Journal of Algebra 375 (2013) 41–58

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Syzygies of differentials of forms

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ARTICLE INFO

Article history:
Received 16 December 2011
Available online 24 November 2012
Communicated by Bernd Ulrich

MSC:
13C40
13D02
13B22
13N05
14M05

Keywords:
Syzygies
Kähler differentials
Polarizability
Jacobian ideal
Homological dimension
Polar map

ABSTRACT

Given a standard graded polynomial ring $R = k[x_1, \dots, x_n]$ over a field k of characteristic zero and a graded k-subalgebra $A = k[f_1, \dots, f_m] \subset R$, one relates the module $\Omega_{A/k}$ of Kähler k-differentials of A to the transposed Jacobian module $\mathcal{D} \subset \sum_{i=1}^{n} R\, dx_i$ of the forms $f_1, \dots, f_m$ by means of a *Leibniz map* $\Omega_{A/k} \to \mathcal{D}$ whose kernel is the torsion of $\Omega_{A/k}$. Letting $\mathfrak{D}$ denote the R-submodule generated by the (image of the) syzygy module of $\Omega_{A/k}$ and $\mathfrak{Z}$ the syzygy module of $\mathcal{D}$, there is a natural inclusion $\mathfrak{D} \subset \mathfrak{Z}$ coming from the chain rule for composite derivatives. The main goal is to give means to test when this inclusion is an equality – in which case one says that the forms $f_1, \dots, f_m$ are *polarizable*. One surveys some classes of subalgebras that are generated by polarizable forms. The problem has some curious connections with constructs of commutative algebra, such as the Jacobian ideal, the conormal module and its torsion, homological dimension in R and syzygies, complete intersections and Koszul algebras. Some of these connections trigger questions which have interest in their own.

Introduction

The subject envisaged turns out to be a special case of a general module approximation situation, which we now state in a precise way.

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[1] Partially supported by the Ministerio de Educación y Ciencia – España (MTM2010-20279-C02-02).
[2] The author thanks the Universities of La Laguna and Valladolid for hospitality and facilities provided during the preparation of this work.

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http://dx.doi.org/10.1016/j.jalgebra.2012.11.008

We are given a field k of characteristic zero and an integral domain A of finite type over k admitting an embedding $A \simeq k[\mathbf{f}] = k[f_1, \dots, f_m] \subset R$, where $R = k[\mathbf{x}] = k[x_1, \dots, x_n]$ is a polynomial ring. Let $\mathcal{D}$ be a finitely generated torsion free R-module with an embedding $\mathcal{D} \subset R^n \simeq A^n \otimes_A R$ that reflects the nature of the above embedding $A \subset R$ – this vagueness is to be clarified in each particular event. The question is whether one can "approximate" a free presentation of $\mathcal{D}$ over R by means of a free presentation of a well-established A-module D.

More precisely, let $0 \to \mathfrak{Z} \to R^m \to \mathcal{D} \to 0$ be a presentation of $\mathcal{D}$ based on the generators of the embedding $\mathcal{D} \subset R^n$. One looks for an appropriate finitely generated A-module D along with a presentation $0 \to \mathrm{Z} \to A^m \to \mathrm{D} \to 0$ such that the R-submodule $\mathfrak{D} := \mathrm{Im}(\mathrm{Z} \otimes_A R) \subset A^m \otimes_A R \simeq R^m$ generated by Z approximates $\mathfrak{Z} \subset R^m$.

Besides looking for a sufficiently ubiquitous D, one ought to conceive a notion of "approximation". Natural features are:

(i) (Inclusion) $\mathfrak{D} \subset \mathfrak{Z}$;
(ii) (Rank) $\mathrm{rank}_R(\mathfrak{D}) = \mathrm{rank}_R(\mathfrak{Z})$;
Further, since $\mathfrak{Z}$ is a reflexive R-module, we may want that:
(iii) (Depth) $\mathfrak{D}$ be a reflexive R-module;
Or else, one may wish that $\mathfrak{Z}$ be closely approximated in low codimension, such as:
(iv) (Low codimension) $\mathfrak{D} = \mathfrak{Z}$ locally in codimension one (i.e., $\mathrm{ht}(\mathfrak{D} : \mathfrak{Z}) \geqslant 2$);
(v) (Contraction) $A^m \cap \mathfrak{Z} = \mathrm{Z}$ in the natural inclusion of A-modules $A^m \subset R^m$ induced by the ring extension $A \subset R$.

As is well-known, conditions (i)–(iii) imply that $\mathfrak{D} :_R \mathfrak{Z}$ is height one unmixed or else $\mathfrak{D} : \mathfrak{Z} = R$, hence adding condition (iv) yields the equality $\mathfrak{D} = \mathfrak{Z}$. In particular, they imply condition (v). On itself, (v) is a minimality condition on the approximation.

In this paper we deal with the following situation: one takes $\mathcal{D}$ as the transposed Jacobian module $\mathcal{D}(\mathbf{f})$ of the forms $\mathbf{f}$, while for D one takes the module $\Omega_{A/k}$ of Kähler k-differentials of A and then compare the respective modules of syzygies $\mathfrak{Z}$ and $\mathfrak{D}$. One should note that while $\Omega_{A/k}$ depends only on the k-algebra A and not on any particular presentation – such as $A \simeq k[\mathbf{f}] \subset k[\mathbf{x}]$ – $\mathcal{D}(\mathbf{f})$ depends on the choice of the forms $\mathbf{f}$. What saves the face of the transposed Jacobian module regarding this instability is a *Leibniz map* $\Omega_{A/k} \to \mathcal{D}(\mathbf{f})$. Although this map depends on $\mathbf{f}$, its kernel is ultimately uniquely defined and coincides with the A-torsion submodule of $\Omega_{A/k}$ (see Theorem 2.1).

We will say that $\mathbf{f}$ is *polarizable* if $\mathfrak{D} = \mathfrak{Z}$ in the foregoing notation.

From a strict point of view, the problem considered in this paper has already been stated in [7,1]. However, while in the latter one took a purely combinatorial approach, here one gets entangled in several module theoretic queries. As will be seen, this makes up for quite some difference between this and the previous references.

We focus on a set of arbitrary forms of degree 2. Actually some of the preliminary results in this paper hold for forms $\mathbf{f}$ of any fixed degree, but as we will see there is little hope to come around polarizability in such generality. From another end, polarizability is hardly a matter of "general" embeddings $A \subset R$. To illustrate this point, consider sets of general quadrics defining some of the well-known rational maps $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{m-1}$: these may fail to be polarizable even when the image of such maps is a smooth variety (see Example 5.2). Thus, there seems to be a strong correlation to sparcity and non-genericity, perhaps nearly the same way that birationality is thus related – in fact, in [1] there are unexpected connections between these two notions and the normality of A in case $\mathbf{f}$ are monomials of degree 2. As we guess, the lack of sparcity often imposes a high initial degree on the defining equations of A, an early obstruction to polarizability due to the relatively small initial degree of the syzygy module $\mathfrak{Z}$.

We next describe the contents of each section.

The first section contains the setup and definitions. The main statement is a result bringing up the Jacobian ideal of A as a tool to approximate the two modules. We state a general result about modules of the same rank and Fitting ideals (Lemma 1.3).

In the second section we introduce the Leibniz map $\lambda : \Omega_{A/k} \to \mathcal{D}(\mathbf{f})$. The main result (Theorem 2.1) shows a tight relationship between the kernel of λ, the torsion of $\Omega_{A/k}$ and the "contraction" of the differential syzygies $\mathfrak{Z}$ to A. We then relate this result to polarizability, by showing that under a certain contractibility hypothesis, polarizability implies the reflexiveness of the torsionfree A-module $P/P^{(2)}$, where $A \simeq k[\mathbf{T}]/P$.

The third section is dedicated to a discussion about when the algebra $A = k[\mathbf{f}]$ is a complete intersection on the presence of polarizability. As it turns out, this is the case nearly exactly when the transposed Jacobian module of $\mathbf{f}$ has homological dimension at most 1. There are some variations on this theme assuming conditions on A of possibly unexpected nature.

In the next section we discuss the relationship between the ordinary syzygies of $\mathbf{f}$ and their differential syzygies, thereby introducing a framework where both fit into a basic exact sequence. A consequence is the ability to give an alternative way of looking at the differential syzygies as related to the ordinary syzygies. As an application we show that by restricting the generating degrees of P, polarizability, and hence, the complete intersection property, in fact follows from the assumption that the homological dimension of $\mathcal{D}(\mathbf{f})$ is at most 1.

The last two sections are dedicated to a detailed analysis of selected examples with varied behavior, and also examples of a more structured nature coming from constructs in algebraic geometry. At the end we state some open questions whose answers look essential for further development of the subject.

1. Preliminaries

We establish the basic setup, drawing upon the notation stated in the introduction. Thus, $A = k[\mathbf{f}] \subset R$ denotes a finitely generated k-subalgebra of the polynomial ring R and $A \simeq k[\mathbf{T}]/P$ by mapping $T_j \mapsto f_j$.

