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Characterisation of the weak lower semicontinuity for a type of nonlocal integral functional: The n-dimensional scalar case

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ABSTRACT

In this work we are going to prove the functional J defined by

$$J(u)=\int_{\Omega\times\Omega} W\big(\nabla u(x),\nabla u(y)\big)\,dx\,dy,$$

is weakly lower semicontinuous in $W^{1,p}(\Omega)$ if and only if W is separately convex. We assume that Ω is an open set in $\mathbb{R}^n$ and W is a real-valued continuous function fulfilling standard growth and coerciveness conditions. The key to state this equivalence is a variational result established in terms of Young measures.

1. Introduction

This work is devoted to study the weak lower semicontinuous property of the functional

$$J(u)=\int_{\Omega\times\Omega} W\big(\nabla u(x),\nabla u(y)\big)\,dx\,dy \tag{1.1}$$

where $u\in W^{1,p}(\Omega;\mathbb{R})$, Ω is a bounded regular domain in $\mathbb{R}^n$, $n\geqslant 1$, $p>1$ and $W:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is a real continuous function satisfying the bounds

$$c\big(|\lambda_1|^p+|\lambda_2|^p-1\big)\leqslant W(\lambda_1,\lambda_2)\leqslant C\big(|\lambda_1|^p+|\lambda_2|^p+1\big) \tag{1.2}$$

and $0<c<C$. Also, due to the definition of J and without lost of generality, the integrand W is assumed to be a symmetric function, i.e. $W(\lambda_1,\lambda_2)=W(\lambda_2,\lambda_1)$ for any $(\lambda_1,\lambda_2)\in\mathbb{R}^{2n}$. The main result of the paper is

Theorem 1.1. *Under the above hypotheses the functional J defined by* (1.1) *is weak lower semicontinuous in $W^{1,p}(\Omega)$ if and only if W is separately convex.*

Even though the separate convexity of W always implies lower semicontinuity for the functional J, the reverse implication has been proved only for the case $n=1$ (see [4]).

The proof of Theorem 1.1 is entirely based on the optimality conditions that the minimizing sequences of the functional J must satisfy. A similar analysis has been employed to study the existence of minimizers of the problem

$$\min\big\{J(u):\ u-u_0\in W_0^{1,p}(\Omega;\mathbb{R})\big\} \tag{1.3}$$

where $u_0\in W^{1,p}(\Omega;\mathbb{R})$ and $n=1$ (see [9]).

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Concerning this type of problems several works have been published. In connection with variational problems of nonlocal nature the reader can consult [5] for problems related to Ferromagnetism, [6] about the regularization of a nonconvex problem, and [3,12] or [13] in order to analyze mechanical problems formulated in the general context of the Nonlocal Elasticity (see also [8]). In [1] and [15] some interesting tools to obtain a full relaxation of specific nonlocal variational problems have been analyzed, and [7] is also remarkable work for a general class of nonlocal integral functionals.

The paper is organized as follows: in Section 2 we give a characterization for the lower semicontinuous envelope of J in terms of Young measures. Section 3 is devoted to state some basic optimality conditions for the Young measure solution in the obtainment of the lower semicontinuous envelope. In Section 4 we prove Theorem 1.1. Finally, in Section 5 we reexamine the procedure carried out when the integrand of J depends also on the variables $(x, y, u(x), u(y))$. We prove a new characterization for the weak lower semicontinuity when the integrand has the format $W = W(x, y, \nabla u(x), \nabla u(y))$ (Theorem 5.1).

2. Preliminaries

Young measures [16] is a classical tool that will play a fundamental role in the study of the integral functional given in (1.1). We start giving a basic version of the Existence Theorem on Young measures (see [2], [10, Theorem 6.2]):

Theorem 2.1. *Let $1 \leqslant p < \infty$, Ω an open regular domain in $\mathbb{R}^n$ and $f_j : \Omega \to \mathbb{R}^m$.*

(1) *If $\{f_j\}$ is a bounded sequence in $L^p(\Omega)$, there exists a subsequence (not relabeled) and a family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$, depending measurably on $x \in \Omega$ (for any continuous function ψ the map $x \to \langle\psi, \nu_x\rangle$ is measurable) such that whenever the sequence $\psi(f_j)$ converges weakly in $L^1(E)$ for some measurable $E \subset \Omega$, we have*

$$\psi(f_j) \rightharpoonup \overline{\psi}(x) = \langle\psi, \nu_x\rangle \doteq \int_{\mathbb{R}^m} \psi(\lambda)\, d\nu_x(\lambda).$$

Moreover

$$\int_\Omega \int_{\mathbb{R}^m} |\lambda|^p \, d\nu_x(\lambda)\, dx < \infty$$

(in such a case $\nu = \{\nu_x\}_{x\in\Omega}$ is said to be the Young measure generated by the sequence $\{f_j\}$).

