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The stability problem for the torque-free gyrostat investigated by using algebraic methods

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ABSTRACT

We apply an algebraic method for studying stability with respect to a set of conserved quantities for the problem of the torque-free gyrostat. If the conditions of this algebraic method are not fulfilled then the Lyapunov stability cannot be decided on using the specified set of conserved quantities.

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1. Introduction

A very important problem in the theory of differential equations is the problem of stability. A very useful tool for determining the stability of an equilibrium point is Lyapunov's direct method connected with the Lyapunov functions. A natural candidate for being a Lyapunov function is a conserved quantity. In a lot of examples coming from mathematical physics, we identify a set $\{F_1, \ldots, F_k\}$ of conserved quantities. In many situations they are not positive definite functions at the equilibrium points of interest. In this situation, a first step towards deciding whether the equilibrium point is stable is searching for a Lyapunov function of the form $\Phi(F_1, \ldots, F_k)$, where $\Phi : \mathbb{R}^k \to \mathbb{R}$ is a smooth function. A function $\Phi(F_1, \ldots, F_k)$ is a Lyapunov function if and only if it is a positive definite function. In stability theory, some important methods for constructing positive definite functions using conserved quantities are known. We recall the so-called "Chetaev method" presented in [1] and some methods which appeared in the context of Hamilton–Poisson systems. In 1965 Arnold's method was introduced; see [2]. At the beginning of eighties the energy–Casimir method was developed (see [3,4]) and in 1998 the paper [5] presented the Ortega–Ratiu method. In [6] the equivalence of Arnold's method, the energy–Casimir method and the Ortega–Ratiu method is proved.

If there exists, for an equilibrium point, a positive definite function of type $\Phi(F_1, \ldots, F_k)$, we say that the equilibrium point is stable with respect to the set of conserved quantities $\{F_1, \ldots, F_k\}$. In 1958 Pozharitsky (see [7]) proved that it is sufficient to study the function $\|(F_1, \ldots, F_k)\|$ in order to decide whether an equilibrium point is stable with respect to the set of conserved quantities $\{F_1, \ldots, F_k\}$ (see [1]). Another method for deciding whether an equilibrium point is stable with respect to the set of conserved quantities $\{F_1, \ldots, F_k\}$ (see [1]) is given by an algebraic method which reduces to studying whether the equilibrium point x_e is isolated in the set of all the solutions of the algebraic system $F_1(x) = F_1(x_e), \ldots, F_k(x) = F_k(x_e)$. We also show that if the equilibrium point x_e is not isolated in the set of all the solutions of the algebraic system given above, then it is impossible to construct a Lyapunov function in x_e using the set of conserved quantities $\{F_1, \ldots, F_k\}$. We apply this algebraic method to decide on the stability of an equilibrium point with respect to a set of conserved quantities for the problem of the torque-free gyrostat. In Section 2 we present some notions and results on stability theory which we apply in the study of our example.

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In Section 3 we present the mathematical model of a torque-free gyrostat and we give a set of two functionally independent conserved quantities. We find the set of uniform rotations and we first study their stability with respect to a single conserved quantity. For the cases where the vector of the gyrostatic moment is situated along a principal axis of inertia of the gyrostat, we study the stability of a uniform rotation with respect to the set of two conserved quantities. We prove that a uniform rotation is stable with respect to the given set of conserved quantities if and only if it is stable in the sense of Lyapunov. It is interesting to see that there exist some singular cases for which we cannot decide on the Lyapunov stability of a uniform rotation using the algebraic method or the linearization method.

In a future study we will apply the algebraic method which is used in this work to the problem of the rotational motion of a gyrostat in the presence of an axisymmetric force field. We take advantage of the set of conserved quantities found in [8].