Recall the well-known conormal sequence

$$P/P^2 \xrightarrow{\delta} \sum_{j=1}^{m} A\,dT_j \to \Omega_{A/k} \to 0,$$

where δ maps a generator of P to its differential modulo P. More exactly, upon choosing a generating set of P, the image of δ is the A-submodule generated by the image of the transposed Jacobian matrix over $k[\mathbf{T}]$ of a generating set of P. Since P is a prime ideal and $\mathrm{char}(k) = 0$, we know the kernel of the leftmost map above. Therefore, we will focus on the exact sequence

$$0 \to P/P^{(2)} \xrightarrow{\delta} \sum_{j=1}^{m} A\,dT_j \to \Omega_{A/k} \to 0, \tag{1}$$

where $P^{(2)}$ stands for the second symbolic power of P.

Throughout, we let $\mathfrak{D} \subset \sum_{j=1}^{m} R\,dT_j$, denote the R-submodule generated by the image $\delta(P/P^{(2)})$ through the embedding $\sum_{j=1}^{m} A\,dT_j \subset \sum_{i=1}^{m} R\,dT_j$ induced by the inclusion $A \subset R$. Then $\mathfrak{D}$ is generated by the vectors $\sum_j \frac{\partial F}{\partial T_j}(\mathbf{f})\,dT_j$, where F runs through a set of generators of P. On the other hand, by the chain rule of composite derivatives, if $F \in P$ then $\sum_{j=1}^{m} \frac{\partial F}{\partial T_j}(\mathbf{f})\,df_j = 0$. This means that $\mathfrak{D} \subset \mathfrak{Z}$, where $\mathfrak{Z}$ is the first syzygy module of the differentials $d\mathbf{f}$. In other words, the elements of $\mathfrak{D}$ are relations of the transposed Jacobian matrix of $\mathbf{f}$. These ideas have also been discussed in a different context in [9, Section 1.1] and even earlier in [2, Main Lemma 2.3(i)]. Borrowing from this line of thought, the present problem asks when a natural short complex involving differentials is exact.

Definition 1.1. The elements of $\mathfrak{D}$ will be called *polar syzygies* of $\mathbf{f}$ (or of the embedding $A \subset R$ if no confusion arises), while $\mathfrak{D}$ itself is named the *polar syzygy module* of $\mathbf{f}$. For the sake of comparison, we call $\mathfrak{Z}$ the *differential syzygy module* of $\mathbf{f}$ – though this is actually the syzygy module of the differentials $d\mathbf{f}$. Accordingly, we say that $\mathbf{f}$ (or the embedding $A \subset R$) is *polarizable* if $\mathfrak{D} = \mathfrak{Z}$.

The definition adopted here is the same as in [1] but slightly departs from the original definition given in [7]. We will have a chance to deal with the slight difference later in this work – see also the comment after Example 3.6. However, in general it remains a bewildering piece.

Example 1.2. To see how the concept may actually depend on the choice of the generators $\mathbf{f}$, consider the following non-extravagant example: $A = k[x^2, x^3] \subset k[x]$. Here, the differentials are $2x\,dx$ and $3x^2\,dx$, respectively, hence the syzygy module of this set of generators is the cyclic module generated by the single vector $(3x, -2)^t$, while the polar module $\mathfrak{D}$ is generated by the vector $(3x^4, -2x^3)^t$ that lies deep inside $\mathfrak{Z}$. Thus, there is an inclusion as it should be, but not an equality. On the other hand, the transposed Jacobian module $\mathcal{D}(\mathbf{f})$ is minimally generated by the differential $2x\,dx$, hence, for this generator, $\mathfrak{Z} = \{0\}$. But, of course, A itself cannot be generated over k by less than 2 elements, so we still have the same $\mathfrak{D} \neq \{0\}$ as before! Thus the theory depends on fixing the set $\mathbf{f}$ of generators of the k-algebra A, fixing $\mathfrak{Z}$ as the kernel of the presentation of $\mathcal{D}(\mathbf{f})$ on the generators $d\mathbf{f}$ and, likewise fixing $\mathfrak{D}$ as the R-submodule generated by those polar syzygies obtained by evaluating a set of generators of P which are polynomial relations of $d\mathbf{f}$.

Of course, some of the particular features of the above example will not be present if one sticks to the truly *unirational* case, i.e., when A is generated by forms of a fixed degree. But even then polarizability may fail to be an invariant property of the k-isomorphism class of A (see [1, 5.5]). Nonetheless a definite advantage of dealing with homogeneous generators, not necessarily of the same degree, is that, as a consequence of the Euler map, their k-linear independence is tantamount to that of the corresponding differentials (see the setup in the proof of Proposition 4.1).

A trivial example of a polarizable embedding is the case when A itself is a polynomial ring over k, i.e., when $\mathbf{f}$ are algebraically independent over k. This is scarcely of any interest for the problem and we will rather look at genuine cases of the notion. For these, an important role will be played by the *Jacobian ideal* $\mathfrak{J} = \mathfrak{J}_A$ of A, which is read off the presentation $A \simeq k[T_1, \ldots, T_m]/P$ by taking the ideal on A generated by the g-minors of the Jacobian matrix of a set of generators of P, where $g = \operatorname{ht} P$. It is well-known that $\mathfrak{J}$ coincides with the dth Fitting ideal of the module of Kähler k-differentials, where $d = \dim A$, hence is independent of the presentation of A and the choice of generators of P. We will write $\mathfrak{J}R \subset R$ for the extended ideal in $R = k[\mathbf{x}]$.

In order to involve the Jacobian ideal we will rely on a lemma of independent general interest, of which we found no explicit mention, much less a proof, though it is conceivable that it may be a piece of folklore.

To place it in a familiar framework, let R be an integral domain and let E be finitely generated R-module with rank r; then some power of the rth Fitting ideal $I \subset R$ of E annihilates its torsion $\tau(E)$. This assertion follows simply from the fact that $\tau(E)_P = \{0\}$ for every prime ideal $P \notin V(I)$. In particular, if $E \twoheadrightarrow F$ is a surjective homomorphism of R-modules of the same rank r then some power of the Fitting ideal of E annihilates $\ker(E \twoheadrightarrow F)$.

The next result shows that if, moreover, E and F are generated by sets of the same cardinality then I itself already annihilates the kernel. We state the result in the following format, which is more appropriate to the subsequent development.

Lemma 1.3. *Let R be an integral domain and let $M \subset N \subset R^m$ be finitely generated submodules of a free module, having the same rank g. Let $I \subset R$ denote the Fitting ideal of order $m - g$ of the cokernel R^m/M. Then $I \subset M : N$.*

Proof. The proof consists of a simple application of Laplace rule for computing determinants. Namely, note that I can be taken to be the ideal generated by the $g \times g$ minors of the matrix whose columns are the generators of M expressed as linear combinations of the canonical basis of R^m. Thus, let $\Delta \in I$ denote a nonzero determinant thereof. We may assume for simplicity that it is the determinant of the $g \times g$ submatrix on the upper left corner. Given any $i = g+1, \ldots, m$, consider the following $(g+1) \times g$ submatrix of the columns generating M:

$$\begin{pmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,g} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,g} \\ \vdots & \vdots & & \vdots \\ u_{g,1} & u_{g,2} & \cdots & u_{g,g} \\ u_{i,1} & u_{i,2} & \cdots & u_{i,g} \end{pmatrix}.$$

Now, given any column generator of N, right border the above matrix with the corresponding entries $v_1, \ldots, v_g, v_i$ of this column to get a $(g+1) \times (g+1)$ matrix whose columns are elements of N. These generate a submodule of N, hence has rank at most g. Therefore the corresponding $(g+1) \times (g+1)$ determinant vanishes. Developing this determinant by Laplace along the bottom row, one finds

$$\Delta v_i = \Delta_{\hat{1}2\ldots gv} u_{i,1} + \Delta_{1\hat{2}3\ldots gv} u_{i,2} + \cdots + \Delta_{12\ldots\hat{g}v} u_{i,g}, \tag{2}$$

where $\Delta_{12\ldots\hat{j}\ldots gv}$ denotes the g-minor obtained by replacing the jth column with column v.

If, on the other hand, $i \in \{1, \ldots, g\}$ then we obtain again a $(g+1) \times (g+1)$ matrix by first bordering the initial $g \times g$ submatrix with the entries $v_1, \ldots, v_g$ and then repeating the ith row of this matrix on the bottom. Clearly, this determinant is zero; developing it as before along the repeated row, we find a similar expression as (2), with the same fixed g-minors as multipliers.