(2) *A family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$, depending measurably on $x \in \Omega$, can be generated by a sequence of functions $\{f_j\}$ such that $\{|f_j|^p\}$ is equiintegrable, if and only if*

$$\int_\Omega \int_{\mathbb{R}^m} |\lambda|^p \, d\nu_x(\lambda)\, dx < \infty.$$

In order to characterize the sequences of pairs $\{(\nabla u_j(x), \nabla u_j(y))\}$ we have:

Theorem 2.2. *(See [11].) Let $1 \leqslant p < \infty$ and Ω an open regular domain in $\mathbb{R}^n$. Let $\Pi = \{\Pi_{(x,y)}\}$ be a family of probability measures supported in $\mathbb{R}^n \times \mathbb{R}^n$. Π is the Young measure generated by a sequence $g_j(x, y) = (\nabla u_j(x), \nabla u_j(y))$, where $\{u_j\}$ is a bonded sequence in $W^{1,p}(\Omega)$ such that $\{|\nabla u_j|^p\}$ is weakly convergent in $L^1(\Omega)$ if and only if*

$$\Pi_{(x,y)} = \nu_x \otimes \nu_y, \quad (x, y) \in \Omega \times \Omega, \tag{2.1}$$

where $\nu = \{\nu_x\}_{x\in\Omega}$ is the Young measure generated by the sequence of gradients $\{\nabla u_j\}$.

Remark 2.1. Concerning the above result it must be pointed out that we have the representation

$$\lim_{j\to\infty} \int_{\Omega\times\Omega} \psi\big(\nabla u_j(x), \nabla u_j(y)\big)\, dx\, dy \int_{\Omega\times\Omega} \int_{\mathbb{R}^n\times\mathbb{R}^n} \psi(\lambda_1, \lambda_2)\, d\nu_x(\lambda_1)\, d\nu_y(\lambda_2)\, dx\, dy \tag{2.2}$$

for any continuous ψ such that $\{\psi(\nabla u_j(x), \nabla u_j(y))\}_j$ converges weakly in $L^1(\Omega \times \Omega)$. In connection with the convergence (2.2) it will be useful to recall that, a family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$ can be generated by sequence of gradients $\{\nabla u_j\}$ such that $\{|\nabla u_j|^p\}$ is weakly convergent in $L^1(\Omega)$, if and only if

$$\int_\Omega \int_{\mathbb{R}^n} |\lambda|^p \, d\nu_x(\lambda)\, dx < \infty \tag{2.3}$$

and there exists $u \in W^{1,p}(\Omega)$ such that

$$\nabla u(x) = \int_{\mathbb{R}^m} \lambda \, d\nu_x(\lambda) \tag{2.4}$$

(see [10, Theorem 8.7]).

Remark 2.2. Another meaningful remark concerns the competing sequences of the problem

$$\min\Big\{\liminf_{n\to\infty} J(u_j)\colon\ u_j \in W^{1,p}(\Omega;\mathbb{R}),\ u_j \rightharpoonup u \text{ in } W^{1,p}(\Omega)\Big\}. \tag{2.5}$$

If $\{v_j\}_j$ is an admissible sequence for (2.5) and $\mu = \{\mu_x\}_{x\in\Omega}$ is its corresponding gradient Young measure (the Young measure generated by the sequence of gradients $\{\nabla v_j\}_j$), then we can find another admissible sequence $\{u_j\}_j$ sharing the same underlying gradient Young measure μ and such that $\{|\nabla u_j|^p\}$ is weakly convergent in $L^1(\Omega)$ (see [10, Lemma 8.15]).

