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Maximal Orders over Krull Domains[1]

ROBERT M. FOSSUM

The University of Illinois, Urbana, Illinois 61801

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INTRODUCTION

Using the notions of divisorial A-lattices for a Krull domain introduced in [7], we present a theory of orders over a Krull domain, which can be used to generalize much of the material presently available only for a noetherian Krull domain (i.e., a noetherian integrally closed integral domain). A brief outline of this article follows.

Firstly, we show, given a Krull domain A with field of quotients K and an A-order in a finite dimensional central simple K-algebra, that there are maximal A-orders containing the given order. Furthermore, a maximal A-order arises as an endomorphism ring of a maximal A-order in the associated division ring. Tame orders are also discussed briefly.

Secondly, we consider certain groups of divisors for a tame order and show, for a maximal order over a Krull domain, that these groups are isomorphic and depend, in a functorial sense, only on the class of the central simple algebra in the Brauer group of K.

Thirdly, we discuss extensions of Krull domains and associated extensions of orders. In particular, we consider subintersections and polynomial extensions,

We note here several conventions. If we say an algebra is central simple over a field K, we will always assume it to be finite dimensional. All modules will be unitary, (and so the rings always have 1). We use the notation and terminology introduced in [7] without further comment, especially that of Section 5 of [7]. If A is a Krull domain, **P** will denote its set of height one prime ideals.

1. MAXIMAL ORDERS

Let A be a Krull domain with field of quotients K. Let Σ be a central simple K-algebra.

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An A-order in Σ is a subring Λ of Σ satisfying the properties

(o1) $A \subseteq \Lambda$.

(o2) $K\Lambda = \Sigma$ (i.e. Λ contains a K-basis of Σ).

(o3) Each element of Λ is integral over A.

(See [8], Definition 2, page 69).

According to Example 4 of ([4], Chapitre VII, page 47), Σ always contains an A-order, which is even an A-lattice in Σ.

Since Σ is central separable over K, the reduced trace $Tr : \Sigma \to K$ induces an isomorphism $t : \Sigma \to \mathrm{Hom}_K(\Sigma, K)$ of K-modules, by $t(x)(y) = Tr(xy)$, for $x, y \in \Sigma$. Let $x_1, \ldots, x_r$ be a basis of Σ. Let $x_1^*, \ldots, x_r^*$ be elements of Σ such that $Tr(x_i^* x_j) = \delta_{ij}$ (Kronecker δ). If L denotes the free A-module generated by the set $\{x_j\}$, denote by L^c the free A-module in Σ generated by the set $\{x_j^*\}$. Note that t restricted to L^c induces an isomorphism of L^c with $L^* = A : L \cong \mathrm{Hom}_A(L, A)$.

PROPOSITION 1.1. *Let Λ be an A-order in Σ. If L is a free A-submodule of Λ, and Γ is an A-order in Σ which contains Λ, then $L^c \supseteq \Gamma$.*

Proof. Another description of L^c is $L^c = \{x \in \Sigma : Tr(xL) \subseteq A\}$. Since $L \subseteq \Gamma$ we know that $\Gamma L \subseteq \Gamma$. Because Γ is integral over A, $Tr(\Gamma) \subseteq A$, and hence $Tr(\Gamma L) \subseteq A$. So $\Gamma \subseteq L^c$.

COROLLARY 1.2. *Each A-order in Σ is an A-lattice in Σ.*

Proof. By condition (o2), an A-order contains a K-basis of Σ, and hence a free A-submodule which spans Σ. By the proposition, it is hence contained in a free A-lattice, so it is a lattice by definition.

We say an A-order Λ is *maximal*, if it is not properly contained in an A-order in Σ. If Λ is an A-order, it is easily verified that $\bigcap_{p \in \mathbf{P}} \Lambda_p$ is an A-order containing Λ. Conditions (o1) and (o2) are immediate, and condition (o3) follows from the fact that $A = \bigcap_{p \in \mathbf{P}} A_p$. We then have as in [1]

PROPOSITION. 1.3. *Let Λ be an A-order in Σ. Λ is a maximal A-order if, and only if,*

(i) *Λ is a divisorial A-lattice in Σ and*

(ii) *Λ_p is a maximal A_p-order for each $p \in \mathbf{P}$.*

We can now use the results of [7], namely, that the divisorial A-lattices in a free A-module satisfy the maximum condition (for $T_1 A$ is a noetherian object in the quotient category $\mathcal{M}/\mathcal{M}_2$) to prove (as in [8])

THEOREM 1.4. *Let Λ be an A-order in Σ. Then there is a maximal A-order in Σ containing Λ.*