This shows that the entire column generator $v \in N$ is conducted by Δ inside M. $\square$

Corollary 1.4. *Keeping the previous notation, let $\mathfrak{J} \subset A$ stand for the Jacobian ideal of A. Then $\mathfrak{J}R \subset \mathfrak{D} :_R \mathfrak{Z}$. Moreover, $\mathbf{f}$ is polarizable if and only if $\mathfrak{D} :_{R^m} \mathfrak{J}R = \mathfrak{D}$.*

Proof. We have $\mathfrak{D} \subset \mathfrak{Z} \subset \sum_{j=1}^m R\,dT_j$ and $\operatorname{rank}_R(\mathfrak{D}) = \operatorname{rank}_A(P/P^{(2)}) = \operatorname{ht} P = \operatorname{rank}_R(\mathfrak{Z})$ (see [1, 2.3]). By definition and the well-known fact that determinants commute with base change, $\mathfrak{J}R$ is the Fitting ideal of $R^m/\mathfrak{D}$ of order $m - \operatorname{rank}_R(\mathfrak{D})$. We apply Lemma 1.3 with $R^m = \sum_{j=1}^m R\,dT_j$, $M = \mathfrak{D}$, $N = \mathfrak{Z}$, where $I = \mathfrak{J}R$. This proves the first statement. The second assertion is a consequence of a general property of modules: if N is the kernel of a homomorphism $\eta : R^m \to R^n$ of free modules, then $N :_{R^m} J = N$ for any nonzero ideal $J \subset R$. Indeed, let $0 \neq a \in J$ and $v \in R^m$ with $av \in N$. Applying η yields $a\eta(v) \in \eta(N) = \{0\}$, hence $\eta(v) = 0$, i.e., $v \in N$, as claimed. Now, to get the second statement of the corollary, apply this fact with $N = \mathfrak{Z}$. $\square$

Of course, in the statement of the corollary one could replace $\mathfrak{J}R$ by any nonzero ideal $\mathfrak{a} \subset R$ such that $\mathfrak{a} \subset \mathfrak{D} :_R \mathfrak{Z}$. The point is that the Jacobian ideal is a *bona fide* test ideal for polarizability that avoids knowing $\mathfrak{Z}$ a priori. In this vein, it would be interesting to know a priori when $\mathfrak{Z} = (\mathfrak{D})^{**}$ (bidual). Of course, it is often the case that $\mathfrak{D}$ is reflexive but $\mathfrak{D} \neq \mathfrak{Z}$ locally in codimension one.

For further insight into the role of the Jacobian ideal vis-à-vis polarizability we refer to the next section, where the role of the Kähler differential forms is emphasized.

2. The Leibniz map

Recall the R-module $\mathfrak{Z} = \ker(\sum_{j=1}^m R\,dT_j \twoheadrightarrow \mathcal{D}(\mathbf{f}))$ and the conormal exact sequence of the module of Kähler k-differentials of A as in (1):

$$0 \to P/P^{(2)} \xrightarrow{\delta} \sum_{j=1}^m A\,dT_j \xrightarrow{\pi} \Omega_{A/k} \to 0.$$

Theorem 2.1. *There is an A-module homomorphism $\lambda : \Omega_{A/k} \to \mathcal{D}(\mathbf{f})$ such that*

$$\pi^{-1}\big(\ker(\lambda)\big) = \pi^{-1}\big(\tau_A(\Omega_{A/k})\big) = (\mathfrak{Z})^c,$$

where $\tau_A(\Omega_{A/k})$ denotes A-torsion submodule of $\Omega_{A/k}$ and $(\mathfrak{Z})^c$ denotes the contraction of $\mathfrak{Z}$ via the natural inclusion $\sum_{j=1}^m A\,dT_j \subset \sum_{j=1}^m R\,dT_j$ induced by $A \subset R$.

Proof. We define the A-module homomorphism

$$\tilde{\lambda}: \sum_{j=1}^m A\,dT_j \twoheadrightarrow \sum_{j=1}^m A\,df_j \subset \sum_{j=1}^m R\,df_j = \mathcal{D}(\mathbf{f}) \subset \sum_{i=1}^n R\,dx_i$$

by the association $\tilde{\lambda}(dT_j) = df_j$. Then the chain rule for composite derivatives shows that $\tilde{\lambda}$ induces an A-map $\lambda : \Omega_{A/k} \to \mathcal{D}(\mathbf{f})$. Note that the image of λ generates $\mathcal{D}(\mathbf{f})$, although the map depends on $\mathbf{f}$ – i.e., on the chosen embedding $A \subset R$, not just on A. Thus, $\Omega_{A/k}$ is the only module in sight that depends only on A.

Now, λ fits in a commutative diagram of A-modules and A-homomorphisms

$$\begin{tikzcd} 0 \arrow[r] & P/P^{(2)} \arrow[r, "\delta"] \arrow[d, phantom, "\bigcap"] & \sum_{j=1}^m A\,dT_j \arrow[r, "\pi"] \arrow[d, phantom, "\bigcap"] & \Omega_{A/k} \arrow[r] \arrow[d, "\lambda"] & 0 \\ 0 \arrow[r] & \mathfrak{Z} \arrow[r] & \sum_{j=1}^m R\,dT_j \arrow[r] & \mathcal{D}(\mathbf{f}) \arrow[r] & 0. \end{tikzcd}$$

From this diagram one readily sees that $(\mathfrak{Z})^c \subset \pi^{-1}(\ker(\lambda))$. On the other hand, one clearly has a surjection $\Omega_{A/k}/\tau_A(\Omega_{A/k}) \twoheadrightarrow \Omega_{A/k}/\ker(\lambda)$ since $\mathcal{D}(\mathbf{f})$ is R-torsionfree, hence also A-torsionfree. Let K (respectively, L) denote the fraction field of A (respectively, of R). Then

$$\begin{aligned} \operatorname{rank}_A\bigl(\Omega_{A/k}/\ker(\lambda)\bigr) &= \dim_K\bigl(\Omega_{A/k}/\ker(\lambda) \otimes_A K\bigr) = \dim_L\bigl(R\Omega_{A/k}/\ker(\lambda) \otimes_R L\bigr) \\ &= \dim_L\bigl(\bigl(\Omega_{A/k}/\ker(\lambda) \otimes_A K\bigr) \otimes_K L\bigr) \\ &= \operatorname{rank}_R\bigl(\mathcal{D}(\mathbf{f})\bigr) = \dim A. \end{aligned}$$

It follows that the kernel of the surjection $\Omega_{A/k}/\tau_A(\Omega_{A/k}) \twoheadrightarrow \Omega_{A/k}/\ker(\lambda)$ has rank 0, hence must be the null module since $\Omega_{A/k}/\tau_A(\Omega_{A/k})$ is torsionfree. Therefore, $\ker(\lambda) = \tau_A(\Omega_{A/k})$.

Now, again since $\Omega_{A/k}$ has rank $\dim A$, then $\operatorname{rank}_A(P/P^{(2)}) = m - \dim A = \operatorname{ht}(P)$. Therefore, the Fitting ideal of order $\dim A$ of $\Omega_{A/k}$ is the Jacobian ideal $\mathfrak{J}$ of A. It follows that $\tau_A(\Omega_{A/k}) = 0 : \mathfrak{J}^\infty$ (see [8, Lemma 5.2]). Writing out this equality in terms of submodules of $\sum_{j=1}^m A\,dT_j$, yields the lifting $\pi^{-1}(\tau_A(\Omega_{A/k})) = P/P^{(2)} : \mathfrak{J}^\infty$. But the embedding $\sum_{j=1}^m A\,dT_j \subset \sum_{j=1}^m R\,dT_j$ induces an embedding $P/P^{(2)} : \mathfrak{J}^\infty \subset \mathfrak{D} : (\mathfrak{J}R)^\infty$ which is preserved after contraction back to $\sum_{j=1}^m A\,dT_j$. On the other hand,

$$\mathfrak{D} : (\mathfrak{J}R)^\infty \subset \mathfrak{Z} : (\mathfrak{J}R)^\infty = \mathfrak{Z},$$

since $\mathfrak{Z}$ is a second syzygy. It follows that $\pi^{-1}(\tau_A(\Omega_{A/k})) \subset (\sum_{j=1}^m A\,dT_j) \cap \mathfrak{Z} = (\mathfrak{Z})^c$. Collecting the pieces, we have

$$\pi^{-1}\bigl(\tau_A(\Omega_{A/k})\bigr) \subset (\mathfrak{Z})^c \subset \pi^{-1}\bigl(\ker(\lambda)\bigr) = \pi^{-1}\bigl(\tau_A(\Omega_{A/k})\bigr),$$

thus proving the statement. □

Definition 2.2. We call the map λ the *Leibniz map.*

The following result is an immediate consequence of Theorem 2.1.

Corollary 2.3. *With the above notation, the following conditions are equivalent*:

(i) *$P/P^{(2)}$ is contracted from $\mathfrak{Z}$.*
(ii) *The Leibniz map λ is injective.*
(iii) *$\Omega_{A/k}$ is torsion free.*

Moreover, any of these conditions implies that $P/P^{(2)}$ is a reflexive A-module. In particular, if $P/P^{(2)}$ is the contraction of its R-extension $\mathfrak{D}$ and if **f** *is polarizable then $P/P^{(2)}$ is a reflexive A-module.*

Additional applications will be given in the next section.

Remark 2.4. It is illuminating to compare the result in the above corollary to the one of Corollary 1.4. The reason why torsionfreeness of $\Omega_{A/k}$ does not imply polarizability is that the former condition means that $P/P^{(2)} :_A \mathfrak{J} = P/P^{(2)}$, and hence $(P/P^{(2)} :_A \mathfrak{J})R = \mathfrak{D}$, while polarizability says that $\mathfrak{D} :_R \mathfrak{J}R = \mathfrak{D}$. Clearly, in general there is an inclusion $(P/P^{(2)} :_A \mathfrak{J})R \subset (P/P^{(2)})R :_R \mathfrak{J}R = \mathfrak{D} :_R \mathfrak{J}R$; thus, knowing that $\Omega_{A/k}$ is torsionfree does not teach us a lot more than the inclusion $\mathfrak{D} \subset \mathfrak{D} :_R \mathfrak{J}R$.

For explicit examples comparing the two properties, see Example 3.3; and, in degree 2, Example 5.4.