Under these circumstances, we have an essential relaxation result:

Theorem 2.3 *(General relaxation). Let m be the minimum of the problem* (2.5) *and* $\overline{m}$ *the infimum of the problem*

$$\inf\Big\{\int_{\Omega\times\Omega}\int_{\mathbb{R}^n\times\mathbb{R}^n} W(\lambda_1,\lambda_2)\, d\mu_x(\lambda_1)\, d\mu_y(\lambda_2)\, dx\, dy\colon\ \mu = \{\mu_x\}_{x\in\Omega} \in \overline{\mathcal{A}}\Big\} \tag{2.6}$$

where $\overline{\mathcal{A}}$ *is the set of young measures* $\mu = \{\mu_x\}_{x\in\Omega}$ *holding* (2.3) *and* (2.4). *Then*

$$m = \overline{m},$$

and $\overline{m}$ *is indeed a minimum.*

Proof. We realize that if ν minimizes (2.6) then by Remark 2.1 we can find a sequence of gradients $\{\nabla u_j\}_j$ such that $\{|\nabla u_j|^p\}$ is weakly convergent in $L^1(\Omega)$. Thus, thanks to the bounds assumed on W (2.2) holds. This implies $m \leqslant \overline{m}$. To see the reverse inequality we use Remark 2.2 in order to ensures the weak convergence in $L^1(\Omega)$ of the sequence $\{W(\nabla u_j(x), \nabla u_j(y))\}_j$ and consequently (2.2) holds. In order to check that $\overline{m}$ is a minimum, take $\{\nabla u_j\}_j$, a minimizing sequence for (2.5). Since this sequence can be selected so that$\{|\nabla u_j|^p\}$ is weakly convergent in $L^1(\Omega)$, then

$$m = \liminf_{n\to\infty} \iint_{\Omega\times\Omega} W\big(\nabla u_j(x), \nabla u_j(y)\big)\, dx\, dy.$$

To conclude the proof, take the Young measure ν generated by this sequence. We get

$$m = \int_{\Omega\times\Omega}\int_{\mathbb{R}^n\times\mathbb{R}^n} W(\lambda_1,\lambda_2)\, d\mu_x(\lambda_1)\, d\mu_y(\lambda_2)\, dx\, dy = \overline{m}.$$

Thus ν is a minimizer to the problem (2.6). □

Within the context of Theorem 2.3 the minimization problem (2.6) is said to be a relaxation of (2.5). (2.6) is indeed an explicit representation of $sc^- J(u)$, the lower semicontinuous of the functional J at u.

3. The basic optimality conditions

Assume Ω is a regular open set in $\mathbb{R}^n$ and $\{u_j\}$ is a sequence solution of the problem

$$\min\Big\{\liminf_{n\to\infty} J(u_j)\colon\ u_j \in W^{1,p}(\Omega;\mathbb{R}),\ u_j \rightharpoonup u \text{ in } W^{1,p}(\Omega)\Big\}. \tag{3.1}$$

Let $\nu = \{\nu_x\}_{x\in\Omega}$ be the Young measure generated by $\{\nabla u_j(x)\}_j$. Let us consider any Young measure σ generated by a sequence of gradients $\{\nabla v_j(x)\}_j$ such that $\{v_j\}_j$ is admissible for the minimization principle (3.1) (without lost of generality we can assume that $\{|\nabla v_j|^p\}$ is weakly convergent in $L^1(\Omega)$). For each $t \geqslant 0$ we define the new Young measure $\mu^t = \{\mu^t_x\}_{x\in\Omega}$ as

$$\mu^t_x = t\sigma_x + (1-t)\nu_x,$$

where $\sigma = \{\sigma_x\}_{x\in\Omega}$ is the Young measure generated by the sequence $\{\nabla v_k(x)\}_k$. Since the action of each μ_x^t on any function ψ is given by the formula

$$\langle \mu_x^t, \psi \rangle = \int_{\mathbb{R}^n} \psi(\lambda)\, d\mu_x^t(\lambda) = t \int_{\mathbb{R}^n} \psi(\lambda)\, d\sigma_x(\lambda) + (1-t) \int_{\mathbb{R}^n} \psi(\lambda)\, d\nu_x(\lambda),$$

then it is clear that μ^t satisfies (2.3) and (2.4), and therefore $\mu^t \in \overline{\mathcal{A}}$. Let g be the function

$$g(t) \doteq \iint_{\Omega\times\Omega} \iint_{\mathbb{R}^n\times\mathbb{R}^n} W(\lambda_1,\lambda_2)\, d\mu_x^t(\lambda_1)\, d\mu_y^t(\lambda_2)\, dx\, dy, \quad t \geqslant 0.$$