Proof. Let L be a free A-submodule of Λ generated by a K-basis of Σ. In the set of divisorial A-orders which are contained in L^c and which contain Λ, pick a maximal such A-order. That there is always a divisorial A-order containing Λ follows from the remarks in the paragraph immediately preceding Proposition 1.3. The order picked must be a maximal order, for by Proposition 1.1, any order containing Λ is contained in L^c. Now if Γ is a maximal divisorial A-order in L^c containing Λ, then Γ_p is a maximal A_p-order. For if not, we can find for each $p \in \mathbf{P}$, a maximal A_p-order $\Gamma(p)$ containing Γ_p, and contained in $L^c{}_p$ $(=L_p{}^c)$, and with $\Gamma(p) = L^c{}_p$ for all but a finite number of prime ideals $p \in \mathbf{P}$. Then $\Gamma' = \bigcap \Gamma(p)$ is a divisorial A-lattice containing Γ. It is a subring of Σ, and it is clear that ($o3$) is satisfied, so Γ' is a divisorial A-order, and hence $\Gamma' = \Gamma$. Thus, $\Gamma_p = \Gamma(p)$ for each $p \in \mathbf{P}$. Hence by Proposition 1.3, Γ is a maximal A-order.

Because the integral closure of a Krull domain in a finite extension of its field of quotients is again a Krull domain ([*4*], Chapter VII, Section 1), we can discuss maximal orders in a finite dimensional semi-simple algebra.

Let B be an integrally closed integral domain with field of quotients L. Let T be a finite dimensional, semi-simple, L-algebra, with simple components Σ_i, $1 \leqslant i \leqslant n$, so $T = \prod_{i=1}^n \Sigma_i$. Let K_i be the center of Σ_i, and let A_i be the integral closure of B in K_i. Then $A = \prod_1^n A_i$ is the integral closure of B in the center, $\prod K_i$, of T. If Λ_i is a B-order in Σ_i (i.e. conditions ($o1$), ($o2$) and ($o3$) are satisfied), then $\Lambda = \prod \Lambda_i$ is a B-order in T, and $\Lambda \cap K_i = \Lambda_i \cap K_i$ is a B-order in K_i, and thus contained in A_i. If Λ is a B-order in T, then $\Lambda_i = A_i \Lambda$ is a B-order in Σ_i, and $\prod \Lambda_i$ is a B-order in T containing Λ. If Λ is a maximal B-order in T, then $\Lambda = \prod \Lambda_i$, $\Lambda \cap K_i = A_i$, and Λ_i is a maximal A_i-order in Σ_i. Thus to build a maximal B-order Γ in T containing a given B-order Λ in T, it is necessary and sufficient to build in each Σ_i a maximal B-order containing $\Lambda_i = A_i \Lambda$. But Λ_i is a maximal B-order in Σ_i if, and only if, Λ_i is a maximal A_i-order in Σ_i.

THEOREM 1.5. (a) *Let B be a Krull domain with field of quotients L. Let T be a finite dimensional semi-simple L-algebra, and let Λ be a B-order in T. Then there is a maximal B-order Γ in T which contains Λ.*

(b) *Γ is a maximal B-order in T if, and only if, Γ is a divisorial B-order in T, and Γ_p is a maximal B_p-order in T, for each prime ideal p in B with $htp = 1$.*

Proof. We retain the notation introduced prior to the statement of the theorem. As noted previously, each A_i is then a Krull domain, so we may apply Theorem 1.4 to conclude that each Σ_i admits a maximal A_i-order Γ_i

containing $A_i\Lambda$. Hence $\Gamma = \prod \Gamma_i$ is the desired maximal order. This proves (*a*) and (b) follows, as did Proposition 1.3.

This slight digression serves to show that it is usually sufficient to consider orders in a central simple algebra. The advantage of the treatment of orders over a Krull domain is seen in the fact that this class of integral domains is closed under the integral closure operation, (in finite extensions of the ground field). Thus one can make the transition from a semi-simple algebra to its central simple components with no fear of losing hold of a Krull domain.

Since noetherian integrally closed integral domains are Krull domains we have

COROLLARY 1.6. *Let B be a noetherian integrally closed integral domain. Let T be a finite dimensional semi-simple algebra over its field of quotients. Then any B-order in T is contained in a maximal B-order in T.*

Suppose again that A is a Krull domain, and Σ is a central simple K-algebra. When A is noetherian, and $\Sigma = \mathrm{Hom}_K(V, V)$ for a finite dimensional vector space V over K, it is shown in ([*1*], Proposition 4.29) that an A-order Λ in Σ is maximal if, and only if, $\Lambda = \mathrm{Hom}_A(E, E)$ for some divisorial A-lattice E in V. In [*9*] we mentioned that this structure theorem has the obvious generalization: If $\Sigma = \mathrm{Hom}_D(V, V)$, where D is a central division K-algebra, V a finite dimensional right D-vector space, then an A-order Λ in Σ is maximal if, and only if, there are a maximal A-order Γ in D and a right Γ-submodule E of V, which is a divisorial A-lattice such that $\Lambda = \mathrm{Hom}_\Gamma(E, E)$. No proof was offered in [*9*], as the method given in [*1*] can be used to obtain this result. We have the same result when A is a Krull domain, and we give a proof since the method varies slightly from [*1*].

