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Analysis of two stationary magnetohydrodynamics systems of equations including Joule heating

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ABSTRACT

We study the coupling of the equations of steady-state magnetohydrodynamics (MHD) with the heat equation when the buoyancy effects due to temperature differences in the flow as well as Joule effect and viscous heating are (all) taken into account. Two models for the gravity force are considered: the first one is the well-known Boussinesq approximation; in the second one density is assumed to be constant except in the gravity force, where it is assumed to be a non-increasing function of the temperature. The equations are posed in a bounded three-dimensional domain. We give existence results of weak solutions to both models under certain conditions on the data. We also give some uniqueness results.

1. Introduction

This work deals with the analysis of the equations of stationary, incompressible magnetohydrodynamics (MHD), including buoyancy forces due to temperature differences in the flow, and both Joule heating and viscous heating. Two different models are considered, which only differ in the treatment of the gravity forces. In the first model we employ the well-known Boussinesq approximation and, in the second one we assume that density is constant except in the gravity force where it is assumed to be a non-increasing function of the temperature.

The analysis of the stationary, incompressible MHD equations, without including thermal effects, has been done in [14]. Existence and uniqueness of solution have been proved under conditions of smallness of the data. The same problem is analyzed in [1] where the smallness conditions are relaxed assuming a tangential velocity boundary condition. The work in [19] deals with the same system of equations under homogeneous boundary conditions, and the magnetic field is sought in an adequate space to the use of edge finite elements. The analysis of the problem including the thermal equation is done in [15], but neglecting Joule and viscous heating. In [16] the authors suggest a formulation in terms of the current density to consider realistic boundary conditions. This formulation is coupled with the heat equation in [17] neglecting again Joule and viscous heating.

The present work summarizes some of the results presented in [22]. In the first part of this paper we consider the same system of equations as in [15], but including viscous heating and Joule effect, which is the main heat source in many real applications. The main difficulty of the analysis is that, due to their quadratic nature, these heat sources merely belong to $L^1(\Omega)$. To deal with this difficulty we employ the concept of solution by transposition as given by Stampacchia in [21]. Moreover, for the magnetic part we use the same spaces as in [19].

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The equations are posed in a bounded simply connected three-dimensional domain which can be either of class $\mathcal{C}^{1,1}$ or a Lipschitz polyhedron, not necessarily convex. The boundary conditions for the velocity and temperature are of Dirichlet type. We also specify the normal component of the magnetic field and the tangential component of the electric field on the boundary.

The outline of the paper is as follows: in Section 2 we introduce the first model by using the Boussinesq approximation and write its non-dimensionalized version. In Section 3 we introduce the notation for some function spaces and recall the tangential trace theory for the space $\mathbf{H}(\mathbf{curl};\Omega)$. The analysis of the coupled MHD model is done in three stages: in Section 4 we analyze the MHD problem assuming that the temperature is known and by introducing a suitable linearization. Then, in Section 5 we analyze the heat equation with sources in $L^1(\Omega)$ and, in Section 6, we prove the existence of solution for the coupled problem under some conditions of smallness of the data. In order to avoid these conditions of smallness, in Section 7 we introduce and analyze a second model for which we assume that density is a non-increasing function of temperature. For this model, we first prove the existence of solution to the coupled problem assuming only the smallness of the velocity at the boundary. Next, following the ideas of [1], we relax this smallness condition by assuming a tangential velocity boundary condition. Finally, in Section 8 we give some results of uniqueness for the two models, that require more severe conditions of smallness of the data.

2. Statement of the problem. Steady MHD using the Boussinesq approximation

The first problem we analyze arises by considering that the model domain is occupied by a fluid with constant homogeneous physical properties, except density in the gravity force which is treated using Boussinesq approximation. We take into account two source terms in the heat equation corresponding to viscous and Joule heatings.

2.1. Equations and non-dimensionalization

Let $\Omega\subset\mathbb{R}^3$ be either a bounded simply connected domain of class $\mathcal{C}^{1,1}$ or a bounded Lipschitz polyhedron. Consider the following system of equations which holds in Ω:

$$\frac{1}{\mu}\,\mathbf{curl}\,\mathbf{B}=\mathbf{J},\qquad \mathbf{J}=\sigma(\mathbf{E}+\mathbf{u}\times\mathbf{B}),\qquad \operatorname{div}\mathbf{B}=0,\qquad \mathbf{curl}\,\mathbf{E}=\mathbf{0},\tag{1}$$

$$-\eta\Delta\mathbf{u}+\rho(\mathbf{grad}\,\mathbf{u})\mathbf{u}+\mathbf{grad}\,p-\mathbf{J}\times\mathbf{B}=\mathbf{f}_0+\rho\mathbf{g},\qquad \operatorname{div}\mathbf{u}=0,\tag{2}$$

$$-k\Delta T+\rho c_p\mathbf{u}\cdot\mathbf{grad}\,T=\frac{1}{\sigma}|\mathbf{J}|^2+\frac{\eta}{2}\left|\mathbf{grad}\,\mathbf{u}+\mathbf{grad}\,\mathbf{u}^t\right|^2+\psi.\tag{3}$$

The unknown variables are the magnetic induction $\mathbf{B}$, the electric field $\mathbf{E}$, the electric current density $\mathbf{J}$, the velocity field $\mathbf{u}$, the pressure p and the temperature T. The physical parameters are the magnetic permeability μ, the electrical conductivity σ, the dynamic viscosity η, the mass density ρ, the thermal conductivity k and the specific heat at constant pressure c_p. Fields $\mathbf{f}_0$ and ψ are a given force and a given heat source, respectively. The term $\rho\mathbf{g}$ represents gravity force, with $\mathbf{g}$ being the gravity acceleration, whereas terms $\frac{1}{\sigma}|\mathbf{J}|^2$ and $\frac{\eta}{2}|\mathbf{grad}\,\mathbf{u}+\mathbf{grad}\,\mathbf{u}^t|^2$ represent Joule heating and viscous heating, respectively.

For the first model we consider Boussinesq approximation. It consists in assuming that density variations are negligible except in the gravity force term, where it is assumed to be of the form $\rho=\rho_r(1-\beta(T-T_r))$, with T_r a reference temperature, ρ_r the density at this reference temperature and β the thermal expansion coefficient at temperature T_r. To analyze the system we eliminate the fields $\mathbf{E}$ and $\mathbf{J}$ from the equations, and write them in terms of the magnetic induction $\mathbf{B}$. Thus we arrive at the following system of equations:

$$\frac{1}{\mu\sigma}\,\mathbf{curl}(\mathbf{curl}\,\mathbf{B})-\mathbf{curl}(\mathbf{u}\times\mathbf{B})=\mathbf{0},\tag{4}$$

$$\operatorname{div}\mathbf{B}=0,\tag{5}$$

$$-\eta\Delta\mathbf{u}+\rho(\mathbf{grad}\,\mathbf{u})\mathbf{u}+\mathbf{grad}\,p'-\frac{1}{\mu}(\mathbf{curl}\,\mathbf{B})\times\mathbf{B}=\mathbf{f}_0-\rho\beta(T-T_r)\mathbf{g},\tag{6}$$

$$\operatorname{div}\mathbf{u}=0,\tag{7}$$

$$-k\Delta T+\rho c_p\mathbf{u}\cdot\mathbf{grad}\,T=\frac{1}{\sigma\mu^2}|\mathbf{curl}\,\mathbf{B}|^2+\frac{\eta}{2}\left|\mathbf{grad}\,\mathbf{u}+\mathbf{grad}\,\mathbf{u}^t\right|^2+\psi,\tag{8}$$

where p' denotes the corrected pressure, $p'=p-\rho_r\mathbf{g}\cdot\mathbf{x}$. Here and in the sequel, in order to alleviate the notation ρ will denote the reference density instead of ρ_r.

Let us now introduce the following non-dimensional quantities: the Hartmann number H_a, the interaction parameter N, the Reynolds number R_e, the magnetic Reynolds number R_m, the Prandtl number P_r, the Grashof number G_r and the Eckert number E_c. They are given by the expressions

$$H_a = \mathcal{B}\mathcal{L}\left(\frac{\sigma}{\eta}\right)^{1/2}, \qquad N = \sigma\mathcal{B}^2\frac{\mathcal{L}}{\rho u}, \qquad R_e = \frac{\rho u\mathcal{L}}{\eta}, \qquad R_m = \mu\sigma u\mathcal{L},$$
$$P_r = \frac{\eta c_p}{k}, \qquad G_r = \frac{\beta g\Delta\mathcal{T}\mathcal{L}^3}{\nu^2}, \qquad E_c = \frac{u^2}{c_p\Delta\mathcal{T}},$$

where $\mathcal{B}$, u, $\mathcal{L}$ and $\Delta\mathcal{T}$ are typical values of magnetic induction, velocity, length and temperature difference, respectively. Moreover, $\nu = \rho/\eta$ is the kinematic viscosity and $g = |\mathbf{g}|$ is the magnitude of gravity acceleration. See further details in [18,15] and references therein.

We replace temperature T by the temperature difference with respect to reference temperature T_r. From now on this difference will be also denoted by T. Then, we normalize equations as follows: magnetic induction $\mathbf{B}$ by $\mathcal{B}$, velocity $\mathbf{u}$ by u, (corrected) pressure p' by $\sigma u\mathcal{B}^2\mathcal{L}$, body force $\mathbf{f}_0$ by $\sigma u\mathcal{B}^2$, temperature difference T by $\Delta\mathcal{T}$ and heat source ψ by $\rho c_p u\Delta\mathcal{T}/\mathcal{L}$ (see [15] and references therein). After this normalization we arrive at the following non-dimensionalized system of equations, which holds in Ω:

$$\frac{1}{R_m}\,\mathbf{curl}(\mathbf{curl}\,\mathbf{B}) - \mathbf{curl}(\mathbf{u}\times\mathbf{B}) = \mathbf{0}, \tag{9}$$
$$\operatorname{div}\mathbf{B} = 0, \tag{10}$$
$$-\frac{1}{H_a^2}\Delta\mathbf{u} + \frac{1}{N}(\mathbf{grad}\,\mathbf{u})\mathbf{u} + \mathbf{grad}\,p - \frac{1}{R_m}(\mathbf{curl}\,\mathbf{B})\times\mathbf{B} = \mathbf{f}_0 - \frac{G_r}{NR_e^2}\frac{\mathbf{g}}{g}T, \tag{11}$$
$$\operatorname{div}\mathbf{u} = 0, \tag{12}$$
$$-\frac{1}{P_rR_e}\Delta T + \mathbf{u}\cdot\mathbf{grad}\,T = \frac{E_c}{R_e}\left[\frac{H_a^2}{R_m^2}|\mathbf{curl}\,\mathbf{B}|^2 + \frac{1}{2}\left|\mathbf{grad}\,\mathbf{u} + \mathbf{grad}\,\mathbf{u}^t\right|^2\right] + \psi. \tag{13}$$

2.2. Boundary conditions

We denote by $\partial\Omega$ the boundary of Ω, and by $\mathbf{n}$ the unit outward-pointing normal vector to $\partial\Omega$. For the hydrodynamic model we impose a Dirichlet boundary condition, namely,

$$\mathbf{u}_{|\partial\Omega} = \mathbf{u}_d, \tag{14}$$

where $\mathbf{u}_d$ is a given vector function on $\partial\Omega$. For the electromagnetic model we impose the boundary conditions

$$(\mathbf{B}\cdot\mathbf{n})_{|\partial\Omega} = l, \tag{15}$$
$$\left[\left(\frac{1}{R_m}(\mathbf{curl}\,\mathbf{B}) - (\mathbf{u}\times\mathbf{B})\right)\times\mathbf{n}\right]_{|\partial\Omega} = \mathbf{k}, \tag{16}$$

where the second equation arises from a condition of the form $\mathbf{E}\times\mathbf{n} = \mathbf{k}$, after an appropriate non-dimensionalization. We notice that l and $\mathbf{k}$ must satisfy some compatibility conditions that will be detailed below. For the temperature, we also impose a Dirichlet boundary condition:

$$T_{|\partial\Omega} = T_d. \tag{17}$$

3. Function spaces

In this section we introduce several function spaces defined on Ω which will be used in different parts of this work. For any real number $p \geqslant 1$, $L^p(\Omega)$ denotes the Lebesgue space of (real or complex) scalar functions the p-th power of which are integrable; its vectorial counterpart is denoted by $\mathbf{L}^p(\Omega)$. These spaces are equipped, respectively, with the norms

$$\|\theta\|_{L^p} := \left(\int_\Omega \left|\theta(\mathbf{x})\right|^p \mathrm{d}\mathbf{x}\right)^{1/p}, \qquad \|\mathbf{u}\|_{\mathbf{L}^p} := \left(\int_\Omega \left|\mathbf{u}(\mathbf{x})\right|^p \mathrm{d}\mathbf{x}\right)^{1/p}.$$

For any non-negative integer m, we denote by $H^m(\Omega)$ the usual m-th order Sobolev space. It is endowed with the usual norm

$$\|\theta\|_m := \left(\sum_{0\leqslant|\alpha|\leqslant m}\left\|D^\alpha\theta\right\|_{L^2}^2\right)^{1/2}.$$

We denote by $\mathbf{H}^m(\Omega) := (H^m(\Omega))^3$ its vector-valued counterpart, and again by $\|\,\boldsymbol{.}\,\|_m$ its norm. We use the convention $H^0(\Omega) = L^2(\Omega)$ and $\mathbf{H}^0(\Omega) = \mathbf{L}^2(\Omega)$. Moreover, for the space $L^2(\Omega)$ (respectively, $\mathbf{L}^2(\Omega)$) we denote its scalar product by

$(\theta,\zeta)_\Omega := \int_\Omega \theta\zeta\,d\mathbf{x}$ (respectively, $(\mathbf{v},\mathbf{w})_\Omega := \int_\Omega \mathbf{v}\cdot\mathbf{w}\,d\mathbf{x}$). We also use this last notation when $\mathbf{v}\in\mathbf{L}^p(\Omega)$ and $\mathbf{w}\in\mathbf{L}^{p'}(\Omega)$ with $p\in[1,\infty]$ and $1/p'=1-1/p$.

Let $H_0^1(\Omega) := \{\theta\in H^1(\Omega):\ \theta_{|\partial\Omega}=0\}$, $\mathbf{H}_0^1(\Omega) := \{\mathbf{w}\in\mathbf{H}^1(\Omega):\ \mathbf{w}_{|\partial\Omega}=\mathbf{0}\}$, $\mathbf{H}_T^1(\Omega) := \{\mathbf{w}\in\mathbf{H}^1(\Omega):\ (\mathbf{w}\cdot\mathbf{n})_{|\partial\Omega}=\mathbf{0}\}$. In order to impose the divergence-free condition in the hydrodynamic problem, we shall make use of the subspaces $\mathbf{Z}(\Omega) := \{\mathbf{w}\in\mathbf{H}^1(\Omega):\ \operatorname{div}\mathbf{w}=0\}$, and $\mathbf{Z}_0(\Omega) := \{\mathbf{w}\in\mathbf{H}_0^1(\Omega):\ \operatorname{div}\mathbf{w}=0\}$. Moreover, the pressure will belong to $L_0^2(\Omega) := \{q\in L^2(\Omega):\ \int_\Omega q\,d\mathbf{x}=0\}$.

We denote by $\mathcal{D}(\Omega)$ the space of infinitely differentiable compactly-supported functions in Ω and by $\mathcal{D}'(\Omega)$ the space of distributions on Ω.

Recall that, for any real number $p\geqslant 1$, $W^{1,p}(\Omega)$ denotes the space of functions belonging to $L^p(\Omega)$ such that their first order distributional derivatives also belong to $L^p(\Omega)$. It is endowed with the norm $\|\theta\|_{1,p} := \|\theta\|_{L^p} + \sum_{i=1}^3 \|\frac{\partial\theta}{\partial x_i}\|_{L^p}$. We define $W_0^{1,p}(\Omega) := \{\theta\in W^{1,p}(\Omega):\ \theta_{|\partial\Omega}=0\}$.

For any linear space V endowed with the norm $\|\cdot\|_V$, we equip its dual space V' with the norm

$$\|f\|_{V'} := \sup_{v\in V,\, v\neq 0} \frac{\langle f,v\rangle}{\|v\|_V},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V and V'. In the following we will use the notations $\langle\cdot,\cdot\rangle_\Omega$ and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ depending on whether the spaces are defined in Ω or on the boundary $\partial\Omega$. In particular, we recall the dual spaces $H^{-1}(\Omega)=(H_0^1(\Omega))'$, $\mathbf{H}^{-1}(\Omega)=(\mathbf{H}_0^1(\Omega))'$ and denote both norms by $\|\cdot\|_{-1}$. We will also need the dual space $W^{-1,p'}(\Omega)=(W_0^{1,p}(\Omega))'$, equipped with the norm $\|\cdot\|_{-1,p'}$, where $1/p'=1-1/p$, for any $p\in[1,\infty)$.

We also need the well-known trace spaces $H^{1/2}(\partial\Omega) := \{\theta_{|\partial\Omega}:\ \theta\in H^1(\Omega)\}$, $\mathbf{H}^{1/2}(\partial\Omega) := \{\mathbf{w}_{|\partial\Omega}:\ \mathbf{w}\in\mathbf{H}^1(\Omega)\}$, endowed with the norms

$$\|q\|_{1/2,\partial\Omega} := \inf_{\theta\in H^1(\Omega),\,\theta_{|\partial\Omega}=q} \|\theta\|_1, \qquad \|\mathbf{q}\|_{1/2,\partial\Omega} := \inf_{\mathbf{w}\in\mathbf{H}^1(\Omega),\,\mathbf{w}_{|\partial\Omega}=\mathbf{q}} \|\mathbf{w}\|_1,$$

and their respective dual spaces $H^{-1/2}(\partial\Omega)=(H^{1/2}(\partial\Omega))'$, $\mathbf{H}^{-1/2}(\partial\Omega)=(\mathbf{H}^{1/2}(\partial\Omega))'$ equipped with the usual norms $\|\cdot\|_{-1/2,\partial\Omega}$.

As a consequence of Poincaré's inequality, it is known that the seminorm $|\theta|_{1,p} := \sum_{i=1}^3 \|\frac{\partial\theta}{\partial x_i}\|_{L^p}$ is in fact a norm in $W_0^{1,p}(\Omega)$ equivalent to the norm $\|\cdot\|_{1,p}$. Moreover,

$$\|\theta\|_{1,p} \leqslant C(p)|\theta|_{1,p}, \tag{18}$$

with $C(p)$ a constant depending on the domain Ω and p. This result implies the equivalence, in the dual space $W^{-1,p'}(\Omega)=(W_0^{1,p}(\Omega))'$, of the norm $\|\cdot\|_{-1,p'}$ and the dual norm of $|\cdot|_{1,p}$, which we denote by $|\cdot|_{-1,p'}$. In the particular case of spaces $H_0^1(\Omega)$ and $\mathbf{H}_0^1(\Omega)$, Poincaré's inequality states the equivalence of the usual norms $\|\cdot\|_1$ and the seminorms $|\cdot|_1$ defined as

$$|\theta|_1 := \|\mathbf{grad}\,\theta\|_0 = \left(\sum_{i=1}^3 \left\|\frac{\partial\theta}{\partial x_i}\right\|_0^2\right)^{1/2}, \qquad |\mathbf{w}|_1 := \|\mathbf{grad}\,\mathbf{w}\|_0 = \left(\sum_{i=1}^3\sum_{j=1}^3 \left\|\frac{\partial w_i}{\partial x_j}\right\|_0^2\right)^{1/2},$$

i.e., there exists a constant C_0 such that

$$\|\theta\|_1 \leqslant C_0|\theta|_1 \quad \forall\theta\in H_0^1(\Omega), \qquad \|\mathbf{w}\|_1 \leqslant C_0|\mathbf{w}|_1 \quad \forall\mathbf{w}\in\mathbf{H}_0^1(\Omega). \tag{19}$$

Moreover, spaces $H_0^1(\Omega)$ and $\mathbf{H}_0^1(\Omega)$ are continuously imbedded in $L^6(\Omega)$ and $\mathbf{L}^6(\Omega)$, respectively,

$$\|\theta\|_{L^6} \leqslant S\|\mathbf{grad}\,\theta\|_0 = |\theta|_1 \quad \forall\theta\in H_0^1(\Omega), \qquad \|\mathbf{w}\|_{\mathbf{L}^6} \leqslant S\|\mathbf{grad}\,\mathbf{w}\|_0 = |\mathbf{w}|_1 \quad \forall\mathbf{w}\in\mathbf{H}_0^1(\Omega), \tag{20}$$

with S a constant independent of the domain Ω. Using Hölder's inequality we also obtain

$$\|\theta\|_{L^4} \leqslant S_4\|\mathbf{grad}\,\theta\|_0 = |\theta|_1 \quad \forall\theta\in H_0^1(\Omega), \qquad \|\mathbf{w}\|_{\mathbf{L}^4} \leqslant S_4\|\mathbf{grad}\,\mathbf{w}\|_0 = |\mathbf{w}|_1 \quad \forall\mathbf{w}\in\mathbf{H}_0^1(\Omega), \tag{21}$$

with $S_4 = meas(\Omega)^{1/12}S$.