Then, thanks to the fact that $\{u_j\}_j$ minimizes (3.1) we have $\frac{d}{dt}[g(t)]_{t=0^+} \geqslant 0$, which read as

$$\frac{d}{dt}\Bigg[t^2 \iint_{\Omega\times\Omega} \iint_{\mathbb{R}^n\times\mathbb{R}^n} W(\lambda_1,\lambda_2)\, d\sigma_x(\lambda_1)\, d\sigma_y(\lambda_2) + 2t(1-t) \iint_{\Omega\times\Omega} \iint_{\mathbb{R}^n\times\mathbb{R}^n} W(\lambda_1,\lambda_2)\, d\sigma_x(\lambda_1)\, d\nu_y(\lambda_2)\, dx\, dy$$
$$+ (1-t)^2 \iint_{\Omega\times\Omega} \iint_{\mathbb{R}^n\times\mathbb{R}^n} W(\lambda_1,\lambda_2)\, d\nu_x(\lambda_1)\, d\nu_y(\lambda_2)\Bigg]_{t=0^+} \geqslant 0.$$

After differentiation we find

$$\iint_{\Omega\times\Omega} \iint_{\mathbb{R}^n\times\mathbb{R}^n} W(\lambda_1,\lambda_2)\, d\sigma_x(\lambda_1)\, d\nu_y(\lambda_2)\, dx\, dy \geqslant \iint_{\Omega\times\Omega} \iint_{\mathbb{R}^n\times\mathbb{R}^n} W(\lambda_1,\lambda_2)\, d\nu_x(\lambda_1)\, d\nu_y(\lambda_2)\, dx\, dy. \tag{3.2}$$

The inequality (3.2) automatically guarantees the thesis of the following proposition:

Proposition 3.1. *If $\nu = \{\nu_x\}_{x\in\Omega}$ is the Young measure generated by a minimizing sequence $\{\nabla u_j\}_j$ for the principle* (3.1)*, then $\nu = \{\nu_x\}_{x\in\Omega}$ is a minimizer for the problem*

$$\min\left\{ \int_{\Omega} \int_{\mathbb{R}^n} G(\lambda_1)\, d\gamma_x(\lambda_1)\, dx\colon\ \gamma = \{\gamma_x\}_{x\in\Omega} \text{ satisfying (2.3) and (2.4)} \right\} \tag{3.3}$$

where

$$G(\lambda_1) \doteq \int_{\Omega} \int_{\mathbb{R}^n} W(\lambda_1,\lambda_2)\, d\nu_y(\lambda_2)\, dy.$$

Moreover, the sequence $\{\nabla u_j\}_j$ minimizes the functional $l : W^{1,p}(\Omega) \to \mathbb{R}$ defined as

$$l(u) = \liminf_{n\to\infty} \left\{ \int_{\Omega} G\big(\nabla z_j(x)\big)\, dx\colon\ z_j \in W^{1,p}(\Omega;\mathbb{R}),\ \text{such that } z_j \rightharpoonup u \text{ in } W^{1,p}(\Omega) \right\}. \tag{3.4}$$

To prove this result it must be taken into account that $G : \mathbb{R}^n \to \mathbb{R}$ is a real continuous function such that

$$c\big(|\lambda|^p - 1\big) \leqslant G(\lambda) \leqslant C\big(|\lambda|^p + 1\big).$$

The proof is obtained following the same lines of the proof of Theorem 2.3. We factually can state that problem (3.3) is a relaxation of (3.4).

We use the generalized Weierstrass condition on the minimum principle (3.3) to assert the following result about generalized optimality conditions (see [14]):

Proposition 3.2. *Let $\nu = \{\nu_x\}_{x\in\Omega}$ be a Young measure solution for* (3.3)*. Then*

$$\operatorname{div} \mathcal{F}(x) = 0 \quad \text{in } W^{-1,p/(p-1)}(\Omega) \tag{3.5}$$

and

$$\int_{\mathbb{R}^n} \big(G(\lambda_1) - \mathcal{F}(x)\cdot\lambda_1\big)\, d\nu_x(\lambda_1) = \min_{s\in\mathbb{R}^n} \mathcal{H}(x,s) \tag{3.6}$$

for a.e. $x \in \Omega$, where

$$\mathcal{F}(x) = \int_{\mathbb{R}^n} \frac{\partial G}{\partial \lambda_1}(\lambda_1)\, d\nu_x(\lambda_1) \in L^{p/(p-1)}\big(\Omega;\mathbb{R}^n\big)$$

and

$$\mathcal{H}(x,s) = G(s) - \mathcal{F}(x)\cdot s. \tag{3.7}$$

Moreover

$$\operatorname{supp}\nu_x \subset \operatorname{Arg\,min}\mathcal{H}(x,\cdot) \tag{3.8}$$

for a.e. $x \in \Omega$.

