_0(G/N) \times_{\mathrm{lt}} G$ and $C^*(G) \times_{\delta|} G/N$ have exactly the same covariant representations, and hence are isomorphic (see, for example, [4, Theorem A.64]). Thus we can view X as a $(C^*(G) \times_{\delta|} G/N)$–$C^*(N)$ bimodule. Theorem 5.16 of [4] (with $A = \mathbb{C}$ and $M = G$) says that, provided G is amenable, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}\,C^*(G) & \xrightarrow{\ \mathrm{Res}\ } & \mathrm{Rep}\,C^*(N) \\ & {\scriptstyle Y\text{-Ind}} \searrow \quad \swarrow {\scriptstyle X\text{-Ind}} & \\ & \mathrm{Rep}(C^*(G) \times_{\delta|} G/N) & \end{array}$$

Since X-Ind is a bijection, it follows that a representation π of $C^*(N)$ extends to a representation of $C^*(G)$ if and only if $X\text{-Ind}\,\pi$ is in the range of Y-Ind.

To deduce Theorem 5 from this, we have to make two observations. First, the representation $X\text{-Ind}\,\pi$ of $C_0(G/N)\times_{\mathrm{lt}} G$ is equivalent to $M \times \mathrm{Ind}\,\pi$. To see this, note that the intertwining unitary isomorphism W of $(X \otimes_{C^*(N)} \mathscr{H}, X\text{-Ind}\,\pi)$ onto $(\mathscr{H}(\mathrm{Ind}\,\pi), \mathrm{Ind}\,\pi)$ constructed in the proof of [17, Theorem C.33] carries the left action of $C_0(G/N)$ into M. Second, we recall from Mansfield's imprimitivity theorem [10, Theorem 28] that a representation μ of $C^*(G)\times_{\delta|} G/N$ has the form $Y\text{-Ind}\,\rho$ if and only if there is a unitary representation Q of G/N on $\mathscr{H}(\mu)$ such that (μ, Q) is covariant for the dual action $(\delta|)^\wedge$ of G/N. Since [4, Theorem A.64] also says that the isomorphism of $C_0(G/N) \times_{\mathrm{lt}} G$ onto $C^*(G) \times_{\delta|} G/N$ carries the action β into $(\delta|)^\wedge$, the result follows. □

Remark 6. Comparing Theorem 5 with Theorem 2, it is natural to ask what happened to the hypothesis "π is G-invariant". Suppose π is G-invariant, so that there exist unitary operators W_s on $\mathscr{H}$ such that $W_s\pi(n)W_s^* = \pi(sns^{-1})$. Then

$$U_s(\xi)(t) = W_s(\xi(ts))\Delta(sN)^{1/2}$$

defines a unitary operator U_s on $\mathscr{H}(\mathrm{Ind}\,\pi)$ which intertwines the covariant representations $(M, \mathrm{Ind}\,\pi)$ and $(M \circ \mathrm{rt}_{sN}, \mathrm{Ind}\,\pi)$. So $R_{sN} := U_{c(sN)}$ defines a map $R : G/N \to U(\mathscr{H}(\mathrm{Ind}\,\pi))$ which formally satisfies the covariance relations but is not necessarily a representation.

Our original extension problem for a G-invariant representation $\pi : N \to U(\mathscr{H})$ therefore reduces to:

> Given a representation ϕ of $C^*(G) \times_{\delta|} G/N$ such that $\phi \circ (\delta|)^\wedge_{sN}$ is equivalent to ϕ for every $sN \in G/N$, is there a representation Q of G/N such that (ϕ, Q) is covariant for $(C^*(G) \times_{\delta|} G/N, G/N, (\delta|)^\wedge)$?