For the electromagnetic problem we shall make use of the following Hilbert spaces:

$$\mathbf{H}(\mathbf{curl};\Omega) := \left\{\mathbf{v}\in\mathbf{L}^2(\Omega):\ \mathbf{curl}\,\mathbf{v}\in\mathbf{L}^2(\Omega)\right\}, \qquad \mathbf{H}(\operatorname{div};\Omega) := \left\{\mathbf{w}\in\mathbf{L}^2(\Omega):\ \operatorname{div}\mathbf{w}\in L^2(\Omega)\right\},$$
$$\mathbf{H}_0(\operatorname{div};\Omega) := \left\{\mathbf{v}\in\mathbf{H}(\operatorname{div};\Omega):\ \mathbf{v}\cdot\mathbf{n}=0 \text{ on } \partial\Omega\right\}.$$

To simplify notation we introduce the spaces

$$\mathbf{X}(\Omega) := \mathbf{H}(\mathbf{curl};\Omega) \cap \mathbf{H}(\mathrm{div};\Omega), \qquad \mathbf{X}_0(\Omega) := \mathbf{H}(\mathbf{curl};\Omega) \cap \mathbf{H}_0(\mathrm{div};\Omega),$$
$$\mathbf{Y}(\Omega) := \mathbf{H}(\mathbf{curl};\Omega) \cap \mathbf{H}(\mathrm{div};\Omega) \cap \mathbf{L}^3(\Omega),$$

equipped with the norms

$$\|\mathbf{C}\|_{\mathbf{X}} := \left(\|\mathbf{C}\|_0^2 + \|\mathbf{curl}\,\mathbf{C}\|_0^2 + \|\mathrm{div}\,\mathbf{C}\|_0^2\right)^{1/2}, \qquad \|\mathbf{D}\|_{\mathbf{Y}} := \left(\|\mathbf{curl}\,\mathbf{D}\|_0^2 + \|\mathrm{div}\,\mathbf{D}\|_0^2\right)^{1/2} + \|\mathbf{D}\|_{\mathbf{L}^3}.$$

For the coupled magnetohydrodynamic problem we need the product spaces

$$\mathcal{W}(\Omega) := \mathbf{H}^1(\Omega) \times \mathbf{X}(\Omega), \qquad \mathcal{W}_0(\Omega) := \mathbf{H}_0^1(\Omega) \times \mathbf{X}_0(\Omega), \qquad \mathcal{Z}_0(\Omega) := \mathbf{Z}_0(\Omega) \times \mathbf{X}_0(\Omega),$$

which are equipped with the usual product norm

$$\left\|(\mathbf{w},\mathbf{D})\right\|_{\mathcal{W}} := \left(\|\mathbf{w}\|_1^2 + \|\mathbf{D}\|_{\mathbf{X}}^2\right)^{1/2}.$$

We also need the product space $\mathcal{Z}(\Omega) := \mathbf{Z}(\Omega) \times \mathbf{Y}(\Omega)$.

If the domain Ω is bounded and simply connected with Lipschitz boundary, then the mapping $\mathbf{w} \mapsto |\mathbf{w}|_{\mathbf{X}} := (\|\mathbf{curl}\,\mathbf{w}\|_0^2 + \|\mathrm{div}\,\mathbf{w}\|_0^2)^{1/2}$ defines a norm in $\mathbf{X}_0(\Omega)$ equivalent to the norm $\|\cdot\|_{\mathbf{X}}$, through the inequality

$$\|\mathbf{w}\|_{\mathbf{X}} \leqslant C_1 |\mathbf{w}|_{\mathbf{X}} \quad \forall \mathbf{w} \in \mathbf{X}_0(\Omega). \tag{22}$$

This result is a consequence of Lemma 3.6 in Chapter 1 of [12]. This last inequality and (19) state that the expression

$$\left|(\mathbf{v},\mathbf{C})\right|_{\mathcal{W}} := \left(|\mathbf{v}|_1^2 + |\mathbf{C}|_{\mathbf{X}}^2\right)^{1/2},$$

defines a norm in $\mathcal{W}_0(\Omega)$, equivalent to the product norm $\|\cdot\|_{\mathcal{W}}$.

Lemma 1. *Let $\delta \in (0, 1/2)$, then the following imbeddings hold*

$$\left\{\mathbf{C} \in \mathbf{X}(\Omega) \colon \mathbf{C} \cdot \mathbf{n}_{|\partial\Omega} \in H^\delta(\partial\Omega)\right\} \subset \mathbf{H}^{1/2+\varepsilon}(\Omega) \Subset \mathbf{H}^{1/2}(\Omega) \subset \mathbf{L}^3(\Omega),$$

where $\Subset$ denotes a compact imbedding, and there exists a constant κ, depending on δ and Ω, such that

$$\|\mathbf{D}\|_{\mathbf{L}^3} \leqslant \kappa\left(\|\mathbf{D}\|_{\mathbf{X}} + \|\mathbf{D}\cdot\mathbf{n}\|_{\delta,\partial\Omega}\right) \quad \forall \mathbf{D} \in \left\{\mathbf{C} \in \mathbf{X}(\Omega) \colon \mathbf{C}\cdot\mathbf{n}_{|\partial\Omega} \in H^\delta(\partial\Omega)\right\}, \tag{23}$$

where $\|\cdot\|_{\delta,\partial\Omega}$ denotes the norm of $H^\delta(\partial\Omega)$.

Proof. The first imbedding is proved in [3, Th. 4.4], the second one can be found, for instance, in [13, Th. 1.4.3.2], whereas the third one is a consequence of [13, Th. 1.4.3.1]. □

Finally we remind two well-known Green's formulas, that will be used throughout this work

$$\int_\Omega \mathbf{u} \cdot \mathbf{grad}\, v \,\mathrm{d}\mathbf{x} + \int_\Omega (\mathrm{div}\,\mathbf{u}) v \,\mathrm{d}\mathbf{x} = \langle \mathbf{u}\cdot\mathbf{n}, v\rangle_{\partial\Omega} \quad \forall \mathbf{u} \in \mathbf{H}(\mathrm{div};\Omega),\ \forall v \in H^1(\Omega), \tag{24}$$

$$\int_\Omega \mathbf{u} \cdot (\mathbf{curl}\,\mathbf{w}) \,\mathrm{d}\mathbf{x} - \int_\Omega (\mathbf{curl}\,\mathbf{u}) \cdot \mathbf{w} \,\mathrm{d}\mathbf{x} = \langle \mathbf{u}\times\mathbf{n}, \mathbf{w}\rangle_{\partial\Omega} \quad \forall \mathbf{u} \in \mathbf{H}(\mathbf{curl};\Omega),\ \forall \mathbf{w} \in \mathbf{H}^1(\Omega). \tag{25}$$

We notice that the second of the previous Green's formulas is not valid when both functions belong to $\mathbf{H}(\mathbf{curl};\Omega)$. In order to extend the use of this formula to a more general situation, it is necessary to characterize the space of tangential traces of $\mathbf{H}(\mathbf{curl};\Omega)$. To this end we define the space $H^{3/2}(\partial\Omega) := \{u_{|\partial\Omega} \colon u \in H^2(\Omega)\}$, and denote its dual by $H^{-3/2}(\partial\Omega)$. We also define the space

$$\mathbf{H}_T^{1/2}(\partial\Omega) := \left\{\mathbf{w} \in \mathbf{H}^{1/2}(\partial\Omega) \colon (\mathbf{w}\cdot\mathbf{n})_{|\partial\Omega} = 0\right\}.$$

In the case of Ω being of class $\mathcal{C}^{1,1}$ we can introduce its dual space, denoted by $\mathbf{H}_T^{-1/2}(\partial\Omega)$, which can be identified to $\{\mathbf{w} \in \mathbf{H}^{-1/2}(\partial\Omega) \colon (\mathbf{w}\cdot\mathbf{n})_{|\partial\Omega} = 0\}$. Then we define the 'tangential gradient operator' $\mathbf{grad}_\Gamma : H^{3/2}(\partial\Omega) \to \mathbf{H}_T^{1/2}(\partial\Omega)$ given by $\mathbf{grad}_\Gamma\,\varphi := \mathbf{n} \times ((\mathbf{grad}\,\tilde{\varphi})_{|\partial\Omega} \times \mathbf{n})$ and this definition can be seen to be independent of the lifting $\tilde{\varphi} \in H^2(\Omega)$ of φ (see, for instance, [1]). Following [2] (see also [4]), we define the 'tangential divergence operator' $\mathrm{div}_\Gamma : \mathbf{H}_T^{-1/2}(\partial\Omega) \to H^{-3/2}(\partial\Omega)$ as the adjoint operator of $-\mathbf{grad}_\Gamma$. In the same paper it is proved that the tangential trace operator γ_τ defined as $\gamma_\tau(\mathbf{u}) := \mathbf{u} \times \mathbf{n}$, is linear, continuous and surjective from $\mathbf{H}(\mathbf{curl};\Omega)$ into $\mathbf{H}_T^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega)$, this last space being defined as

$$\mathbf{H}_T^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega) := \left\{\boldsymbol{\lambda} \in \mathbf{H}_T^{-1/2}(\partial\Omega) \colon \mathrm{div}_\Gamma\,\boldsymbol{\lambda} \in H^{-1/2}(\partial\Omega)\right\}.$$

If Ω is a Lipschitz polyhedron the scalar product $\boldsymbol{\lambda}\cdot\mathbf{n}$ for $\boldsymbol{\lambda}\in\mathbf{H}^{-1/2}(\partial\Omega)$, and the tangential divergence operator are not meaningful anymore. In order to define the space of tangential traces of $\mathbf{H}(\mathbf{curl};\Omega)$, in [7,8] some of the previous definitions and results are generalized, considering the definitions of some spaces and operators face by face and imposing certain compatibility conditions at the edges of the polyhedron. In the following we summarize some of the main results of these papers and refer the reader to the articles for the full rigorous proofs and details.

Let Ω be a Lipschitz polyhedron such that its boundary $\partial\Omega$ is split into M open faces Γ_j, $j=1,\dots,M$, so that $\partial\Omega=\bigcup_{j=1}^{M}\overline{\Gamma_j}$. When Γ_i and Γ_j are two adjacent faces, we denote by e_{ij} their 'common' edge. Moreover, for a given face Γ_i, $\mathcal{S}_i$ will denote the set of indices j such that the faces Γ_j have a 'common' edge with Γ_i. Finally, we denote by $\mathbf{u}_i$ the trace $\mathbf{u}_{|\Gamma_i}$ (in particular $\mathbf{n}_i=\mathbf{n}_{|\Gamma_i}$), by $\boldsymbol{\tau}_{ij}$ the unit vector in the direction of the edge e_{ij}, and set $\boldsymbol{\tau}_i=\boldsymbol{\tau}_{ij}\times\mathbf{n}_i$ so that $(\boldsymbol{\tau}_i,\boldsymbol{\tau}_{ij},\mathbf{n}_i)$ is an orthonormal basis of $\mathbb{R}^3$.

Let us introduce the spaces

$$\mathbf{L}^2_t(\partial\Omega):=\left\{\boldsymbol{\phi}\in\mathbf{L}^2(\partial\Omega):\ \boldsymbol{\phi}\cdot\mathbf{n}_{|\partial\Omega}=0\right\},\qquad \mathbf{H}^{1/2}_{-}(\partial\Omega):=\left\{\boldsymbol{\lambda}\in\mathbf{L}^2_t(\partial\Omega):\ \boldsymbol{\lambda}_j\in\mathbf{H}^{1/2}(\Gamma_j),\ 1\leqslant j\leqslant M\right\}.$$

For $\psi_i\in H^{1/2}(\Gamma_i)$, $\psi_j\in H^{1/2}(\Gamma_j)$ we denote

$$C(\psi_i,\psi_j):=\int_{\Gamma_i}\int_{\Gamma_j}\frac{|\psi_i(\mathbf{x})-\psi_j(\mathbf{y})|^2}{\|\mathbf{x}-\mathbf{y}\|^3}\,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y}).$$

Let us introduce the Hilbert spaces

$$\mathbf{H}^{1/2}_{\|}(\partial\Omega):=\left\{\boldsymbol{\phi}\in\mathbf{H}^{1/2}_{-}(\partial\Omega):\ \|\boldsymbol{\phi}\|_{\|,1/2,\partial\Omega}:=\left(\sum_{j=1}^{M}\|\boldsymbol{\phi}\|^2_{1/2,\Gamma_j}+\sum_{j=1}^{M}\sum_{i\in\mathcal{S}_j}C(\boldsymbol{\phi}_i\cdot\boldsymbol{\tau}_{ij},\boldsymbol{\phi}_j\cdot\boldsymbol{\tau}_{ij})\right)^{1/2}<+\infty\right\},$$

$$\mathbf{H}^{1/2}_{\perp}(\partial\Omega):=\left\{\boldsymbol{\phi}\in\mathbf{H}^{1/2}_{-}(\partial\Omega):\ \|\boldsymbol{\phi}\|_{\perp,1/2,\partial\Omega}:=\left(\sum_{j=1}^{M}\|\boldsymbol{\phi}\|_{1/2,\Gamma_j}+\sum_{j=1}^{M}\sum_{i\in\mathcal{S}_j}C(\boldsymbol{\phi}_i\cdot\boldsymbol{\tau}_{i},\boldsymbol{\phi}_j\cdot\boldsymbol{\tau}_{j})\right)^{1/2}<+\infty\right\}.$$

The mappings $\pi_\tau(\mathbf{u}):=\mathbf{n}\times(\mathbf{u}\times\mathbf{n})_{|\partial\Omega}$ and $\gamma_\tau(\mathbf{u}):=\mathbf{u}\times\mathbf{n}_{|\partial\Omega}$, constructed face by face, are linear, continuous and surjective from $\mathbf{H}^1(\Omega)$ onto $\mathbf{H}^{1/2}_{\|}(\partial\Omega)$ and $\mathbf{H}^{1/2}_{\perp}(\partial\Omega)$, respectively. The dual spaces of $\mathbf{H}^{1/2}_{\|}(\partial\Omega)$ and $\mathbf{H}^{1/2}_{\perp}(\partial\Omega)$ will be denoted by $\mathbf{H}^{-1/2}_{\|}(\partial\Omega)$ and $\mathbf{H}^{-1/2}_{\perp}(\partial\Omega)$, respectively.

For Lipschitz polyhedra, the tangential gradient operator $\mathbf{grad}_\Gamma\in\mathcal{L}(H^2(\Omega),\mathbf{H}^{1/2}_{\|}(\partial\Omega))$ and the tangential curl operator $\mathbf{curl}_\Gamma\in\mathcal{L}(H^2(\Omega),\mathbf{H}^{1/2}_{\perp}(\partial\Omega))$ can be defined face by face (see [7] and [8]). These operators satisfy, for all $u\in H^2(\Omega)$

$$\mathbf{grad}_\Gamma u=\pi_\tau(\mathbf{grad}\,u),\qquad \mathbf{curl}_\Gamma u=\gamma_\tau(\mathbf{grad}\,u).$$

Next, for any $\varphi\in H^{3/2}(\partial\Omega)$ we define $\mathbf{grad}_\Gamma\varphi:=\mathbf{grad}_\Gamma\tilde{\varphi}$, where $\tilde{\varphi}\in H^2(\Omega)$ is such that $\tilde{\varphi}_{|\partial\Omega}=\varphi$. The definition of $\mathbf{grad}_\Gamma\varphi$ can be seen to be independent of $\tilde{\varphi}$, and $\mathbf{grad}_\Gamma\in\mathcal{L}(H^{3/2}(\partial\Omega),\mathbf{H}^{1/2}_{\|}(\partial\Omega))$. Analogously, we define $\mathbf{curl}_\Gamma\varphi:=\mathbf{curl}_\Gamma\tilde{\varphi}$, and $\mathbf{curl}_\Gamma\in\mathcal{L}(H^{3/2}(\partial\Omega),\mathbf{H}^{1/2}_{\perp}(\partial\Omega))$.

We also define the 'tangential divergence operator' $\operatorname{div}_\Gamma:\mathbf{H}^{-1/2}_{\|}(\partial\Omega)\to H^{-3/2}(\partial\Omega)$ as the adjoint operator of $-\mathbf{grad}_\Gamma$, and the operator $\operatorname{curl}_\Gamma:\mathbf{H}^{-1/2}_{\perp}(\partial\Omega)\to H^{-3/2}(\partial\Omega)$ as the adjoint of $\mathbf{curl}_\Gamma$. Now let us set

$$\mathbf{H}^{-1/2}_{\|}(\operatorname{div}_\Gamma,\partial\Omega):=\left\{\boldsymbol{\lambda}\in\mathbf{H}^{-1/2}_{\|}(\partial\Omega):\ \operatorname{div}_\Gamma(\boldsymbol{\lambda})\in H^{-1/2}(\partial\Omega)\right\},\tag{26}$$

$$\mathbf{H}^{-1/2}_{\perp}(\operatorname{curl}_\Gamma,\partial\Omega):=\left\{\boldsymbol{\lambda}\in\mathbf{H}^{-1/2}_{\perp}(\partial\Omega):\ \operatorname{curl}_\Gamma(\boldsymbol{\lambda})\in H^{-1/2}(\partial\Omega)\right\}.\tag{27}$$

From Theorems 3.9 and 3.10 in [7] and Theorem 5.4 in [8] we know that the mappings γ_τ and π_τ are linear, continuous and surjective from $\mathbf{H}(\mathbf{curl};\Omega)$ into $\mathbf{H}^{-1/2}_{\|}(\operatorname{div}_\Gamma,\partial\Omega)$ and $\mathbf{H}^{-1/2}_{\perp}(\operatorname{curl}_\Gamma,\partial\Omega)$, respectively. Moreover, in the same papers it is proved that $(\mathbf{H}^{-1/2}_{\perp}(\operatorname{curl}_\Gamma,\partial\Omega))'=\mathbf{H}^{-1/2}_{\|}(\operatorname{div}_\Gamma,\partial\Omega)$, and denoting by $\langle\cdot,\cdot\rangle_{\partial\Omega}$ their duality pairing the following Green's formula holds:

$$\int_\Omega(\mathbf{curl}\,\mathbf{w})\cdot\mathbf{u}-\mathbf{w}\cdot(\mathbf{curl}\,\mathbf{u})\,\mathrm{d}\mathbf{x}=\left\langle\gamma_\tau(\mathbf{u}),\pi_\tau(\mathbf{w})\right\rangle_{\partial\Omega}\quad\forall\mathbf{u},\mathbf{w}\in\mathbf{H}(\mathbf{curl};\Omega).\tag{28}$$

Remark 1. In order to avoid the use of new constants we shall consider $\mathbf{H}^{-1/2}_{\|}(\operatorname{div}_\Gamma,\partial\Omega)$ endowed with the norm, denoted $\|\cdot\|_{\mathbf{H}^{-1/2}_{\|}(\operatorname{div}_\Gamma,\partial\Omega)}$, defined as the dual norm of the graph norm of $\mathbf{H}^{-1/2}_{\perp}(\operatorname{curl}_\Gamma,\partial\Omega)$.

Finally, we will also make use of the following lemma:

Lemma 2. *The space* $\{\pi_\tau(\mathbf{u})\colon\ \mathbf{u}\in\mathbf{X}_0(\Omega)\}$ *is dense in* $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\partial\Omega)$.

Proof. It is similar to the proof of Lemma 2.4 in [9]. Let us denote by $\Sigma=\bigcup_{i,j=1}^M e_{ij}$ the union of all edges of the polyhedron and by $\mathbf{H}_S^1(\Omega)$ the functions in $\mathbf{H}^1(\Omega)$ compactly supported in $\overline{\Omega}\setminus\Sigma$. The first step is to prove that the space of tangential component traces for $\mathbf{H}_S^1(\Omega)$ is contained in the one for $\mathbf{H}^1(\Omega)\cap\mathbf{X}_0(\Omega)$. Let $\mathbf{v}\in\mathbf{H}_S^1(\Omega)$; its normal component $(\mathbf{v}\cdot\mathbf{n})\mathbf{n}$ $(=\mathbf{v}-\pi_\tau\mathbf{v})$ belongs to $\mathbf{H}^{1/2}(\Gamma_j)$ on each face and, since $\mathbf{v}\in\mathbf{H}_S^1(\Omega)$, it also belongs to $\mathbf{H}^{1/2}(\partial\Omega)$, the space of traces for $\mathbf{H}^1(\Omega)$ (see [7, Th. 2.5]). Hence, there exists $\mathbf{w}\in\mathbf{H}^1(\Omega)$ such that its trace is equal to the normal component of $\mathbf{v}$ and, if we define $\mathbf{z}:=\mathbf{v}-\mathbf{w}$, it holds that $\mathbf{z}\in\mathbf{H}^1(\Omega)\cap\mathbf{X}_0(\Omega)$ and $\pi_\tau\mathbf{z}=\pi_\tau\mathbf{v}$.

Now, we recall that $\mathbf{H}_S^1(\Omega)$ is dense in $\mathbf{H}^1(\Omega)$ and that $\mathbf{H}^1(\Omega)$ is dense in $\mathbf{H}(\mathbf{curl};\Omega)$. Since π_τ is continuous and surjective from $\mathbf{H}(\mathbf{curl};\Omega)$ into $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\partial\Omega)$ the result follows. □

Remark 2. Some of the results of this article can be improved if the domain Ω is assumed to be bounded and of class $\mathcal{C}^{1,1}$ or a bounded convex polyhedron. In these cases, $\mathbf{H}_T^1(\Omega)=\mathbf{H}(\mathbf{curl};\Omega)\cap\mathbf{H}_0(\mathrm{div};\Omega)=\mathbf{X}_0(\Omega)$ and for any function $\mathbf{w}\in\mathbf{X}_0(\Omega)$, we have the inequality,

$$\|\mathbf{w}\|_1\leqslant C_2\big(\|\mathbf{curl}\,\mathbf{w}\|_0^2+\|\mathrm{div}\,\mathbf{w}\|_0^2\big)^{1/2},\tag{29}$$

with C_2 some constant dependent on Ω. Hence the norms $\|\,\boldsymbol{.}\,\|_1$ and $\|\,\boldsymbol{.}\,\|_{\mathbf{X}}$ are equivalent in $\mathbf{H}_T^1(\Omega)$. This result is presented in Chapter 1 of [12]. It is a consequence of Theorem 3.8 and Lemma 3.6 for the case of a bounded domain with $\mathcal{C}^{1,1}$ boundary, and of Theorem 3.9 and Lemma 3.6 for the case of a convex polyhedron. Since our test functions will belong to the space $\mathbf{X}_0(\Omega)$, in the right-hand side of (28) we can use the equality

$$\big\langle\gamma_\tau(\mathbf{u}),\pi_\tau(\mathbf{w})\big\rangle_{\partial\Omega}=\langle\mathbf{u}\times\mathbf{n},\mathbf{w}\rangle_{\partial\Omega},$$

where the angles in the left-hand side denote the duality pairing between $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\partial\Omega)$ and $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\partial\Omega)$, and the ones in the right-hand side denote the duality pairing between $\mathbf{H}^{-1/2}(\partial\Omega)$ and $\mathbf{H}^{1/2}(\partial\Omega)$. In fact, this is valid for any $\mathbf{u}\in\mathbf{H}(\mathbf{curl};\Omega)$ and $\mathbf{w}\in\mathbf{H}^1(\Omega)$.