From (3.6) we have

$$\int_{\Omega}\int_{\mathbb{R}^n}\big(W(s,\lambda_2) - \mathcal{F}(x)\cdot s\big)\,d\nu_y(\lambda_2)\,dy \geqslant \int_{\mathbb{R}^n}\int_{\Omega}\int_{\mathbb{R}^n}\big(W(\lambda_1,\lambda_2) - \mathcal{F}(x)\cdot\lambda_1\big)\,d\nu_y(\lambda_2)\,dy\,d\nu_x(\lambda_1)$$

for any $s \in \mathbb{R}^n$. In particular

$$\int_{\mathbb{R}^n}\int_{\Omega}\int_{\mathbb{R}^n} W(s,\lambda_2)\,d\nu_y(\lambda_2)\,dy\,d\gamma_x(s) \geqslant \int_{\mathbb{R}^n}\int_{\Omega}\int_{\mathbb{R}^n} W(\lambda_1,\lambda_2)\,d\nu_y(\lambda_2)\,dy\,d\nu_x(\lambda_1) \tag{3.9}$$

where γ_x is any probability measure such that

$$\nabla u(x) = \int_{\mathbb{R}^n} s\,d\gamma_x(s), \quad \text{a.e. } x \in \Omega, \qquad \int_{\Omega}\int_{\mathbb{R}^n} |s|^p\,d\gamma_x(s)\,dx < \infty.$$

We are in position to state the main result of this section.

Theorem 3.3. *If the sequence* $\{u_j\} \in W^{1,p}(\Omega)$, $u_j \rightharpoonup u$, *is a solution to the minimization problem* (3.1) *then*

$$C_1\left(\int_{\Omega}\int_{\mathbb{R}^n} W\big(\nabla u(x),\lambda_2\big)\,d\nu_y(\lambda_2)\,dy\right) = \int_{\mathbb{R}^n}\left(\int_{\Omega}\int_{\mathbb{R}^n} W(\lambda_1,\lambda_2)\,d\nu_y(\lambda_2)\,dy\right)d\nu_x(\lambda_1) \tag{3.10}$$

a.e. $x \in \Omega$, *where* ν *is the Young measure generated by* $\{\nabla u_j\}$ *and the l.s.t. of* (3.10) *is the convex envelope of the function*

$$\lambda_1 \to \int_{\Omega}\int_{\mathbb{R}^n} W(\lambda_1,\lambda_2)\,d\nu_y(\lambda_2)\,dy$$

evaluated upon $\lambda_1 = \nabla u(x)$.

Proof. The proof is just formula (3.9). □

Note that if $\overline{m}$ denotes the minimum of problem (3.1) then

$$\overline{m} = \int_{\Omega} C_1\left(\int_{\Omega}\int_{\mathbb{R}^n} W\big(\nabla u(x),\lambda_2\big)\,d\nu_y(\lambda_2)\,dy\right)dx. \tag{3.11}$$

4. Lower semicontinuity

If J is l.s.c in $W^{1,p}(\Omega)$ then the sequence $\{u_n\}$, where $u_n = u$ for any n, solves the minimization problem

$$sc^- J(u) \doteq \min\left\{\liminf_{n\to\infty} J(u_j):\ u_j \in W^{1,p}(\Omega),\ u_j \rightharpoonup u \text{ in } W^{1,p}(\Omega)\right\}$$

and therefore (3.10) ensures that

$$C_1\left(\int_{\Omega} W\big(\nabla u(x),\nabla u(y)\big)\,dy\right) = \int_{\Omega} W\big(\nabla u(x),\nabla u(y)\big)\,dy. \tag{4.1}$$

Regarding the lower semicontinuity on affine function we have the following result:

Theorem 4.1. J *is weak lower semicontinuous at the affine function* $u_0(x) \equiv \gamma\cdot x$, *where* γ *is any vector from* $\mathbb{R}^n$ *if and only if*

$$C_1\big(W(\gamma,\gamma)\big) = W(\gamma,\gamma). \tag{4.2}$$

Proof. In view of (4.1) $C_1(W(\gamma,\gamma)) = W(\gamma,\gamma)$ is a necessary condition. In order to check (4.2) is a sufficient condition we consider any sequence $\{v_j\}$ from $W^{1,p}(\Omega)$ such that $v_j \rightharpoonup \gamma \cdot x$ in $W^{1,p}(\Omega)$. Then, due to (3.10) and (3.11), we have

$$\lim_j J(v_j) \geqslant \min\Big\{\liminf_{n\to\infty} J(u_j)\colon\ u_j \in W^{1,p}(\Omega),\ u_j \rightharpoonup \gamma\cdot x \text{ in } W^{1,p}(\Omega)\Big\} = |\Omega| C_1\bigg(\int_\Omega \int_{\mathbb{R}^n} W(\gamma,\lambda_2)\, d\nu_y(\lambda_2)\, dy\bigg)$$

where $\nu = \{\nu_x\}_{x\in\Omega}$ is any Young measure solution of (3.1) generated by a minimizing sequence $\{u_j\}$, and such that $u_j \rightharpoonup \gamma\cdot x$ in $W^{1,p}(\Omega)$. Now, let $\bar\nu$ be the probability measure, with barycenter γ, obtained from the homogenization of $\nu = \{\nu_x\}_{x\in\Omega}$, i.e.

$$\langle\bar\nu, g\rangle = \int_{\mathbb{R}^n} g(\lambda)\, d\bar\nu(\lambda) \doteq \frac{1}{|\Omega|}\int_\Omega\int_{\mathbb{R}^n} g(\lambda_2)\, d\nu_y(\lambda_2)\, dy.$$

Then

$$\int_\Omega\int_{\mathbb{R}^n} W(\gamma,\lambda_2)\, d\nu_y(\lambda_2)\, dy = |\Omega|\big\langle\bar\nu, W(\gamma,\cdot)\big\rangle = |\Omega|\int_{\mathbb{R}^n} W(\gamma,\lambda)\, d\bar\nu(\lambda) \geqslant |\Omega| C_2 W(\gamma,\gamma)$$

where $C_2 W(\gamma,\gamma)$ is the convex envelope of $W(\gamma,\cdot)$ at γ. By using $C_1(W(\gamma,\gamma)) = W(\gamma,\gamma)$ twice we have

$$\lim_j J(v_j) \geqslant |\Omega| C_1\big(|\Omega| C_2 W(\gamma,\gamma)\big) = |\Omega|^2 C_1\big(W(\gamma,\gamma)\big) = |\Omega|^2 W(\gamma,\gamma) = J(\gamma\cdot x).$$

This completes the proof. □

Corollary 4.2. *The affine function $u_0(x) = \gamma\cdot x$ is a solution to the minimization problem*

$$\min\big\{J(u)\colon\ u - u_0 \in W_0^{1,p}(\Omega;\mathbb{R})\big\} \tag{4.3}$$

if and only if $C_1(W(\gamma,\gamma)) = W(\gamma,\gamma)$.

Proof. We recall (see [10, Theorem 8.3]) that for any weakly convergent sequence $\{u_j\}_j$ such that $u_j \rightharpoonup u_0$ in $W^{1,p}(\Omega)$ we can find a new sequence $\{v_j\}_j$ such that $v_j - u_0 \in W_0^{1,p}(\Omega)$, $v_j \rightharpoonup u_0$ in $W^{1,p}(\Omega)$ and such that the two sequences of gradients, $\{\nabla u_j\}_j$ and $\{\nabla v_j\}_j$, have the same underlying Young measure. Thus, if we assume u_0 is a solution to problem (4.3) then $\{u_j\}$, where $u_j = u_0$ for all j, is a minimizing sequence for (3.1) with $u = u_0$. Then by (4.1) we get $C_1(W(\gamma,\gamma)) = W(\gamma,\gamma)$. To prove the only if part assume $\nu = \{\nu_x\}_{x\in\Omega}$ is the Young measure given by a minimizing sequence $\{\nabla u_j\}$ such that u is the weak limit of $\{u_j\}_j$ in $W^{1,p}(\Omega)$. Now, let $\bar\nu$ be the probability measure obtained from the homogenization of $\nu = \{\nu_x\}_{x\in\Omega}$. Then there exists a sequence v_j, bounded in $W^{1,p}(\Omega)$ with the same boundary values, such that the corresponding Young measure is $\bar\nu$. In such a case the infimum $\bar m$ of the minimization problem (4.3) is

$$\bar m = \int_{\mathbb{R}^n}\bigg(\int_{\mathbb{R}^n} W(\lambda_1,\lambda_2)\, d\bar\nu(\lambda_2)\bigg)\, d\bar\nu(\lambda_1).$$