3. Complete intersections

In this part we focus on the case where A is a complete intersection. The first result is a criterion, in terms of polarizability, for a normal almost complete intersection A with “small” invariants to be a complete intersection.

Proposition 3.1. *With the above notation, suppose that*:

(a) *A is a normal almost complete intersection*;
(b) $\dim A = \operatorname{ecod} A = 2$, *where* ecod *denotes embedding codimension.*

If the embedding $A \subset R$ is polarizable and $P/P^{(2)}$ is the contraction of its R-extension then A is a complete intersection in this embedding (*i.e., P is generated by* 2 *elements*).

Proof. Let $A \simeq k[\mathbf{T}]/P$ as before. Since $\dim A = 2$ and A is normal (hence, satisfies (S_2)), then A is Cohen–Macaulay. Since A is an almost complete intersection then $P^2 = P^{(2)}$ (see [10, (4.4)]). Then, by Corollary 2.3, $P/P^2 = P/P^{(2)}$ is reflexive, hence Cohen–Macaulay because $\dim A = 2$. By [6, 2.4], A is a complete intersection. □

Question 3.2. In general the assumption that P have deviation at most 1 is essential for triggering the torsionfreeness of P/P^2 even if the conditions of (b) hold. However, in the present case, P is a particular prime ideal, so one asks whether in the present context the assumption of being almost complete intersection is superfluous.

Example 3.3. Let $A = k[x^3, x^2y, xy^2, y^3] \subset R = k[x, y]$ be the parameters of the rational normal cubic. Then $A \simeq k[T_1, T_2, T_3, T_4]/P$, with $P = I_2(H)$, where

$$H = \begin{pmatrix} T_1 & T_2 & T_3 \\ T_2 & T_3 & T_4 \end{pmatrix}.$$

Therefore, A is a normal Cohen–Macaulay almost complete intersection of codimension 2, hence P/P^2 is torsionfree, but not reflexive. In fact, since A is non-obstructed, P/P^2 is a proper reduction of the double dual $(P/P^2)^{**}$ [10, Corollary 4.5]. Also, $\Omega_{A/k}$ has nontrivial torsion submodule. Note that P/P^2 is contracted from its R-extension. Therefore, $A \subset R$ is not polarizable.

Of a different flavor is the next result, naturally extending [7, Theorem 2.5, (a) $\Leftrightarrow$ (c)].

Proposition 3.4. *With the above notation, suppose that* $\mathbf{f}$ *are homogeneous of the same degree and that the embedding* $A = k[\mathbf{f}] \subset R$ *is polarizable. Writing* $A \simeq k[\mathbf{T}]/P$, *assume that* $\mu(P) = \mu(\mathfrak{D})$. *The following conditions are equivalent*:

(a) *The homological dimension of* $\mathcal{D}(\mathbf{f})$ *is at most one*;
(b) P *is generated by a regular sequence* (*i.e.,* A *is a complete intersection*).

Proof. (a) $\Rightarrow$ (b) By a well-known stability preliminary, the homological assumption means that $\mathfrak{Z}$ is a projective module, hence free. Since $\mathbf{f}$ is assumed to be polarizable, $\mathfrak{D}$ is free as well (of rank $\operatorname{ht} P$ [1, 2.3]). Now, by assumption $\mu(P) = \mu(\mathfrak{D})$. On the other hand, $\mu_A(P/P^2) = \mu(P)$ because P is homogeneous. Then one gets $\mu_A(P/P^2) = \operatorname{ht} P = \operatorname{rank}_A(P/P^2)$. It follows that P/P^2 is a free A-module and hence, P is generated by a regular sequence of forms by the Ferrand–Vasconcelos theorem.

(b) $\Rightarrow$ (a) Since P is generated by a regular sequence then P/P^2 is a free A-module. The assumption $\mu(P) = \mu(\mathfrak{D})$ then yields $\mu(\mathfrak{D}) = \operatorname{ht} P = \operatorname{rank}_R(\mathfrak{D})$. By polarizability, $\mu(\mathfrak{Z}) = \operatorname{rank}_R(\mathfrak{Z})$, hence $\mathfrak{Z}$ is free. This shows that the homological dimension of $\mathcal{D}(\mathbf{f})$ is at most one. $\square$

The previous result stands on the assumption that $\mu(P) = \mu(\mathfrak{D})$. Here is one situation where this equality holds.

Proposition 3.5. *Suppose that* $\mathbf{f}$ *are* k*-linearly independent forms of the same degree and, moreover, that* P *is generated in the same degree. Then* $\mu(P) = \mu(\mathfrak{D})$.

Proof. Morally, this comprises two steps. First, $\mu(P) = \mu(P/P^{(2)})$, i.e., P does not loose minimal generators in the passage to the symbolic conormal module. Namely, we claim that the kernel of the following map of k-vector spaces

$$\frac{P}{(\mathbf{T})P} \twoheadrightarrow \frac{P/P^{(2)}}{(\mathbf{T})P/P^{(2)}}$$

vanishes. Indeed, by the Zariski–Nagata differential criterion for symbolic powers in characteristic zero, the elements of $P^{(2)}$ satisfy the property that all its partial derivatives belong to P. Also, since $P^{(2)}$ is the P-primary component of the homogeneous ideal P^2, it is homogeneous. Therefore, the Euler formula as applied to individual elements of $P^{(2)}$ yields $P^{(2)} \subset (\mathbf{T})P$, as required.

Second step: $\mu(P) = \mu(\mathfrak{D})$, i.e., P – or $P/P^{(2)}$, which now amounts to the same – does not loose minimal generators while extending to an R-module. For this to hold we need the second assumption. Thus, let P be minimally generated by forms $\{F_1, \dots, F_r\}$ of the same degree. In any case $\mathfrak{D}$ is generated by the evaluated differentials $\{dF_1(\mathbf{f}), \dots, dF_r(\mathbf{f})\}$ as an R-module. But by the assumption on $\mathbf{f}$ these generators have the same standard R-degree, so the only way they can fail to be minimal is that they be linearly dependent over the base field k. However, such a dependence immediately implies that there are $\lambda_1, \dots, \lambda_r \in k$, not all zero, such that

$$\sum_{1 \leqslant l \leqslant r} \lambda_l \frac{\partial F_l}{\partial T_j} \in P, \quad 1 \leqslant j \leqslant m.$$

Since $\deg(\partial F_l/\partial T_j) < \deg(F_l)$ for $\partial F_l/\partial T_j \neq 0$, we arrive at a contradiction. $\square$

The hypothesis on P seems to be needed in general, as the following example indicates.

Example 3.6. Let $R = k[x_1, \ldots, x_6]$ and let

$$\mathbf{f} = \{x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_1, x_1x_3, x_3x_5, x_5x_1\}. \tag{3}$$

Then the presentation ideal of $k[\mathbf{f}]$ is a codimension 3 almost complete intersection in which the quadrics generate a maximal regular sequence, while the fourth generator lives in degree 3. It can be shown that $\mathbf{f}$ is polarizable and that $\mathfrak{D}$ is a free module generated by the differentials of the three quadrics (see [1, Example 5.21]).

It would be interesting to know, under the assumption that $\mathbf{f}$ are homogeneous of the same degree and perhaps also under the assumption of polarizability, when the equality $\mu(P) = \mu(\mathfrak{D})$ holds. By a quirk this is the case if $\mathbf{f}$ happens to be the set of degree 2 monomials corresponding to a connected bipartite graph – see [7, Theorem 2.3], where this falls in an indirect way from the main result.

Remark 3.7. To see how subtle the problem is, one can take the following bipartite graph, whose edge-ideal is ideal theoretically entirely analogous to the above:

$$\mathbf{f} = \{x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_1, x_1x_7, x_3x_7, x_5x_7\}. \tag{4}$$

Both this and (3) are Cohen–Macaulay ideals of codimension 3, with same graded Betti numbers. However, here $\mu(P) = \mu(\mathfrak{D})$ (by [7] or by direct computation).

We observe that in both (3) and (4) the ideal P is the homogeneous defining ideal of an arithmetically normal projective variety. Thus, in both cases one can deduce polarizability from [1, Theorem 5.10].

We have the following immediate consequence.

Corollary 3.8. *Suppose that* $\mathbf{f}$ *are homogeneous of the same degree and that the embedding* $k[\mathbf{f}] \subset R$ *is polarizable. If* P *is generated in the same degree and the homological dimension of* $\mathcal{D}(\mathbf{f})$ *is at most one then* A *is a complete intersection.*

Proof. One applies Proposition 3.4 along with Proposition 3.5. $\square$

We emphasize the role of polarizability in the last corollary. Thus, e.g., if $\mathbf{f}$ are the parameters defining the twisted cubic in $\mathbb{P}^3$, the hypotheses in the above statement are satisfied, nevertheless A is not a complete intersection, and indeed it is not polarizable (see Example 3.3).

At the other end of the spectrum, even under the hypothesis of polarizability the result is false in higher homological dimension.

Example 3.9. Let $R = k[x_1, x_2, x_3]$ and let $A = R^{(2)} \subset R$ be the 2-Veronese of R. Then this embedding is polarizable by [1, 5.1] and A is defined by quadrics, while the projective dimension of $\mathcal{D}(\mathbf{f})$ is 2.