3.1. Compatibility and regularity conditions for source and boundary data

In this subsection we specify the precise compatibility and regularity for the boundary conditions and the given sources, in order to obtain a weak formulation of the problem. First, for the Navier–Stokes equations we assume

$$\mathbf{f}_0\in\mathbf{H}^{-1}(\Omega),\tag{30}$$

$$\mathbf{u}_d\in\mathbf{H}^{1/2}(\partial\Omega)\quad\text{with}\quad\int_{\partial\Omega}\mathbf{u}_d\cdot\mathbf{n}\,\mathbf{dx}=0,\tag{31}$$

the compatibility condition for the boundary data being needed because the velocity field is divergence-free.

Next, for the electromagnetic data, we have the following conditions

$$l\in H^\delta(\partial\Omega)\quad\text{with}\quad\int_{\partial\Omega}l\,\mathbf{dx}=0\quad\text{and}\quad 0<\delta<1/2,\tag{32}$$

$$\mathbf{k}\in\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\partial\Omega)\quad\text{and}\quad\mathrm{div}_\Gamma\,\mathbf{k}=0.\tag{33}$$

The compatibility condition for l is a consequence of $\mathbf{B}$ being a divergence-free field. Furthermore, we impose $0<\delta<1/2$ in order to obtain a magnetic induction field $\mathbf{B}\in\mathbf{L}^3(\Omega)$. The first condition of (33) is a direct consequence of the boundary condition (16): if we define the non-dimensionalized electric field $\mathbf{E}:=\frac{1}{R_m}\,\mathbf{curl}\,\mathbf{B}-\mathbf{u}\times\mathbf{B}$, since we shall require $\mathbf{u}\in\mathbf{H}^1(\Omega)\subset\mathbf{L}^6(\Omega)$ and $\mathbf{B}\in\mathbf{Y}(\Omega)$, we will have that $\mathbf{E}\in\mathbf{L}^2(\Omega)$. Moreover, from (9) we also know that $\mathbf{curl}\,\mathbf{E}=\mathbf{0}$, so $\mathbf{E}\in\mathbf{H}(\mathbf{curl};\Omega)$ and its tangential trace $\mathbf{E}\times\mathbf{n}=\gamma_\tau(\mathbf{E})\in\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\partial\Omega)$. The second condition in (33) is a consequence of $\mathbf{E}$ being an irrotational field: for any $\varphi\in H^{3/2}(\partial\Omega)$ we consider any extension $\tilde{\varphi}\in H^2(\Omega)$ such that $\tilde{\varphi}_{|\partial\Omega}=\varphi$. From the definitions of the tangential divergence and of the tangential gradient we have

$$\langle\mathrm{div}_\Gamma\,\mathbf{k},\varphi\rangle_{\partial\Omega}=-\langle\mathbf{k},\mathbf{grad}_\Gamma\,\tilde{\varphi}\rangle_{\partial\Omega}=-\big\langle\mathbf{k},\pi_\tau(\mathbf{grad}\,\tilde{\varphi})\big\rangle_{\partial\Omega}=-\big\langle\gamma_\tau(\mathbf{E}),\pi_\tau(\mathbf{grad}\,\tilde{\varphi})\big\rangle_{\partial\Omega},$$

and from Green's formula (28) we conclude that $\mathrm{div}_\Gamma\,\mathbf{k}=0$, because $\mathbf{curl}\,\mathbf{E}=\mathbf{0}$.

Remark 3. If $\mathbf{k} \in \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \partial\Omega)$ and $\operatorname{div}_\Gamma \mathbf{k} = 0$, it also holds that

$$\langle \mathbf{k}, \pi_\tau(\mathbf{grad}\,\psi)\rangle_{\partial\Omega} = 0 \quad \forall \psi \in H^1(\Omega).$$

In fact, this equality holds for any $\psi \in H^2(\Omega)$ just by the definition of the surface divergence. Since $H^2(\Omega)$ is dense in $H^1(\Omega)$, $\mathbf{grad} \in \mathcal{L}(H^1(\Omega), \mathbf{H}(\mathbf{curl};\Omega))$ and $\pi_\tau \in \mathcal{L}(\mathbf{H}(\mathbf{curl};\Omega), \mathbf{H}_{\perp}^{-1/2}(\operatorname{curl}_\Gamma, \partial\Omega))$, the equality is true for any $\psi \in H^1(\Omega)$.

For the heat transfer equation, the heat source and the boundary data must satisfy

$$\psi \in L^1(\Omega), \qquad T_d \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega). \tag{34}$$

In what follows we are going to consider the thermal-magnetohydrodynamic problem (9)–(17) along with the conditions (30)–(34).

4. Magnetohydrodynamic problem

This section is devoted to the study of the pure magnetohydrodynamic problem, where the temperature is supposed to be known. We begin by introducing a weak formulation of the problem and proving some properties of the forms involved. Then we prove that this weak formulation is equivalent to the strong formulation (i.e. involving partial differential equations, understood in the sense of distributions) and we finish by analyzing a linearized version of the problem that will help us to study the coupled problem.

4.1. Weak formulation

Before presenting the weak formulation of problem (9)–(17), we introduce some forms that will allow us to simplify the notation. Let $a_0 : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}$, $a_1 : \mathbf{X}(\Omega) \times \mathbf{X}(\Omega) \to \mathbb{R}$, $c_0 : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}$, $c_1 : \mathbf{X}(\Omega) \times \mathbf{Y}(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}$, $b : \mathbf{H}^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$, be defined by

$$a_0(\mathbf{u}, \mathbf{v}) := \frac{1}{H_a^2} \int_\Omega \mathbf{grad}\,\mathbf{u} : \mathbf{grad}\,\mathbf{v}\,\mathrm{d}\mathbf{x},$$

$$a_1(\mathbf{B}, \mathbf{C}) := \frac{1}{R_m^2} \int_\Omega \big[(\mathbf{curl}\,\mathbf{B}) \cdot (\mathbf{curl}\,\mathbf{C}) + (\operatorname{div}\mathbf{B})(\operatorname{div}\mathbf{C})\big]\,\mathrm{d}\mathbf{x},$$

$$c_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{N} \int_\Omega (\mathbf{grad}\,\mathbf{v})\mathbf{u} \cdot \mathbf{w}\,\mathrm{d}\mathbf{x},$$

$$c_1(\mathbf{B}, \mathbf{C}, \mathbf{u}) := \frac{1}{R_m} \int_\Omega (\mathbf{curl}\,\mathbf{B}) \times \mathbf{C} \cdot \mathbf{u}\,\mathrm{d}\mathbf{x},$$

$$b(\mathbf{u}, p) := -\int_\Omega p(\operatorname{div}\mathbf{u})\,\mathrm{d}\mathbf{x},$$

and let $F : \mathbf{H}_0^1(\Omega) \times \mathbf{X}_0(\Omega) \to \mathbb{R}$ be given by

$$F((\mathbf{v}, \mathbf{C})) := \langle \mathbf{f}_0, \mathbf{v}\rangle_\Omega + \frac{1}{R_m}\langle \mathbf{k}, \pi_\tau(\mathbf{C})\rangle_{\partial\Omega}.$$

We will also make use of the linear mapping $G : L^{6/5}(\Omega) \to \mathbf{H}^{-1}(\Omega)$, defined by

$$\langle G(T), \mathbf{v}\rangle_\Omega = \frac{G_r}{N R_e^2} \int_\Omega T \frac{\mathbf{g}}{g} \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega).$$

The integral is meaningful because of the Sobolev imbedding $\mathbf{H}^1(\Omega) \subset \mathbf{L}^6(\Omega)$.

Lemma 3. *The linear form $F(\cdot)$, the bilinear forms $a_0(\cdot,\cdot)$, $a_1(\cdot,\cdot)$ and the trilinear forms $c_0(\cdot,\cdot,\cdot)$ and $c_1(\cdot,\cdot,\cdot)$ are continuous in the spaces where they have been defined. Moreover, $a_0(\cdot,\cdot)$, $a_1(\cdot,\cdot)$ are coercive on $\mathbf{H}_0^1(\Omega)$ and $\mathbf{X}_0(\Omega)$, respectively.*

Proof. Most of the inequalities can be proved in a standard way as it is done in [1] and [14], but we reproduce them here in detail because the continuity constants will be important in the forthcoming results. The following inequalities are easily obtained from the definitions of the forms:

$$
\begin{aligned}
&\left|a_0(\mathbf{u},\mathbf{v})\right| \leqslant \frac{1}{H_a^2}|\mathbf{u}|_1|\mathbf{v}|_1 \quad \forall \mathbf{u},\mathbf{v}\in \mathbf{H}^1(\Omega),\\
&\left|a_1(\mathbf{B},\mathbf{C})\right| \leqslant \frac{1}{R_m^2}\big(\|\mathbf{curl}\,\mathbf{B}\|_0\|\mathbf{curl}\,\mathbf{C}\|_0+\|\mathrm{div}\,\mathbf{B}\|_0\|\mathrm{div}\,\mathbf{C}\|_0\big) \leqslant \frac{1}{R_m^2}|\mathbf{B}|_{\mathbf{X}}|\mathbf{C}|_{\mathbf{X}} \quad \forall \mathbf{B},\mathbf{C}\in \mathbf{X}(\Omega),\\
&\left|c_0(\mathbf{u},\mathbf{v},\mathbf{w})\right| \leqslant \frac{1}{N}\|\mathbf{u}\|_{\mathbf{L}^4}|\mathbf{v}|_1\|\mathbf{w}\|_{\mathbf{L}^4} \quad \forall \mathbf{u},\mathbf{v},\mathbf{w}\in \mathbf{H}^1(\Omega),\\
&\left|c_1(\mathbf{B},\mathbf{C},\mathbf{u})\right| \leqslant \frac{1}{R_m}\|\mathbf{curl}\,\mathbf{B}\|_0\|\mathbf{C}\|_{\mathbf{L}^3}\|\mathbf{u}\|_{\mathbf{L}^6} \leqslant \frac{1}{R_m}|\mathbf{B}|_{\mathbf{X}}\|\mathbf{C}\|_{\mathbf{L}^3}\|\mathbf{u}\|_{\mathbf{L}^6} \quad \forall \mathbf{B}\in \mathbf{X}(\Omega),\ \forall \mathbf{C}\in \mathbf{Y}(\Omega),\ \forall \mathbf{u}\in \mathbf{H}^1(\Omega),\\
&\left|F((\mathbf{v},\mathbf{C}))\right| \leqslant \lambda_F\left|(\mathbf{v},\mathbf{C})\right|_{\mathcal{W}} \quad \forall (\mathbf{v},\mathbf{C})\in \mathcal{W}_0(\Omega),
\end{aligned}
$$

where $\lambda_F := (C_0^2\|\mathbf{f}_0\|_{-1}^2 + (C_1^2/R_m^2)\|\mathbf{k}\|^2_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\partial\Omega)}\|\pi_\tau\|^2_{\mathcal{L}(\mathbf{H}(\mathbf{curl};\Omega),\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\partial\Omega))})^{1/2}$, with C_0 and C_1 the constants appearing in the inequalities (19) and (22), respectively. The coerciveness results read

$$
\begin{aligned}
&a_0(\hat{\mathbf{u}},\hat{\mathbf{u}}) = \frac{1}{H_a^2}\int_\Omega |\mathbf{grad}\,\hat{\mathbf{u}}|^2\,d\mathbf{x} = \frac{1}{H_a^2}\|\mathbf{grad}\,\hat{\mathbf{u}}\|_0^2 = \frac{1}{H_a^2}|\hat{\mathbf{u}}|_1^2 \quad \forall \hat{\mathbf{u}}\in \mathbf{H}_0^1(\Omega),\\
&a_1(\hat{\mathbf{B}},\hat{\mathbf{B}}) = \frac{1}{R_m^2}\int_\Omega \big(|\mathbf{curl}\,\hat{\mathbf{B}}|^2+|\mathrm{div}\,\hat{\mathbf{B}}|^2\big)\,d\mathbf{x} = \frac{1}{R_m^2}\big(\|\mathbf{curl}\,\hat{\mathbf{B}}\|_0^2+\|\mathrm{div}\,\hat{\mathbf{B}}\|_0^2\big) = \frac{1}{R_m^2}|\hat{\mathbf{B}}|_{\mathbf{X}}^2 \quad \forall \hat{\mathbf{B}}\in \mathbf{X}_0(\Omega). \qquad \square
\end{aligned}
$$

Finally, the following result will also be helpful:

$$
\left|\big(G(T),\mathbf{v}\big)_\Omega\right| \leqslant \frac{G_r}{NR_e^2}\|T\|_{L^{6/5}}\|\mathbf{v}\|_{\mathbf{L}^6} \leqslant \frac{SG_r}{NR_e^2}\|T\|_{L^{6/5}}|\mathbf{v}|_1 \quad \forall T\in L^{6/5}(\Omega),\ \forall \mathbf{v}\in \mathbf{H}_0^1(\Omega).
$$

For the sake of simplicity, in the sequel we will use the previous notation λ_F and also $\lambda_G := SG_r/(NR_e^2)$.

Lemma 4. *Let $\mathbf{u}\in \mathbf{Z}(\Omega)$, $\mathbf{v},\mathbf{w}\in \mathbf{H}^1(\Omega)$ and assume that at least one of $\mathbf{u},\mathbf{v},\mathbf{w}$ belongs to $\mathbf{H}_0^1(\Omega)$. Then*

$$
c_0(\mathbf{u},\mathbf{v},\mathbf{w}) = -c_0(\mathbf{u},\mathbf{w},\mathbf{v}). \tag{35}
$$

Proof. The result is well known and a proof can be found in [14]. It relies on the fact that **u** is divergence-free and on the use of a Green's formula. $\square$

We can now introduce the weak formulation of the magnetohydrodynamic problem.
Given $\mathbf{f}_0$, $\mathbf{u}_d$, l and $\mathbf{k}$ satisfying (30)–(33)*, and $T\in L^{6/5}(\Omega)$ find*

$$
(\mathbf{u},\mathbf{B})\in \mathcal{Z}(\Omega), \tag{36}
$$

such that

$$
a_0(\mathbf{u},\mathbf{v}) + a_1(\mathbf{B},\mathbf{C}) + c_0(\mathbf{u},\mathbf{u},\mathbf{v}) - c_1(\mathbf{B},\mathbf{B},\mathbf{v}) + c_1(\mathbf{C},\mathbf{B},\mathbf{u}) = F((\mathbf{v},\mathbf{C})) - \big(G(T),\mathbf{v}\big)_\Omega \quad \forall (\mathbf{v},\mathbf{C})\in \mathcal{Z}_0(\Omega), \tag{37}
$$

$$
\mathbf{u}_{|\partial\Omega} = \mathbf{u}_d, \qquad (\mathbf{B}\cdot\mathbf{n})_{|\partial\Omega} = l. \tag{38}
$$

We now proceed to show that, for any given $T\in L^{6/5}(\Omega)$, any pair $(\mathbf{u},\mathbf{B})$ satisfying (36)–(38) is also a solution of Eqs. (9)–(12) along with boundary conditions (14)–(16). To do that we shall make use of the following lemma, whose proof essentially follows the ideas of the proof of Lemma 3.2 in [14].

Lemma 5. *Let $\mathbf{B}\in \mathbf{H}(\mathrm{div};\Omega)$ with $(\mathbf{B}\cdot\mathbf{n})_{|\partial\Omega} = l$, l satisfying* (32)*. Then there exists a unique scalar function $\chi\in H^1(\Omega)\cap L_0^2(\Omega)$ such that*

$$
\begin{cases}
\Delta\chi = \mathrm{div}\,\mathbf{B},\\
(\mathbf{grad}\,\chi\cdot\mathbf{n})_{|\partial\Omega} = 0.
\end{cases} \tag{39}
$$

As a consequence, $\mathbf{grad}\,\chi\in \mathbf{X}_0(\Omega)$.

Proposition 6. *Given $T\in L^{6/5}(\Omega)$, if $(\mathbf{u},\mathbf{B})$ is a pair satisfying* (36)–(38) *then there exists a unique $p\in L_0^2(\Omega)$ such that $((\mathbf{u},\mathbf{B}),p)$ satisfies* (9)–(12) *and boundary conditions* (14)–(16)*.*

Proof. We follow essentially the proof of Proposition 3.1 in [14]. Let $\mathbf{l} \in \mathbf{H}^{-1}(\Omega)$ be defined as

$$\langle \mathbf{l}, \mathbf{v}\rangle_\Omega := \frac{1}{H_a^2}\int_\Omega \mathbf{grad}\,\mathbf{u} : \mathbf{grad}\,\mathbf{v}\,d\mathbf{x} + \frac{1}{N}\int_\Omega (\mathbf{grad}\,\mathbf{u})\mathbf{u}\cdot\mathbf{v}\,d\mathbf{x} - \frac{1}{R_m}\int_\Omega (\mathbf{curl}\,\mathbf{B})\times\mathbf{B}\cdot\mathbf{v}\,d\mathbf{x}$$
$$- \langle \mathbf{f}_0, \mathbf{v}\rangle_\Omega + \frac{G_r}{NR_e^2}\int_\Omega T\frac{\mathbf{g}}{g}\cdot\mathbf{v}\,d\mathbf{x} \quad \forall \mathbf{v}\in \mathbf{H}_0^1(\Omega).$$

Taking $\mathbf{C} = \mathbf{0}$ in (37) we get $\langle \mathbf{l}, \mathbf{v}\rangle_\Omega = 0 \quad \forall \mathbf{v} \in \mathbf{Z}_0(\Omega)$. Hence there exists a unique $p \in L_0^2(\Omega)$ such that $\mathbf{l} = -\,\mathbf{grad}\,p$ in $\mathbf{H}^{-1}(\Omega)$ (see [12, Chapter I, Lemma 2.1]). Thus, Eq. (11) is satisfied in $\mathbf{H}^{-1}(\Omega)$.

As $\mathbf{B} \in \mathbf{Y}(\Omega) \subset \mathbf{H}(\mathrm{div};\Omega)$ and $(\mathbf{B}\cdot\mathbf{n})_{|\partial\Omega} = l$ we are in the hypotheses of Lemma 5, so we can take $\mathbf{C} = \mathbf{grad}\,\chi$ and $\mathbf{v} = \mathbf{0}$ in (37) to obtain

$$\frac{1}{R_m^2}\int_\Omega (\mathrm{div}\,\mathbf{B})(\mathrm{div}\,\mathbf{B})\,d\mathbf{x} = \frac{1}{R_m^2}\int_\Omega (\mathrm{div}\,\mathbf{B})(\mathrm{div}\,\mathbf{grad}\,\chi)\,d\mathbf{x} = \frac{1}{R_m}\langle \mathbf{k}, \pi_\tau(\mathbf{grad}\,\chi)\rangle_{\partial\Omega}. \tag{40}$$

Moreover, since $\mathbf{k}$ satisfies the hypotheses of Remark 3 we can affirm that $\mathrm{div}\,\mathbf{B} = 0$ a.e. in Ω and Eq. (10) holds. Taking $\mathbf{v} = \mathbf{0}$ in (37), using (10) and the identity $(\mathbf{curl}\,\mathbf{C})\times\mathbf{B}\cdot\mathbf{u} = -(\mathbf{u}\times\mathbf{B})\cdot(\mathbf{curl}\,\mathbf{C})$ we get

$$\frac{1}{R_m^2}\int_\Omega (\mathbf{curl}\,\mathbf{B})\cdot(\mathbf{curl}\,\mathbf{C})\,d\mathbf{x} - \frac{1}{R_m}\int_\Omega (\mathbf{u}\times\mathbf{B})\cdot(\mathbf{curl}\,\mathbf{C})\,d\mathbf{x} = \frac{1}{R_m}\langle \mathbf{k}, \pi_\tau(\mathbf{C})\rangle_{\partial\Omega} \quad \forall \mathbf{C}\in\mathbf{X}_0(\Omega), \tag{41}$$

and since $\mathbf{H}_0^1(\Omega) \subset \mathbf{X}_0(\Omega)$ we get that (9) is valid in $\mathbf{H}^{-1}(\Omega)$.

If we define the electric field by $\mathbf{E} := \frac{1}{R_m}\mathbf{curl}\,\mathbf{B} - \mathbf{u}\times\mathbf{B}$ we know that $\mathbf{E} \in \mathbf{L}^2(\Omega)$ and, from Eq. (9), $\mathbf{curl}\,\mathbf{E} = \mathbf{0}$. Hence, $\mathbf{E} \in \mathbf{H}(\mathbf{curl};\Omega)$ and $\mathbf{E}\times\mathbf{n}_{|\partial\Omega} = \gamma_\tau(\mathbf{E}) \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega)$. From the previous equation, using Green's formula (28) and Lemma 2 it follows that $\mathbf{k} = \gamma_\tau(\mathbf{E})$ in $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega)$. □

Remark 4. The inverse result can also be proved. Assume that we have $T \in L^{6/5}(\Omega)$, $(\mathbf{u}, \mathbf{B}) \in \mathbf{H}^1(\Omega)\times\mathbf{Y}(\Omega)$ and $p \in L_0^2(\Omega)$ satisfying (9)–(12), along with boundary conditions (14)–(16) and compatibility conditions (30)–(33). Then $(\mathbf{u}, \mathbf{B})$ is a solution of (36)–(38). This is obtained by: (i) multiplying (11) by a test function $\mathbf{v} \in \mathbf{Z}_0(\Omega)$, (ii) multiplying (9) by $\mathbf{C} \in \mathbf{X}_0(\Omega)$ and using Green's formula (28) and boundary condition (16), (iii) summing up the resulting equations and taking into account Eq. (10).

