Some Results on Finite Semigroups

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Herein we state and prove various propositions about finite semigroups. Nearly all of these results were discovered during a seminar entitled "Finite Semigroups and Machines" [7] given by the author at the University of California at Berkeley, 1964-1965. In general, the proofs are applications of the methods introduced by Kenneth Krohn and the author [2, 3, 4] or of standard semigroup theory, particularly the Green relations and the Rees theorem, as presented in [6] and [1]. In Section 5, we use some techniques introduced by Paul Zeiger [8]. In Section 6 we require a lemma proved by Dennis Allen.

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In the following, all semigroups are of finite order unless the contrary is explicitly stated. Undefined notation follows [1] or [4].

We say that a semigroup $C$ is combinatorial iff each subgroup of $C$ has order one. In Section 1 we give several characterizations of combinatorial semigroups. Proposition 1.1(f) is new and surprising; Proposition 1.1(d) is old and surprising. Proposition 1.2 proves that maximal combinatorial subsemigroups are self left (and right) idealizing.

In Section 2 we state Green's results which prove that $\mathcal{J} = \mathcal{D}$ for torsion semigroups. In Section 3 we use this result to extend to arbitrary $\mathcal{J}$-classes some standard theorems concerning the behavior of homomorphisms on regular $\mathcal{J}$-classes.

Proposition 4.1 proves that an element $x$ of a semigroup $S$ is nilpotent if every nontrivial irreducible complex character of $S$ vanishes at $x$.

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In [3] and [4] the important concept of complexity was introduced (see also Section 5 of this paper). Let \( F_n(X_n) \) denote the semigroup of all mappings on the set \( X_n = \{1, \ldots, n\} \) under the multiplication \( (f \cdot g)(x) = g(f(x)) \). In Section 5 we prove that \( C(F_n(X_n)) = (2(n-1), G) \) when \( n \geq 2 \).

Let \( \alpha \) be any of the Green relations \( \mathcal{L}, \mathcal{R}, \) or \( \mathcal{H} \) on a semigroup \( S \). Then \( \phi : S \rightarrow T \) is an \( \alpha \)' homomorphism iff \( \phi(s_1) = \phi(s_2) \) for regular elements \( s_1, s_2, \in S \) implies that \( s_1 \alpha s_2 \). We say \( \phi \) is a \( \gamma \) homomorphism iff \( \phi \) is 1:1 when restricted to any subgroup of \( S \). These concepts arise as follows: Let \( S_2, S_1 \) be semigroups, \( Y : S_1 \rightarrow \text{Endo}_k(S_2) \) a homomorphism, and \( S_2 \times_{\gamma} S_1 \) the corresponding semidirect product. Then the natural projection \( S_2 \times_{\gamma} S_1 \rightarrow S_1 \) is a \( \gamma \) homomorphism if \( S_2 \) is combinatorial and an \( \mathcal{L}' \) homomorphism if \( S_2 \) is a group.

In Section 6 we prove the following homomorphism theorem: Let \( \phi : S \rightarrow T \) be an arbitrary homomorphism. Then \( \phi = \phi_n \phi_{n-1} \ldots \phi_1 \) where \( \phi_1, \phi_2, \ldots \) are \( \gamma \)-homomorphisms and \( \phi_2, \phi_4, \ldots \) are \( \mathcal{H}' \)-homomorphisms or vice versa.

Further, in the regular case, every \( \mathcal{H}' \)-homomorphism \( \psi \), when restricted to a maximal subgroup of \( S \), has a kernel, and this collection of normal subgroups determines \( \psi \) and is thus, in a very strong sense, a kernel for \( \psi \).

Thus for "half" of the homomorphisms of semigroups there is a respectable homomorphism theorem. The other half should be considered via the cohomology theory of semigroups.

We also prove that any semigroup \( S \) has a minimal (in the functorial sense) \( \gamma \) homomorphic image \( S' \). Equivalently, we prove that if \( S \rightarrow S/Q_k \) is a \( \gamma \) homomorphism for \( k = 1, 2 \), then the homomorphism \( S \rightarrow S/Q_k \rightarrow S/(Q_1 \vee Q_2) \) is a \( \gamma \) homomorphism. Also, \( S \) has a functorially minimal \( \mathcal{L}' \) homomorphic image \( S^{\mathcal{L}'} \). \( S | T \) implies \( S^{\mathcal{L}'} | T^{\mathcal{L}'} \) (see Notation 1.1).