2. Lyapunov's direct method based on solving algebraic equations

We consider an open set $D \subset \mathbb{R}^n$ and the locally Lipschitz function $f : D \to \mathbb{R}^n$ which generates the differential equation

$$\dot{x} = f(x). \tag{2.1}$$

We denote by $x(\cdot, x_0)$ the maximal solution of the above differential equation which verifies the initial condition $x(0, x_0) = x_0$. A point $x_e \in D$ is an equilibrium point of (2.1) if and only if $f(x_e) = 0$. An equilibrium point $x_e \in D$ is stable (or stable in the sense of Lyapunov) if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all y in the ball $B(x_e, \delta)$ and $t \geq 0$ we have $\|x(t, y) - x_e\| < \varepsilon$ (see [9]). The most important result for proving stability of an equilibrium point is given by Lyapunov's direct method.

Theorem 2.1. *If there exists a continuous function $V : D \to \mathbb{R}$ satisfying the conditions: $V(x_e) = 0$, $V(x) > 0$ for x in a neighborhood of x_e with $x \neq x_e$ and $t \to V(x(t, y))$ is a decreasing function for all $y \in D$, then the equilibrium point x_e is stable.*

A continuous function which satisfies the conditions (i) and (ii) is called a positive definite function at the equilibrium point x_e. A continuous function V satisfying the hypotheses of the above theorem is called a Lyapunov function at the equilibrium point x_e. We introduce the following notion of stability.

Definition 2.1. The equilibrium point x_e of (2.1) is stable with respect to the set of conserved quantities $\{F_1, \ldots, F_k\}$ if there exists a continuous function $\Phi : \mathbb{R}^k \to \mathbb{R}$ such that $x \to \Phi(F_1, \ldots, F_k)(x) - \Phi(F_1, \ldots, F_k)(x_e)$ is a positive definite function in x_e.

Under the conditions of the above definition, the function $x \to \Phi(F_1, \ldots, F_k)(x) - \Phi(F_1, \ldots, F_k)(x_e)$ is a Lyapunov function at the equilibrium point x_e. We have the obvious consequences.

Theorem 2.2. *Let x_e be an equilibrium point and $\{F_1, \ldots, F_k\}$ be a set of conserved quantities for (2.1).*

(i) *If x_e is stable with respect to the set $\{F_1, \ldots, F_k\}$ then it is stable in the sense of Lyapunov.*
(ii) *Let $q \in \{1, \ldots, k\}$ be an integer number. If x_e is stable with respect to $\{F_1, \ldots, F_q\}$, then it is stable with respect to $\{F_1, \ldots, F_k\}$.*

We have the following equivalent conditions for the stability of an equilibrium point with respect to a set of conserved quantities.

Theorem 2.3. *Let x_e be an equilibrium point of (2.1) and $\{F_1, \ldots, F_k\}$ a set of conserved quantities. The following statements are equivalent:*

(i) *x_e is stable with respect to the set of conserved quantities $\{F_1, \ldots, F_k\}$;*
(ii) *$x \to \|(F_1, \ldots, F_k)(x) - (F_1, \ldots, F_k)(x_e)\|$ is a positive definite function in x_e;*
(iii) *the system $F_1(x) = F_1(x_e), \ldots, F_k(x) = F_k(x_e)$ has no root besides x_e in some neighborhood of x_e.*

In 1958, Pozharitsky proved the equivalence between (i) and (ii); see [7,1] p. 130. Equivalence between (ii) and (iii) appears in [1] p. 151. In the paper [10], Aeyels presented an interesting proof for the implication "(iii) $\Rightarrow x_e$ is Lyapunov stable".

Theorem 2.3(iii) gives an algebraic method for establishing Lyapunov stability of an equilibrium point. Moreover, it also shows that if the equilibrium point x_e is not isolated in the set of solutions for the algebraic system of equations then it is impossible to construct a Lyapunov function in x_e using the set of conserved quantities $\{F_1, \ldots, F_k\}$. We will apply this algebraic method to study the stability of uniform rotations for a torque-free gyrostat.

For the case of one conserved quantity, i.e. $k = 1$, we have the well known result.