4.2. Reduction to homogeneous boundary conditions

Now we split the unknowns into two parts: the first one satisfying the inhomogeneous boundary conditions and the second one satisfying homogeneous boundary conditions.

If the domain Ω is a Lipschitz polyhedron, supposing that boundary conditions $\mathbf{u}_d$ and l satisfy Eqs. (31) and (32), respectively, there exist extensions $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$ and $\mathbf{B}_0 \in \mathbf{Y}(\Omega)$ satisfying

$$\mathbf{u}_{0|\partial\Omega} = \mathbf{u}_d \quad \text{with} \quad \mathrm{div}\,\mathbf{u}_0 = 0 \quad \text{and} \quad \|\mathbf{u}_0\|_1 \leqslant \Lambda_1\|\mathbf{u}_d\|_{1/2,\partial\Omega}, \tag{42}$$

$$(\mathbf{B}_0\cdot\mathbf{n})_{|\partial\Omega} = l \quad \text{with} \quad \mathrm{div}\,\mathbf{B}_0 = 0, \quad \mathbf{curl}\,\mathbf{B}_0 = \mathbf{0} \quad \text{and} \quad \|\mathbf{B}_0\|_{\mathbf{L}^3} \leqslant \Lambda_2\|l\|_{\delta,\partial\Omega}, \tag{43}$$

where Λ_1 and Λ_2 are two constants depending on Ω.

The construction of $\mathbf{u}_0$ is well known and can be found, for instance, in [14] or [12]. The construction of $\mathbf{B}_0$ is the same as in [14] and is based on the solution of the Neumann problem

$$\begin{cases} -\Delta\chi = 0, \\ (\mathbf{grad}\,\chi\cdot\mathbf{n})_{|\partial\Omega} = l. \end{cases}$$

Due to compatibility condition (32) this problem has a unique solution $\chi \in H^1(\Omega)\cap L_0^2(\Omega)$. Taking $\mathbf{B}_0 = \mathbf{grad}\,\chi$ it is clear that $\mathrm{div}\,\mathbf{B}_0 = 0$ and $\mathbf{curl}\,\mathbf{B}_0 = \mathbf{0}$, hence $\mathbf{B}_0 \in \mathbf{X}(\Omega)$. Moreover, $(\mathbf{B}_0\cdot\mathbf{n})_{|\partial\Omega} = l \in H^\delta(\Omega)$ and $\|\mathbf{B}_0\|_{\mathbf{X}} = \|\mathbf{B}_0\|_0 \leqslant \kappa_1\|l\|_{-1/2,\partial\Omega}$. As a consequence of Lemma 1 we know that $\mathbf{B}_0 \in \mathbf{Y}(\Omega)$ and taking into account that $\mathbf{B}_0$ is curl and divergence-free we have $\|\mathbf{B}_0\|_{\mathbf{Y}} = \|\mathbf{B}_0\|_{\mathbf{L}^3} \leqslant \kappa(\|\mathbf{B}_0\|_{\mathbf{X}} + \|\mathbf{B}_0\cdot\mathbf{n}\|_{\delta,\partial\Omega}) \leqslant \kappa(\kappa_1\|l\|_{-1/2,\partial\Omega} + \|l\|_{\delta,\partial\Omega}) \leqslant \kappa(\kappa_1\kappa_2 + 1)\|l\|_{\delta,\partial\Omega} = \Lambda_2\|l\|_{\delta,\partial\Omega}$ where κ is the constant introduced in (23) and κ_2 is the constant of the imbedding $H^\delta(\partial\Omega) \subset H^{-1/2}(\partial\Omega)$. Both κ and κ_2 depend on δ and Ω, and κ_1 depends on Ω.

Once we have constructed these fields we can split the unknowns as follows: $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}_0$, with $\hat{\mathbf{u}} \in \mathbf{Z}_0(\Omega)$ and $\mathbf{B} = \hat{\mathbf{B}} + \mathbf{B}_0$ with $\hat{\mathbf{B}} \in \mathbf{X}_0(\Omega)$. Thus we can rewrite problem (36)–(38):

Given $\mathbf{u}_0 \in \mathbf{Z}(\Omega)$, $\mathbf{B}_0 \in \mathbf{Y}(\Omega)$ *satisfying* (42)–(43), *and* $T \in L^{6/5}(\Omega)$ *find*

$$(\hat{\mathbf{u}}, \hat{\mathbf{B}}) \in \mathcal{Z}_0(\Omega) \tag{44}$$

such that

$$\begin{aligned} &a_0(\hat{\mathbf{u}}, \mathbf{v}) + a_1(\hat{\mathbf{B}}, \mathbf{C}) + c_0(\hat{\mathbf{u}}, \hat{\mathbf{u}}, \mathbf{v}) + c_0(\hat{\mathbf{u}}, \mathbf{u}_0, \mathbf{v}) + c_0(\mathbf{u}_0, \hat{\mathbf{u}}, \mathbf{v}) \\ &\quad - c_1(\hat{\mathbf{B}}, \hat{\mathbf{B}}, \mathbf{v}) - c_1(\hat{\mathbf{B}}, \mathbf{B}_0, \mathbf{v}) + c_1(\mathbf{C}, \hat{\mathbf{B}}, \hat{\mathbf{u}}) + c_1(\mathbf{C}, \hat{\mathbf{B}}, \mathbf{u}_0) + c_1(\mathbf{C}, \mathbf{B}_0, \hat{\mathbf{u}}) \\ &\quad = F((\mathbf{v}, \mathbf{C})) - \big(G(T), \mathbf{v}\big)_\Omega - a_0(\mathbf{u}_0, \mathbf{v}) - c_0(\mathbf{u}_0, \mathbf{u}_0, \mathbf{v}) - c_1(\mathbf{C}, \mathbf{B}_0, \mathbf{u}_0) \quad \forall (\mathbf{v}, \mathbf{C}) \in \mathcal{Z}_0(\Omega). \end{aligned} \tag{45}$$

Since $\mathbf{X}_0(\Omega) \subset \mathbf{Y}(\Omega)$ all the terms concerning the trilinear form $c_1(\cdot,\cdot,\cdot)$ make sense. It is easily seen that $(\hat{\mathbf{u}}, \hat{\mathbf{B}})$ is a solution of (44)–(45) if and only if $(\mathbf{u}, \mathbf{B}) = (\hat{\mathbf{u}}, \hat{\mathbf{B}}) + (\mathbf{u}_0, \mathbf{B}_0)$ is a solution of problem (36)–(38).

4.3. Linearized MHD problem

Now we introduce a linearized version of the MHD problem that will be helpful to prove the existence of solution to the coupled problem via a fixed point theorem. The linearized problem reads as follows.

Given $(\hat{\mathbf{w}}, \hat{\mathbf{D}}) \in \mathcal{Z}_0(\Omega)$, $T \in L^{6/5}(\Omega)$ *and* $(\mathbf{u}_0, \mathbf{B}_0) \in \mathcal{Z}(\Omega)$ *with* $\mathbf{curl}\, \mathbf{B}_0 = \mathbf{0}$, *find*

$$(\hat{\mathbf{u}}, \hat{\mathbf{B}}) \in \mathcal{Z}_0(\Omega) \tag{46}$$

such that

$$\begin{aligned} &a_0(\hat{\mathbf{u}}, \mathbf{v}) + a_1(\hat{\mathbf{B}}, \mathbf{C}) + c_0(\hat{\mathbf{w}}, \hat{\mathbf{u}}, \mathbf{v}) + c_0(\hat{\mathbf{u}}, \mathbf{u}_0, \mathbf{v}) + c_0(\mathbf{u}_0, \hat{\mathbf{u}}, \mathbf{v}) \\ &\quad - c_1(\hat{\mathbf{B}}, \hat{\mathbf{D}}, \mathbf{v}) - c_1(\hat{\mathbf{B}}, \mathbf{B}_0, \mathbf{v}) + c_1(\mathbf{C}, \hat{\mathbf{D}}, \hat{\mathbf{u}}) + c_1(\mathbf{C}, \hat{\mathbf{B}}, \mathbf{u}_0) + c_1(\mathbf{C}, \mathbf{B}_0, \hat{\mathbf{u}}) \\ &\quad = F((\mathbf{v}, \mathbf{C})) - \big(G(T), \mathbf{v}\big)_\Omega - a_0(\mathbf{u}_0, \mathbf{v}) - c_0(\mathbf{u}_0, \mathbf{u}_0, \mathbf{v}) - c_1(\mathbf{C}, \mathbf{B}_0, \mathbf{u}_0) \quad \forall (\mathbf{v}, \mathbf{C}) \in \mathcal{Z}_0(\Omega). \end{aligned} \tag{47}$$

Proposition 7. *Whenever*

$$\alpha := \min\left\{ \frac{1}{H_a^2} - \frac{S_4}{N} \|\mathbf{u}_0\|_{\mathbf{L}^4}, \frac{1}{R_m^2} - \frac{\kappa C_1}{R_m} \|\mathbf{u}_0\|_{\mathbf{L}^6} \right\} > 0, \tag{48}$$

there exists a unique solution $(\hat{\mathbf{u}}, \hat{\mathbf{B}})$ *to problem* (46)–(47). *Moreover,*

$$\big|(\hat{\mathbf{u}}, \hat{\mathbf{B}})\big|_{\mathcal{W}} \leqslant \frac{1}{\alpha}\left(\lambda_F + \lambda_G \|T\|_{L^{6/5}} + \frac{1}{H_a^2}|\mathbf{u}_0|_1 + \frac{1}{N}\|\mathbf{u}_0\|_{\mathbf{L}^4}^2 + \frac{1}{R_m}\|\mathbf{B}_0\|_{\mathbf{L}^3}\|\mathbf{u}_0\|_{\mathbf{L}^6} \right),$$

where λ_F *and* λ_G *are the constants introduced above.*

Proof. Since $(\mathbf{u}_0, \mathbf{B}_0)$, $(\hat{\mathbf{w}}, \hat{\mathbf{D}})$ and T are given, we can define the bilinear form $\tilde{a} : \mathcal{Z}_0(\Omega) \times \mathcal{Z}_0(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} \tilde{a}\big((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \mathbf{C})\big) := {} & a_0(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{B}, \mathbf{C}) + c_0(\hat{\mathbf{w}}, \mathbf{u}, \mathbf{v}) + c_0(\mathbf{u}, \mathbf{u}_0, \mathbf{v}) + c_0(\mathbf{u}_0, \mathbf{u}, \mathbf{v}) \\ & - c_1(\mathbf{B}, \hat{\mathbf{D}}, \mathbf{v}) - c_1(\mathbf{B}, \mathbf{B}_0, \mathbf{v}) + c_1(\mathbf{C}, \hat{\mathbf{D}}, \mathbf{u}) + c_1(\mathbf{C}, \mathbf{B}, \mathbf{u}_0) + c_1(\mathbf{C}, \mathbf{B}_0, \mathbf{u}), \end{aligned}$$

and the linear form $\widetilde{F} : \mathcal{Z}_0(\Omega) \to \mathbb{R}$ by setting $\widetilde{F}((\mathbf{v}, \mathbf{C}))$ as the right-hand side of Eq. (47).

Due to the inequalities presented in (21) and in Lemma 3, and the antisymmetry property for $c_0(\cdot,\cdot,\cdot)$ stated in Lemma 4 we obtain

$$\begin{aligned} a_0(\mathbf{u}, \mathbf{u}) + c_0(\mathbf{u}, \mathbf{u}_0, \mathbf{u}) &\geqslant \frac{1}{H_a^2}|\mathbf{u}|_1^2 - \frac{1}{N}\|\mathbf{u}_0\|_{\mathbf{L}^4}\|\mathbf{u}\|_{\mathbf{L}^4}|\mathbf{u}|_1 \geqslant \left(\frac{1}{H_a^2} - \frac{S_4}{N}\|\mathbf{u}_0\|_{\mathbf{L}^4} \right)|\mathbf{u}|_1^2 \quad \forall \mathbf{u} \in \mathbf{Z}_0(\Omega), \\ a_1(\mathbf{B}, \mathbf{B}) + c_1(\mathbf{B}, \mathbf{B}, \mathbf{u}_0) &\geqslant \frac{1}{R_m^2}|\mathbf{B}|_{\mathbf{X}}^2 - \frac{1}{R_m}|\mathbf{B}|_{\mathbf{X}}\|\mathbf{B}\|_{\mathbf{L}^3}\|\mathbf{u}_0\|_{\mathbf{L}^6} \geqslant \left(\frac{1}{R_m^2} - \frac{\kappa C_1}{R_m}\|\mathbf{u}_0\|_{\mathbf{L}^6} \right)|\mathbf{B}|_{\mathbf{X}}^2 \quad \forall \mathbf{B} \in \mathbf{X}_0(\Omega). \end{aligned}$$

From these minorations and the antisymmetry property for $c_0(\cdot,\cdot,\cdot)$ we infer that condition (48) guarantees the coerciveness of the bilinear form $\tilde{a}(\cdot,\cdot)$. Then the result follows from Lax–Milgram lemma. □

Assuming that (48) holds, we can define the mapping $\mathcal{G}_1 : \mathcal{Z}_0(\Omega) \times L^{6/5}(\Omega) \to \mathcal{Z}_0(\Omega)$ which maps any given pair $(\hat{\mathbf{w}}, \hat{\mathbf{D}}) \in \mathcal{Z}_0(\Omega)$ and a given temperature $T \in L^{6/5}(\Omega)$ into the solution $(\hat{\mathbf{u}}, \hat{\mathbf{B}})$ of the MHD linearized problem (46)–(47). This mapping satisfies the following property:

Lemma 8. *Let* $(\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n) \rightharpoonup (\hat{\mathbf{w}}, \hat{\mathbf{D}})$ *weakly in* $\mathcal{Z}_0(\Omega)$*,* $T_n \to T$ *strongly in* $L^{6/5}(\Omega)$ *and assume that* (48) *is satisfied. Then* $\mathcal{G}_1((\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n), T_n) \to \mathcal{G}_1((\hat{\mathbf{w}}, \hat{\mathbf{D}}), T)$ *strongly in* $\mathcal{Z}_0(\Omega)$*.*

Proof. Let us denote $(\hat{\mathbf{u}}_n, \hat{\mathbf{B}}_n) = \mathcal{G}_1((\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n), T_n)$ and $(\hat{\mathbf{u}}, \hat{\mathbf{B}}) = \mathcal{G}_1((\hat{\mathbf{w}}, \hat{\mathbf{D}}), T)$. Writing Eq. (47) of the MHD linearized problem for both solutions, subtracting the resulting equations and adding and subtracting the terms $c_0(\hat{\mathbf{w}}_n, \hat{\mathbf{u}}, \mathbf{v})$, $c_1(\hat{\mathbf{B}}, \hat{\mathbf{D}}_n, \mathbf{v})$ and $c_1(\mathbf{C}, \hat{\mathbf{D}}_n, \hat{\mathbf{u}})$ we get

$$\begin{aligned} & a_0(\hat{\mathbf{u}}_n - \hat{\mathbf{u}}, \mathbf{v}) + a_1(\hat{\mathbf{B}}_n - \hat{\mathbf{B}}, \mathbf{C}) + c_0(\hat{\mathbf{w}}_n, \hat{\mathbf{u}}_n - \hat{\mathbf{u}}, \mathbf{v}) + c_0(\hat{\mathbf{w}}_n - \hat{\mathbf{w}}, \hat{\mathbf{u}}, \mathbf{v}) + c_0(\hat{\mathbf{u}}_n - \hat{\mathbf{u}}, \mathbf{u}_0, \mathbf{v}) \\ &\quad + c_0(\mathbf{u}_0, \hat{\mathbf{u}}_n - \hat{\mathbf{u}}, \mathbf{v}) - c_1(\hat{\mathbf{B}}_n - \hat{\mathbf{B}}, \hat{\mathbf{D}}_n, \mathbf{v}) - c_1(\hat{\mathbf{B}}, \hat{\mathbf{D}}_n - \hat{\mathbf{D}}, \mathbf{v}) - c_1(\hat{\mathbf{B}}_n - \hat{\mathbf{B}}, \mathbf{B}_0, \mathbf{v}) \\ &\quad + c_1(\mathbf{C}, \hat{\mathbf{D}}_n, \hat{\mathbf{u}}_n - \hat{\mathbf{u}}) + c_1(\mathbf{C}, \hat{\mathbf{D}}_n - \hat{\mathbf{D}}, \hat{\mathbf{u}}) + c_1(\mathbf{C}, \hat{\mathbf{B}}_n - \hat{\mathbf{B}}, \mathbf{u}_0) + c_1(\mathbf{C}, \mathbf{B}_0, \hat{\mathbf{u}}_n - \hat{\mathbf{u}}) \\ &= -\big(G(T_n - T), \mathbf{v}\big)_\Omega \quad \forall (\mathbf{v}, \mathbf{C}) \in \mathcal{Z}_0(\Omega). \end{aligned}$$

Let us choose $(\mathbf{v}, \mathbf{C}) = (\hat{\mathbf{u}}_n, \hat{\mathbf{B}}_n) - (\hat{\mathbf{u}}, \hat{\mathbf{B}})$ as test function. Taking into account the continuity and the antisymmetry results stated in Lemmas 3 and 4, and using the coerciveness of $\tilde{a}(\boldsymbol{.},\boldsymbol{.})$ we obtain

$$\alpha \big|(\hat{\mathbf{u}}_n, \hat{\mathbf{B}}_n) - (\hat{\mathbf{u}}, \hat{\mathbf{B}})\big|_{\mathcal{W}} \leqslant \lambda_G \|T_n - T\|_{L^{6/5}} + \frac{1}{N}\|\hat{\mathbf{w}}_n - \hat{\mathbf{w}}\|_{\mathbf{L}^4}\|\hat{\mathbf{u}}\|_{\mathbf{L}^4} + \frac{S}{R_m}|\hat{\mathbf{B}}|_{\mathbf{X}}\|\hat{\mathbf{D}}_n - \hat{\mathbf{D}}\|_{\mathbf{L}^3} + \frac{1}{R_m}\|\hat{\mathbf{D}}_n - \hat{\mathbf{D}}\|_{\mathbf{L}^3}\|\hat{\mathbf{u}}\|_{\mathbf{L}^6}.$$

Due to the compact imbeddings $\mathbf{H}^1(\Omega) \Subset \mathbf{L}^4(\Omega)$ and $\mathbf{X}_0(\Omega) \Subset \mathbf{L}^3(\Omega)$, the second one obtained as a consequence of Lemma 1, and since $(\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n) \rightharpoonup (\hat{\mathbf{w}}, \hat{\mathbf{D}})$ in $\mathcal{Z}_0(\Omega) = \mathbf{Z}_0(\Omega) \times \mathbf{X}_0(\Omega)$, we know that $\hat{\mathbf{w}}_n \to \hat{\mathbf{w}}$ and $\hat{\mathbf{D}}_n \to \hat{\mathbf{D}}$ strongly in $\mathbf{L}^4(\Omega)$ and $\mathbf{L}^3(\Omega)$, respectively. Thus, considering also the strong convergence $T_n \to T$ in $L^{6/5}(\Omega)$, we obtain from the previous inequality that $|(\hat{\mathbf{u}}_n, \hat{\mathbf{B}}_n) - (\hat{\mathbf{u}}, \hat{\mathbf{B}})|_{\mathcal{W}} \to 0$. Finally, due to the equivalence of this seminorm to the usual norm in $\mathcal{W}_0(\Omega)$ the result follows. □

We have proved existence and uniqueness of solution to the linearized MHD problem. Moreover, we have proved, in the preceding lemma, that mapping $\mathcal{G}_1$ is sequentially continuous from $(\mathbf{Z}_0(\Omega) - \text{weak}) \times (L^{6/5}(\Omega) - \text{strong})$ into $(\mathbf{Z}_0(\Omega) - \text{strong})$. This property will be necessary to prove the existence of solution to the coupled problem. In the next subsection we analyze the thermal subproblem.

Remark 5. In the case of Ω being of class $\mathcal{C}^{1,1}$ we can require the compatibility condition $l \in H^{1/2}(\partial\Omega)$. The construction of $\mathbf{B}_0$ is analogous to the one presented in Section 4.2 but, in this case, the field χ is known to be in $H^2(\Omega)$. Therefore, $\mathbf{B}_0 \in \mathbf{H}^1(\Omega)$ and we can find an estimate of the form $\|\mathbf{B}_0\|_1 \leqslant \hat{\Lambda}_2\|l\|_{1/2,\partial\Omega}$.

In the case of Ω being a bounded convex polyhedron we can require the compatibility condition $l\mathbf{n} \in \mathbf{H}^{1/2}(\partial\Omega)$. A different construction of $\mathbf{B}_0$, also given in [14], guarantees that $\mathbf{B}_0 \in \mathbf{H}^1(\Omega)$ along with an estimate of the form $\|\mathbf{B}_0\|_1 \leqslant \hat{\Lambda}_2\|l\|_{1/2,\partial\Omega}$.

In both cases, since $\mathbf{B}_0 \in \mathbf{H}^1(\Omega)$ and $\mathbf{X}_0(\Omega) = \mathbf{H}^1_T(\Omega)$, the magnetic induction field satisfies $\mathbf{B} \in \mathbf{H}^1(\Omega)$.

Remark 6. In the case where Ω is $\mathcal{C}^{1,1}$ the boundary data $\mathbf{k}$ belongs to the space $\mathbf{H}_T^{-1/2}(\partial\Omega)$ and we can take as continuity constant for the linear form $F(\boldsymbol{.})$ the number

$$\lambda_F := \big(C_0^2\|\mathbf{f}_0\|_{-1}^2 + \big(C_2^2/R_m^2\big)\|\mathbf{k}\|^2_{\mathbf{H}_T^{-1/2}(\partial\Omega)}\big)^{1/2},$$

where C_2 is the constant appearing in (29).