Finally, as an application of the above techniques and the main theorem of [4] we prove the following theorem. Let \( \#(S) \) denote the complexity number of \( S \). Let \( S \) be a union of groups, \( \phi : S \rightarrow T \) an arbitrary homomorphism, so \( \#(T) = k \leq n = \#(S) \). Then there exist \( S_n, \ldots, S_1 \) so that \( S \rightarrow S_n \rightarrow S_{n-1} \rightarrow \cdots \rightarrow S_{k+1} \rightarrow S_k = T \rightarrow S_{k-1} \rightarrow \cdots \rightarrow S_1 = \{1\} \) with \( \#(S_j) = j \) for \( 1 \leq j \leq n \).

1. **Combinatorial Semigroups**

**Notation 1.1.** Let \( A, B, C, \) and \( D \) denote finite non-empty sets and let \( C, S, T, \) and \( U \) denote semigroups. We say \( S \) divides \( T \), denoted \( S \mid T \) iff \( S \) is a homomorphic image of a subsemigroup \( T \subseteq T. \Sigma A = \{(a_1, \ldots, a_n) : a_k \in A \text{ for } 1 \leq k \leq n\} \) is the free (non-commutative) semigroup (without identity) with generators \( A \). A machine is any function \( f : \Sigma A \rightarrow B \). We denote
by \( f^e : \sum A \to \sum B \) the extended machine such that \( f^e(a_1, \ldots, a_n) = (f(a_1), f(a_2), \ldots, f(a_k), \ldots, f(a_n)) \). If \( g : \sum C \to D \) is another machine, then \( f \times g : \sum (A \times C) \to \sum (B \times D) \) is given by \( (f \times g)((a_1, c_1), \ldots, (a_n, c_n)) = (f(a_1, \ldots, a_n), g(c_1, \ldots, c_n)) \). For a semigroup, the machine of \( S \), \( S^f : \sum S \to \sum S \) is defined by \( S^f(s_1, \ldots, s_n) = \prod_{i=1}^n s_i \). \( A^r \) (respectively \( A^l \)) is the semigroup with elements \( A \) and multiplication \( ab = b \) (respectively \( ab = a \)) for all \( a, b \in A \).

Let \( S \) be a semigroup then \( (S^f)^r : \sum S \to \sum S \) is the extended machine \( S^fS^f \), and \( (S^f)^n \) denotes \( S^f(S^f)^{n-1} \) for \( n \geq 3 \).

\( F(A, B) \) denotes the collection of all mappings of \( A \) into \( B \), and \( F(A, S) \) denotes the semigroup of all mappings of \( A \) into \( S \) under pointwise multiplication.

Let \( Y : S \to \text{Endo}(S) \) be a homomorphism. Denote \( (Y(s_1))(s_2) \) by \( s_1^Y(s_2) \) so \( s_1^Y(s_2) = s_1^Y(s_2) \). The semidirect product \( S \times S \) is the semigroup with elements \( S \times S \) and multiplication \( (s_1, s_2) \cdot (s_1', s_2') = (s_1s_1', s_2s_2') \).

The wreath product \( S \circ S \) is the semigroup \( F(S, S) \times S \) where \( (s(f))(s_1') = f(s_1s_1') \). Let \( U_3^n = U_3, U_3^{n+1} = U_3^{n+1} \) for \( n \geq 3 \). (Note the order in which the product is formed, since \( S \circ S \) is not isomorphic with \( S \circ (S \circ S) \) under the above definition of \( \circ \).

Let \( T \) be a subset of \( S \). Then \( R(S)(T) = \{s \in S : Ts \subseteq T \}, L(S)(T) = \{s \in S : sT \subseteq T \}, \) and \( I(S)(T) = R(S)(T) \cap L(S)(T) \).

Let \( S^l \) be the semigroup obtained from \( S \) by adjoining an identity \( 1 \) if \( S \) has no identity. Let \( \pi A = F((1, 2, 3, \ldots), A) \) and let \( X \in \pi S \). Then \( P_X : \sum S \to \sum S \) is defined by \( P_X(s_1, \ldots, s_n) = (x_1s_1, x_2s_2, \ldots, x_ns_n) \), where \( X = (x_1, x_2, \ldots, x_n) \).