The question naturally arises as to when the two conditions in Corollary 3.8 actually imply that $k[\mathbf{f}] \subset R$ is polarizable (hence a complete intersection). In the next section we give an answer to this question under an additional condition (cf. Proposition 4.4).

4. The Euler–Jacobi–Koszul exact sequence

In this section we still assume that $\mathbf{f}$ are forms, but not necessarily of the same degree.

We wish to relate more closely the polar and differential syzygies of $\mathbf{f}$ to its ordinary syzygies. Henceforth, for any set $\mathbf{h} \subset R$, $Z(\mathbf{h})$ denotes the first syzygy module of $\mathbf{h}$. Set $d_j = \deg(f_j)$, $1 \leqslant j \leqslant m$, and $\tilde{\mathbf{f}} = \{d_1 f_1, \ldots, d_m f_m\}$. Since $\operatorname{char}(k) = 0$, clearly $(\mathbf{f})R = (\tilde{\mathbf{f}})R$.

Proposition 4.1. *There is an exact sequence of R-modules*

$$0 \to \mathfrak{Z} \to Z(\tilde{\mathbf{f}}) \xrightarrow{\jmath} \mathcal{D}(\mathbf{f}) \cap Z(\mathbf{x}) \to 0, \tag{5}$$

where $\jmath$ sends a syzygy $(g_1, \ldots, g_m)$ of $\tilde{\mathbf{f}}$ to the element $\sum_{j=1}^m g_j\, df_j$. In particular, if $\mathbf{f}$ is polarizable then there is an exact sequence of R-modules

$$0 \to \mathfrak{D} \to Z(\tilde{\mathbf{f}}) \to \mathcal{D}(\mathbf{f}) \cap Z(\mathbf{x}) \to 0. \tag{6}$$

Proof. The restriction of the Euler map $\epsilon : \sum_{i=1}^n R\, dx_i \to R$, where $\epsilon(dx_i) = x_i$, induces an exact sequence of R-modules

$$0 \to \mathcal{D}(\mathbf{f}) \cap Z(\mathbf{x}) \to \mathcal{D}(\mathbf{f}) = \sum_{j=1}^m R\, df_j \to (\tilde{\mathbf{f}})R = (\mathbf{f})R \to 0.$$

This in turn fits in a snake diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & \mathcal{D}(\mathbf{f}) \cap Z(\mathbf{x}) \arrow[d] & \\
0 \arrow[r] & \mathfrak{Z} \arrow[r] \arrow[d] & \sum_{i=1}^m R\, dT_j \arrow[r, "dT_j \mapsto df_j"] \arrow[d, equal] & \mathcal{D}(\mathbf{f}) \arrow[r] \arrow[d, "\epsilon"] & 0 \\
0 \arrow[r] & Z(\tilde{\mathbf{f}}) \arrow[r] \arrow[d] & \sum_{i=1}^m R\, dT_j \arrow[r, "dT_j \mapsto d_j f_j"] \arrow[d] & (\mathbf{f})R \arrow[r] \arrow[d] & 0 \\
 & Z(\tilde{\mathbf{f}})/\mathfrak{Z} \arrow[d] & 0 & 0 & \\
 & 0, & & &
\end{tikzcd}$$

Then $\jmath$ is the lifting to $Z(\tilde{\mathbf{f}})$ of the inverse of the connecting isomorphism in the kernel–cokernel sequence by the snake lemma. To make this map explicit, note that an element $\sum_{j=1}^m g_j\, df_j \in \mathcal{D}(\mathbf{f}) \cap Z(\mathbf{x})$ is characterized by the equation $\mathbf{g} \cdot \Theta(\mathbf{f}) \cdot \mathbf{x}^t = 0$ or by its transpose, where Θ denotes the Jacobian matrix of $\mathbf{f}$. This readily gives the way $\jmath$ acts. $\square$

Definition 4.2. The exact sequence (5) could be called the *Euler–Jacobi–Koszul syzygy sequence*, while $\mathcal{D}(\mathbf{f}) \cap Z(\mathbf{x})$ could accordingly be dubbed the *module of Euler–Jacobian syzygies* of $\mathbf{f}$ and $\jmath$ the *Jacobi map* of $\mathbf{f}$.

Both exact sequences are ways of showing, in characteristic zero, that the syzygies of a polarizable set of forms of the same degree have a fixed structure in differential terms.

Here is a computational view of the above syzygy sequence in terms of the involved matrices, emphasizing the Jacobi map. For the sake of simplicity we assume that $\mathbf{f}$ are forms of the same degree and for the sake of lighter reading we write $R^n = \sum_{i=1}^m R\, dx_i$, $R^m = \sum_{j=1}^m R\, dT_j$.

Denoting by Θ^t the transposed Jacobian matrix of $\mathbf{f}$, one has maps

$$\begin{tikzcd}
R^q \arrow[dr, "\psi"'] & & & & & \\
& R^p \arrow[rr, "\varphi"] \arrow[dr, "\Theta^t\cdot\varphi"'] & & R^m \arrow[r] \arrow[dl, "\Theta^t"] & (\mathbf{f}) \arrow[r] & 0 \\
& & R^n & & &
\end{tikzcd}$$

where φ denotes the syzygy matrix of $\mathbf{f}$ and ψ denotes the syzygy matrix of $\Theta^t\cdot\varphi$. Then

$$\operatorname{im}\bigl(\Theta^t\cdot\varphi\bigr)\subset\operatorname{im}\bigl(\Theta^t\bigr)\cap\operatorname{im}\left(\bigwedge^2 R^n \xrightarrow{\kappa} R^n\right),$$

with κ denoting the first map of the Koszul complex on $\mathbf{x}$. Thus, the columns of the matrix $\Theta^t\cdot\varphi$ generate the image of the map $\mathfrak{j}$. Viewed this way, $\mathfrak{Z}$ is also the submodule of R^m generated by the columns of the product matrix $\varphi\cdot\psi$, though not minimally – in fact, quite often some columns may be null.

Certainly, this is not how one would like to compute generators of $\mathfrak{Z}$, as it depends on computing syzygies of an even more involved module. Its purpose is mostly theoretical, indicating the intermediation of the syzygies of $\mathbf{f}$, thus leading us to make assumptions on those syzygies.

Let $\operatorname{indeg}_T(E)$ the initial degree of a graded module over a graded ring T. One has the easy

Lemma 4.3. *Let $\mathbf{f}\subset R$ be k-linearly independent forms of the same degree $\geqslant 2$. Then $\operatorname{indeg}_R(\mathfrak{D})\geqslant 2$ and $\operatorname{indeg}_R(\mathfrak{Z})\geqslant 2$. Moreover, suppose that $\mathfrak{D}$ is generated in fixed degree d, that $\operatorname{indeg}_R(\mathfrak{Z})\geqslant d$ and that $\mu(\mathfrak{D})\geqslant\mu(\mathfrak{Z})$. Then $\mathbf{f}$ is polarizable.*

Proof. Since $\mathbf{f}$ are k-linearly independent forms of the same degree, one has an isomorphism of graded k-algebras $k[\mathbf{T}]/P\simeq A=k[\mathbf{f}]\subset R$, where P is a homogeneous ideal generated in degree $\geqslant 2$. Clearly then the image of P/P^2 in $\sum_j A\,dT_j$ is a graded A-module generated in degree $\geqslant 1$. Recall that $\mathfrak{D}$ is generated by these generators further evaluated on $\mathbf{f}\subset R$. Since $\deg(\mathbf{f})\geqslant 2$, we clearly have $\operatorname{indeg}_R(\mathfrak{D})\geqslant 2$.

As for $\mathfrak{Z}$, as seen above the entire syzygy module $\mathfrak{Z}$ is generated by the columns of the product matrix $\varphi\cdot\psi$, where φ is the syzygy matrix of $\mathbf{f}$, hence its entries have R-degree at least 1. Taking a minimal set of generators of $\operatorname{Im}(\Theta^t\cdot\varphi)$, the entries of the syzygy matrix ψ of $\Theta^t\cdot\varphi$ will be forms of positive degree. It follows that the columns of $\varphi\cdot\psi$ have degree at least 2, hence $\mathfrak{Z}$ is generated in degree at least 2.

The last assertion is clear since then all the minimal generators of $\mathfrak{D}$ must be minimal generators of $\mathfrak{Z}$ and the latter can thereby have no other minimal generators. □

An example of application of this sort of ideas is as follows.

Proposition 4.4. *With the above notation, suppose that $\mathbf{f}$ are k-linearly independent forms of degree 2 and set $k[\mathbf{f}]\simeq k[\mathbf{T}]/P$ as before. Assume that:*

(a) *P is minimally generated by forms of degree 2;*
(b) *The homological dimension of $\mathcal{D}(\mathbf{f})$ is at most one.*

Then A is a polarizable complete intersection.

Proof. By (b) $\mathfrak{Z}$ is a free module, i.e., $\mu(\mathfrak{Z}) = \operatorname{rank}\mathfrak{Z}$. Since $\operatorname{rank}\mathfrak{D} = \operatorname{rank}\mathfrak{Z}$, we have $\mu(\mathfrak{D}) \geqslant \mu(\mathfrak{Z})$. By Lemma 4.3, $\mathfrak{D} = \mathfrak{Z}$ must be the case. That A is besides a complete intersection follows from Corollary 3.8. □

Remark 4.5. Assumption (a) in the previous proposition is essential – see Example 5.3 below. This shows that, in general, normal complete intersections are not polarizable.