Theorem 2.4. *Let x_e be an equilibrium point of (2.1) and F a conserved quantity. The following statements are equivalent:*

(i) *x_e is stable with respect to the conserved quantity F;*
(ii) *x_e is a strict local extremum of F.*

3. The stability of the uniform rotations

For the problem of the torque-free gyrostat we find the set of uniform rotations and we study their stability with respect to a conserved quantity. For the cases where the vector of the gyrostatic moment is situated along a principal axis of inertia of the gyrostat, we study the stability of a uniform rotation with respect to the set formed by two conserved quantities. Except for two singular cases, the Lyapunov stability problem for the torque-free gyrostat can be decided using the algebraic method with two conserved quantities and the linearization method. For the singular cases we decide the Lyapunov stability by studying the dynamics in an invariant set. The equation for rotation of a torque-free gyrostat is given by (see [8,11])

$$\mathbb{I}\dot{\vec{\omega}} = (\mathbb{I}\vec{\omega} + \vec{\mu}) \times \vec{\omega}, \tag{3.1}$$

where $\vec{\omega}$ is the angular velocity, and $\mathbb{I}$ is the inertia tensor and $\vec{\mu}$ is the constant vector of the gyrostatic moment. We denote by I_1, I_2 and I_3 the principal moments of inertia and suppose that $I_1 > I_2 > I_3$. If we use the angular momentum vector $\vec{M} = \mathbb{I}\vec{\omega}$, then the equation becomes

$$\dot{\vec{M}} = (\vec{M} + \vec{\mu}) \times \mathbb{I}^{-1}\vec{M}. \tag{3.2}$$

For the above dynamics we have two conserved quantities $F_1 = \frac{1}{2}\vec{M} \cdot \mathbb{I}^{-1}\vec{M}, F_2 = \frac{1}{2}(\vec{M} + \vec{\mu}) \cdot (\vec{M} + \vec{\mu})$. Next, we find the set of the uniform rotations. In the paper [12] there was considered the differential equation $\dot{\vec{N}} = \vec{N} \times \mathbb{I}^{-1}\vec{N} + \vec{a} \times \vec{N}$, where $\vec{a} \in \mathbb{R}^3$. This equation is equivalent to the torque-free gyrostat equation (3.2) where $\vec{a} = -\mathbb{I}^{-1}\vec{\mu}$ and one makes the change of variable $\vec{M} = \vec{N} - \vec{\mu}$. According to [12] the equilibrium points of (3.2) are of the types:

$$\vec{M}_1 = (-\mu_1, -\mu_2, -\mu_3); \qquad \vec{M}_2 = \left(\frac{\lambda I_1}{1 - \lambda I_1}\mu_1, \frac{\lambda I_2}{1 - \lambda I_2}\mu_2, \frac{\lambda I_3}{1 - \lambda I_3}\mu_3\right), \quad \lambda \in \mathbb{R} \setminus \left\{\frac{1}{I_1}, \frac{1}{I_2}, \frac{1}{I_3}\right\};$$

$$\vec{M}_3 = \left(\beta, \frac{I_2}{I_1 - I_2}\mu_2, \frac{I_3}{I_1 - I_3}\mu_3\right) \text{ if } \mu_1 = 0, \beta \in \mathbb{R}; \quad \vec{M}_4 = \left(\frac{I_1}{I_2 - I_1}\mu_1, \beta, \frac{I_3}{I_2 - I_3}\mu_3\right) \text{ if } \mu_2 = 0, \beta \in \mathbb{R};$$

$$\vec{M}_5 = \left(\frac{I_1}{I_3 - I_1}\mu_1, \frac{I_2}{I_3 - I_2}\mu_2, \beta\right) \text{ if } \mu_3 = 0, \beta \in \mathbb{R}.$$

Analogous considerations are given in [11], pp. 78–80, for finding the uniform rotations of (3.1).

First we study the stability of a uniform rotation with respect to one conserved quantity.