THEOREM 1.7. *Let A be a Krull domain with field of quotients K. Let Σ be a central simple K-algebra. Suppose $\Sigma = \mathrm{Hom}_D(V, V)$, where D is a central division K-algebra, and V is a finite dimensional right D vector space. An A-order Λ in Σ is maximal if, and only if, there are a maximal A-order Γ in D and a right Γ-submodule E of V, which is a divisorial A-lattice such that $\Lambda \cong \mathrm{Hom}_\Gamma(E, E)$.*

Remark. It will follow that $\Gamma \cong \mathrm{Hom}_\Lambda(E, E)$.

Proof. We first remark that all functions considered are in $\mathrm{Hom}_K(V, V)$, so in particular D and Σ^{op} are subalgebras of $\mathrm{Hom}_K(V, V)$ with $\Sigma^{op} = \{f \in \mathrm{Hom}_K(V, V) : fd = df, \text{ all } d \in D\}$ and $D = \{d : fd = df, \text{ all } f \in \Sigma^{op}\}$. Now V is a simple Σ-module. Let $v \in V$ with $v \neq 0$ and let $E' = \Lambda v$, where Λ is a maximal A-order in Σ. Then E' is an A-lattice in V. Let $E = \bigcap_{p\in \mathbf{P}} E'_p$. Since each E'_p is a Λ_p-module, E is a $\Lambda = \bigcap_{p\in\mathbf{P}} \Lambda_p$ module. By the results in ([*7*], Section 5), $\Gamma = D \cap (E : E)$ is a divisorial A-order in D and

$\Gamma_p = D \cap (E_p : E_p) \cong \mathrm{Hom}_{\Lambda_p}(E_p, E_p)$. By Proposition 2.8 and Theorem 3.6 of [1], Γ_p is a maximal A-order in D for each $p \in \mathbf{P}$. It is clear that $\Gamma \cong \mathrm{Hom}_\Lambda(E, E)$. Now $\Sigma^{op} \cap (E : E)$ is an A-order in Σ^{op} which contains Λ^{op}. Hence $\Lambda^{op} = \Sigma^{op} \cap (E : E)$. But, it is clear that this is just isomorphic to $\mathrm{Hom}_\Gamma(E, E)^{op}$, so we have proved one implication.

As for the other implication, suppose Γ is a maximal A-order in E, and E is a divisorial A-lattice in V. Then $\mathrm{Hom}_D(V, V)^{op} \cap (E : E)$ in $\mathrm{Hom}_K(V, V)$ is a divisorial A-order in $\mathrm{Hom}_D(V, V)^{op} = \Sigma^{op}$. By the same arguments as above, it is a maximal order. The proof is complete.

A generalization of maximal order is a tàme order (cf. [12]). Let A, K, and Σ be as usual. We say an A-order Λ in Σ is *tame* if it is divisorial and Λ_p is an hereditary A_p-order in Σ for each $p \in \mathbf{P}$. It is clear from ([7], Section 5) that if Γ is a maximal A-order containing the tame order Λ, then $\Gamma_p = \Lambda_p$ for all but a finite number of p. Thus, much of what is said in [5] and ([12], Chapter I) can be said for tame A-orders when A is a Krull domain. Here we are interested in two inheritance properties which tame orders enjoy.

THEOREM 1.8. *Let A be a Krull domain with field of quotients K. Let Σ be a central simple K-algebra. Suppose Λ is a tame (resp. maximal) A-order in Σ. Let $\mathbf{P}'$ be a subset of $\mathbf{P}$ and set $B = \bigcap_{p \in \mathbf{P}'} A_p$ (a subintersection). Then $\Lambda' = \bigcap_{p \in \mathbf{P}'} \Lambda_p$ is a tame (resp. maximal) B-order in Σ.*

Proof. B is again a Krull domain, and Λ' is a B-order which is divisorial. That Λ' is tame (resp. maximal) is clear by localizing at each $p \in \mathbf{P}'$.

COROLLARY 1.9. *Let S be a multiplicatively closed subset of A. If Λ is a tame (resp. maximal) A-order in Σ, then Λ_S is a tame (resp. maximal) A_S-order in Σ.*

Proof. $A_S = \bigcap_{p \in \mathbf{P}'} A_p$, where $\mathbf{P}' = \{p \in \mathbf{P} : p \cap S = \phi\}$. $\Lambda_S = \bigcap_{p \in \mathbf{P}'} \Lambda_p$.