Let $\bar\mu = t\bar\sigma + (1-t)\bar\nu$ be any convex variation of the homogeneous Young measure solution $\bar\nu$. By performing the same analysis from the beginning of Section 3 we arrive at the analogous inequality of (3.2):

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} W(\lambda_1,\lambda_2)\, d\bar\nu(\lambda_2)\, d\bar\sigma(\lambda_1) \geqslant \iint_{\mathbb{R}^n\times\mathbb{R}^n} W(\lambda_1,\lambda_2)\, d\bar\nu(\lambda_2)\, d\bar\nu(\lambda_1)$$

for any homogeneous Young measure $\bar\sigma$ such that $\int_{\mathbb{R}^n}\lambda\, d\bar\sigma(\lambda) = \gamma$ and $\int_{\mathbb{R}^n}|\lambda|^p\, d\bar\sigma(\lambda) < \infty$. This implies

$$\bar m = C_1\bigg(\int_{\mathbb{R}^n} W(\gamma,\lambda_2)\, d\bar\nu(\lambda_2)\bigg).$$

Since $\int_{\mathbb{R}^n} W(\gamma,\lambda_2)\, d\bar\nu(\lambda_2) \geqslant C_2 W(\gamma,\gamma)$ and $C_1(W(\gamma,\gamma)) = W(\gamma,\gamma)$ we have

$$\bar m \geqslant C_1\big(C_2 W(\gamma,\gamma)\big) = W(\gamma,\gamma) = J(u_0). \qquad \square$$

As in the previous results the proof of Theorem 1.1 only requires the optimality condition (4.1).

Proof of Theorem 1.1. The proof is elementary: assume J is weak lower semicontinuous at any $u \in \mathcal{A}$. Let A and B be any couple of vectors from $\mathbb{R}^n$ and apply (4.1) to any piecewise linear function u such that $\nabla u(x) = A$ in Ω_α and $= B$ in $\Omega - \Omega_\alpha$ where $\alpha = |\Omega_\alpha| \in (0,1)$ is arbitrary. Then

$$C_1\left(\int_\Omega W\big(\nabla u(x), \nabla u(y)\big)\,dy\right) = C_1\big(\alpha W\big(\nabla u(x), A\big) + (1-\alpha)W\big(\nabla u(x), B\big)\big)$$
$$= \alpha W\big(\nabla u(x), A\big) + (1-\alpha)W\big(\nabla u(x), B\big).$$

Since ∇u can be A or B, we have the identities

$$C_1\big(\alpha W(A, A) + (1-\alpha)W(A, B)\big) = \alpha W(A, A) + (1-\alpha)W(A, B),$$
$$C_1\big(\alpha W(B, A) + (1-\alpha)W(B, B)\big) = \alpha W(B, A) + (1-\alpha)W(B, B).$$

In particular the first of the above equations serves to assert the function

$$f(s) \doteq \alpha W(s, A) + (1-\alpha)W(s, B), \quad s \in \mathbb{R}^n,$$

is convex at $s = A$. Then for any A_1, A_2 from $\mathbb{R}^n$ and $\beta \in (0, 1)$ such that $\beta A_1 + (1-\beta)A_2 = A$ we have

$$f(A) \leqslant \beta\big(\alpha W(A_1, A) + (1-\alpha)W(A_1, B)\big) + (1-\beta)\big(\big(\alpha W(A_2, A) + (1-\alpha)W(A_2, B)\big)\big)$$
$$= \alpha\big(\beta W(A_1, A) + (1-\beta)W(A_2, A)\big) + (1-\alpha)\big(\beta W(A_1, B) + (1-\beta)W(A_2, B)\big).$$

By letting $\alpha \downarrow 0$ we obtain

$$W(A, B) \leqslant \beta W(A_1, B) + (1-\beta)W(A_2, B).$$

Since B is arbitrary, from the above inequality follows that $W(\cdot, A)$ is convex. Proceeding analogously with the second identity we prove $W(B, \cdot)$ is convex. □