Theorem 3.1. *For the uniform rotations of a torque-free gyrostat we have:*

 (i) *The unique uniform rotation which is stable with respect to F_1 is $(0, 0, 0)$. This uniform rotation is of type $\vec{M}_2$ obtained for $\lambda = 0$.*
(ii) *The unique uniform rotation which is stable with respect to F_2 is $\vec{M}_1 = (-\mu_1, -\mu_2, -\mu_3)$.*

Proof. The uniform rotation $(0, 0, 0)$ is the unique strict local extremum of the conserved quantity F_1. The uniform rotation $(-\mu_1, -\mu_2, -\mu_3)$ is the unique strict local extremum of the conserved quantity F_2. Using Theorem 2.4 we obtain the results. $\square$

The uniform rotations found in the above theorem are the only uniform rotations for which Lyapunov stability can be proved by using only one of the conserved quantities. For the rest of the uniform rotations it is necessary to consider both conserved quantities. Next, we study the stability of the uniform rotations with respect to the set of conserved quantities $\{F_1, F_2\}$. In what follows we restrict ourselves to the cases for which the vector of gyrostatic moment $\vec{\mu}$ is situated along a principal axis of inertia of the gyrostat.

3.1. The case $\mu_2 = \mu_3 = 0$

In this case we have the following kinds of uniform rotations:

$$\vec{M}_{1-2} = (q, 0, 0), \quad q \in \mathbb{R}; \qquad \vec{M}_4 = \left(\frac{I_1}{I_2 - I_1}\mu_1, q, 0\right), \quad q \in \mathbb{R}^*; \qquad \vec{M}_5 = \left(\frac{I_1}{I_3 - I_1}\mu_1, 0, q\right), \quad q \in \mathbb{R}^*.$$

First, we study the solutions of the algebraic system $F_1(\vec{M}) = F_1(\vec{M}_e), F_2(\vec{M}) = F_2(\vec{M}_e)$, where $\vec{M}_e$ is a uniform rotation. The above system of algebraic equations has the form

$$\begin{cases} \dfrac{M_2^2}{I_2} + \dfrac{M_3^2}{I_3} = \dfrac{M_{1e}^2}{I_1} + \dfrac{M_{2e}^2}{I_2} + \dfrac{M_{3e}^2}{I_3} - \dfrac{M_1^2}{I_1} \\ M_2^2 + M_3^2 = M_{1e}^2 + M_{2e}^2 + M_{3e}^2 - M_1^2 - 2\mu_1 M_1 \end{cases} \tag{3.3}$$

where the unknowns are M_1, M_2 and M_3. The system has at least the solution (M_{1e}, M_{2e}, M_{3e}). We want to see whether this solution is isolated in the set of all the solutions of the algebraic system. For our study it is preferable to change the variable M_1 with $x = M_1 - M_{1e}$. The algebraic system (3.3) becomes

$$\begin{cases} \dfrac{M_2^2}{I_2} + \dfrac{M_3^2}{I_3} = \dfrac{M_{2e}^2}{I_2} + \dfrac{M_{3e}^2}{I_3} - \dfrac{x^2 + 2xM_{1e}}{I_1} \\ M_2^2 + M_3^2 = M_{2e}^2 + M_{3e}^2 - (x^2 + 2xM_{1e}) - 2\mu_1(x + M_{1e}) \end{cases} \tag{3.4}$$

with the unknowns x, M_2 and M_3. The system has at least the solution $(0, M_{2e}, M_{3e})$. The solution (M_{1e}, M_{2e}, M_{3e}) of (3.3) is isolated in the set of all the solutions of this system if and only if the solution $(0, M_{2e}, M_{3e})$ of (3.4) is isolated in the set of corresponding solutions. If we use the unknowns M_2^2 and M_3^2, then we have a linear system. Using Cramer's rule we can find the solutions of (3.4).