5. Thermal problem: solution by transposition

The main difficulty in the analysis of the thermal problem are the quadratic source terms, which belong to $L^1(\Omega)$. For the treatment of this problem, we will make use of the concept of solution by transposition, as studied by Stampacchia in [21]. Throughout this section we will seek a temperature $T \in W^{1,q}(\Omega)$, with $6/5 \leqslant q < 3/2$, even if for the coupled problem we shall always work with $T \in W^{1,6/5}(\Omega)$.

The heat sources in Eq. (13) are the Joule effect, the viscous heating and the given volumetric source ψ. However, in this section we consider the thermal problem with an arbitrary heat source $f \in L^1(\Omega)$:

Given $\mathbf{u} \in \mathbf{Z}(\Omega)$*,* $f \in L^1(\Omega)$ *and* $T_d \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$*, find T satisfying*

$$\begin{cases} L_{\mathbf{u}} T = f, \\ T_{|\partial\Omega} = T_d, \end{cases} \tag{49}$$

where the linear differential operator $L_{\mathbf{u}} : W^{1,6/5}(\Omega) \to \mathcal{D}'(\Omega)$ is defined by

$$L_{\mathbf{u}} T := -\frac{1}{P_r R_e} \Delta T + \mathbf{u} \cdot \mathbf{grad}\, T. \tag{50}$$

We first recall the results given in [21] for the analysis of the problem (49) with L^1 source and homogeneous Dirichlet boundary condition. Later on we shall give a proper meaning to the solution of such problem with L^1 sources and non-homogeneous boundary condition.

We define the bilinear form $a_{\mathbf{u}} : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ as

$$a_{\mathbf{u}}(T, z) := \frac{1}{P_r R_e} \int_\Omega \mathbf{grad}\, T \cdot \mathbf{grad}\, z \, d\mathbf{x} + \int_\Omega \mathbf{u} \cdot \mathbf{grad}\, T z \, d\mathbf{x}. \tag{51}$$

The operator $L_{\mathbf{u}} \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ is associated with the bilinear form $a_{\mathbf{u}}(\cdot,\cdot)$ according to the equality,

$$\langle L_{\mathbf{u}} T, z \rangle_{H^{-1}, H_0^1} = a_{\mathbf{u}}(T, z) \quad \forall T, z \in H_0^1(\Omega).$$

If we introduce the bilinear form $a_{\mathbf{u}}^*(T, z) := a_{\mathbf{u}}(z, T)$ and denote by $L_{\mathbf{u}}^*$ its associated operator, which is the formal adjoint of $L_{\mathbf{u}}$, since $\mathbf{u} \in \mathbf{Z}(\Omega)$ it is easily seen that

$$L_{\mathbf{u}}^* T = -\frac{1}{P_r R_e} \Delta T - \operatorname{div}(\mathbf{u} T) = -\frac{1}{P_r R_e} \Delta T - \mathbf{u} \cdot \mathbf{grad}\, T = L_{-\mathbf{u}} T. \tag{52}$$

Let us define the Green's operator $G_{\mathbf{u}} : H^{-1}(\Omega) \to H_0^1(\Omega)$ which maps any $f \in H^{-1}(\Omega)$ into the unique element $G_{\mathbf{u}} f \in H_0^1(\Omega)$ such that $L_{\mathbf{u}}(G_{\mathbf{u}} f) = f$ in $H^{-1}(\Omega)$. Analogously, let us define the Green's operator $G_{\mathbf{u}}^*$ associated to $L_{\mathbf{u}}^*$. From [21, Th. 4.2] we know that

$$G_{\mathbf{u}}^* \in \mathcal{L}\big(W^{-1,p}(\Omega), L^\infty(\Omega)\big) \quad \forall p > 3,$$

denoting by $G_{\mathbf{u}}^{*t}$ its adjoint operator and, since $L^1(\Omega) \subset (L^\infty(\Omega))'$, it can be considered as $G_{\mathbf{u}}^{*t} : L^1(\Omega) \to W^{1,q}(\Omega)$ with $q = p' < 3/2$.

Given $f \in L^1(\Omega)$ the solution by transposition to the homogeneous Dirichlet problem

$$\begin{cases} L_{\mathbf{u}} T = f & \text{in } \Omega, \\ T = 0 & \text{on } \partial\Omega, \end{cases} \tag{53}$$

is $T = G_{\mathbf{u}}^{*t} f$. For any $q \in (1, 3/2)$, it is characterized by

$$\begin{cases} T \in W_0^{1,q}(\Omega), \\ \displaystyle\int_\Omega T \varphi \, d\mathbf{x} = \int_\Omega f \big(G_{\mathbf{u}}^* \varphi\big) \, d\mathbf{x} \quad \forall \varphi \in \mathcal{D}(\Omega), \end{cases} \tag{54}$$

which is equivalent to

$$\begin{cases} T \in W_0^{1,q}(\Omega), \\ \displaystyle\int_\Omega T \big(L_{\mathbf{u}}^* \psi\big) \, d\mathbf{x} = \int_\Omega f \psi \, d\mathbf{x} \quad \forall \psi \in H_0^1(\Omega) \cap L^\infty(\Omega) \text{ such that } L_{\mathbf{u}}^* \psi \in \mathcal{D}(\Omega). \end{cases} \tag{55}$$

In fact, $T \in \bigcap_{1<q<3/2} W_0^{1,q}(\Omega)$. Moreover, from (52) we know that $G_{\mathbf{u}}^* = G_{-\mathbf{u}}$.

In order to introduce the concept of solution by transposition with non-homogeneous boundary condition, let us first consider the homogeneous Dirichlet problem with right-hand side $f \in L^1(\Omega) + H^{-1}(\Omega)$. We have two different kinds of solution: the standard weak solution for $f \in H^{-1}(\Omega)$ and the solution by transposition for $f \in L^1(\Omega)$. These two solutions coincide in the intersection, i.e., for $f \in L^1(\Omega) \cap H^{-1}(\Omega)$ it holds $G_{\mathbf{u}} f = G_{\mathbf{u}}^{*t} f$. Indeed, let us denote $T = G_{\mathbf{u}} f \in H_0^1(\Omega)$ the standard weak solution. We must prove that T also satisfies (54). It is obvious that $T \in H_0^1(\Omega) \subset W_0^{1,q}(\Omega)$ for any $q < 3/2$. Moreover, for any $\varphi \in \mathcal{D}(\Omega)$ we have

$$\int_\Omega T \varphi \, d\mathbf{x} = a_{\mathbf{u}}\big(T, G_{\mathbf{u}}^* \varphi\big) = \big\langle f, G_{\mathbf{u}}^* \varphi \big\rangle_{H^{-1}, H_0^1} = \int_\Omega f \big(G_{\mathbf{u}}^* \varphi\big) \, d\mathbf{x}, \tag{56}$$

because $f \in L^1(\Omega) \cap H^{-1}(\Omega)$ and $G_{\mathbf{u}}^* \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Therefore, we have proved that $G_{\mathbf{u}} f = G_{\mathbf{u}}^{*t} f$ for any $f \in L^1(\Omega) \cap H^{-1}(\Omega)$.

Now, let $f \in L^1(\Omega) + H^{-1}(\Omega)$, which is expressed as $f = f_1 + f_2$, with $f_1 \in L^1(\Omega)$ and $f_2 \in H^{-1}(\Omega)$. We define the solution to the corresponding problem as $S_{\mathbf{u}} f := G^{*t}_{\mathbf{u}} f_1 + G_{\mathbf{u}} f_2 \in W^{1,q}_0(\Omega)$. Thanks to the above result this solution is independent of the decomposition chosen for f.

Now we address the problem (49) with L^1 sources and non-homogeneous boundary condition $T_d \in H^{1/2}(\partial\Omega)$. Let $\widetilde{T}_d \in H^1(\Omega)$ be a lifting of T_d. By writing $T = \widetilde{T}_d + \hat{T}$, problem (49) is transformed into a problem like (53) for $\hat{T}$ but with source term $f - L_{\mathbf{u}}\widetilde{T}_d \in L^1(\Omega) + H^{-1}(\Omega)$. Thus we define the solution by transposition to problem (49) as

$$T = \widetilde{T}_d + S_{\mathbf{u}}(f - L_{\mathbf{u}}\widetilde{T}_d),$$

which is easily seen to be independent of the choice of $\widetilde{T}_d$. Note that $T \in W^{1,q}(\Omega)$ $\forall q \in (1, 3/2)$.

In order to construct the solution to problem (49) by transposition, we choose the lifting $\widetilde{T}_d$ as the unique function in $H^1(\Omega)$ such that

$$\begin{cases} L_{\mathbf{u}}\widetilde{T}_d = 0 & \text{in } \Omega, \\ \widetilde{T}_d = T_d & \text{on } \partial\Omega. \end{cases} \tag{57}$$

Using standard arguments, we obtain

$$\|\widetilde{T}_d\|_1 \leqslant C\big(1 + \|\mathbf{u}\|_1\big)\|T_d\|_{1/2,\partial\Omega}, \tag{58}$$

with C a constant depending on Ω and the non-dimensional numbers P_r and R_e.

Assuming $T_d \in L^\infty(\partial\Omega)$ and due to Theorems 3.6 and 3.7 in [21] we also have

$$\|\widetilde{T}_d\|_{L^\infty(\Omega)} \leqslant \|T_d\|_{L^\infty(\partial\Omega)}. \tag{59}$$

Let us now introduce the mapping, $G_D : \mathbf{Z}(\Omega) \to H^1(\Omega)$ defined as $G_D(\mathbf{u}) = \widetilde{T}_d$. Using (58) it is easily proved that G_D is continuous.

Now we focus on the analysis of problem (53), whose solution is given by $\hat{T} = G^{*t}_{\mathbf{u}} f$.

Proposition 9. *Given $f \in L^1(\Omega)$, $\mathbf{u} \in \mathbf{Z}(\Omega)$ and $L_{\mathbf{u}}$ the operator defined in* (50)*, then the solution $\hat{T}$ to problem* (53) *satisfies*

$$\|\hat{T}\|_{1,q} \leqslant K\|f\|_{L^1}, \tag{60}$$

with $K \equiv K(q)$ a constant independent of the velocity $\mathbf{u}$ and of the right-hand side f.

Proof. It consists on checking the steps of the proof of Theorem 4.1 in [21] for the case $N = 3$. The constant K has the expression

$$K = C(q) S P_r R_e 2^{\frac{1/2-1/q'}{1/3-1/q'}}\, meas(\Omega)^{1/3-1/q'}\sqrt{3}, \tag{61}$$

with $C(q)$ and S the constants appearing in (18) and (20), respectively. □

Let the mapping $\widetilde{G} : \mathbf{Z}(\Omega) \times L^1(\Omega) \to W^{1,q}_0(\Omega)$ be defined as $\widetilde{G}(\mathbf{u}, f) := G^{*t}_{\mathbf{u}} f$. The following lemma can be easily proved

Lemma 10. *If $\mathbf{u}_n \to \mathbf{u}$ strongly in $\mathbf{Z}(\Omega)$ and $g \in L^2(\Omega)$, then $\widetilde{G}(\mathbf{u}_n, g) \to \widetilde{G}(\mathbf{u}, g)$ strongly in $W^{1,q}(\Omega)$.*

Proposition 11. *The mapping $\widetilde{G}$ is continuous in $\mathbf{Z}(\Omega) \times L^1(\Omega)$.*

Proof. Let $\mathbf{u}_n \to \mathbf{u}$ strongly in $\mathbf{Z}(\Omega)$ and $f_n \to f$ strongly in $L^1(\Omega)$. First, we have

$$\widetilde{G}(\mathbf{u}_n, f_n) - \widetilde{G}(\mathbf{u}, f) = \widetilde{G}(\mathbf{u}_n, f_n) - \widetilde{G}(\mathbf{u}_n, f) + \widetilde{G}(\mathbf{u}_n, f) - \widetilde{G}(\mathbf{u}, f), \tag{62}$$

and using Proposition 9 we know that $\widetilde{G}(\mathbf{u}_n, f_n) - \widetilde{G}(\mathbf{u}_n, f)$ tends to zero in $W^{1,q}_0(\Omega)$. Next, for any $k > 0$ we consider the truncated function $\tau_k(f)$ where $\tau_k(x) = \min(k, \max(x, -k))$. It is obvious that

$$\widetilde{G}(\mathbf{u}_n, f) - \widetilde{G}(\mathbf{u}, f) = \widetilde{G}(\mathbf{u}_n, f) - \widetilde{G}\big(\mathbf{u}_n, \tau_k(f)\big) + \widetilde{G}\big(\mathbf{u}_n, \tau_k(f)\big) - \widetilde{G}\big(\mathbf{u}, \tau_k(f)\big) + \widetilde{G}\big(\mathbf{u}, \tau_k(f)\big) - \widetilde{G}(\mathbf{u}, f),$$

and from Proposition 9 we obtain

$$\big\|\widetilde{G}(\mathbf{u}_n, f) - \widetilde{G}(\mathbf{u}, f)\big\|_{1,q} \leqslant 2K\|f - \tau_k f\|_{L^1} + \big\|\widetilde{G}\big(\mathbf{u}_n, \tau_k(f)\big) - \widetilde{G}\big(\mathbf{u}, \tau_k(f)\big)\big\|_{1,q}.$$

The first term in the right-hand side tends to zero as $k \to \infty$ due to the Lebesgue dominated converge theorem, whereas, for fixed $k > 0$, the second term tends to zero as $n \to \infty$ due to Lemma 10. □

Remark 7. Owing to Theorems 1 and 3 in [5] (see also [6]), the problem (53) has a unique renormalized solution. Taking into account the continuity of the transposition solution with respect to the right-hand side (that is, the continuity of the operator $G_{\mathbf{u}}^{*t}: L^1(\Omega) \to W^{1,q}(\Omega)$ with $q < 3/2$) and the usual procedure to prove the existence of a renormalized solution, it is clear that the transposition solution $G_{\mathbf{u}}^{*t} f$ is a renormalized solution. Since this one is unique, both kinds of solution coincide.

6. Coupled problem

In order to prove the existence of a solution to our coupled problem via a fixed point theorem, a mapping from $\mathcal{Z}_0(\Omega)$ into itself will be introduced, and then we will prove the existence of a fixed point for that mapping. To do that, we first introduce the mapping $\mathcal{G}_2 : \mathbf{Z}_0(\Omega) \times L^1(\Omega) \to W^{1,6/5}(\Omega)$ defined as $\mathcal{G}_2(\hat{\mathbf{w}}, f) := \widetilde{G}(\mathbf{w}, f) + G_D(\mathbf{w})$, with $\mathbf{w} = \hat{\mathbf{w}} + \mathbf{u}_0$, and mappings G_D and $\widetilde{G}$ have been introduced above. We also introduce $\mathcal{G}_3 : \mathcal{Z}_0(\Omega) \to L^1(\Omega)$ defined as $\mathcal{G}_3((\hat{\mathbf{w}}, \hat{\mathbf{D}})) := \frac{E_c}{R_e}[\frac{H_a^2}{R_m^2}|\mathbf{curl}\,\hat{\mathbf{D}}|^2 + \frac{1}{2}|\mathbf{grad}\,\mathbf{w} + \mathbf{grad}\,\mathbf{w}^t|^2] + \psi$, with $\mathbf{w}$ defined as before. Thus, $\mathcal{G}_3$ maps any pair $(\hat{\mathbf{w}}, \hat{\mathbf{D}})$ into its corresponding heat source.

To find a solution of our problem it suffices to find a fixed point of the mapping $\mathcal{G} : \mathcal{Z}_0(\Omega) \to \mathcal{Z}_0(\Omega)$ defined as $\mathcal{G}((\hat{\mathbf{w}}, \hat{\mathbf{D}})) := \mathcal{G}_1((\hat{\mathbf{w}}, \hat{\mathbf{D}}), \mathcal{G}_2(\hat{\mathbf{w}}, \mathcal{G}_3((\hat{\mathbf{w}}, \hat{\mathbf{D}}))))$, with $\mathcal{G}_1$ the mapping introduced in Section 4.3.

Let $\mathcal{E}_e(\mathbf{u}_0, \mathbf{B}_0, T_d, \mathbf{k}, \mathbf{f}_0, \psi)$ be defined, depending on the boundary and source data, by

$$\mathcal{E}_e(\mathbf{u}_0, \mathbf{B}_0, T_d, \mathbf{k}, \mathbf{f}_0, \psi) = \lambda_F + \lambda_G\big(8Kk_{f2}|\mathbf{u}_0|_1^2 + K\|\psi\|_{L^1} + meas(\Omega)^{5/6}\|T_d\|_{L^\infty(\partial\Omega)}\big) + \frac{1}{H_a^2}|\mathbf{u}_0|_1 + \frac{1}{N}\|\mathbf{u}_0\|_{\mathbf{L}^4}^2 + \frac{1}{R_m}\|\mathbf{B}_0\|_{\mathbf{L}^3}\|\mathbf{u}_0\|_{\mathbf{L}^6},$$

where $k_f = \max\{k_{f1}, 8k_{f2}\}$, with $k_{f1} = E_c H_a^2/(R_e R_m^2)$, $k_{f2} = E_c/(2R_e)$, and K is the constant appearing in Proposition 9 for $q = 6/5$. Unless explicitly addressed, in the sequel we will always take $K \equiv K(6/5)$.

We have the following lemma:

Lemma 12. *If $(\mathbf{u}_0, \mathbf{B}_0) \in \mathcal{Z}_0(\Omega)$, with $\mathbf{curl}\,\mathbf{B}_0 = \mathbf{0}$, $T_d \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$ and $\psi \in L^1(\Omega)$, assuming that $\mathbf{u}_0$ satisfies* (48) *and*

$$\mathcal{E}_e(\mathbf{u}_0, \mathbf{B}_0, T_d, \mathbf{k}, \mathbf{f}_0, \psi) \leqslant \frac{\alpha^2}{4\lambda_G K k_f}, \tag{63}$$

there exists a constant $R > 0$ such that if $|(\hat{\mathbf{w}}, \hat{\mathbf{D}})|_{\mathcal{W}} \leqslant R$ then $|\mathcal{G}((\hat{\mathbf{w}}, \hat{\mathbf{D}}))|_{\mathcal{W}} \leqslant R$.

Proof. Let $R > 0$ be a real number and $(\hat{\mathbf{w}}, \hat{\mathbf{D}}) \in \mathcal{Z}_0(\Omega)$ such that $|(\hat{\mathbf{w}}, \hat{\mathbf{D}})|_{\mathcal{W}} \leqslant R$. According to the definition of k_{f1}, k_{f2} and k_f, we easily get

$$\big\|\mathcal{G}_3((\hat{\mathbf{w}}, \hat{\mathbf{D}}))\big\|_{L^1} \leqslant k_f\big|(\hat{\mathbf{w}}, \hat{\mathbf{D}})\big|_{\mathcal{W}}^2 + 8k_{f2}|\mathbf{u}_0|_1^2 + \|\psi\|_{L^1}. \tag{64}$$

Let us introduce $f := \mathcal{G}_3((\hat{\mathbf{w}}, \hat{\mathbf{D}}))$; then

$$\big\|\mathcal{G}_2(\hat{\mathbf{w}}, f)\big\|_{L^{6/5}} \leqslant \big\|\widetilde{G}(\mathbf{w}, f)\big\|_{L^{6/5}} + \big\|G_D(\mathbf{w})\big\|_{L^{6/5}} \leqslant K\|f\|_{L^1} + meas(\Omega)^{5/6}\|T_d\|_{L^\infty(\partial\Omega)}, \tag{65}$$

with $K \equiv K(6/5) = 4C(6/5)SP_rR_e\,meas(\Omega)^{1/6}\sqrt{3}$, as given in Proposition 9. The value $meas(\Omega)^{5/6}$ appears as a consequence of (59).

For the third step, since we have assumed that (48) is valid, denoting $T = \mathcal{G}_2(\hat{\mathbf{w}}, f)$ and recalling Proposition 7 we get

$$\alpha\big|\mathcal{G}_1\big((\hat{\mathbf{w}}, \hat{\mathbf{D}}), T\big)\big|_{\mathcal{W}} \leqslant \lambda_F + \lambda_G\|T\|_{L^{6/5}} + \frac{1}{H_a^2}|\mathbf{u}_0|_1 + \frac{1}{N}\|\mathbf{u}_0\|_{\mathbf{L}^4}^2 + \frac{1}{R_m}\|\mathbf{B}_0\|_{\mathbf{L}^3}\|\mathbf{u}_0\|_{\mathbf{L}^6}. \tag{66}$$

Joining the three inequalities we obtain

$$\begin{aligned}
\alpha\big|\mathcal{G}((\hat{\mathbf{w}}, \hat{\mathbf{D}}))\big|_{\mathcal{W}} &\leqslant \lambda_F + \lambda_G\|T\|_{L^{6/5}} + \frac{1}{H_a^2}|\mathbf{u}_0|_1 + \frac{1}{N}\|\mathbf{u}_0\|_{\mathbf{L}^4}^2 + \frac{1}{R_m}\|\mathbf{B}_0\|_{\mathbf{L}^3}\|\mathbf{u}_0\|_{\mathbf{L}^6} \\
&\leqslant \lambda_F + \lambda_G\big(K\|f\|_{L^1} + meas(\Omega)^{5/6}\|T_d\|_{L^\infty(\partial\Omega)}\big) + \frac{1}{H_a^2}|\mathbf{u}_0|_1 + \frac{1}{N}\|\mathbf{u}_0\|_{\mathbf{L}^4}^2 + \frac{1}{R_m}\|\mathbf{B}_0\|_{\mathbf{L}^3}\|\mathbf{u}_0\|_{\mathbf{L}^6} \\
&\leqslant \lambda_F + \lambda_G\big[K\big(k_f\big|(\hat{\mathbf{w}}, \hat{\mathbf{D}})\big|_{\mathcal{W}}^2 + 8k_{f2}|\mathbf{u}_0|_1^2 + \|\psi\|_{L^1}\big) + meas(\Omega)^{5/6}\|T_d\|_{L^\infty(\partial\Omega)}\big] \\
&\quad + \frac{1}{H_a^2}|\mathbf{u}_0|_1 + \frac{1}{N}\|\mathbf{u}_0\|_{\mathbf{L}^4}^2 + \frac{1}{R_m}\|\mathbf{B}_0\|_{\mathbf{L}^3}\|\mathbf{u}_0\|_{\mathbf{L}^6} = k_0 + k_2\big|(\hat{\mathbf{w}}, \hat{\mathbf{D}})\big|_{\mathcal{W}}^2,
\end{aligned} \tag{67}$$

with $k_0 = \mathcal{E}_e(\mathbf{u}_0, \mathbf{B}_0, T_d, \mathbf{k}, \mathbf{f}_0, \psi) \geqslant 0$, which depends on the source and boundary data, and $k_2 = \lambda_G K k_f > 0$, which only depends on the physical parameters and on the domain Ω.