5. Illustrative examples

The following is a selection of examples to help visualize the theory.

Example 5.1. If the degrees of the generators **f** of A are higher than 2, polarizability is a rare phenomenon even if A is a homogeneous isolated singularity. To see this, consider a homogeneous version of Example 1.2: $A = k[x_1^3, x_1^2x_2, x_2^3] \subset k[x_1, x_2]$. A simple calculation shows that $\mathfrak{Z}$ is (cyclic) generated in R-degree 3, while $\mathfrak{D}$ is (cyclic) generated in R-degree 6 because the presentation ideal of A over k is generated by a form of degree 3 in the presentation variables. Perhaps more conceptual is the case of the parameters of the twisted cubic as shown in Example 3.3.

Example 5.2. If **f** are general forms of degree 2 then polarizability may fail. Indeed, if **f** are 5 general quadrics in $k[x_0, x_1, x_2]$ then $k[\mathbf{f}]$ is (up to degree renormalization) the homogeneous coordinate ring of a general projection V to $\mathbb{P}^4$ of the 2-Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. It is known and classical that V is a smooth variety cut out by cubics, hence the polar syzygy module $\mathfrak{D}$ is minimally generated in degree 4. But the differential syzygy module $\mathfrak{Z}$ is generated in degree 3.

This example teaches us quite a bit:

(1) A is an isolated singularity (i.e., $\operatorname{Proj}(A)$ is smooth), but it is not normal nor Cohen–Macaulay as $\operatorname{depth}(k[\mathbf{f}]) = 1$;
(2) **f** are the Pfaffians of a skew-symmetric 5×5 matrix, whose entries are then necessarily linear forms. Therefore, the ideal $(\mathbf{f})$ is linearly presented. This is in direct contrast with the situation where **f** are monomials, in which case **f** is always polarizable (see [1, Proposition 5.18]).

In any case, one may observe that through the Jacobi map of Proposition 4.1 the linear syzygies of **f** correspond to the degree 2 component of the kernel of the Euler map $\mathcal{D}(\mathbf{f}) \to (\mathbf{f})$. Thus, if **f** linearly presented then this kernel is generated in degree 2, a condition which would be interesting to understand. Unfortunately, this condition does not seem to have immediate impact on the degrees of $\mathfrak{Z}$ as the generators of the latter will be non-minimal syzygies of **f**.

Example 5.3. (See [1, Theorem 5.10].) A normal hypersurface is not polarizable in general. Let

$$A = k\big[x_1^2, x_1x_2, x_2x_3, x_3x_4, x_2x_4\big] \subset k[x_1, x_2, x_3, x_4].$$

Then A is a normal hypersurface of degree 3 with defining equation $T_2^2T_4 - T_1T_3T_5$. Obviously, $\mathfrak{D}$ is free (of rank one). A is not polarizable since the transposed Jacobian matrix of its generators has a nonzero relation in degree 3 while $\mathfrak{D}$ is generated in degree 4. Actually, $\mathfrak{Z}$ is cyclic, hence free. Thus, though $\mathfrak{D}$ is isomorphic to $\mathfrak{Z}$ as abstract R-modules, it is not a second syzygy in its natural embedding (the inclusion $\mathfrak{D} \subset \mathfrak{Z}$ is not an equality in codimension one exactly along the prime (x_2)).

A computation shows that P/P^2 is the contraction of $\mathfrak{Z}$, hence is contracted from its R-extension as well. Thus, contractibility and reflexivity together do not imply polarizability, hence the last assertion in Corollary 2.3 admits no weak converse. One may wonder whether contractibility is always the case for a normal complete intersection parameterized by 2-forms. Note, however, that there may exist ideals in A which are not contracted from their extensions in R, as is here the case – e.g., $I = (x_3x_4)A$ is not contracted from its extension since $x_2^2 \cdot x_3x_4 \in IR \cap A \setminus A$.

Example 5.4. (See [7, p. 992].) A is now generated by square-free monomials and $\mathfrak{D}$ is not a reflexive R-module. Namely, one takes

$$A = k[x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_1, x_2x_4, x_2x_6] \subset k[x_1, x_2, x_3, x_4, x_5, x_6].$$

Here one can show that A is normal, hence Cohen–Macaulay, i.e., P is a codimension two perfect ideal. Moreover, it can easily be seen that P is an almost complete intersection. It follows that $P^2 = P^{(2)}$ (see [10, (4.4)]). On the other hand, A is locally regular in codimension 2, hence $\Omega_{A/k}$ is torsionfree (see [10, 4.6]). It follows from Corollary 2.3 that P/P^2 is both reflexive and contracted. It was pointed out in [7] that A is not polarizable. An additional calculation gives $\mathrm{ht}(\mathfrak{D} : \mathfrak{Z}) \geqslant 2$, hence the two modules coincide in codimension one. Of course $\mathfrak{D}$ fails to be reflexive since A is not polarizable.

Example 5.5. Next is an example where nearly everything goes wrong although the k-generators $\mathbf{f}$ of A in the embedding $A \subset R$ have some inner symmetry:

$$A = k\left[x_1^2, x_3^2, x_1x_2, x_2x_3, x_3x_4, x_1x_4\right] \subset k[x_1, x_2, x_3, x_4].$$

The presentation ideal P of A is a codimension two ideal generated by one quadric and three cubics. Clearly, the quadric forbids the generators to be minors. Therefore, A is not Cohen–Macaulay, hence not normal either (although it is locally regular in codimension one). Here, $P^2 = P^{(2)}$ and P/P^2 is not a reflexive A-module. A is not polarizable, hence $\mathfrak{D}$ is not a reflexive R-module either – as a direct computational check, e.g., one can see that $\mathrm{ht}(\mathfrak{D} : \mathfrak{Z}) \geqslant 2$.

6. Structured "geometric" classes

In the previous section most examples had no structured nature. In this part we collect a few examples of a more structured geometric nature.

Previously known results are as follows:

VERONESE EMBEDDINGS OF ORDER 2. It has been proved in [1, Corollary 5.1] that the edge algebra of a complete graph with a loop at every vertex is polarizable. This means that the 2-Veronese embedding of $\mathbb{P}^n$ is given by a polarizable parameterization.

SEGRE EMBEDDING AND ITS COORDINATE PROJECTIONS. It was proved in [1, Corollary 5.2] that the edge algebra of a connected bipartite graph is polarizable. This means that the homogeneous coordinate rings of the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$ and of all its "connected" coordinate projections are polarizable.

6.1. Grassmannian of lines and scrollar parameterizations

In this part we prove two new results.

Proposition 6.1. *Let* $\mathbf{X} := (x_{kl})$ $(1 \leqslant k \leqslant 2,\ 1 \leqslant l \leqslant m \geqslant 3)$ *be a* $2 \times m$ *generic matrix over* k. *Let* $R = k[x_{kl}]$ *and* $A = k[\mathbf{f}] \subset R$ *where* $\mathbf{f}$ *is the set of the* 2×2 *minors of* $\mathbf{X}$. *Then* $\mathbf{f}$ *is polarizable.*

Proof. It is classical and well-known that a defining ideal $P \subset k[\mathbf{T}]$ of A over k is generated by the Grassmann–Plücker (quadratic) relations of these minors and such P is minimally generated by $\binom{m}{4}$ such relations quadratic $\mathbf{T}$-forms, one for each 2×4 submatrix of $\mathbf{X}$. By Proposition 3.5, $\mathfrak{D}$ is minimally generated by $\binom{m}{4}$ elements. We will show that any element in $\mathfrak{Z}$ can be expressed as an R-combination of these $\binom{m}{4}$ polar syzygies, i.e., $\mathbf{f}$ is polarizable.

For this we will proceed by induction on the number m of columns (not on the number $2m$ of variables!). The assertion is (vacuously) verified for $m = 3$ since the minors are algebraically independent in this case and $\mathfrak{D} = \mathfrak{Z} = 0$.

Thus, assume that $m \geqslant 4$. To stay clear, we write $\mathbf{X}(m)$ for $\mathbf{X}$, $\mathbf{f}(m)$ for $\mathbf{f}$ and, accordingly, write $P(m)$ for P and $\mathfrak{Z}(m)$ for $\mathfrak{Z}$. Note that the equality $\binom{m}{4} = \binom{m-1}{4} + \binom{m-1}{3}$ is the numerical counterpart of saying that $P(m) = (P(m-1), Q(m-1))$, where $Q(m-1)$ denotes the set of Grassmann–Plücker relations obtained from all 2×4 submatrices of $\mathbf{X}(m)$ involving the last column.