I. *The uniform rotation of type $\vec{M}_{1-2}$.* The solutions of (3.4) verify

$$\begin{cases} M_2^2 = -\dfrac{2xI_2}{I_1(I_2 - I_3)} \left(\dfrac{1}{2}x(I_1 - I_3) + q(I_1 - I_3) + I_1\mu_1 \right) \\ M_3^2 = \dfrac{2xI_3}{I_1(I_2 - I_3)} \left(\dfrac{1}{2}x(I_1 - I_2) + q(I_1 - I_2) + I_1\mu_1 \right). \end{cases} \tag{3.5}$$

I.1. If $q = -\frac{I_1\mu_1}{I_1 - I_3}$, then the system (3.5) becomes

$$\begin{cases} M_2^2 = -\dfrac{x^2 I_2(I_1 - I_3)}{I_1(I_2 - I_3)} \\ M_3^2 = \dfrac{2xI_3}{I_1(I_2 - I_3)} \left(\dfrac{1}{2}x(I_1 - I_2) + \dfrac{I_1(I_2 - I_3)}{I_1 - I_3}\mu_1 \right). \end{cases} \tag{3.6}$$

By our hypotheses we have $I_1 > I_2 > I_3$ and if (x, M_2, M_3) is a solution of (3.6), then $M_2^2 \leq 0$. We deduce that $(0, 0, 0)$ is the unique solution of the above system and consequently, it is isolated in the set of all the solutions.

I.2. If $q = -\frac{I_1\mu_1}{I_1 - I_2}$, then the system (3.5) becomes

$$\begin{cases} M_2^2 = -\dfrac{2xI_2}{I_1(I_2 - I_3)} \left(\dfrac{1}{2}x(I_1 - I_3) + \dfrac{I_1(I_2 - I_3)}{I_1 - I_2}\mu_1 \right) \\ M_3^2 = \dfrac{x^2 I_3(I_1 - I_2)}{I_1(I_2 - I_3)}. \end{cases} \tag{3.7}$$

For $|x|$ sufficiently small we have $\operatorname{sgn}\left(-\frac{2xI_2}{I_1(I_2 - I_3)} \left(\frac{1}{2}x(I_1 - I_3) + \frac{I_1(I_2 - I_3)}{I_1 - I_2}\mu_1 \right) \right) = -\operatorname{sgn}(x) \cdot \operatorname{sgn}(\mu_1)$. For every $|x|$ sufficiently small such that $\operatorname{sgn}(x) = -\operatorname{sgn}(\mu_1)$, we obtain a solution of (3.7) and consequently, we have that $(0, 0, 0)$ is not an isolated solution in the set of all the solutions.

I.3. The case when $q \neq -\frac{I_1\mu_1}{I_1 - I_2}$ and $q \neq -\frac{I_1\mu_1}{I_1 - I_3}$. For $|x|$ sufficiently small the terms in the right hand side of the system (3.5) have the properties $\operatorname{sgn}\left(-\frac{2xI_2}{I_1(I_2 - I_3)} \left(\frac{1}{2}x(I_1 - I_3) + q(I_1 - I_3) + I_1\mu_1 \right) \right) = -\operatorname{sgn}(x) \cdot \operatorname{sgn}(q(I_1 - I_3) + I_1\mu_1)$, and $\operatorname{sgn}\left(\frac{2xI_3}{I_1(I_2 - I_3)} \left(\frac{1}{2}x(I_1 - I_2) + q(I_1 - I_2) + I_1\mu_1 \right) \right) = \operatorname{sgn}(x) \cdot \operatorname{sgn}(q(I_1 - I_2) + I_1\mu_1)$. If $\operatorname{sgn}(q(I_1 - I_3) + I_1\mu_1) \cdot \operatorname{sgn}(q(I_1 - I_2) + I_1\mu_1) > 0$, then there exists $r > 0$ such that a solution of the form (x, M_2, M_3) which verifies $x \neq 0$ has the property $|x| > r$. In this case, the solution $(0, 0, 0)$ of the system (3.5) is an isolated solution in the set of all the solutions. If $\operatorname{sgn}(q(I_1 - I_3) + I_1\mu_1) \cdot \operatorname{sgn}(q(I_1 - I_2) + I_1\mu_1) < 0$, then for every $|x|$ sufficiently small we have that the solutions of the system (3.5) are of the form (x, M_2, M_3). We obtain that the solution $(0, 0, 0)$ of the system (3.5) is not isolated in the set of all the solutions.