\( F(A, A) \) is the set \( F(A, A) \) with multiplication \( (f \cdot g)(a) = f(g(a)) \). \( F(A) \) is the reverse semigroup of \( F(A, A) \), i.e., the semigroup with elements \( F(A, A) \) and multiplication \( (f \cdot g)(a) = g(f(a)) \).

Let \( L_\mathfrak{U} : (\Sigma A)^l \to (\Sigma A)^l \) is given by \( L_\mathfrak{U}(r) = tr \).

**Proposition 1.1.** Let \( S \) be a finite semigroup. Then the following are equivalent.

(a) If \( G \) is a subgroup of \( S \), then \( G \cong \{1\} \).

(b) There exists a positive integer \( r = r(S) \) such that \( s^r = s^{r+k} \) for all \( k \geq 1 \) and all \( s \in S \).

(c) Each \( \mathcal{H} \)-class of \( S \) consists of exactly one element.

(d) There exists a positive integer \( n = m(S) \) such that

\[
S \mid U^{(n)}_3
\]

(e) There exists a positive integer \( m = m(S) \) such that

\[
(S^f)^m = (S^f)^{m+1}
\]
(f) There exists a positive integer \( q = q(S) \) such that for all \( X \in \mathcal{S} \)

\[
(P_X S^o)^q = (P_X S^o)^{q+1}
\]  

and

\[
(S^o P_X)^q = (S^o P_X)^{q+1}.
\]

We say that \( S \) is \textit{combinatorial} iff it satisfies the above. The class of combinatorial semigroups is closed under the operations of division, finite direct products, \( X \), and \( w \). Thus we have the following:

\textbf{Principle of Induction for Combinatorial Semigroups.}

Let \( P \) be any property of finite semigroups such that

1. \( U_3^{(n)} \) satisfies \( P \).
2. \( P \) is closed under division.

Then every combinatorial semigroup satisfies \( P \).

\textbf{Proof of Proposition 1.1.} Clearly (a) and (b) are equivalent. (c) implies (a), since any two elements of a subgroup of \( S \) are \( \mathcal{H} \)-equivalent. Conversely, if \( H \) is an \( \mathcal{H} \)-class of \( S \) with \( |H| \geq 2 \), it is well known ([1], Section 2.4) that \( \phi : RI_3(H) \rightarrow F_{\mathcal{H}}(H) \) with \( \phi(s)(h) = hs \) is a homomorphism with image a group \( G \) of order \( |H| \). Now let \( T \) be a subsemigroup of \( RI_3(H) \) minimal with respect to the property that \( \phi(T) = G \). Then \( T \) is a subgroup of \( S \), and \( |T| \geq |G| = |H| \geq 2 \). Thus (a) and (c) are equivalent.

The equivalence of (a) and (d) and the closure properties of semigroups satisfying (a) is proved in [2] (see also [3]). The principle of induction stated above now follows from (d). Clearly (f) implies (e) by taking for \( X \) the sequence \((1, 1, 1, \ldots)\). It is easy to see that no nontrivial group \( S \) satisfies (e). Thus (e) implies (a), so it is sufficient to show that (a) implies (f). For this we use the above principle of induction for combinatorial semigroups.

It is easy to verify that the class of semigroups satisfying (f) is closed under division and finite direct products. Thus, we need only show that \( U_3^{(n)} \) satisfies (f) for \( n = 1, 2, \ldots \). Since \( (xy)^n = (xy)^{n+1} \) implies \( (yx)^{n+1} = y(xy)^n x = y(xy)^{n+1} x = (yx)^{n+2} \), it is sufficient to show that \( U_3^{(n)} \) satisfies (1.3).

It is trivial to verify that \( (P_X U_3^{(n)})^q = (P_X U_3^{(n)})^{q+1} \) for all \( X \in \mathcal{S} \).