We have to prove that there exists a certain constant $R > 0$ such that $|(\hat{\mathbf{w}}, \hat{\mathbf{D}})|_{\mathcal{W}} \leqslant R$ implies $|\mathcal{G}((\hat{\mathbf{w}}, \hat{\mathbf{D}}))|_{\mathcal{W}} \leqslant R$. In view of the last inequality, it is enough to prove that there exists a constant $R > 0$ such that

$$k_0 - \alpha R + k_2 R^2 \leqslant 0. \tag{68}$$

The case $k_0 = 0$ is trivial, thus we assume $k_0 > 0$. The roots of the corresponding quadratic equation are given by

$$R_\pm = \frac{\alpha \pm \sqrt{\alpha^2 - 4k_0k_2}}{2k_2}, \tag{69}$$

and since $R_+R_- = k_0/k_2 > 0$, the two roots are either positive, or negative or complex. From the expression of k_0 and k_2 it is clear that (63) is equivalent to

$$\Delta = \alpha^2 - 4k_0k_2 \geqslant 0, \tag{70}$$

and, since we have assumed (48), we know that $\alpha > 0$. Hence the two conditions guarantee that the equation has two real positive roots and therefore we are able to find a real positive constant R such that $\mathcal{G}(\overline{B(0,R)}) \subset \overline{B(0,R)}$. □

Now we are in a position to prove the main result of this section, which gives the existence of a solution to the coupled thermal-magnetohydrodynamic problem.

Theorem 13. *Under the hypotheses of Lemma* 12 *the mapping $\mathcal{G}$ has at least one fixed point.*

Proof. The result is a consequence of applying Schauder fixed point theorem to mapping $\mathcal{G}$. Under the assumed hypotheses we have already proved in Lemma 12 that there exists a constant $R > 0$ such that $\mathcal{G}$ maps the ball $\overline{B(0,R)} \subset \mathcal{Z}_0(\Omega)$ into itself. Moreover, mapping $\mathcal{G}$ is continuous because of the continuity of mappings $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$. In order to apply the Schauder fixed point theorem, we must prove the compactness of $\mathcal{G}$.

Let $B \subset \mathcal{Z}_0(\Omega)$ be a bounded set. We must prove that for any sequence $\{(\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n)\} \subset B$, $\{\mathcal{G}((\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n))\}$ has a convergent subsequence. Since $\mathcal{Z}_0(\Omega)$ is a reflexive Banach space, there exists a subsequence still denoted by $\{(\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n)\}$ such that $(\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n) \rightharpoonup (\hat{\mathbf{w}}, \hat{\mathbf{D}})$ weakly in $\mathcal{Z}_0(\Omega)$. Due to the definition of $\mathcal{G}_3$, the sequence $f_n = \mathcal{G}_3((\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n))$ is bounded in $L^1(\Omega)$. Moreover, due to Proposition 9 we also have that $T_n = \mathcal{G}_2(\hat{\mathbf{w}}_n, f_n)$ is bounded in $W^{1,6/5}(\Omega)$. Hence, there is a subsequence that we still denote by T_n such that $T_n \to T$ strongly in $L^{6/5}(\Omega)$. Finally, due to the result proved in Lemma 8, it holds that $\mathcal{G}_1((\hat{\mathbf{w}}_n, \hat{\mathbf{D}}_n), T_n) = (\hat{\mathbf{u}}_n, \hat{\mathbf{B}}_n) \to \mathcal{G}_1((\hat{\mathbf{w}}, \hat{\mathbf{D}}), T) = (\hat{\mathbf{u}}, \hat{\mathbf{B}})$ strongly in $\mathcal{Z}_0(\Omega)$. Thus $\mathcal{G}(B)$ is relatively compact, which completes the proof. □

7. Steady MHD equations without using the Boussinesq approximation

In the previous section we have considered the stationary MHD equations using the Boussinesq approximation. Under this approximation we were constrained to impose very strict conditions on the given data to ensure the existence of a solution to our problem (see Lemma 12). As a first step, in order to study a more complicated model and following some of the ideas appearing in [10], we propose a different mathematical model for which, instead of using Boussinesq approximation, we assume that the density appearing in the gravity force is a function of temperature satisfying certain properties. For this density function we are able to obtain an *a priori* bound for the solution of the model, which will lead to prove the existence of solution under less severe constraints on the data.

7.1. Mathematical model

We consider the steady MHD equations coupled with the heat transfer equation. Following some ideas appeared in [10], instead of the Boussinesq approximation we consider that density is constant in the left-hand side terms and that it is a function of temperature in the gravity force term: $\rho = \hat{\rho}(T)$. Moreover, we assume that function $\hat{\rho} : (0, +\infty) \to (0, +\infty)$ is strictly positive, continuous and non-increasing. Notice that these assumptions do not hold for the Boussinesq approximation. The equations, before the non-dimensionalization, are the same that appeared in Section 2.1, except (6) which is replaced by

$$-\eta\Delta\mathbf{u} + \rho(\mathbf{grad}\,\mathbf{u})\mathbf{u} + \mathbf{grad}\,p - \frac{1}{\mu}(\mathbf{curl}\,\mathbf{B}) \times \mathbf{B} = \mathbf{f}_0 + \hat{\rho}(T)\mathbf{g}. \tag{71}$$

For the non-dimensionalization $\mathcal{B}$, u and $\mathcal{L}$ are as in Section 2.1. All the fields are normalized as in that section, but replacing $\Delta\mathcal{T}$ by $\mathcal{T}$, which stands for the typical value of temperature. The density function $\hat{\rho}$ is normalized by $\sigma u \mathcal{B}^2$. After this normalization, the buoyancy term is expressed in the form $\hat{\varrho}(\hat{T}) = \frac{1}{\sigma u \mathcal{B}^2}\hat{\rho}(T) = \frac{1}{\sigma u \mathcal{B}^2}\hat{\rho}(\mathcal{T}\hat{T})$. Now, maintaining the same notation for the normalized fields we arrive at the following non-dimensionalized system of equations, which holds in Ω:

$$\frac{1}{R_m}\,\mathbf{curl}(\mathbf{curl}\,\mathbf{B}) - \mathbf{curl}(\mathbf{u}\times\mathbf{B}) = \mathbf{0}, \tag{72}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{73}$$

$$-\frac{1}{H_a^2}\Delta\mathbf{u} + \frac{1}{N}(\mathbf{grad}\,\mathbf{u})\mathbf{u} + \mathbf{grad}\,p - \frac{1}{R_m}(\mathbf{curl}\,\mathbf{B})\times\mathbf{B} = \mathbf{f}_0 + \hat{\rho}(T)\mathbf{g}, \tag{74}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{75}$$

$$-\frac{1}{P_r R_e}\Delta T + \mathbf{u}\cdot\mathbf{grad}\,T = \frac{1}{R_e}\frac{u^2}{c_p\mathcal{T}}\left[\frac{H_a^2}{R_m^2}|\mathbf{curl}\,\mathbf{B}|^2 + \frac{1}{2}\left|\mathbf{grad}\,\mathbf{u} + \mathbf{grad}\,\mathbf{u}^t\right|^2\right] + \psi. \tag{76}$$

This system of partial differential equations is completed with boundary conditions (14)–(17), but noticing that now both T and T_d represent normalized temperatures, and not the difference with respect to a reference temperature.

The compatibility and regularity conditions for the given data are the same as those previously introduced in Section 3.1, but we also assume the heat source ψ to be non-negative and the temperature on the boundary to be strictly positive, i.e.

$$\psi \in L^1(\Omega), \qquad \psi(\mathbf{x}) \geqslant 0 \quad \text{a.e. in } \Omega, \tag{77}$$

$$T_d \in H^{1/2}(\partial\Omega)\cap L^\infty(\partial\Omega), \qquad T_d(\mathbf{x}) \geqslant T_{\min} > 0 \quad \text{a.e. on } \partial\Omega. \tag{78}$$

Moreover, the non-dimensionalized response function $\hat{\rho}$ satisfies the properties we have mentioned above.

7.2. An a priori bound for the solutions

In the equations considering the Boussinesq approximation it was not possible to obtain an *a priori* bound for the solution, due to the buoyancy force term appearing in the right-hand side of the Navier–Stokes equations. Now, since Joule effect and viscous heating are non-negative, due to condition (77) and as a consequence of Theorem 3.7 in [21], we know that any temperature T solution to (76) with boundary condition (17) satisfies

$$\operatorname*{ess\,inf}_{\mathbf{x}\in\Omega} T(\mathbf{x}) \geqslant \operatorname*{ess\,inf}_{\mathbf{x}\in\partial\Omega} T_d(\mathbf{x}) = T_{\min} > 0. \tag{79}$$

Hence, since $\hat{\rho}$ is continuous and non-increasing, we have

$$0 < \hat{\rho}\big(T(\mathbf{x})\big) \leqslant \hat{\rho}(T_{\min}) = \hat{\rho}_{\max}, \tag{80}$$

and the product $\hat{\rho}(T)\mathbf{g}$ satisfies

$$\big\|\hat{\rho}(T)\mathbf{g}\big\|_{-1} \leqslant g\big\|\hat{\rho}(T)\big\|_{L^\infty}\, meas(\Omega)^{1/2} \leqslant g\hat{\rho}_{\max}\, meas(\Omega)^{1/2},$$

where $g = |\mathbf{g}|$ is the modulus of gravity acceleration.

Reasoning as in Proposition 6 it can be seen that $(\mathbf{u},\mathbf{B}) \in \mathcal{Z}(\Omega)$ is a solution to (72)–(75) along with boundary conditions (14)–(16) if and only if it is a solution to the problem:

Given $\mathbf{f}_0$, $\mathbf{u}_d$, l *and* $\mathbf{k}$ *satisfying* (30)–(33) *and* $T:\Omega\to\mathbb{R}$ *a measurable function such that* $T(\mathbf{x}) \geqslant T_{\min} > 0$ *a.e. in* Ω, *find*

$$(\mathbf{u},\mathbf{B}) \in \mathcal{Z}(\Omega), \tag{81}$$

satisfying

$$\begin{aligned} &a_0(\mathbf{u},\mathbf{v}) + a_1(\mathbf{B},\mathbf{C}) + c_0(\mathbf{u},\mathbf{u},\mathbf{v}) - c_1(\mathbf{B},\mathbf{B},\mathbf{v}) + c_1(\mathbf{C},\mathbf{B},\mathbf{u}) \\ &\quad = F((\mathbf{v},\mathbf{C})) + \int_\Omega \hat{\rho}(T)\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} \quad \forall(\mathbf{v},\mathbf{C}) \in \mathcal{Z}_0(\Omega), \end{aligned} \tag{82}$$

$$\mathbf{u}_{|\partial\Omega} = \mathbf{u}_d, \qquad (\mathbf{B}\cdot\mathbf{n})_{|\partial\Omega} = l. \tag{83}$$

We notice that this problem is the same as (36)–(38) except for the gravity force term.

Using the splittings $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}_0$, $\mathbf{B} = \hat{\mathbf{B}} + \mathbf{B}_0$ the problem can be reduced to another one with homogeneous boundary conditions which, taking into account that $\mathbf{B}_0$ is irrotational, can be rewritten in the form:

Given $\mathbf{u}_0 \in \mathbf{Z}(\Omega)$, $\mathbf{B}_0 \in \mathbf{Y}(\Omega)$ *satisfying* (42)–(43), *and* $T:\Omega\to\mathbb{R}$ *a measurable function such that* $T(\mathbf{x}) \geqslant T_{\min} > 0$ *a.e. in* Ω, *find*

$$(\hat{\mathbf{u}},\hat{\mathbf{B}}) \in \mathcal{Z}_0(\Omega) \tag{84}$$

such that

$$\begin{aligned}
&a_0(\hat{\mathbf{u}},\mathbf{v}) + a_1(\hat{\mathbf{B}},\mathbf{C}) + c_0(\hat{\mathbf{u}},\hat{\mathbf{u}},\mathbf{v}) + c_0(\hat{\mathbf{u}},\mathbf{u}_0,\mathbf{v}) + c_0(\mathbf{u}_0,\hat{\mathbf{u}},\mathbf{v}) - c_1(\hat{\mathbf{B}},\hat{\mathbf{B}},\mathbf{v}) \\
&\quad - c_1(\hat{\mathbf{B}},\mathbf{B}_0,\mathbf{v}) + c_1(\mathbf{C},\hat{\mathbf{B}},\hat{\mathbf{u}}) + c_1(\mathbf{C},\hat{\mathbf{B}},\mathbf{u}_0) + c_1(\mathbf{C},\mathbf{B}_0,\hat{\mathbf{u}}) \\
&\quad = F((\mathbf{v},\mathbf{C})) + \int_\Omega \hat{\rho}(T)\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} - a_0(\mathbf{u}_0,\mathbf{v}) - c_0(\mathbf{u}_0,\mathbf{u}_0,\mathbf{v}) - c_1(\mathbf{C},\mathbf{B}_0,\mathbf{u}_0) \quad \forall(\mathbf{v},\mathbf{C}) \in \mathcal{Z}_0(\Omega). \qquad (85)
\end{aligned}$$

The next proposition gives us an *a priori* bound for any solution of the MHD problem. The proof is analogous to the one of Proposition 7.

Proposition 14. *If $\mathbf{u}_0 \in \mathbf{Z}(\Omega)$ is a lifting of the boundary condition $\mathbf{u}_d$ satisfying* (48) *then, for any solution $(\hat{\mathbf{u}},\hat{\mathbf{B}})$ to problem* (84)–(85) *the following inequality holds*:

$$\big|(\hat{\mathbf{u}},\hat{\mathbf{B}})\big|_{\mathcal{W}} \leqslant \frac{k_0}{\alpha}, \qquad (86)$$

with $k_0 := (\lambda_F + \lambda_{\hat{G}} + \frac{1}{H_a^2}|\mathbf{u}_0|_1 + \frac{1}{N}\|\mathbf{u}_0\|^2_{\mathbf{L}^4} + \frac{1}{R_m}\|\mathbf{B}_0\|_{\mathbf{L}^3}\|\mathbf{u}_0\|_{\mathbf{L}^6})$ and $\lambda_{\hat{G}} := Sg\hat{\rho}_{\max}\,meas(\Omega)^{5/6}$.

From this *a priori* bound for the term $(\hat{\mathbf{u}},\hat{\mathbf{B}})$, and using inequality (59) and Proposition 9, it is also possible to find an *a priori* estimate for the temperature T in $W^{1,q}(\Omega)$, with $q < 3/2$ and, in particular, for $q = 6/5$.

7.3. Linearized version of the MHD problem

First of all, and due to the lower bound given in (79), it is convenient to introduce the convex set

$$L^{6/5}_{\min}(\Omega) := \big\{\theta \in L^{6/5}(\Omega)\colon\ \theta(\mathbf{x}) \geqslant T_{\min} \text{ a.e. in } \Omega\big\}.$$

We can now introduce the linearized version of the MHD problem, which differs from the one presented in Section 4 only in the buoyancy term. This linearized version of the problem reads:

Given $(\hat{\mathbf{w}},\hat{\mathbf{D}}) \in \mathcal{Z}_0(\Omega)$, $T \in L^{6/5}_{\min}(\Omega)$ and $(\mathbf{u}_0,\mathbf{B}_0) \in \mathcal{Z}(\Omega)$ with $\mathbf{curl}\,\mathbf{B}_0 = \mathbf{0}$, find

$$(\hat{\mathbf{u}},\hat{\mathbf{B}}) \in \mathcal{Z}_0(\Omega) \qquad (87)$$

such that

$$\begin{aligned}
&a_0(\hat{\mathbf{u}},\mathbf{v}) + a_1(\hat{\mathbf{B}},\mathbf{C}) + c_0(\hat{\mathbf{w}},\hat{\mathbf{u}},\mathbf{v}) + c_0(\mathbf{u}_0,\hat{\mathbf{u}},\mathbf{v}) + c_0(\hat{\mathbf{u}},\mathbf{u}_0,\mathbf{v}) - c_1(\hat{\mathbf{B}},\hat{\mathbf{D}},\mathbf{v}) \\
&\quad - c_1(\hat{\mathbf{B}},\mathbf{B}_0,\mathbf{v}) + c_1(\mathbf{C},\hat{\mathbf{D}},\hat{\mathbf{u}}) + c_1(\mathbf{C},\hat{\mathbf{B}},\mathbf{u}_0) + c_1(\mathbf{C},\mathbf{B}_0,\hat{\mathbf{u}}) \\
&\quad = F((\mathbf{v},\mathbf{C})) + \int_\Omega \hat{\rho}(T)\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} - a_0(\mathbf{u}_0,\mathbf{v}) - c_0(\mathbf{u}_0,\mathbf{u}_0,\mathbf{v}) - c_1(\mathbf{C},\mathbf{B}_0,\mathbf{u}_0) \quad \forall(\mathbf{v},\mathbf{C}) \in \mathcal{Z}_0(\Omega). \qquad (88)
\end{aligned}$$

In the following proposition we prove the existence of a unique solution to this problem, and give an estimate for this solution independent of the temperature field T. The proof is analogous to that of Proposition 7.

Proposition 15. *Assuming* (48)*, there exists a unique solution $(\hat{\mathbf{u}},\hat{\mathbf{B}})$ to problem* (87)–(88)*. Moreover,*

$$\big|(\hat{\mathbf{u}},\hat{\mathbf{B}})\big|_{\mathcal{W}} \leqslant \frac{k_0}{\alpha}, \qquad (89)$$

where k_0 is the constant appearing in Proposition 14*.*

We can now introduce the mapping $\hat{\mathcal{G}}_1 : \mathcal{Z}_0(\Omega) \times L^{6/5}_{\min}(\Omega) \to \mathcal{Z}_0(\Omega)$, which maps any pair $((\hat{\mathbf{w}},\hat{\mathbf{D}}),T)$ into $\hat{\mathcal{G}}_1((\hat{\mathbf{w}},\hat{\mathbf{D}}),T) := (\hat{\mathbf{u}},\hat{\mathbf{B}})$, the corresponding solution to problem (87)–(88).

Lemma 16. *Assume that* (48) *is satisfied. Let $(\hat{\mathbf{w}}_n,\hat{\mathbf{D}}_n) \rightharpoonup (\hat{\mathbf{w}},\hat{\mathbf{D}})$ weakly in $\mathcal{Z}_0(\Omega)$ and $\{T_n\} \subset L^{6/5}_{\min}(\Omega)$ such that $T_n \to T$ strongly in $L^{6/5}(\Omega)$. Then $T \in L^{6/5}_{\min}(\Omega)$ and $\hat{\mathcal{G}}_1((\hat{\mathbf{w}}_n,\hat{\mathbf{D}}_n),T_n) \to \hat{\mathcal{G}}_1((\hat{\mathbf{w}},\hat{\mathbf{D}}),T)$ strongly in $\mathcal{Z}_0(\Omega)$.*

Proof. The fact that $T \in L^{6/5}_{\min}(\Omega)$ is clear, because $L^{6/5}_{\min}(\Omega)$ is a closed subset of $L^{6/5}(\Omega)$. The rest of the proof is analogous to that of Lemma 8, but substituting the term $(G(T_n - T),\mathbf{v})_\Omega$ by $\int_\Omega (\hat{\rho}(T_n) - \hat{\rho}(T))\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}$ and taking into account that $\hat{\rho}(T_n) \to \hat{\rho}(T)$ in $L^p(\Omega)$ for any $1 < p < +\infty$. $\square$

7.4. Coupled problem

The thermal subproblem is identical to the one analyzed in Section 5. Similar to what we did in Section 6 we introduce the mappings $\hat{\mathcal{G}}_2 : \mathbf{Z}_0(\Omega) \times L^1_+(\Omega) \to W^{1,6/5}(\Omega) \cap L^{6/5}_{\min}(\Omega)$, with $\hat{\mathcal{G}}_2(\hat{\mathbf{w}}, f) := \widetilde{G}(\mathbf{w}, f) + G_D(\mathbf{w})$, and $\hat{\mathcal{G}}_3 : \mathcal{Z}_0(\Omega) \to L^1_+(\Omega)$ defined as $\hat{\mathcal{G}}_3((\hat{\mathbf{w}}, \hat{\mathbf{D}})) := \frac{1}{R_e} \frac{u^2}{c_p \mathcal{T}} [\frac{H_a^2}{R_m^2} |\mathbf{curl}\, \hat{\mathbf{D}}|^2 + \frac{1}{2} |\mathbf{grad}\, \mathbf{w} + \mathbf{grad}\, \mathbf{w}^t|^2] + \psi$, where $\mathbf{w} = \hat{\mathbf{w}} + \mathbf{u}_0$ and we have denoted $L^1_+(\Omega) = \{f \in L^1(\Omega);\ f \geqslant 0 \text{ a.e. in } \Omega\}$. The temperature $\hat{\mathcal{G}}_2(\hat{\mathbf{w}}, f)$ belongs to $L^{6/5}_{\min}(\Omega)$ thanks to (77) and (78). As in the previous case, in order to prove the existence of a solution to our problem it suffices to prove the existence of a fixed point of the mapping $\hat{\mathcal{G}} : \mathcal{Z}_0(\Omega) \to \mathcal{Z}_0(\Omega)$ defined as $\hat{\mathcal{G}}((\hat{\mathbf{w}}, \hat{\mathbf{D}})) := \hat{\mathcal{G}}_1((\hat{\mathbf{w}}, \hat{\mathbf{D}}), \hat{\mathcal{G}}_2(\hat{\mathbf{w}}, \hat{\mathcal{G}}_3((\hat{\mathbf{w}}, \hat{\mathbf{D}}))))$.