Since there are by hypothesis unitary operators R_{sN} such that $\phi \circ (\delta|)^\wedge_{sN} = \mathrm{Ad}R_{sN} \circ \phi$, we can repeat the analysis of Theorem 2 to see that there is a twisted action (β, ω) of G/N on the commutant of the range of ϕ, such that (β, ω) is exterior equivalent to an ordinary action if and only if we can adjust the R_{sN} to obtain the required representation Q. Thus $\beta_{sN} = \mathrm{Ad}R_{sN}$ and, for $\xi \in \mathscr{H}(\mathrm{Ind}\,\pi)$,

$$\omega(rN, sN)(\xi)(t) = R_{rN}R_{sN}R^*_{rsN}(\xi)(t) = U_{c(rN)}U_{c(sN)}U^*_{c(rsN)}(\xi)(t)$$

$$= W_{c(rN)}W_{c(sN)}W^*_{c(rsN)}(\xi(tc(rN)c(sN)c(rsN)^{-1}))$$

$$= W_{c(rN)}W_{c(sN)}W^*_{c(rsN)}\pi(c(rN)c(sN)c(rsN)^{-1})^{-1}(\xi(t))$$

$$= W_{c(rN)}W_{c(sN)}W^*_{c(rsN)}\pi(c(rsN)c(sN)^{-1}c(rN)^{-1})(\xi(t)).$$

We claim that the obstruction (β,ω) is essentially the same as the obstruction (α,σ) to extending π from Theorem 2. To see this, we first identify $\pi(N)'$ with $\phi(C^*(G)\times_{\delta|}G/N)'$ when $\phi = M\times \operatorname{Ind}\pi$. If $T\in\pi(N)'$, then the formula $1\otimes T(\xi)(t) = T(\xi(t))$ defines an operator in $\phi(C^*(G)\times_{\delta|}G/N)'$. When we view $\mathscr{H}(\operatorname{Ind}\pi)$ as $X\otimes_{C^*(N)}\mathscr{H}$, then we recover $\mathscr{H}$ as $\widetilde{X}\otimes_{C^*(G)\times G/N}(X\otimes_{C^*(N)}\mathscr{H})$, where $\widetilde{X}$ is the dual imprimitivity bimodule, and the natural isomorphism of $\mathscr{H}$ onto $\widetilde{X}\otimes_{C^*(G)\times G/N}(X\otimes_{C^*(N)}\mathscr{H})$ takes T to $1\otimes 1\otimes T$. Thus $T\mapsto 1\otimes T$ is an isomorphism of $\pi(N)'$ onto $\phi(C^*(G)\times_{\delta|}G/N)'$.

With V as in Eq. (1), the cocycle σ in the twisted action (α,σ) satisfies

$$\begin{aligned}\sigma(rN,sN) &= V_rV_sV^*_{rs}\\ &= W_{c(rN)}\pi(c(rN)^{-1}r)W_{c(sN)}\pi(c(sN)^{-1}s)\pi(c(rsN)^{-1}rs)^{-1}W^*_{c(rsN)}\\ &= W_{c(rN)}W_{c(sN)}\pi(c(sN)^{-1}c(rN)^{-1}rc(sN))\pi(c(sN)^{-1}s)\pi(s^{-1}r^{-1}c(rsN))W^*_{c(rsN)}\\ &= W_{c(rN)}W_{c(sN)}\pi(c(sN)^{-1}c(rN)^{-1}c(rsN))W^*_{c(rsN)}\\ &= W_{c(rN)}W_{c(sN)}W^*_{c(rsN)}\pi(c(rsN)c(sN)^{-1}c(rN)^{-1}).\end{aligned}$$

Thus with this choice of R_{sN}, we have $\omega(rN,sN) = 1\otimes\sigma(rN,sN)$, and for $T\in\pi(N)'$,

$$\begin{aligned}\beta_{sN}(1\otimes T)(\xi)(t) &= R_{sN}(1\otimes T)R^*_{sN}(\xi)(t)\\ &= W_{c(sN)}TW^*_{c(sN)}(\xi(t))\\ &= (1\otimes V_sTV_s^*)(\xi)(t)\\ &= (1\otimes \alpha_s(T))(\xi)(t).\end{aligned}$$

So the isomorphism of $\pi(N)'$ onto $\phi(C^*(G)\times_{\delta|}G/N)'$ carries (α,σ) into the twisted action (β,ω) which obstructs the existence of Q. Thus, reassuringly, the cohomological obstruction to finding Q is identical to the obstruction to extending π.

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