If A is a Krull domain with field of quotients K, and X is an inderminant, then $A[X]$ is a Krull domain with field of quotients $K(X)$. If Σ is a central simple K-algebra, then $\Sigma(X) = K(X) \otimes_K \Sigma$ is a central simple $K(X)$-algebra. Furthermore, if Λ is an A-order in Σ, then $\Lambda[X] = A[X] \otimes_A \Lambda$ is an A-order in $\Sigma(X)$. We have

THEOREM 1.11. *Suppose A is a Krull domain with field of quotients K. Let Σ be a central simple K-algebra, and Λ a tame (resp. maximal) A-order in K. Then $\Lambda[X]$ is a tame (resp. maximal) $A[X]$-order in $\Sigma(X)$.*

Proof. In view of Proposition 1.3 and the definitions, we must show $\Lambda[X]$ is divisorial, and that $\Lambda[X]_P$ is an hereditary $A[X]_P$-order (resp. maximal order) for each prime ideal P of $A[X]$ with $htP = 1$. There are two types of these prime ideals, those with $P \cap A = 0$, and those with $P \cap A = p \neq 0$.

In either case let $p = P \cap A$. Then $A[X]_P = A_P[X]_Q$, where $Q = PA_p[X]$. If $P \cap A = 0$, then $A[X]_P = K[X]_Q$.

To see that $\Lambda[X]$ is divisorial, we write

$$\bigcap_{htP=1} \Lambda[X]_P = \Big(\bigcap_{P\cap A=0} \Lambda[X]_P\Big) \cap \Big(\bigcap_{P\cap A\neq 0} \Lambda[X]_P\Big).$$

Let $\mathbf{I}$ be the set of monic irreducible polynomials in $K[X]$. For each $f \in \mathbf{I}$, let (f) denote the prime ideal it generates. When $P \cap A = 0$ we have $A[X]_P = K[X]_{(f)}$ where $PK[X] = (f)$. Hence $\bigcap_{P\cap A=0} \Lambda[X]_P = \bigcap_{f\in \mathbf{I}} \Sigma[X]_{(f)}$. Now $K[X]$ is noetherian, so we may use the lattice theory in ([*4*], Chap. VIII) toconclude that $\Sigma[X] = \Sigma[X]^{**} = \bigcap_{f\in I} \Sigma[X]_{(f)}$. Hence $\bigcap_{htP=1} \Lambda[X]_P = \Sigma[X] \cap \bigcap_{P\cap A\neq 0} \Lambda[X]_P$. Suppose that $h \in \bigcap_{htP=1} \Lambda[X]_P$. Then we may write $h = g/s$, where $g \in \Lambda[X]$ and $0 \neq s \in A$. Since $f \in \Lambda[X]_P$ for each prime ideal P with $htP = 1$, we get each coefficient of f, say λ_i/s, in Λ_p, where $p = P \cap A$. But $\Lambda = \bigcap_{htP=1} \Lambda_p$, so $\lambda_i/s \in \Lambda$ for each i, hence $f \in \Lambda[X]$ which is what we set out to establish.

We now show that $\Lambda[X]_P$ is hereditary (maximal) for each P. If $P \cap A = 0$, then $\Sigma[X]_P = \Sigma[X]_{(f)}$. Now Σ is a central separable K-algebra, hence $\Sigma[X]$ is a central separable $K[X]$-algebra and so a maximal order in $\Sigma(X)$ as seen in [*2*]. Thus we have that $\Sigma[X]_P$ is a maximal order for those prime ideals which meet A at 0.

Now suppose $P \cap A = p \neq 0$. Then $\Lambda[X]_P = A_P[X]_Q \otimes_A \Lambda = A_P[X]_Q \otimes_{A_p} \Lambda_p$. We know Λ_p is an hereditary (maximal) A_p-order. Since $P = pA[X]$, the ring $A_p[X]_Q = A_p(X)$ as defined in [*11*]. Once we prove the next lemma the proof of the theorem will be complete.

LEMMA 1.12. *Let $\mathfrak{O}$ be a discrete rank one valuation ring and Λ an hereditary $\mathfrak{O}$-order. Then $\mathfrak{O}(X) \otimes_{\mathfrak{O}} \Lambda$ is an hereditary $\mathfrak{O}$-order. If Λ is a maximal $\mathfrak{O}$-order, then $\Lambda(X)$ is a maximal $\mathfrak{O}(X)$-order.*