On the other hand, the transposed Jacobian matrix of $\mathbf{f}(m)$ has the following shape

$$
{}^t\Theta\big(\mathbf{f}(m)\big) = \left(\begin{array}{c|ccccc}
 & x_{2m} & 0 & 0 & \cdots & 0\\
 & 0 & x_{2m} & 0 & \cdots & 0\\
{}^t\Theta(\mathbf{f}(m-1))_1^{m-1} & 0 & 0 & x_{2m} & \cdots & 0\\
 & \vdots & \vdots & \vdots & \ldots & \vdots\\
 & 0 & 0 & 0 & \cdots & x_{2m}\\ \hline
\mathbf{0} & -x_{21} & -x_{22} & -x_{23} & \cdots & -x_{2m-1}\\ \hline
 & -x_{1m} & 0 & 0 & \cdots & 0\\
 & 0 & -x_{1m} & 0 & \cdots & 0\\
{}^t\Theta(\mathbf{f}(m-1))_{m+1}^{2m-1} & 0 & 0 & -x_{1m} & \cdots & 0\\
 & \vdots & \vdots & \vdots & \ldots & \vdots\\
 & 0 & 0 & 0 & \cdots & -x_{1m}\\ \hline
\mathbf{0} & x_{11} & x_{12} & x_{13} & \cdots & x_{1m-1}
\end{array}\right),
$$

where the left vertical block, interspersed with two rows of zeros, is the transposed Jacobian matrix of $\mathbf{f}(m-1)$. By the inductive hypothesis, $\mathfrak{Z}(m-1)$ is generated by $\binom{m-1}{4}$ polar syzygies. Let's denote by $\widetilde{\mathfrak{Z}(m-1)}$ the submodule of $\mathfrak{Z}(m)$ minimally generated by these $\binom{m-1}{4}$ polar syzygies each stacked over a column of $m-1$ zeros. Observe that $\widetilde{\mathfrak{Z}(m-1)}$ is the submodule of $\mathfrak{Z}(m)$ generated by the polar syzygies arising from the minimal generators of $P(m-1)$, and that any element in $\mathfrak{Z}(m)$ not involving the right vertical block above (i.e., whose last $m-1$ entries are zero) belongs to $\widetilde{\mathfrak{Z}(m-1)}$.

Let $\mathbf{h} \in R^{m-1}$ denote the vector whose coordinates are the last $m-1$ coordinates of an arbitrarily given vector $\mathbf{s} \in \mathfrak{Z}(m)$. By the shape of the matrix, $\mathbf{h}$ is a syzygy of the matrix formed with the mth and last rows of the matrix right vertical block. Up to a permutation of these rows and a sign, this matrix is just $\mathbf{X}(m-1)$. We know the minimally generating syzygies of this generic matrix from the Buchsbaum–Rim complex. To wit, for every 2×3 submatrix of $\mathbf{X}(m-1)$ with columns $1 \leqslant i < j < k \leqslant m-1$, take its 2×2 signed minors. Then let $\mathbf{m}_{ijk} \in R^{m-1}$ denote the vector whose coordinates are these minors in places i, j and k, and zero elsewhere. The set of these vectors minimally generate the module of syzygies of $\mathbf{X}(m-1)$.

Accordingly, write $\mathbf{h} = \sum_{1 \leqslant i<j<k \leqslant m-1} q_{ijk}\mathbf{m}_{ijk}$ for suitable $q_{ijk} \in R$.

From the other end, choosing a 2×3 submatrix of $\mathbf{X}(m-1)$ with columns i, j, k amounts to picking a 2×4 submatrix of $\mathbf{X}(m)$ involving the last column. The latter gives rise to a Grassmann–Plücker equation $\mathbf{p}_{ijk}$ inducing a polar syzygy whose nonzero coordinates in the last $m-1$ slots are precisely the (signed) minors of the 2×3 submatrix in places i, j and k as above.

It follows that the last $m-1$ coordinates of $\mathbf{s} - \sum_{1 \leqslant i<j<k \leqslant m-1} q_{ijk}\mathbf{p}_{ijk}$ are zero, hence this vector belongs to $\widetilde{\mathfrak{Z}(m-1)}$. Therefore, $\mathbf{s}$ is an R-combination of the polar syzygies $\mathbf{p}_{ijk}$ and of the polar syzygies that generate $\widetilde{\mathfrak{Z}(m-1)}$. This shows the contention. $\square$

We next treat a well-known class of scrollar parameterizations. The approach of [4] is suited here, whereby any d-catalecticant $2 \times m$ generic matrix can be written as a scrollar matrix. As remarked in [4] this gives a subclass of all scrollar matrices for which the corresponding subalgebras are Koszul algebras, hence defined by quadratic relations (that include the appropriate Grassmann–Plücker relations).

Explicitly, fixing an integer $1 \leqslant d \leqslant m$, let now

$$\mathbf{X}(m,d) := \begin{pmatrix} x_1 & x_2 & \cdots & x_{1+d} & \cdots & x_{m-1} & x_m \\ x_{1+d} & x_{2+d} & \cdots & x_{1+2d} & \cdots & x_{m-1+d} & x_{m+d} \end{pmatrix}.$$

Note that the extreme values $d = 1$ and $d = m$ correspond, respectively, to the ordinary Hankel matrix and the generic matrix. As in the previous example, we let $\mathbf{f}(m,d)$ denote the 2×2 minors of $\mathbf{X}(m,d)$ and consider the k-subalgebra $A = k[\mathbf{f}(m,d)] \subset R = k[x_1, \ldots, x_{m+d}]$. It has also been shown that $\dim A = \min\{2m-3, m+d\}$ (see [4, Theorem 5.2]). Therefore, $\dim A = \dim R$ if and only if $d \leqslant m-3$; for $d > m-3$, A is isomorphic to the homogeneous coordinate ring of the Grassmannian $\mathbb{G}(1, m+d-1)$ discussed in the previous example.

Proposition 6.2. *Let $\mathbf{f}(m,d)$ denote the 2×2 minors of the d-catalecticant $2 \times m$ generic matrix $\mathbf{X}(m,d)$. Then $\mathbf{f}(m,d)$ is polarizable.*

Proof. To see that $\mathbf{f}(m,d)$ is polarizable, we would like to apply the same recipe as in the generic case. This is fine insofar that removing the last column of the above matrix preserves its catalecticant shape with the same leap d, thus allowing to induct on the number of columns as in the completely generic case.

However, there are some numerical changes when $d \leqslant m-4$. Firstly, the ideal $P(m,d)$ is now generated by the Grassmann–Plücker relations and additional quadratic relations containing more than 3 terms. According to [4, Corollary 3.4] these latter relations – called *N-relations* – constitute a set whose cardinality is the number $\nu(m,d)$ of indices i, j, k, l such that $1 \leqslant i < j < k < l \leqslant m$ and $k - j > d$. Therefore, $\nu(m,d) = \binom{m-d}{4}$, and hence $\mu(P(m,d)) = \binom{m}{4} + \binom{m-d}{4}$.

A set of generators of $P(m,d)$ as given in [4, Corollary 3.4] can be further slightly modified so that each N-relation is replaced by a quadratic relation with exactly 6 terms, which for convenience we will call an *M-relation*.

The advantage of switching to M-relations stems from the fact that, while an N-relation as in [4] comes naturally from a parallel relation at the level of the initial ideal, an M-relation is seen to be structurally defined. Namely, for every choice of indices i, j, k, l such that $d+1 \leqslant i < j < k < l \leqslant m$, pick the 2×4 submatrix of $\mathbf{X}(m,d)$ whose entries on the *first* row are x_i, x_j, x_k, x_l. (Clearly, the total number of such different choices is $\nu(m,d) = \binom{m-d}{4}$.) Then pick the unique 2×4 submatrix of $\mathbf{X}(m,d)$ whose entries on the *second* row are x_i, x_j, x_k, x_l. Next stack these two matrices together to form a 4×4 matrix $M(i,j,k,l)$ with one repeated row, so its determinant vanishes. Expanding this determinant by the Laplacian rule along the 2-minors of the first two rows yields a quadratic relation between a subset of the 2-minors of $M(i,j,k,l)$ which, by construction, gives a relation between a subset of the 2-minors of the original matrix $\mathbf{X}(m,d)$. In this way, one gets exactly one M-relation for each choice of a 2×4 submatrix of $\mathbf{X}(m,d)$ involving four of the last $m-d$ columns of $\mathbf{X}(m,d)$.

Now one argues in a similar way as in the generic case by a slight adaptation of the procedure. □

6.2. A class of polar maps

This example is inspired by a renowned construction of Gordan and Noether in connection with the Hesse problem (see [5]). We will actually look only at the simplest situation of their construction as follows: let $f_1, \ldots, f_{n-r}$ $(n \geqslant r+1)$ be forms of the same degree $\geqslant 2$ in the polynomial ring $R := k[x_1, \ldots, x_r]$ over a field k – in [5] the authors assume that the given forms are algebraically dependent over k, e.g., when $n \geqslant 2r+1$, but will make no such restriction at the outset.

Consider the k-subalgebra $A := k[f_1, \ldots, f_{n-r}] \subset R$. Take new variables $x_{r+1}, \ldots, x_n$, and let F be the following form

$$F := x_{r+1} f_1 + \cdots + x_n f_{n-r} \in S := k[x_1, \ldots, x_r, x_{r+1}, \ldots, x_n],$$

which can suggestively be thought of as the generic member of the linear system spanned by $\mathbf{f} := \{f_1, \ldots, f_{n-r}\}$.

Let B be the k-subalgebra of S generated by the partial derivatives of F. Write

$$B := k\left[\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_r}, \frac{\partial F}{\partial x_{r+1}} = f_1, \ldots, \frac{\partial F}{\partial x_n} = f_{n-r}\right] \subset S.$$

For simplicity, call these generators $\mathbf{g} = \{g_1, \ldots, g_r, g_{r+1}, \ldots, g_n\}$.