II. *The uniform rotation of type $\vec{M}_4$.* In this case the system (3.4) is equivalent to the system

$$\begin{cases} M_2^2 = q^2 - \dfrac{I_2(I_1 - I_3)}{I_1(I_2 - I_3)}x^2 + \dfrac{2I_2\mu_1}{I_1 - I_2}x \\ M_3^2 = \dfrac{x^2 I_3(I_1 - I_2)}{I_1(I_2 - I_3)}. \end{cases} \tag{3.8}$$

As before, for every $|x|$ sufficiently small we have a solution of the above system which is of the form (x, M_2, M_3) and consequently, $(0, q, 0)$ is not isolated in the set of all the solutions of (3.8).

III. *The uniform rotation of type* $\vec{M}_5$. The system (3.4) is equivalent to the following system:

$$\begin{cases} M_2^2 = -\dfrac{x^2 I_2(I_1 - I_3)}{I_1(I_2 - I_3)} \\ M_3^2 = q^2 + \dfrac{I_3(I_1 - I_2)}{I_1(I_2 - I_3)}x^2 + \dfrac{2I_3\mu_1}{I_1 - I_3}x. \end{cases} \tag{3.9}$$

The solutions of the above system are $(0, 0, q)$ and $(0, 0, -q)$. The solution $(0, 0, q)$ is isolated in the set of all the solutions of (3.9).

Summarizing, we obtain the following result.

Theorem 3.2. *For a vector of a gyrostatic moment along the first axis of inertia we have the results:*

(i) *A uniform rotation of type* $\vec{M}_{1-2} = (q, 0, 0)$ *is stable with respect to the set of conserved quantities* $\{F_1, F_2\}$ *if and only if* $q \in (-\infty, -\frac{I_1\mu_1}{I_1 - I_2}) \cup [-\frac{I_1\mu_1}{I_1 - I_3}, \infty)$ *and* $\mu_1 > 0$ *or* $q \in (-\infty, -\frac{I_1\mu_1}{I_1 - I_3}] \cup (-\frac{I_1\mu_1}{I_1 - I_2}, \infty)$ *and* $\mu_1 < 0$.

(ii) *A uniform rotation of type* $\vec{M}_4 = (\frac{I_1}{I_2 - I_1}\mu_1, q, 0)$ *with* $q \neq 0$ *is not stable with respect to the set of conserved quantities* $\{F_1, F_2\}$.

(iii) *A uniform rotation of type* $\vec{M}_5 = (\frac{I_1}{I_3 - I_1}\mu_1, 0, q)$ *with* $q \neq 0$ *is stable with respect to the set of conserved quantities* $\{F_1, F_2\}$.

We prove that the uniform rotations which are not stable with respect to the set $\{F_1, F_2\}$ are Lyapunov unstable. According to the paper [12], for $q \in (-\frac{I_1\mu_1}{I_1 - I_2}, -\frac{I_1\mu_1}{I_1 - I_3})$ and $\mu_1 > 0$ or for $q \in (-\frac{I_1\mu_1}{I_1 - I_3}, -\frac{I_1\mu_1}{I_1 - I_2})$ and $\mu_1 < 0$, a uniform rotation of type $\vec{M}_{1-2}$ is spectrally unstable and consequently, it is unstable in the sense of Lyapunov. A uniform rotation of type $\vec{M}_4$ is spectrally unstable and also it is unstable in the sense of Lyapunov.