Proof. To show that $\Lambda(X)$ is hereditary, it is sufficient in light of ([*1*], Corollary page 5), to show that the radical of $\Lambda(X)$ is a projective left $\Lambda(X)$-module. To show this, it is enough to show that $\text{Rad}(\Lambda(X)) = \text{Rad}(\Lambda)(X)$ (where here $\text{Rad}(\Lambda)$, denotes the Jacobson radical of the ring Λ). Let m be the maximal ideal of $\mathfrak{O}$ and k the residue class field. Then $\Lambda(X)/m\Lambda(X) \cong (\Lambda/m) \otimes_k k(X)$, so

$$\text{Rad}((\Lambda/m\Lambda) \otimes_k k(X)) = (\text{Rad}(\Lambda/m\Lambda)) \otimes_k k(X) = (\text{Rad}\, \Lambda/m\Lambda) \otimes_k k(X).$$

But $\text{Rad}(\Lambda(X)/m\Lambda(X)) = \text{Rad}\, \Lambda(X)/m\Lambda(X)$. Hence $\text{Rad}(\Lambda(X)) = (\text{Rad}\, \Lambda)(X)$. Since $\text{Rad}\, \Lambda$ is projective as a Λ-module, it follows that $\text{Rad}\, \Lambda(X)$ is a projective $\Lambda(X)$-module.

Also we get that $\Lambda(X)/\operatorname{Rad}\Lambda(X) = (\Lambda/\operatorname{Rad}\Lambda) \otimes_k k(X)$. If $\Lambda/\operatorname{Rad}\Lambda$ is simple (i.e. Λ is a maximal order), then $(\Lambda/\operatorname{Rad}\Lambda) \otimes_k k(X)$ is a simple $k(X)$-algebra since $k(X)$ is a separable extension of k. Hence $\operatorname{Rad}\Lambda(X)$ is a maximal twosided ideal in $\Lambda(X)$, and so $\Lambda(X)$ is a maximal $\mathfrak{O}(X)$-order.

2. Groups of Divisors

Let A be a Krull domain with field of quotients K. Let Λ be a tame A-order in a central simple K-algebra Σ. We call an A-lattice M in Σ, which is a two sided Λ-module, a *Λ-divisor* (or simply a divisor) if it is divisorial as an A-lattice, and if for each $p \in \mathbf{P}$, M_p is an invertible Λ_p-module. That is to say $\Lambda_p = M_p M_p^{-1} = M_p^{-1} M_p$ for each $p \in \mathbf{P}$. Let $D(\Lambda)$ denote the set of Λ-divisors. A product in $D(\Lambda)$ is given by $M \cdot N = (MN)^{**}$. Just as in [*5*, *10*, *12*], $D(\Lambda)$ is then an abelian group, which is free on the ideals $M(p) = \operatorname{Rad}(\Lambda_p) \cap \Lambda$. We call it the *group of divisors* of Λ. We have as in [*9*].

Theorem 2.1. *Let A be a Krull domain with field of quotients K. Let Σ be a central simple K-algebra. If Λ_1, Λ_2 are two maximal A-orders in Σ, then the conductor $\Lambda_1 : \Lambda_2$ induces an isomorphism $d(\Lambda_2, \Lambda_1) : D(\Lambda_1) \to D(\Lambda_2)$ of the groups of divisors. This isomorphism is natural in the sense that if Λ_3 is a third maximal A-order in Σ, then $d(\Lambda_3, \Lambda_2)\, d(\Lambda_2, \Lambda_1) = d(\Lambda_3, \Lambda_1)$.*

Proof. This is established as in [*10*]. An outline is: Let $\Lambda_1 : \Lambda_2 = \{x \in \Sigma : \Lambda_2 x \subseteq \Lambda_1\}$. Then $\Lambda_1 : \Lambda_2$ is a divisorial A-lattice in Σ, which is a left Λ_2 and a right Λ_1 module. Furthermore $((\Lambda_2 : \Lambda_3)(\Lambda_1 : \Lambda_2))^{**} = \Lambda_1 : \Lambda_3$. Now define $d(\Lambda_2, \Lambda_1)(M) = ((\Lambda_1 : \Lambda_2)\, M(\Lambda_2 : \Lambda_1))^{**}$. It is easy to verify that the statements of the theorem now follow.