Proposition 6.3. *With the above notation, if* $\dim A = r$ *then B is polarizable if (and only if) A is polarizable.*

Proof. Note that the hypothesis of the claim implies that $n \geqslant 2r$.

Actually, we prove something more precise. Namely, look at the transposed Jacobian matrix of $\mathbf{g}$ which is nothing but the Hessian matrix of F. A careful inspection of $\mathbf{g}$ leads to the following block shape

$${}^t\Theta(\mathbf{g}) = \left(\begin{array}{c|c} \mathcal{H} & {}^t\Theta(\mathbf{f}) \\ \hline \Theta(\mathbf{f}) & 0 \end{array}\right),$$

where $\Theta(\mathbf{f})$ is the Jacobian matrix of the originally given forms $\mathbf{f}$ and $\mathcal{H}$ is the $r \times r$ Hessian matrix of F regarded as a polynomial in $x_1, \ldots, x_r$ with coefficients in $k[x_{r+1}, \ldots, x_n]$. Since $\dim A = r \leqslant n - r$ then the rank of $\Theta(\mathbf{f})$ is r, hence one readily obtains that the first r coordinates of any syzygy of the above matrix are all zero, while the last $n - r$ coordinates constitute a syzygy of ${}^t\Theta(\mathbf{f})$.

To translate this outcome in a more formal fashion, let $\mathfrak{Z}_T(\mathbf{h})$ stand for the differential syzygy module of a set of forms $\mathbf{h}$ in a polynomial ring T over k. By definition, we have the exact sequence of S-modules

$$0 \to \mathfrak{Z}_R(\mathbf{f}) \to R^{n-r} \xrightarrow{{}^t\Theta(\mathbf{f})} \sum_{i=1}^{r} R\, dx_i. \tag{7}$$

Since $R \subset S$ is a flat (free) extension, we get an exact sequence of R-modules

$$0 \to \mathfrak{Z}_R(\mathbf{f}) \otimes_R S \to S^{n-r} \xrightarrow{{}^t\Theta(\mathbf{f}) \otimes_R S} \sum_{i=1}^{r} S\, dx_i.$$

Taking R-duals in (7), observing that $\operatorname{coker}({}^t\Theta(\mathbf{f}))$ is torsion because the rank of ${}^t\Theta(\mathbf{f})$ is r, and tensoring with S over R, yields a short exact complex

$$0 \to \sum_{i=1}^{r} S \frac{\partial}{\partial x_i} \xrightarrow{\Theta(\mathbf{f}) \otimes_R S} \left(S^{n-r}\right)^*. \tag{8}$$

Consider the map of complexes over S

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{Z}_R(\mathbf{f}) \otimes_R S & \longrightarrow & S^{n-r} & \xrightarrow{{}^t\Theta(\mathbf{f}) \otimes_R S} & \sum_{i=1}^{r} S\, dx_i & & \\
 & & & & \uparrow & & \uparrow{\scriptstyle \mathcal{H}} & & \\
 & & & & 0 & \longrightarrow & \sum_{i=1}^{r} S \frac{\partial}{\partial x_i} & \xrightarrow{\Theta(\mathbf{f}) \otimes_R S} & (S^{n-r})^*.
\end{array}$$

The mapping cone of this map is the exact complex

$$0 \to \mathfrak{Z}_R(\mathbf{f}) \otimes_R S \to S^{n-r} \oplus \sum_{i=1}^{r} S \frac{\partial}{\partial x_i} \xrightarrow{\Psi} \sum_{i=1}^{r} S\, dx_i \oplus \left(S^{n-r}\right)^*,$$

where Ψ is represented, in suitable bases, by the matrix

$$\left(\begin{array}{c|c} {}^t\Theta(\mathbf{f}) & \mathcal{H} \\ \hline 0 & \Theta(\mathbf{f}) \end{array}\right).$$

Therefore, up to change of basis in the free modules, one gets an isomorphism $\mathfrak{Z}_R(\mathbf{f}) \otimes_R S \simeq \mathfrak{Z}_S(\mathbf{g}) \subset S^n$.

In particular, these two S-modules have the same rank. Now, the S-rank of $\mathfrak{Z}_R(\mathbf{f}) \otimes_R S$ equals the R-rank of $\mathfrak{Z}_R(\mathbf{f})$ and the latter is $n - r - \dim A$. But the rank of the S-module $\mathfrak{Z}_S(\mathbf{g})$ is $n - \dim B$. It follows that $\dim B = \dim A + r$. But since B is generated over A by r elements, it follows that B is a polynomial ring over A (of dimension $2r$).

We now deal with the respective modules of polar syzygies. Let $A \simeq k[T_1, \ldots, T_{n-r}]/P$ and $B \simeq k[T_1, \ldots, T_n]/\boldsymbol{P}$ denote respective polynomial presentations of the two algebras. One has an inclusion $Pk[T_1, \ldots, T_n] \subset \boldsymbol{P}$. Since $\mathrm{ht}(P) = n - r - \dim A = n - 2r = n - \dim B = \mathrm{ht}(\boldsymbol{P})$, we have an equality. This implies a similar relation as above between the respective associated modules of polar syzygies. Namely, B is a polynomial ring over A, hence $A \subset B$ is a free extension. Therefore, by the same token, the equality $Pk[T_1, \ldots, T_n] = \boldsymbol{P}$ translates into a natural isomorphism

$$\mathrm{Im}\left(P/P^2\right) \otimes_A B \simeq \mathrm{Im}\left(\boldsymbol{P}/\boldsymbol{P}^2\right) \subset \sum_{\ell=1} B\, dT_\ell.$$

This shows the contention. $\square$

Illustration. Take A to be a d-catalecticant parameterization as in (6.2), with $d \leqslant m - 3$. Then, as remarked there, A has maximal dimension, i.e., $\dim A = m + d$. The polar map of the corresponding hypersurface F has an image of (geometric) dimension $2m + 2d - 1$ in $\mathbb{P}^{\binom{m}{2}+m+d-1}$. As a slight check on the numbers, since we are assuming $d \leqslant m - 3$ then $m + d \leqslant 2m - 3$. Clearly then $\binom{m}{2} < m + d \leqslant 2m - 3$ would entail $(m-2)(m-3) = m^2 - 5m + 6 < 0$, which is impossible as $m \geqslant 3$.

In the simplest case, with $d = 1$ and $m = 4$, we get the polarizable forms

$$\begin{aligned} &x_3x_6 + x_4x_7 + x_5x_8,\, -2x_2x_6 - x_3x_7 - x_4x_8 + x_4x_9 + x_5x_{10},\, x_1x_6 - x_2x_7 - 2x_3x_9 - x_4x_{10} + x_5x_{11}, \\ &\quad x_1x_7 - x_2x_8 + x_2x_9 - x_3x_{10} - 2x_4x_{11},\, x_1x_8 + x_2x_{10} + x_3x_{11},\, -x_2^2 + x_1x_3,\, -x_2x_3 + x_1x_4, \\ &\quad -x_2x_4 + x_1x_5,\, -x_3^2 + x_2x_4,\, -x_3x_4 + x_2x_5,\, -x_4^2 + x_3x_5 \end{aligned}$$

parameterizing a Plücker quadric hypersurface in $\mathbb{P}^{10}$.

Remark 6.4. We observe that this procedure is in principle iterative, but the numbers and the result proper will be different.

We close with some general questions.

Question 6.5. Let $\mathbf{f} \subset R$ be a polarizable set of forms of degree 2. Is $k[\mathbf{f}] \subset R$ normal?

Question 6.6. Let $\mathbf{f} \subset R$ be a polarizable set of forms of degree 2. Is $k[\mathbf{f}] \subset R$ Cohen–Macaulay?

In [1] the answer to the first of these questions is affirmative when $\mathbf{f}$ are monomials, hence so is the answer to the second question under the same assumption.

In general, if the answer to the second of these questions is affirmative – or, if at least $k[\mathbf{f}]$ satisfies Serre's property (S_2) – a loose strategy for the first question would be to show that $\operatorname{ht}\mathfrak{J}R \leqslant \operatorname{ht}\mathfrak{J}$. This would entail $\operatorname{ht}\mathfrak{J} \geqslant 2$ since with $\mathfrak{D} = \mathfrak{Z}$ the ideal $\mathfrak{J}R$ would be the Fitting of a second syzygy. Thus, a preliminary question along this line is to understand when the Jacobian ideal is height decreasing through $k[\mathbf{f}] \subset R$. A sufficient condition for this to happen is that the contraction $\mathfrak{J}R \cap k[\mathbf{f}]$ is contained in some minimal prime of $\mathfrak{J}$ of minimal height.

Affirmative answers to these questions would give another proof of the result of [4] to the effect that the algebra parameterized by the 2×2 minors of a d-catalecticant matrix is normal and Cohen–Macaulay. Of course, this is not a totally impressive outcome since we have drawn upon the ideas of [4] for the structure of such objects in order to prove their polarizability.

Question 6.7. Suppose that P is generated by forms of degree 2 constituting a Gröbner basis for some monomial order. Is $\mathbf{f}$ polarizable?

The assumption forces the k-algebra $A = k[\mathbf{f}] \subset R$ to be a Koszul algebra according to [3, Theorem 2.2]. Note that most geometric examples in this section are Koszul algebras.

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