Now assume there exists \( g_k \) so that (1.3) holds for \( S = U_3^{(k)} \) (hence also for \( S = F(U_3, U_3^{(n)}) \)). Let \( 1_k \) denote the identity of \( U_3^{(k)} \), and write \( 1 \) for \( 1_1 \). Let \( C \) denote the constant function always taking the value 1, and let \( F = F(U_3, U_3^{(n)}) \) \( X \). Let \( X \in \mathcal{S} \) with \( X = ((g_1, b_1), \ldots, (g_p, b_p), \ldots) \), \( b_p \in U_3 \), \( g_p : U_3 \rightarrow U_3^{(n)} \) for \( k \geq 1 \). Let \( Y \in \sum U_3^{(n+1)} \) with \( Y = ((f_1, a_1), \ldots, (f_k, a_k)) \). Let
\( \emptyset = P_X(U^{(n+1)}/\pi) \). Let \( r \) be the largest non-negative integer such that \( b_1, \ldots, b_{r-1}, a_1, \ldots, a_{r-1} \) are all equal to 1. Let \( w \geq 1 \). Let \( C^{2n+1}(Y) = \{(h_1, 1), \ldots, (h_r, c_r), (h_k, c_k)\} \). Then \( c_r, \ldots, c_k \in \{a, b\} \), so \( x_1 = c_1 \) and \( x_2 = c_2 \) for \( f \in F(U_3, U_3^{(n)}) \), \( x \in U_3 \), and \( r \leq l \leq k \). Further, \( g_1 h_1 = h_1, g_2 h_1 = h_2, \ldots, g_{r-1} h_1 = h_{r-1} \), and \( g_r h) = h_r \), i.e., \( C^{2n}(Y') = C^{2n+1}(Y) \), where \( Y' = ((f_1, a_1), \ldots, (f_r, a_r)) \). To see this, let \( C^{2n}(Y) = ((h_1', 1), \ldots, (h_{r-1}', 1), (h_r', c_r), (h_k', c_k)) \). Then by a computation using the induction assumption for \( F \), \( g_1 h_1' = h_1' = h_1, \ldots, g_1 h_1' = h_{r-1}' = h_{r-1} \), and

\[
\begin{align*}
h_r &= g_r [(h_1 \cdots h_{r-1} g_r)^{g_n}(h_1 \cdots h_{r-1} h_r')] \\
&= g_r [(h_1 \cdots h_r)^{g_{n+1}}(h_1 \cdots h_{r-1} h_r')],
\end{align*}
\]

i.e., \( C^{2n+2}(Y') = C^{2n+1}(Y) \).

It now follows by direct computation that \( C^{2n+1+w+2} = \)

\[
P_{X^2}(F(U_3, U_3^{(n)})^r \times U_3^r) = P_{X^2}(F(U_3, U_3^{(n)})^r \times U_3^r)^{g_{n+1}} \emptyset^{2n+1}(Y),
\]

where \( f_1 : \Sigma(F(U_3, U_3^{(n)}) \times U_3) \rightarrow F(U_3, U_3^{(n)}) \times U_3 \), and \( X_2, X_3 \in \pi(F(U_3, U_3^{(n)}) \times U_3) \) are defined as follows: \( f_i((k_i, d_i), \ldots, (k_i, d_i)) = (d_{i-1}(k_i), d_i) \) if \( l \geq 2 \), \( d_i \neq 1 \), and \( (C_1, d_1) \) otherwise. \( X_2 = (j_1, 1), \ldots, (j_r, 1), (\alpha g_{r+1}^{(g_{r+1})^2} h_1 \cdots h_r, 1), \ldots, (\alpha g_{r+1}^{(g_{r+1})^2} h_1 \cdots h_r, 1), (j_{k+1}, 1), \ldots, (j_{m}, 1), \ldots \) and \( X_3 = ((h_1, 1), \ldots, (h_r, 1), (g_{r+1}^{(g_{r+1})^2} h_1 \cdots h_r, 1), \ldots, (g_{k+1}^{(g_{k+1})^2} h_1 \cdots h_r, 1), \ldots, (g_{m}^{(g_{m})^2} h_1 \cdots h_r, 1), \ldots) \), where \( C_1, j_1 = j_2 = \ldots = j_r = \ldots = j_{k+1} = \ldots = j_{m} = \ldots \).