As in [*7*] we can obtain another group for Λ, which is similar to the group of divisors, but arises from the modules rather than the ideals. Let $\mathscr{M} = \mathscr{M}(\Lambda)$ denote the category of left Λ-modules. Denote by $\mathscr{M}_1 = \mathscr{M}_1(\Lambda)$ and $\mathscr{M}_2 = \mathscr{M}_2(\Lambda)$ the Serre subcategories of $\mathscr{M}$ consisting of those left Λ-modules M, such that $K \otimes_A M = 0$ and $A_p \otimes_A M = 0$ for all $p \in \mathbf{P}$, respectively. Let T_0 and T_1 denote the canonical functors $T_i : \mathscr{M} \to \mathscr{M}/\mathscr{M}_{i+1}$, $i = 0, 1$. As in [*7*], $T_i \Lambda$ is a noetherian object and a generator in each $\mathscr{M}/\mathscr{M}_{i+1}$. Furthermore $\mathscr{M}_1/\mathscr{M}_2$ is in a natural way a Serre subcategory of $\mathscr{M}/\mathscr{M}_2$ with quotient category naturally equivalent to $\mathscr{M}/\mathscr{M}_1$. If $(\ )^{\#}$ denotes the (exact abelian) full subcategory of noetherian objects of $(\ )$, then we have that $(\mathscr{M}_1/\mathscr{M}_2)^{\#}$ is a Serre subcategory of $(\mathscr{M}/\mathscr{M}_2)^{\#}$ with quotient category $(\mathscr{M}/\mathscr{M}_1)^{\#}$. Let $G(\Lambda)$ denote the Grothendieck group of the category $(\mathscr{M}_1/\mathscr{M}_2)^{\#}$, and let $\mathscr{G}(\Lambda)$ denote the Grothendieck group of the category $(\mathscr{M}/\mathscr{M}_2)^{\#}$;

Proposition 2.2. *Let Λ be a tame A-order in the central simple K-algebra*

Σ. For each $p \in \mathbf{P}$, let $r(p)$ denote the number (finite) of maximal A_p-orders which contain Λ_p. Then $G(\Lambda) = \prod_{p \in \mathbf{P}} \mathbf{Z}^{r(p)}$.

Proof. Each object in $(\mathcal{M}_1/\mathcal{M}_2)^{\#}$ has finite length, so it is sufficient to compute the Grothendieck group of the associated semi-simple subcategory generated by the simple object. As in [7], it is seen that an object is simple if, and only if, it is a simple object of $T_1(\Lambda_p/\text{Rad } \Lambda_p)$. By results in [5], there are exactly $r(p)$ isomorphism classes of these, and the proposition is proved.

In [9] we established the fact that $D(\Lambda)$ and $G(\Lambda)$ are isomorphic in some natural sense, when A is a noetherian Krull domain and Λ is a maximal order. This result is still true in the more general setting in force here.

We suppose that A is a Krull domain, and that Λ is a maximal A-order. If M is a divisor of Λ, which we suppose is contained in Λ, then $T_1(\Lambda/M)$ is an object in $(\mathcal{M}_1/\mathcal{M}_2)^{\#}$ and hence has a class in $G(\Lambda)$. However, the map $M \to [T_1(\Lambda/M)]$ does not induce the desired isomorphism from $D(\Lambda)$ to $G(\Lambda)$. We can modify this as follows[2]: For each $p \in \mathbf{P}$, let $s(p)$ be the *complete degree* of Σ at p. This integer is obtained by forming the p-adic completion of A_p, $\hat{A}_p$, then writing $\hat{\Sigma}_p = \hat{A}_p \otimes_A \Sigma = \text{Hom}_{D(p)}(V(p), V(p))$ for a suitable division ring $D(p)$. Then $s(p) = \dim_{D(p)} V(p)$. Now $\hat{A}_p \otimes_A \Lambda$ is a maximal order in $\hat{\Sigma}_p$ (cf. [1]), and so can be written in the form $\text{Hom}_\Gamma(E, E)$, where Γ is the (unique) maximal $\hat{A}_p$-order in $D(p)$. Then, if $R(p)$ denotes the radical of $\hat{A}_p \otimes_A \Lambda$, we have $\hat{A}_p \otimes_A \Lambda/R(p) = \text{Hom}_{\Gamma/\Gamma\pi}(E/E\pi, E/E\pi)$ (where $\Gamma\pi$ is the radical of Γ). Now $\Gamma/\Gamma\pi$ is a division ring, and $E/E\pi$ is a vector space of dimension $s(p)$ over $\Gamma/\Gamma\pi$. Hence $\hat{A}_p \otimes_A (\Lambda/\text{Rad } \Lambda_p \cap \Lambda) = \hat{A}_p \otimes_A \Lambda/R(p)$ has length $s(p)$ as a Λ-module.

We now define an isomorphism $f(\Lambda) : D(\Lambda) \to G(\Lambda)$. Take

$$M = \prod_{p \in \mathbf{P}} M(p)^{t(p)}$$

an element of $D(\Lambda)$, and define

$$f(\Lambda)(M) = \sum_{p \in \mathbf{P}} t(p) \frac{[\Lambda/M(p)]}{s(p)},$$

where we have used [ ] to denote the class of $T(\ )$ in $G(\Lambda)$. This map is well defined and is manifestly an isomorphism.