The Lyapunov stability or instability of $\vec{M}_e = (-\frac{I_1\mu_1}{I_1 - I_2}, 0, 0)$ cannot be decided on using the set of conserved quantities $\{F_1, F_2\}$ or using the linearization method. This uniform rotation is spectrally stable and it is not stable with respect to the set of conserved quantities $\{F_1, F_2\}$. The instability in the sense of Lyapunov of this uniform rotation will be proved by studying the dynamics on the invariant set

$$\mathcal{M} = \{\vec{M} \mid F_1(\vec{M}) = F_1(\vec{M}_e), F_2(\vec{M}) = F_2(\vec{M}_e)\}.$$

Theorem 3.3. *The uniform rotation* $(-\frac{I_1\mu_1}{I_1 - I_2}, 0, 0)$ *is unstable in the sense of Lyapunov.*

Proof. The projection of the equation (3.2) on the first axis using the variables x, M_2 and M_3 is

$$\dot{x} = \left(\frac{1}{I_3} - \frac{1}{I_2}\right) M_2 M_3.$$

By using (3.7) we have $\dot{x}^2 = -\frac{2x^3(I_1 - I_2)}{I_1^2(I_2 - I_3)^2}\left(\frac{1}{2}x(I_1 - I_3) + \frac{I_1(I_2 - I_3)}{I_1 - I_2}\mu_1\right)$. First we consider the case $\mu_1 > 0$. Suppose that we have $x(0) > -\frac{2I_1(I_2 - I_3)}{(I_1 - I_2)(I_1 - I_3)}\mu_1$ and $M_2(0)M_3(0) < 0$. Consequently, we obtain that $\dot{x}(0) < 0$. In this case there exists $t^* > 0$ such that $x(t^*) = -\frac{2I_1(I_2 - I_3)}{(I_1 - I_2)(I_1 - I_3)}\mu_1$ which implies that our uniform rotation is unstable. In the case $\mu_1 < 0$ we have analogous considerations. $\square$

For a vector of a gyrostatic moment along the first axis of inertia a uniform rotation is stable with respect to the set of conserved quantities $\{F_1, F_2\}$ if and only if it is stable in the sense of Lyapunov.

3.2. The case $\mu_1 = \mu_3 = 0$

We have the following kinds of uniform rotations:

$$\vec{M}_{1-2} = (0, q, 0), \quad q \in \mathbb{R}; \qquad \vec{M}_3 = \left(q, \frac{I_2}{I_1 - I_2}\mu_2, 0\right), \quad q \in \mathbb{R}^*; \qquad \vec{M}_5 = \left(0, \frac{I_2}{I_3 - I_2}\mu_2, q\right), \quad q \in \mathbb{R}^*.$$

Using the method of the previous section and by analogous calculations we obtain the following result.

Theorem 3.4. *For a vector of a gyrostatic moment along the second axis of inertia we have the results:*

(i) *A uniform rotation of type* $\vec{M}_{1-2} = (0, q, 0)$ *is stable with respect to the set of conserved quantities* $\{F_1, F_2\}$ *if and only if* $q \in [-\frac{I_2\mu_2}{I_2 - I_3}, \frac{I_2\mu_2}{I_1 - I_2}]$ *and* $\mu_2 > 0$ *or* $q \in [\frac{I_2\mu_2}{I_1 - I_2}, -\frac{I_2\mu_2}{I_2 - I_3}]$ *and* $\mu_2 < 0$.

(ii) *A uniform rotation of type* $\vec{M}_3 = (q, \frac{I_2}{I_1 - I_2}\mu_2, 0)$ *with* $q \neq 0$ *is stable with respect to the set of conserved quantities* $\{F_1, F_2\}$.

(iii) *A uniform rotation of type* $\vec{M}_5 = (0, \frac{I_2}{I_3 - I_2}\mu_2, q)$ *with* $q \neq 0$ *is stable with respect to the set of conserved quantities* $\{F_1, F_2\}$.