Theorem 17. *Under assumption* (48), *if* $(\mathbf{u}_0, \mathbf{B}_0) \in \mathcal{Z}_0(\Omega)$, *with* $\mathbf{curl}\, \mathbf{B}_0 = \mathbf{0}$, ψ *satisfies* (77) *and* T_d *satisfies* (78), *then the mapping* $\hat{\mathcal{G}}$ *has at least one fixed point.*

Proof. The proof is analogous to that of Theorem 13, just noticing that Proposition 15 yields $|\hat{\mathcal{G}}((\hat{\mathbf{w}}, \hat{\mathbf{D}}))|_{\mathcal{W}} \leqslant k_0/\alpha\ \forall (\hat{\mathbf{w}}, \hat{\mathbf{D}}) \in \mathcal{Z}_0(\Omega)$ and then using Lemma 16. □

Remark 8. We notice that, under assumption (48), we know that any solution to (81)–(83) belongs to the closed ball $\overline{B(0, k_0/\alpha)}$ from Proposition 14. Thus, from the previous theorem we also know that there exists at least one solution in the mentioned ball.

7.5. Existence of solution without assuming smallness of the data

The existence results of the two coupled problems analyzed above, given in Theorems 13 and 17, rely on some smallness of the given data. Now, following the ideas of [1] we can prove that, for a tangential boundary condition $\mathbf{u}_d$, it is always possible to construct a lifting $\mathbf{u}_0$ satisfying (48), which leads to an existence result for the second coupled problem independently of the size of the given data.

Lemma 18. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with Lipschitz boundary and* $\mathbf{u}_d \in \mathbf{H}^{1/2}_T(\partial\Omega)$. *Then, for every number* $\varepsilon > 0$ *there exists a vector* $\mathbf{u}_\varepsilon \in \mathbf{H}^1_T(\Omega)$ *such that,*

$$\operatorname{div} \mathbf{u}_\varepsilon = 0 \quad \text{in } \Omega, \tag{90}$$

$$\mathbf{u}_\varepsilon = \mathbf{u}_d \quad \text{on } \partial\Omega, \tag{91}$$

$$\|\mathbf{u}_\varepsilon\|_{\mathbf{L}^6} \leqslant \varepsilon. \tag{92}$$

Proof. The proof follows essentially the lines of the one of Lemma 2.2 in [1]. Let us denote by $\mathbf{u}_0$ the standard extension of $\mathbf{u}_d$ to Ω satisfying $\|\mathbf{u}_0\|_1 \leqslant \hat{\Lambda}_1 \|\mathbf{u}_d\|_{1/2,\partial\Omega}$, with $\hat{\Lambda}_1$ independent of $\mathbf{u}_d$. For each real number $\varepsilon_0 > 0$ we introduce the truncation function $\theta_{\varepsilon_0} \in \mathcal{C}^1(\overline{\Omega})$ defined as in [11, Lemma III.6.2] (see also [12, Lemma IV.2.4]), satisfying the following conditions: $|\theta_{\varepsilon_0}(\mathbf{x})| \leqslant 1$ in $\overline{\Omega}$, $\theta_{\varepsilon_0}(\mathbf{x}) = 1$ in a neighborhood of $\partial\Omega$ and $\theta_{\varepsilon_0}(\mathbf{x}) = 0$ for $\operatorname{dist}(\mathbf{x}, \partial\Omega) \geqslant 2\delta(\varepsilon_0)$, with $\delta(\varepsilon_0) = e^{-1/\varepsilon_0}$. Setting $\mathbf{w}_{\varepsilon_0} = \theta_{\varepsilon_0} \mathbf{u}_0$ it is obvious that $\mathbf{w}_{\varepsilon_0} \in \mathbf{H}^1_T(\Omega)$ and $\mathbf{w}_{\varepsilon_0} = \mathbf{u}_d$ on $\partial\Omega$. Then, due to the properties of θ_{ε_0} we know that $\mathbf{w}_{\varepsilon_0}(\mathbf{x}) \to 0$ as $\varepsilon_0 \to 0$ a.e. in Ω, and $|\mathbf{w}_{\varepsilon_0}(\mathbf{x})| \leqslant |\mathbf{u}_0(\mathbf{x})|$ a.e. in Ω. As a consequence of Lebesgue dominated convergence theorem, for any $\varepsilon > 0$ there exists $\varepsilon_1 > 0$ such that $\|\mathbf{w}_{\varepsilon_1}\|_{\mathbf{L}^6} \leqslant \frac{\varepsilon}{1+\kappa_6}$ where $\kappa_6 = \kappa_6(\Omega)$ is a suitable constant to be precised later. Since $\mathbf{w}_{\varepsilon_1} \in \mathbf{H}^1_T(\Omega)$, using the results of [11, Chapter III] there exists $\mathbf{v}_{\varepsilon_1} \in \mathbf{H}^1_0(\Omega)$ such that $\operatorname{div} \mathbf{v}_{\varepsilon_1} = \operatorname{div} \mathbf{w}_{\varepsilon_1}$ and $\|\mathbf{v}_{\varepsilon_1}\|_{\mathbf{L}^6} \leqslant \kappa_6 \|\mathbf{w}_{\varepsilon_1}\|_{\mathbf{L}^6}$. The function $\mathbf{u}_\varepsilon = \mathbf{w}_{\varepsilon_1} - \mathbf{v}_{\varepsilon_1}$ fulfills all the requirements. □

The previous lemma allows us to prove the existence of solution, for a tangential velocity boundary condition, without assuming smallness of the data.

Theorem 19. *Let* $\mathbf{f}_0$, l, $\mathbf{k}$, ψ *and* T_d *satisfying* (30), (32), (33), (77) *and* (78), *respectively. Let* $\hat{\rho} : (0, +\infty) \to (0, +\infty)$ *be continuous and non-increasing and* $\mathbf{u}_d \in \mathbf{H}^{1/2}_T(\partial\Omega)$. *Then there exists at least one solution* $((\mathbf{u}, \mathbf{B}), T)$ *to problem* (72)–(76) *with boundary conditions* (14)–(17).

Proof. It is well known that for any $\mathbf{u} \in \mathbf{L}^6(\Omega)$ we have $\|\mathbf{u}\|_{\mathbf{L}^4} \leqslant meas(\Omega)^{1/12} \|\mathbf{u}\|_{\mathbf{L}^6}$. Since $\mathbf{u}_d \in \mathbf{H}^{1/2}_T(\partial\Omega)$ we can apply Lemma 18 with $\varepsilon < \min\{meas(\Omega)^{-1/12} N/(H_a^2 S_4), 1/(R_m \kappa C_1)\}$ to construct a divergence-free field $\mathbf{u}_0 \in \mathbf{H}^1_T(\Omega)$ such that $\mathbf{u}_{0|\partial\Omega} = \mathbf{u}_d$ and satisfying (48). The result follows from Theorem 17 and the fact that any fixed point of $\hat{\mathcal{G}}$ is also a solution to the mentioned problem. □

8. Uniqueness results

In the previous sections we have proved existence results for the two models presented in this article. We are now going to prove uniqueness results under more severe restrictions on the given data and on the domain. In particular, in the sequel we will assume that Ω is a bounded domain of class $\mathcal{C}^1$.

8.1. Equivalence of the solution by transposition and the weak solution

The technique we will use to prove the uniqueness under smallness of the data will require some results of Lipschitz continuity on bounded sets for the mappings appearing in the definition of mapping $\mathcal{G}$. In particular we will make use of the Lipschitz continuity on bounded sets of the mapping $\widetilde{G}$, defined in Section 5, with respect to $\mathbf{u}$ and f. In order to prove this result it is not convenient to write the thermal problem with L^1 sources in the form of (55), because the test functions depend on the velocity $\mathbf{u}$. Instead, we will rewrite the problem in a weak formulation, using a theorem presented in [20] to show that both formulations are equivalent. This theorem requires a smooth domain, therefore from now on we assume that Ω is bounded and of class $\mathcal{C}^1$. The result is proved for a general temperature $T \in W^{1,q}(\Omega)$ with $q < 3/2$, but later on we will only require $T \in W^{1,6/5}(\Omega)$.

For $\mathbf{u} \in H^1(\Omega)$ we have already introduced the bilinear form $a_{\mathbf{u}} : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ in (51). Now, for $6/5 \leqslant q < 3/2$ we introduce the bilinear form $a_{q,\mathbf{u}} : W_0^{1,q}(\Omega) \times W_0^{1,q'}(\Omega) \to \mathbb{R}$ defined as

$$a_{q,\mathbf{u}}(T, z) := \frac{1}{P_r R_e} \int_\Omega \mathbf{grad}\, T \cdot \mathbf{grad}\, z \,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{u} \cdot \mathbf{grad}\, T\, z \,\mathrm{d}\mathbf{x}.$$

We notice that the bilinear forms $a_{\mathbf{u}}(\cdot,\cdot)$ and $a_{q,\mathbf{u}}(\cdot,\cdot)$ only differ in the spaces on which they are defined. Moreover, it holds that $a_{\mathbf{u}}(T, z) = a_{q,\mathbf{u}}(T, z)\ \forall T \in H_0^1(\Omega),\ \forall z \in W_0^{1,q'}(\Omega)$.

Let $f \in L^1(\Omega)$ and $6/5 \leqslant q < 3/2$, we say that T is a weak solution in $W_0^{1,q}(\Omega)$ to problem (53) if

$$\begin{cases} T \in W_0^{1,q}(\Omega), \\ a_{q,\mathbf{u}}(T, z) = \displaystyle\int_\Omega f z \,\mathrm{d}\mathbf{x} \quad \forall z \in \mathcal{D}(\Omega). \end{cases} \tag{93}$$

Notice that the space of test functions can be replaced by $W_0^{1,q'}(\Omega)$.

Proposition 20. *The solution by transposition to problem* (53) *is also a weak solution in $W_0^{1,q}(\Omega)$ to the same problem.*

Proof. Let $f \in L^1(\Omega)$ and $T = G_{\mathbf{u}}^{*t} f$ be the corresponding solution by transposition. Let $\{f_n\} \subset L^2(\Omega)$ such that $f_n \to f$ strongly in $L^1(\Omega)$ and denote by T_n their corresponding weak solutions in $H_0^1(\Omega)$. We have $T_n = G_{\mathbf{u}}^{*t} f_n$, as we have established in Section 5. Since $G_{\mathbf{u}}^{*t}$ is continuous we have $T_n \to T$ strongly in $W_0^{1,q}(\Omega)$. Moreover, as $T_n \in H_0^1(\Omega)$ we have

$$a_{q,\mathbf{u}}(T_n, z) = a_{\mathbf{u}}(T_n, z) = \int_\Omega f_n z \,\mathrm{d}\mathbf{x} \quad \forall z \in W_0^{1,q'}(\Omega),$$

and due to the continuity of the bilinear form $a_{q,\mathbf{u}}(\cdot,\cdot)$ we obtain that T is a solution to (93). $\square$

Similar to the definition of $L_{\mathbf{u}}$ given in (50), we introduce the operator $L_{\mathbf{0}} : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined as $L_{\mathbf{0}} T := -\frac{1}{P_r R_e} \Delta T$. Assuming that Ω is a bounded domain of class $\mathcal{C}^1$, the operator $L_{\mathbf{0}} : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ is an isomorphism for $p \in (1, +\infty)$ (see [20, Th. 4.6]).

Lemma 21. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $\mathcal{C}^1$ and $\mathbf{u} \in \mathbf{Z}(\Omega)$. If $\psi \in H_0^1(\Omega)$ is such that $L_{-\mathbf{u}}\psi \in \mathcal{D}(\Omega)$ then $\psi \in W^{1,p}(\Omega)$ for any $p < +\infty$.*

Proof. Let us denote $g = L_{-\mathbf{u}}\psi \in \mathcal{D}(\Omega)$. Then, function ψ is the unique solution to the problem

$$\begin{cases} \psi \in H_0^1(\Omega), \\ L_{\mathbf{0}}\psi = \tilde{g} \quad \text{in } \Omega, \end{cases} \tag{94}$$

with $\tilde{g} = g + \mathbf{u} \cdot \mathbf{grad}\, \psi$. From the Sobolev injection, it is easy to see that $L^q(\Omega) \subset W^{-1,q^*}(\Omega)$ with $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{3}$ for any $q \in (1, 3]$. Since $\mathbf{u} \in \mathbf{H}^1(\Omega) \subset \mathbf{L}^6(\Omega)$ and $\mathbf{grad}\, \psi \in \mathbf{L}^2(\Omega)$, we know that $\tilde{g} \in L^{3/2}(\Omega) \subset W^{-1,3}(\Omega)$. From the above lemma,

$\psi \in W_0^{1,3}(\Omega)$. Reasoning as before, $\tilde{g} \in L^2(\Omega) \subset W^{-1,6}(\Omega)$ and then $\psi \in W_0^{1,6}(\Omega)$. Repeating the process again, we infer that $\tilde{g} \in L^3(\Omega) \subset W^{-1,p}(\Omega)$ and then $\psi \in W_0^{1,p}(\Omega)$ for all $p < +\infty$. □

The following lemma proves that a weak solution in $W_0^{1,q}(\Omega)$ is in fact the solution by transposition.

Lemma 22. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $\mathcal{C}^1$, $\mathbf{u} \in \mathbf{Z}(\Omega)$, $f \in L^1(\Omega)$ and q a given real number such that $6/5 \leqslant q < 3/2$. Then T is a solution to (55) if and only if it is a solution to (93).*

Proof. The first implication has been proved in Proposition 20. The other implication is a consequence of the previous lemma. Indeed, let T be a solution of (93), and $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be such that $L_{-\mathbf{u}}\psi \in \mathcal{D}(\Omega)$. From the previous lemma we know that $\psi \in W^{1,q'}(\Omega)$ $\forall q < 3/2$. Then, for any $q \in [6/5, 3/2)$ we have

$$\int_\Omega (L_{-\mathbf{u}}\psi)\varphi \,\mathrm{d}\mathbf{x} = \langle L_{-\mathbf{u}}\psi, \varphi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = a_{\mathbf{u}}(\varphi, \psi) = a_{q,\mathbf{u}}(\varphi, \psi) \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{95}$$

Now, since $\mathcal{D}(\Omega)$ is dense in $W_0^{1,q}(\Omega)$, $L_{-\mathbf{u}}\psi \in \mathcal{D}(\Omega) \subset L^\infty(\Omega)$, $\psi \in W^{1,q'}(\Omega)$ and $a_{q,\mathbf{u}}(\cdot,\cdot)$ is a bilinear and continuous form, the following Green's formula holds:

$$\int_\Omega \varphi(L_{-\mathbf{u}}\psi) \,\mathrm{d}\mathbf{x} = a_{q,\mathbf{u}}(\varphi, \psi) \quad \forall \varphi \in W_0^{1,q}(\Omega).$$

In particular, taking $\varphi = T$ we have

$$\int_\Omega T(L_{-\mathbf{u}}\psi) \,\mathrm{d}\mathbf{x} = a_{q,\mathbf{u}}(T, \psi) = \int_\Omega f\psi \,\mathrm{d}\mathbf{x},$$

because T is a solution to (93). As the result is valid for any arbitrary ψ, we have proved that T is solution to (55). □

Remark 9. In fact, it can be seen that T belongs to the intersection $\bigcap_{1<q<3/2} W_0^{1,q}(\Omega)$.

Once we have proved the equivalence of both formulations, we can prove the Lipschitz continuity result using the weak formulation.

Lemma 23. *Let $f \in L^1(\Omega)$ and $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{Z}(\Omega)$. Let us consider the mapping $\widetilde{G}$ defined in Section 5. Then the following estimate holds*

$$\big\|\widetilde{G}(\mathbf{u}_1, f) - \widetilde{G}(\mathbf{u}_2, f)\big\|_{1,q} \leqslant K(q)\|\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2\|_{\mathbf{L}^6} \big\|\widetilde{G}(\mathbf{u}_2, f)\big\|_{1,6/5}, \tag{96}$$

where $K(q)$ is the constant appearing in Proposition 9. In particular, $K(q)$ is independent of the velocities $\mathbf{u}_1$, $\mathbf{u}_2$ and of the source term f.

Proof. Let us denote $\widetilde{T}_i = \widetilde{G}(\mathbf{u}_i, f)$ for $i = 1, 2$. Lemma 22 states that fields $\widetilde{T}_i \in W_0^{1,q}(\Omega)$ satisfy

$$\frac{1}{P_r R_e}\int_\Omega \mathbf{grad}\,\widetilde{T}_i \cdot \mathbf{grad}\,\varphi \,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{u}_i \cdot \mathbf{grad}\,\widetilde{T}_i\varphi \,\mathrm{d}\mathbf{x} = \int_\Omega f\varphi \,\mathrm{d}\mathbf{x} \quad \forall \varphi \in \mathcal{D}(\Omega),\ i = 1, 2, \tag{97}$$

and subtracting the two equations we get that $\widetilde{T}_1 - \widetilde{T}_2 \in W_0^{1,q}(\Omega)$ and satisfies

$$\frac{1}{P_r R_e}\int_\Omega \mathbf{grad}(\widetilde{T}_1 - \widetilde{T}_2) \cdot \mathbf{grad}\,\varphi \,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{u}_1 \cdot \mathbf{grad}(\widetilde{T}_1 - \widetilde{T}_2)\,\varphi \,\mathrm{d}\mathbf{x} = -\int_\Omega (\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{grad}\,\widetilde{T}_2\,\varphi \,\mathrm{d}\mathbf{x} \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Since $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{Z}(\Omega) \subset \mathbf{L}^6(\Omega)$ and $\mathbf{grad}\,\widetilde{T}_2 \in \mathbf{L}^{6/5}(\Omega)$ we have $(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{grad}\,\widetilde{T}_2 \in L^1(\Omega)$. Moreover, $\mathbf{u}_1 \in \mathbf{Z}(\Omega)$ and, as a consequence of Lemma 22, we deduce

$$\widetilde{T}_1 - \widetilde{T}_2 = \widetilde{G}\big(\mathbf{u}_1, -(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{grad}\,\widetilde{T}_2\big).$$

Hence, we can apply Proposition 9 to obtain

$$\|\widetilde{T}_1 - \widetilde{T}_2\|_{1,q} \leqslant K(q)\big\|(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{grad}\,\widetilde{T}_2\big\|_{L^1} \leqslant K(q)\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{L}^6}\|\widetilde{T}_2\|_{1,6/5}. \qquad \square$$

8.2. Uniqueness result for the model with the Boussinesq approximation

The result of uniqueness for the model using the Boussinesq approximation is given in the following theorem.
Let

$$\mathcal{E}_u(\mathbf{u}_0, \mathbf{B}_0, T_d, \mathbf{k}, \mathbf{f}_0, \psi) = \tilde{k}_1\sqrt{\frac{k_0}{k_2}} + \frac{\tilde{k}_2}{k_2}k_0 + \tilde{k}_0,$$

with k_0 and k_2 the constants appearing in Lemma 12, and $\tilde{k}_0$, $\tilde{k}_1$, $\tilde{k}_2$ having the following expressions:

$$\tilde{k}_0 = \lambda_G\left(meas(\Omega)S^2P_rR_e\|T_d\|_{L^\infty(\partial\Omega)} + 4K\frac{E_c}{R_e}|\mathbf{u}_0|_1 + SK^2\big(8k_{f2}|\mathbf{u}_0|_1^2 + \|\psi\|_{L^1}\big)\right), \tag{98}$$

$$\tilde{k}_1 = 2\lambda_G K\frac{E_c}{R_e}\max\left\{\frac{H_a^2}{R_m^2}, 2\right\} + \sqrt{2}\max\left\{\frac{1}{N}S_4^2, \frac{1}{R_m}SC_1\kappa\right\}, \tag{99}$$

$$\tilde{k}_2 = \lambda_G SK^2k_f. \tag{100}$$

Notice that, similar to $\mathcal{E}_e$, $\mathcal{E}_u$ only depends on the boundary and source data.

Theorem 24. *If $(\mathbf{u}_0, \mathbf{B}_0) \in \mathcal{Z}_0(\Omega)$, with $\mathbf{curl}\,\mathbf{B}_0 = \mathbf{0}$, $T_d \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$ and $\psi \in L^1(\Omega)$, assuming* (48) *and the conditions*

$$0 < \mathcal{E}_e(\mathbf{u}_0, \mathbf{B}_0, T_d, \mathbf{k}, \mathbf{f}_0, \psi) \leqslant \frac{\alpha^2}{4\lambda_G Kk_f}, \tag{101}$$

$$\mathcal{E}_u(\mathbf{u}_0, \mathbf{B}_0, T_d, \mathbf{k}, \mathbf{f}_0, \psi) < \alpha, \tag{102}$$

there exists a constant $R > 0$ such that there is only one fixed point of mapping $\mathcal{G}$ in the ball $\overline{B(0, R)}$.

Proof. Let us assume that $(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1)$, $(\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)$ are two fixed points of mapping $\mathcal{G}$. We want to prove that there exists $R > 0$ and a constant $L < 1$ such that if $(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1)$, $(\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2) \in \overline{B(0, R)}$ then

$$\big|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\big|_{\mathcal{W}} = \big|\mathcal{G}\big((\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1)\big) - \mathcal{G}\big((\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\big)\big|_{\mathcal{W}} \leqslant L\big|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\big|_{\mathcal{W}}. \tag{103}$$

In order to obtain the previous inequality, we must prove some results of Lipschitz continuity for the mappings appearing in the definition of $\mathcal{G}$.