PROPOSITION 2.3. *Let Λ be a maximal A-order. Then $D(\Lambda)$ and $G(\Lambda)$ are isomorphic.*

If Γ is another maximal A-order in Σ, then $(\Lambda : \Gamma) \otimes_\Lambda - : \mathcal{M}(\Lambda) \to \mathcal{M}(\Gamma)$. As in [9], $(\Lambda : \Gamma) \otimes_\Lambda$ —induces equivalences of categories

$$\mathcal{M}(\Lambda)/\mathcal{M}_i(\Lambda) \to \mathcal{M}(\Lambda)/\mathcal{M}_i(\Lambda) \qquad \textit{for} \quad i = 1, 2,$$

[2] Professor Reiner made several suggestions which led to this treatment.

and

$$\mathscr{M}_1(\Lambda)/\mathscr{M}_2(\Lambda) \to \mathscr{M}_1(\Gamma)/\mathscr{M}_2(\Gamma).$$

Thus we can state

THEOREM 2.4. *Let Λ, Γ be maximal A-orders, A a Krull domain. Then the conductor $\Lambda : \Gamma$ induces isomorphisms $g(\Gamma, \Lambda) : G(\Lambda) \to G(\Gamma)$, and $c(\Gamma, \Lambda) : \mathscr{G}(\Lambda) \to \mathscr{G}(\Gamma)$, which are natural in the sense that if Λ' is another maximal A-order, then $g(\Gamma, \Lambda)\, g(\Lambda, \Lambda') = g(\Gamma, \Lambda')$, and $c(\Gamma, \Lambda)\, c(\Lambda, \Lambda') = c(\Gamma, \Lambda')$. Also the diagram*

$$\begin{array}{ccc} D(\Lambda) & \xrightarrow{f(\Lambda)} & G(\Lambda) \\ {\scriptstyle d(\Gamma,\Lambda)}\downarrow & & \downarrow{\scriptstyle g(\Gamma,\Lambda)} \\ D(\Gamma) & \xrightarrow{f(\Gamma)} & G(\Gamma) \end{array}$$

is commutative.

Proof. The first part of the theorem follows exactly as in [*9*]. For the second part, we note that the complete degree of Σ at p does not depend on the maximal order in question. To establish commutativity of the diagram, it is sufficient to consider a generator $M_\Lambda(p)$ of $D(\Lambda)$ and to follow it about. Now

$$\begin{aligned} g(\Gamma, \Lambda) f(\Lambda)\, M_\Lambda(p)) &= \frac{1}{s(p)} [(\Lambda_p : \Gamma_p) \otimes_{\Lambda_p} (\Lambda_p/\text{Rad } \Lambda_p)] \\ &= \frac{1}{s(p)} [(\Lambda_p : \Gamma_p)/(\Lambda_p : \Gamma_p)\, w_p], \end{aligned}$$

where $w_p \Lambda_p = \Lambda_p w_p = \text{Rad } \Lambda_p$. Also $\Gamma_p : \Lambda_p = \Lambda_p c_p$, where c_p is a unit in Σ. So as left Γ_p-modules we have isomorphisms $(\Lambda_p : \Gamma_p)/(\Lambda_p : F_p)\, w_p \cong (\Lambda_p : \Gamma_p)\, c_p/(\Lambda_p : \Gamma_p)\, w_p \Lambda_p c_p \cong (\Lambda_p : \Gamma_p)(\Gamma_p : \Lambda_p)/(\Lambda_p : \Gamma_p) \text{Rad } \Lambda_p(\Gamma_p : \Lambda_p) \cong \Gamma_p/\text{Rad } \Gamma_p$. Hence in $G(\Gamma)$, we have

$$\begin{aligned} g(\Gamma, \Lambda) f(\Lambda)(M_\Lambda(p)) &= \frac{1}{s(p)} [\Gamma_p/\text{Rad } \Gamma_p] = f(\Gamma)(\text{Rad } \Gamma_p \cap \Gamma) \\ &= f(\Gamma)\, d(\Gamma, \Lambda)(M_\Lambda(p)). \end{aligned}$$

We are also able to obtain the corollary to Theorem 2 of [*9*] since the structure theorem of Section 1 is available. As the technique of proof is exactly the same as in [*9*], we omit the details and only state the result.

THEOREM 2.5. *Let A be a Krull domain with field of quotients K. Let D be a central division K-algebra, V a finite dimensional right D-module and*

$\Sigma = \mathrm{Hom}_D(V, V)$. *Let Γ be a maximal A-order in D, E a divisorial A-lattice in V which is a right Γ-module and $\Lambda = \mathrm{Hom}_\Gamma(E, E)$, the corresponding maximal order in Σ. Then*

$$E \otimes_\Gamma - : \mathscr{M}(\Gamma) \to \mathscr{M}(\Lambda) \text{ and } \mathrm{Hom}_\Gamma(E, \Gamma) \otimes_\Lambda - : \mathscr{M}(\Lambda) \to \mathscr{M}(\Gamma)$$

induce inverse isomorphisms

$$G(\Gamma) \xrightarrow{g(\Lambda,\Gamma)} G(\Lambda) \xrightarrow{g(\Gamma,\Lambda)} G(\Gamma)$$

and

$$\mathscr{G}(\Gamma) \xrightarrow{c(\Lambda,\Gamma)} \mathscr{G}(\Lambda) \xrightarrow{c(\Gamma,\Lambda)} \mathscr{G}(\Gamma)$$

which commute with the conductor induced isomorphisms in both Σ and D.