According to the paper [12], for $q \in \mathbb{R} \setminus [-\frac{I_2\mu_2}{I_2-I_3}, \frac{I_2\mu_2}{I_1-I_2}]$ and $\mu_2 > 0$ or $q \in \mathbb{R} \setminus [\frac{I_2\mu_2}{I_1-I_2}, -\frac{I_2\mu_2}{I_2-I_3}]$ and $\mu_2 < 0$, a uniform rotation of type $\vec{M}_{1-2} = (0, q, 0)$ is spectrally unstable and also it is unstable in the sense of Lyapunov.

In this case the stability (in the sense of Lyapunov) can be decided on using the stability with respect to the set of conserved quantities $\{F_1, F_2\}$ and the linearization method. A uniform rotation is stable with respect to the set of conserved quantities $\{F_1, F_2\}$ if and only if it is stable in the sense of Lyapunov.

3.3. The case $\mu_1 = \mu_2 = 0$

We have the following kinds of uniform rotations:

$$\vec{M}_{1-2} = (0, 0, q), \quad q \in \mathbb{R}; \qquad \vec{M}_3 = \left(q, 0, \frac{I_3}{I_1 - I_3}\mu_3 \right), \quad q \in \mathbb{R}^*; \qquad \vec{M}_4 = \left(0, q, \frac{I_3}{I_2 - I_3}\mu_3 \right), \quad q \in \mathbb{R}^*.$$

Theorem 3.5. *For a vector of gyrostatic moment along the third axis of inertia we have the results:*

(i) *A uniform rotation of type $\vec{M}_{1-2} = (0, 0, q)$ is stable with respect to the set of conserved quantities $\{F_1, F_2\}$ if and only if $q \in (-\infty, \frac{I_3\mu_3}{I_1-I_3}] \cup (\frac{I_3\mu_3}{I_2-I_3}, \infty)$ and $\mu_3 > 0$ or $q \in (-\infty, \frac{I_3\mu_3}{I_2-I_3}) \cup [\frac{I_3\mu_3}{I_1-I_3}, \infty)$ and $\mu_3 < 0$.*

(ii) *A uniform rotation of type $\vec{M}_3 = (q, 0, \frac{I_3}{I_1-I_3}\mu_3)$ with $q \neq 0$ is stable with respect to the set of conserved quantities $\{F_1, F_2\}$.*

(iii) *A uniform rotation of type $\vec{M}_4 = (0, q, \frac{I_3}{I_2-I_3}\mu_3)$ with $q \neq 0$ is not stable with respect to the set of conserved quantities $\{F_1, F_2\}$.*

According to the paper [12], for $q \in (\frac{I_3\mu_3}{I_1-I_3}, \frac{I_3\mu_3}{I_2-I_3})$ and $\mu_3 > 0$ or for $q \in (\frac{I_3\mu_3}{I_2-I_3}, \frac{I_3\mu_3}{I_1-I_3})$ and $\mu_3 < 0$, a uniform rotation of type $\vec{M}_{1-2}$ is spectrally unstable and consequently, it is unstable in the sense of Lyapunov. A uniform rotation of type $\vec{M}_4$ is spectrally unstable and also it is unstable in the sense of Lyapunov. The Lyapunov stability or instability of $\vec{M}_e = (0, 0, \frac{I_3\mu_3}{I_2-I_3})$ cannot be decided using the set of conserved quantities $\{F_1, F_2\}$ or using the linearization method. This uniform rotation is spectrally stable and it is not stable with respect to the set of conserved quantities $\{F_1, F_2\}$. The instability in the sense of Lyapunov of this uniform rotation will be proved by studying the dynamics on the invariant set $\mathcal{M}$. The proof is analogous to the proof of the Theorem 3.3.

Theorem 3.6. *The uniform rotation $(0, 0, \frac{I_3\mu_3}{I_2-I_3})$ is unstable in the sense of Lyapunov.*

A uniform rotation is stable with respect to the set of conserved quantities $\{F_1, F_2\}$ if and only if it is stable in the sense of Lyapunov.

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