5. Lower semicontinuity and inhomogeneity

The procedure we have developed in Section 3 applies without major changes when the functional is non-homogeneous,[1] i.e.

$$J(u) = \int_{\Omega\times\Omega} W\big(x, y, u(x), u(y), \nabla u(x), \nabla u(y)\big)\,dx\,dy.$$

If the sequence $\{u_j\} \in W^{1,p}(\Omega)$, $u_j \rightharpoonup u$, is a solution to the minimization problem

$$\min\Big\{\liminf_{n\to\infty} J(u_j)\colon\ u_j \in W^{1,p}(\Omega),\ u_j \rightharpoonup u \text{ in } W^{1,p}(\Omega)\Big\}$$

then

$$C_1\left(\int_\Omega \int_{\mathbb{R}^n} W\big(x, y, u(x), u(y), \nabla u(x), \lambda_2\big)\,d\nu_y(\lambda_2)\,dy\right)$$
$$= \int_{\mathbb{R}^n}\left(\int_\Omega \int_{\mathbb{R}^n} W\big(x, y, u(x), u(y), \lambda_1, \lambda_2\big)\,d\nu_y(\lambda_2)\,dy\right)d\nu_x(\lambda_1) \tag{5.1}$$

a.e. $x \in \Omega$, where ν is the Young measure generated by $\{\nabla u_j\}$. In particular, if $u_j = u$ is a minimizing sequence then (5.1) give rise to

$$C_1\left(\int_\Omega W\big(x, y, u(x), u(y), \nabla u(x), \nabla u(y)\big)\,dy\right) = \int_\Omega W\big(x, y, u(x), u(y), \nabla u(x), \nabla u(y)\big)\,dy. \tag{5.2}$$

Unfortunately, from (5.2) we were not able to deduce separate convexity for the integrand $W(x, y, u, v, A, B)$ in the variables A and B. Nevertheless, (5.2) is again the key point to state the following characterization result of the lower semicontinuity:

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[1] W is assumed to verify the following property of symmetry:

$$W\big(x, y, u(x), u(y), \nabla u(x), \nabla u(y)\big) = W\big(y, x, u(y), u(x), \nabla u(y), \nabla u(x)\big).$$

Theorem 5.1. *Assume $W = W(x, y, \nabla u(x), \nabla u(y))$ is the integrand in the definition of J. Then, J is weak lower semicontinuous if and only if the function*

$$G : (A, B) \in \mathbb{R}^n \times \mathbb{R}^n \to G(A, B) \doteq \int_\Omega W(x, y, A, B)\, dx$$

is separately convex, for any $y \in \Omega$.

Proof. If G is separately convex then

$$G_\nu(B) \doteq \int_\Omega \int_{\mathbb{R}^n} W(x, y, \lambda_1, B)\, d\nu_x(\lambda_1)\, dx, \quad y \in \Omega,$$

$$H(A) \doteq \int_\Omega W(x, y, A, B)\, dy, \quad x \in \Omega \ \big(\text{for any fixed } B \in \mathbb{R}^n\big)$$

and

$$H_\nu(A) \doteq \int_\Omega \int_{\mathbb{R}^n} W(x, y, A, \lambda_2)\, d\nu_y(\lambda_2)\, dy, \quad x \in \Omega,$$

are convex and consequently we can proceed as follows:

$$\begin{aligned}
\int_\Omega \left(\int_\Omega W\big(x, y, \nabla u(x), \nabla u(y)\big)\, dx \right) dy &\leqslant \int_\Omega \left(\int_{\mathbb{R}^n} \left(\int_\Omega W\big(x, y, \nabla u(x), \lambda_2\big)\, dx \right) d\nu_y(\lambda_2) \right) dy \\
&= \int_\Omega \left(\int_{\mathbb{R}^n} \int_\Omega W\big(x, y, \nabla u(x), \lambda_2\big)\, d\nu_y(\lambda_2)\, dy \right) dx \\
&\leqslant \int_\Omega \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \int_\Omega W\big(x, y, \nabla u(x), \lambda_2\big)\, d\nu_y(\lambda_2)\, dy \right) d\nu_x(\lambda_1)\, dx
\end{aligned}$$

which means W is lower semicontinuous. The *if part* follows along the same lines of the proof of Theorem 1.1. □

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