First of all, for mapping $\mathcal{G}_3$ we have the following Lipschitz continuity result:

$$\begin{aligned}&\big\|\mathcal{G}_3\big((\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1)\big) - \mathcal{G}_3\big((\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\big)\big\|_{L^1}\\ &\quad\leqslant \frac{E_c}{R_e}\left(\max\left\{\frac{H_a^2}{R_m^2}, 2\right\}\big(\big|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1)\big|_{\mathcal{W}} + \big|(\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\big|_{\mathcal{W}}\big)\big|(\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2, \hat{\mathbf{B}}_1 - \hat{\mathbf{B}}_2)\big|_{\mathcal{W}} + 4|\mathbf{u}_0|_1|\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2|_1\right).\end{aligned} \tag{104}$$

Now we recall that mapping $\mathcal{G}_2$ was defined as $\mathcal{G}_2(\hat{\mathbf{u}}, f) := \widetilde{G}(\mathbf{u}, f) + G_D(\mathbf{u})$, with $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}_0$. Writing problem (57) for $\mathbf{u}_1$ and $\mathbf{u}_2$, subtracting the two equations and using (59), it is easily seen that

$$\big|G_D(\mathbf{u}_1) - G_D(\mathbf{u}_2)\big|_1 \leqslant P_rR_e\, meas(\Omega)^{1/3}\|T_d\|_{L^\infty(\partial\Omega)}\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{L}^6}. \tag{105}$$

Denoting $f_i = \mathcal{G}_3(\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i)$ and $\widetilde{T}_i = \widetilde{G}(\mathbf{u}_i, f_i)$ for $i = 1, 2$, from Proposition 9, Lemma 23 and inequality (20) we obtain

$$\begin{aligned}&\big\|\widetilde{G}(\mathbf{u}_1, f_1) - \widetilde{G}(\mathbf{u}_2, f_2)\big\|_{1,6/5}\\ &\quad\leqslant \big\|\widetilde{G}(\mathbf{u}_1, f_1 - f_2)\big\|_{1,6/5} + \big\|\widetilde{G}(\mathbf{u}_1, f_2) - \widetilde{G}(\mathbf{u}_2, f_2)\big\|_{1,6/5}\\ &\quad\leqslant K\|f_1 - f_2\|_{L^1} + K\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{L}^6}\|\widetilde{T}_2\|_{1,6/5} \leqslant K\|f_1 - f_2\|_{L^1} + SK|\mathbf{u}_1 - \mathbf{u}_2|_1\|\widetilde{T}_2\|_{1,6/5}.\end{aligned} \tag{106}$$

Now, since $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ are two fixed points of mapping $\mathcal{G}$, from the definition of $\mathcal{G}$ we have $(\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i) = \mathcal{G}((\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i)) = \mathcal{G}_1((\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i), T_i)$, for $i = 1, 2$, where $T_i = \mathcal{G}_2(\hat{\mathbf{u}}_i, f_i)$.

Since $(\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i)$ are the solutions to the corresponding linearized MHD problems, subtracting the equations of the two problems and reasoning as in Lemma 8 we arrive at

$$\begin{aligned}&a_0(\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2) + a_1(\hat{\mathbf{B}}_1 - \hat{\mathbf{B}}_2, \hat{\mathbf{B}}_1 - \hat{\mathbf{B}}_2) + c_0(\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2) + c_0(\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2, \mathbf{u}_0, \hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2)\\ &\quad - c_1(\hat{\mathbf{B}}_2, \hat{\mathbf{B}}_1 - \hat{\mathbf{B}}_2, \hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2) + c_1(\hat{\mathbf{B}}_1 - \hat{\mathbf{B}}_2, \hat{\mathbf{B}}_1 - \hat{\mathbf{B}}_2, \hat{\mathbf{u}}_2) + c_1(\hat{\mathbf{B}}_1 - \hat{\mathbf{B}}_2, \hat{\mathbf{B}}_1 - \hat{\mathbf{B}}_2, \mathbf{u}_0)\\ &\quad = -\big(G(T_1 - T_2), \hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2\big)_\Omega.\end{aligned} \tag{107}$$

For any $(\hat{\mathbf{u}}, \hat{\mathbf{B}}), (\hat{\mathbf{v}}, \hat{\mathbf{C}}), (\hat{\mathbf{w}}, \hat{\mathbf{D}}) \in \mathcal{Z}_0(\Omega)$ we have

$$\begin{aligned}\left|c_0(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}) - c_1(\hat{\mathbf{C}}, \hat{\mathbf{B}}, \hat{\mathbf{w}}) + c_1(\hat{\mathbf{D}}, \hat{\mathbf{B}}, \hat{\mathbf{v}})\right| &\leqslant \frac{1}{N}\|\hat{\mathbf{u}}\|_{\mathbf{L}^4}\|\hat{\mathbf{v}}\|_{\mathbf{L}^4}|\hat{\mathbf{w}}|_1 + \frac{1}{R_m}|\hat{\mathbf{C}}|_{\mathbf{X}}\|\hat{\mathbf{B}}\|_{\mathbf{L}^3}\|\hat{\mathbf{w}}\|_{\mathbf{L}^6} + \frac{1}{R_m}|\hat{\mathbf{D}}|_{\mathbf{X}}\|\hat{\mathbf{B}}\|_{\mathbf{L}^3}\|\hat{\mathbf{v}}\|_{\mathbf{L}^6} \\ &\leqslant \frac{1}{N}S_4^2|\hat{\mathbf{u}}|_1|\hat{\mathbf{v}}|_1|\hat{\mathbf{w}}|_1 + \frac{1}{R_m}SC_1\kappa|\hat{\mathbf{B}}|_{\mathbf{X}}\left|(\hat{\mathbf{v}}, \hat{\mathbf{C}})\right|_{\mathcal{W}}\left|(\hat{\mathbf{w}}, \hat{\mathbf{D}})\right|_{\mathcal{W}} \\ &\leqslant \sqrt{2}\max\left\{\frac{1}{N}S_4^2, \frac{1}{R_m}SC_1\kappa\right\}\left|(\hat{\mathbf{u}}, \hat{\mathbf{B}})\right|_{\mathcal{W}}\left|(\hat{\mathbf{v}}, \hat{\mathbf{C}})\right|_{\mathcal{W}}\left|(\hat{\mathbf{w}}, \hat{\mathbf{D}})\right|_{\mathcal{W}}, \end{aligned} \tag{108}$$

and using in (107) this last inequality and condition (48) we obtain

$$\alpha\left|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}} \leqslant \lambda_G\|T_1 - T_2\|_{L^{6/5}} + \sqrt{2}\max\left\{\frac{1}{N}S_4^2, \frac{1}{R_m}SC_1\kappa\right\}\left|(\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}}\left|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}}.$$

Then, using inequalities (20), (21), (105) and (106) and estimates (59) and (60) we get

$$\begin{aligned}\alpha\left|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}} &\leqslant \lambda_G\big(meas(\Omega)^{2/3}S\left|G_D(\mathbf{u}_1) - G_D(\mathbf{u}_2)\right|_1 + SK(6/5)|\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2|_1\big\|\widetilde{G}(\mathbf{u}_2, f_2)\big\|_{1,6/5} \\ &\quad + K(6/5)\|f_1 - f_2\|_{L^1}\big) + \sqrt{2}\max\left\{\frac{1}{N}S_4^2, \frac{1}{R_m}SC_1\kappa\right\}\left|(\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}}\left|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}} \\ &\leqslant \lambda_G\big(meas(\Omega)S^2P_rR_e\|T_d\|_{L^\infty(\partial\Omega)}|\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2|_1 + SK^2|\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2|_1\|f_2\|_{L^1} + K\|f_1 - f_2\|_{L^1}\big) \\ &\quad + \sqrt{2}\max\left\{\frac{1}{N}S_4^2, \frac{1}{R_m}SC_1\kappa\right\}\left|(\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}}\left|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}}.\end{aligned}$$

Finally, using the inequalities (64) and (104) we have

$$\begin{aligned}\alpha\left|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}} &\leqslant \lambda_G\Big[meas(\Omega)S^2P_rR_e\|T_d\|_{L^\infty(\partial\Omega)}|\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2|_1 \\ &\quad + SK^2\big(k_f\left|(\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}}^2 + 8k_{f2}|\mathbf{u}_0|_1^2 + \|\psi\|_{L^1}\big)|\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2|_1 \\ &\quad + K\frac{E_c}{R_e}\left(\max\left\{\frac{H_a^2}{R_m^2}, 2\right\}\big(\left|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1)\right|_{\mathcal{W}} + \left|(\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}}\big) + 4|\mathbf{u}_0|_1\right)\left|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}}\Big] \\ &\quad + \sqrt{2}\max\left\{\frac{1}{N}S_4^2, \frac{1}{R_m}SC_1\kappa\right\}\left|(\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}}\left|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}}.\end{aligned}$$

As we have assumed that $|(\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i)|_{\mathcal{W}} \leqslant R$ for $i = 1, 2$, it is possible to find a constant $L < 1$ such that (103) is satisfied if

$$\tilde{k}_2R^2 + \tilde{k}_1R + (\tilde{k}_0 - \alpha) < 0, \tag{109}$$

with $\tilde{k}_0$, $\tilde{k}_1$ and $\tilde{k}_2$ the constants given in the statement.

Under assumptions (48) and (101) we have already proved in Theorem 13 that there exists at least one solution in the closed ball $\overline{B(0, R_-)}$, with

$$R_- = \frac{\alpha - \sqrt{\alpha^2 - 4k_0k_2}}{2k_2} = \frac{2k_0}{\alpha + \sqrt{\alpha^2 - 4k_0k_2}} \leqslant \frac{2k_0}{\alpha}.$$

Moreover, since $R_-R_+ = k_0/k_2 > 0$ and $0 < R_- \leqslant R_+$, we have $R_- \leqslant \sqrt{k_0/k_2}$. Replacing R with R_- in (109) and using that $R_-^2 = (-k_0 + \alpha R_-)/k_2$ we get the following inequality:

$$\left(\tilde{k}_1 + \frac{\tilde{k}_2}{k_2}\alpha\right)R_- + \tilde{k}_0 - \alpha - \frac{\tilde{k}_2}{k_2}k_0 < 0.$$

This will be guaranteed whenever

$$\tilde{k}_1\sqrt{\frac{k_0}{k_2}} + \frac{\tilde{k}_2}{k_2}2k_0 + \tilde{k}_0 - \alpha - \frac{\tilde{k}_2}{k_2}k_0 < 0.$$

This condition is equivalent to (102). Hence, in the closed ball $\overline{B(0, R_-)}$ the fixed points satisfy (103), and uniqueness in that ball follows. □

Summarizing, the above theorem states that, under certain conditions of smallness on the boundary and source data, there exists a solution of the problem in a certain closed ball $\overline{B(0, R)}$ contained in the space $\mathcal{Z}_0(\Omega)$, and the solution is unique in the mentioned ball.

8.3. Uniqueness result for the second model

In the previous section we have proved a uniqueness result for the MHD model under the Boussinesq approximation. Now we are going to prove a uniqueness result for the model introduced in Section 7. The result requires further assumptions on the response function $\hat{\rho}$.

Theorem 25. *Let $(\mathbf{u}_0, \mathbf{B}_0) \in \mathcal{Z}_0(\Omega)$, with $\mathbf{curl}\,\mathbf{B}_0 = \mathbf{0}$, and let $\psi \in L^1(\Omega)$ and $T_d \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$ satisfying* (77) *and* (78), *respectively. Assuming that the response function $\hat{\rho}$ is Lipschitz continuous, namely*

$$\left|\hat{\rho}(\theta_1) - \hat{\rho}(\theta_2)\right| \leqslant \Lambda_\rho |\theta_1 - \theta_2| \quad \forall \theta_1, \theta_2 \in [T_{\min}, +\infty), \tag{110}$$

then under the assumptions (48) *and*

$$\tilde{k}_2 k_0^2 + \alpha \tilde{k}_1 k_0 + (\tilde{k}_0 - \alpha)\alpha^2 < 0, \tag{111}$$

there exists a unique fixed point of the mapping $\hat{\mathcal{G}} : \mathcal{Z}_0(\Omega) \to \mathcal{Z}_0(\Omega)$. Moreover, it belongs to the closed ball $\overline{B(0, R_0)}$, with $R_0 = k_0/\alpha$.

The constant k_0 has been introduced in Proposition 14, *and $\tilde{k}_2$, $\tilde{k}_1$, $\tilde{k}_0$ are given by*

$$\tilde{k}_2 = S^2 g \Lambda_\rho K^2 k_f,$$
$$\tilde{k}_1 = 2 S g \Lambda_\rho K \frac{1}{R_e} \frac{u^2}{c_p \mathcal{T}} \max\left\{\frac{H_a^2}{R_m^2}, 2\right\} + \sqrt{2} \max\left\{\frac{1}{N} S_4^2, \frac{1}{R_m} S C_1 \kappa\right\},$$
$$\tilde{k}_0 = S g \Lambda_\rho \left(meas(\Omega) S^2 P_r R_e \|T_d - T_{\min}\|_{L^\infty(\partial\Omega)} + 4K \frac{1}{R_e} \frac{u^2}{c_p \mathcal{T}} |\mathbf{u}_0|_1 + S K^2 \left(8 k_{f2} |\mathbf{u}_0|_1^2 + \|\psi\|_{L^1}\right)\right),$$

where now $k_f = \max\{u^2 H_a^2/(c_p \mathcal{T} R_e R_m^2), 8k_{f2}\}$ and $k_{f2} = u^2/(2 c_p \mathcal{T} R_e)$.

The assumption (48) clearly imposes a condition of smallness on the lifting $\mathbf{u}_0$. Concerning the assumption (111) and recalling the definition of constants k_0 and $\tilde{k}_0$, this condition first imposes smallness on the source data $\mathbf{f}$, $\mathbf{k}$, and ψ, and on the lifting $\mathbf{u}_0$ and $\mathbf{B}_0$. It also requires a small difference between the minimum and maximum temperature on the boundary, and a condition of smallness on the maximum density $\hat{\rho}_{\max}$. If we assume that density function $\hat{\rho}$ tends to zero as the temperature tends to infinity, then these two conditions can be fulfilled at the same time and therefore condition (111) can also be satisfied.

As we have already noticed in Remark 8, under condition (48) and setting $R_0 = k_0/\alpha$, we know that every solution $(\hat{\mathbf{u}}, \hat{\mathbf{B}}) \in \mathcal{Z}_0(\Omega)$ belongs to the closed ball $\overline{B(0, R_0)}$ and there exists at least one solution in that ball. The theorem states that under condition (111) this solution is unique.

Proof of Theorem 25. The proof is similar to the one of Theorem 24. Firstly, concerning the mapping $\hat{\mathcal{G}}_3$ we notice that (104) is replaced by a similar inequality, substituting Eckert number E_c by $u^2/(c_p \mathcal{T})$. Secondly, the estimate (64) holds with $k_{f2} = u^2/(2 c_p \mathcal{T} R_e)$ and $k_f = \max\{u^2 H_a^2/(c_p \mathcal{T} R_e R_m^2), 8k_{f2}\}$.

The definition of mapping $\hat{\mathcal{G}}_2$ is essentially identical to that of mapping $\mathcal{G}_2$, hence the same estimates proved in the previous section for this mapping, or for mappings G_D and $\widetilde{G}$, remain valid. However, the result given in (105) can be improved in the sense that, instead of the norm $\|T_d\|_{L^\infty(\partial\Omega)}$, it is the difference between the maximum and the minimum temperatures on the boundary what is involved. To obtain the result, we first observe that the solution $G_D(\mathbf{u})$ does depend on the boundary condition T_d. Therefore, introducing a slight abuse of notation, we will refer to $G_D(\mathbf{u})$ as $G_D(\mathbf{u}, T_d)$. Since problem (57) is linear, we know that $G_D(\mathbf{u}, T_d) = G_D(\mathbf{u}, T_d - c) + G_D(\mathbf{u}, c) = G_D(\mathbf{u}, T_d - c) + c$ for every constant $c \in \mathbb{R}$. In particular, taking $c = T_{\min}$ (the value defined in (78)), the result obtained in (105) is transformed into

$$\left|G_D(\mathbf{u}_1) - G_D(\mathbf{u}_2)\right|_1 \leqslant P_r R_e\, meas(\Omega)^{1/3} \|T_d - T_{\min}\|_{L^\infty(\partial\Omega)} \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{L}^6}. \tag{112}$$

Finally, let us assume that $(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1), (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2) \in \mathcal{Z}_0(\Omega)$ are two fixed points of mapping $\hat{\mathcal{G}}$ and such that $|(\hat{\mathbf{u}}_i, \hat{\mathbf{B}}_i)|_{\mathcal{W}} \leqslant R$, $i = 1, 2$. Reasoning as in Theorem 24 and using the Lipschitz continuity of $\hat{\rho}$ we obtain

$$\begin{aligned}
&\alpha \left|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}} \\
&\quad \leqslant S g \Lambda_\rho \Bigg[meas(\Omega) S^2 P_r R_e \|T_d - T_{\min}\|_{L^\infty(\partial\Omega)} |\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2|_1 \\
&\qquad + K \frac{1}{R_e} \frac{u^2}{c_p \mathcal{T}} \left(\max\left\{\frac{H_a^2}{R_m^2}, 2\right\} \left(\left|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1)\right|_{\mathcal{W}} + \left|(\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}} \right) + 4 |\mathbf{u}_0|_1 \right) \left|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}} \\
&\qquad + S K^2 \left(k_f \left|(\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}}^2 + 8 k_{f2} |\mathbf{u}_0|_1^2 + \|\psi\|_{L^1} \right) |\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2|_1 \Bigg] \\
&\qquad + \sqrt{2} \max\left\{\frac{1}{N} S_4^2, \frac{1}{R_m} S C_1 \kappa\right\} \left|(\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}} \left|(\hat{\mathbf{u}}_1, \hat{\mathbf{B}}_1) - (\hat{\mathbf{u}}_2, \hat{\mathbf{B}}_2)\right|_{\mathcal{W}}.
\end{aligned}$$

Therefore, a condition for $\hat{\mathcal{G}}$ analogous to (103) is satisfied whenever $\tilde{k}_2 R^2 + \tilde{k}_1 R + (\tilde{k}_0 - \alpha) < 0$, with the constants $\tilde{k}_j$ defined in the statement. Clearly, taking $R = R_0 = k_0/\alpha$ this condition is equivalent to (111), which ends the proof. □

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References

[1] G.V. Alekseev, Solvability of control problems for stationary equations of the magnetohydrodynamics of a viscous fluid, Sibirsk. Mat. Zh. 45 (2004) 243–263.
[2] A. Alonso, A. Valli, Some remarks on the characterization of the space of tangential traces of $H(\text{rot}; \Omega)$ and the construction of an extension operator, Manuscripta Math. 89 (1996) 159–178.
[3] A. Alonso, A. Valli, An optimal domain decomposition preconditioner for low-frequency time-harmonic Maxwell equations, Math. Comp. 68 (1999) 607–631.
[4] C. Bègue, C. Conca, F. Murat, O. Pironneau, Les équations de Stokes et de Navier–Stokes avec des conditions aux limites sur la pression, in: Nonlinear Partial Differential Equations and Their Applications, Collège de France Seminar, vol. IX, Paris, 1985–1986, in: Pitman Res. Notes Math. Ser., vol. 181, Longman Sci. Tech., Harlow, 1988, pp. 179–264.
[5] M. Ben Cheik Ali, O.O. Guibé, Résultats d'existence et d'unicité pour une classe de problèmes non coercifs, C. R. Math. Acad. Sci. Paris Sér. I 329 (1999) 967–972.
[6] M. Ben Cheik Ali, O.O. Guibé, Nonlinear and non-coercive elliptic problems with integrable data, Adv. Math. Sci. Appl. 16 (2006) 275–297.
[7] A. Buffa, P. Ciarlet, On traces for functional spaces related to Maxwell's equations, part I: An integration by parts formula in Lipschitz polyhedra, Math. Methods Appl. Sci. 24 (2001) 9–30.
[8] A. Buffa, P. Ciarlet, On traces for functional spaces related to Maxwell's equations, part II: Hodge decompositions on the boundary of Lipschitz polyhedra and applications, Math. Methods Appl. Sci. 24 (2001) 31–48.
[9] A. Buffa, R. Hiptmair, A coercive combined field integral equation for electromagnetic scattering, SIAM J. Numer. Anal. 42 (2004) 621–640.
[10] J.I. Díaz, J.-M. Rakotoson, P.G. Schmidt, A parabolic system involving a quadratic gradient term related to the Boussinesq approximation, RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 101 (2007) 113–118.
[11] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations, vol. I: Linearized Steady Problems, Springer Tracts Natural Philosophy, vol. 38, Springer-Verlag, New York, 1994.
[12] V. Girault, P.-A. Raviart, Finite Element Methods for Navier–Stokes Equations. Theory and Algorithms, Springer Ser. Comput. Math., vol. 5, Springer-Verlag, Berlin, 1986.
[13] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Monogr. Studies in Math., vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985.
[14] M.D. Gunzburger, A.J. Meir, J.S. Peterson, On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics, Math. Comp. 56 (1991) 523–563.
[15] A.J. Meir, Thermally coupled, stationary, incompressible MHD flow; existence, uniqueness, and finite element approximation, Numer. Methods Partial Differential Equations 11 (1995) 311–337.
[16] A.J. Meir, P.G. Schmidt, Variational methods for stationary MHD flow under natural interface conditions, Nonlinear Anal. 26 (1996) 659–689.
[17] A.J. Meir, P.G. Schmidt, On electromagnetically and thermally driven liquid-metal flows, in: Proceedings of the Third World Congress of Nonlinear Analysts, Part 5, Catania, 2000, vol. 47, 2001, pp. 3281–3294.
[18] H. Schlichting, Teoría de la capa límite, Urmo, 1972, translated from German by F.M. Samaniego.
[19] D. Schötzau, Mixed finite element methods for stationary incompressible magneto-hydrodynamics, Numer. Math. 96 (2004) 771–800.
[20] C.G. Simader, On Dirichlet's Boundary Value Problem. An L^p-Theory Based on a Generalization of Gårding's Inequality, Lecture Notes in Math., vol. 268, Springer-Verlag, Berlin, 1972.
[21] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, Ann. Inst. Fourier (Grenoble) 15 (1965) 189–258.
[22] R. Vázquez, Contributions to the mathematical study of some problems in magnetohydrodynamics and induction heating, PhD thesis, Universidade de Santiago de Compostela, 2008.