Suppose that Λ is a tame A-order in Σ. We define the class group $W(\Lambda)$ (or $W_1(\Lambda)$ to conform with the notation in [*6*, *7*]) to be the image of $G(\Lambda)$ in $\mathscr{G}(\Lambda)$. Since we have the exact sequence of abelian groups.

$$K^1(\Sigma) \xrightarrow{\delta} G(\Lambda) \xrightarrow{\lambda} \mathscr{G}(\Lambda) \xrightarrow{\mu} K^0(\Sigma) \longrightarrow 0$$

$W(\Lambda)$ can be described as any of the groups Im λ, Ker μ or $G(\Lambda)/\mathrm{Im}\ \delta$. Since $K^0(\Sigma)$ is just $\mathbf{Z}$, $W(\Lambda)$ is a direct summand of $\mathscr{G}(\Lambda)$. The functorial properties of $G(\Lambda)$ and $\mathscr{G}(\Lambda)$ expressed in Theorems 2.4 and 2.5 yield

Proposition 2.6. *Under the same hypotheses as in Theorem* 2.5 $g(\Gamma, \Lambda)$ *induces an isomorphism* $w(\Gamma, \Lambda) : W(\Lambda) \to W(\Gamma)$, *which is functorial in the sense of Theorems* 2.4 *and* 2.5.

Remark. It is not the case that $W(\Lambda) = 0$ implies, even for a maximal order, that each divisorial A-order, which is a left Λ ideal, is principal.

3. Extensions

Let A be a Krull domain with field of quotients K. If B is a Krull domain which contains A, and which satisfies the property

(PDE) If P is a prime ideal of B with $htP = 1$, then $ht(PA) \leq 1$.

We said in [*7*] that B is 1-flat over A and we then defined, via $B \otimes_A -$, homomorphisms $D_1(A) \to D_1(B)$ and $W_1(A) \to W_1(B)$, which are the same as those defined in [*4*]. We extend this definition to tame orders over Krull domains.

Let Λ be a tame order in a central simple K-algebra Σ. Let B be a Krull domain containing A, which is 1-flat over A. Let Γ be a tame B-order, which contains $B \otimes_A \Lambda$ in $L \otimes_K \Sigma$.

LEMMA 3.1. *If P is a prime ideal of B with $htP = 1$, then Γ_P is a flat Λ_p-module where $p = P \cap A$.*

Proof. We know that Γ_P is an A_p-torsion free Λ_p-module. Λ_p is such that if E is a finitely generated Λ_p-module, then $hd_{\Lambda_p} E = hd_{A_p} E$ by Theorem 2.2 of [1]. Now Γ_P as a Λ_p-module is the direct limit of its finitely generated Λ_p-sub-modules which are A_p-torsion free and hence projective. Thus Γ_P is the direct limit of Λ_p-projective modules, and hence is the direct limit of flat Λ_p-modules, so is flat.

Thus $\Gamma \otimes_\Lambda$ — induces homomorphism $G(\Lambda) \to G(\Gamma)$, $\mathcal{G}(\Lambda) \to \mathcal{G}(\Gamma)$, and $W(\Lambda) \to W(\Gamma)$ (see [7], Section 7). We are most interested in the homomorphism $W(\Lambda) \to W(\Gamma)$, when B is a subintersection or a polynomial extension and $\Gamma = B \otimes_A \Lambda$.

PROPOSITION 3.2. *Let A be a Krull domain and Λ a tame A-order. Let $\mathbf{P}'$ be a subset of $\mathbf{P}$, and let $\Gamma = \bigcap_{p \in P'} \Lambda_p$. Then the homomorphism $W(\Lambda) \to W(\Gamma)$ is an epimorphism.*

Proof. $G(\Lambda) \to G(\Gamma)$ is an epimorphism.

COROLLARY 3.3. *Let S be a multiplicatively closed subset of the Krull domain. A. Let Λ be a tame A-order. Then the homomorphsim $W(\Lambda) \to W(\Lambda_S)$ is an epimorphism.*

PROPOSITION 3.4. *Let Λ be a tame A-order. Then there is a homomorphism $W(\Lambda) \to W(\Lambda[X])$ induced by $\Lambda[X] \otimes_\Lambda$ — .*

We do not know whether this homomorphism is an epimorphism but conjecture that it is (see [6, 7]).

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