Let \( Q_{X^2} = P_{X^2}(F(U_3, U_3^{(n)})^r \times U_3^r)^{g_{n+1}} \). Then by the induction assumption, \( Q_{X^2} = Q_{X^2+1} \) for all \( X_2 \). Thus (1.5) implies that (1.3) holds for \( S = U_3^{(n+1)} \) with \( q = q_{n+1} = 3q_n + 2 \). Thus (a) implies (f) and Proposition 1.1 is proved.

\[
\begin{array}{cccc}
S^1 & S_1 & S_2 & S_3 \\
x_1 & x_1 s_1 = a_{11} & x_1 s_2 = a_{12} & \cdots \\
x_2 & x_2 x_1 s_1 = a_{21} & x_2 x_1 s_2 = a_{22} & \cdots \\
x_3 & \cdot & \cdot & \alpha \\
\vdots & \alpha & \cdot & \alpha \beta \\
\end{array}
\]
Remark 1.1. A way to visualize $S^{1/0}$ is by the "Pascal Array" of the multiplication table of $S^1$. Let $X_1 = (s_1, ..., s_n) \in \pi S^1$ and $X_2 = (x_1, ..., x_n) \in \pi S^1$. Then consider Figure 1.1. Let $f_k = (S^{1/0}_{L_{a_k}})$, where $L_{a_k} (s_1, s_2, ..., s_n) = (x_k, s_1, s_2, ..., s_n)$. Then $a_{ij} = f_{j} f_{j-1} ... f_2 (s_1, ..., s_k)$. Proposition 1.1(c) asserts that $S$ is a combinatorial semigroup if and when $x_1 = x_2 = ... = x_n = 1$ in Fig. 1.1., there exists a positive integer $m = m(S)$ such that for all $X_1 \in \pi S^1$, the $m, m + 1, m + 2, ...$ rows are identical.

We have the following "duality." Transposition in Fig. 1.1 about the line $a_{11}, a_{22}, ..., i.e., the transformation taking $a_{ij}$ to $a_{ji}$, may be effected by interchanging $X_2$ and $X_1$ and replacing $S^1$ by $r(S^1)$, the reverse semigroup of $S^1$. The dual result of Proposition 1.1(e) is $(e')$ $S$ is a combinatorial semigroup if and there exists a positive integer $m = m(S)$ such that for any $x_1, ..., x_j \in S^1$, if

$$b_k = (S^{1/0}_{L_{x_1}})(S^{1/0}_{L_{x_2}})^r ... (S^{1/0}_{L_{x_j}})^r(k1)$$

where $(k1)$ is the $k$-long sequence of 1's, $(1, ..., 1)$, then $b_m = b_{m+1} = ...$.

**Proposition 1.2.** Let $C$ be a maximal combinatorial subsemigroup of $S$, i.e., $C \subseteq T \subseteq S$ with $T$ a combinatorial subsemigroup of $S$ implies $T = C$. Then $I_0(C) = R I_0(C) = L I_0(C) = C$.

**Proof.** It suffices by duality to prove $L I_0(C) = C$. Suppose not. Let $T$ be a subsemigroup of $S$ minimal with respect to the property that $C \subseteq T \subseteq L I_0(C)$. Then $T$ is noncombinatorial, and $C$ is a maximal subsemigroup of $T$ which is also a combinatorial left ideal of $T$. Let $J$ be a $\mathcal{J}$-class of $T$ minimal (in the order $J_1 \leq J_2$ iff $T_1 J_1 T^1 \subseteq T^1 J_2 T^1$) with respect to not being contained in $C$. Clearly $J$ exists, and $C \cup J$ is a subsemigroup, so $C \cup J = T$, since $C$ is a maximal subsemigroup of $T$. It follows that $J$ contains a non-trivial subgroup of $T$ and is thus a regular $\mathcal{J}$-class. Now since $C$ is a left ideal of $T$, if $C \cap J \neq \phi$, then $C$ contains an $\mathcal{L}$-class of $J$, hence (by Rees Theorem) $C$ contains a nontrivial subgroup of $T$, which is impossible, since $C$ is combinatorial. Thus $J \cap C = \phi$. Let $L$ be an $\mathcal{L}$-class of $T$ contained in $J$. Now by the definition of $J$ and from Proposition 2.2(b) of this paper, we see that $C \cup L$ is a subsemigroup of $T$, hence equals $T$. Thus, $J = L$ is a left simple subsemigroup of $T$ (by Rees Theorem), so $L \cap L = \phi$ (since $C$ is a left ideal and $L \cap C = \phi$) which implies that $CL \cap L = \phi$, since for $l_1, l_2 \in L$, $c \in C$, if $cl_2 \in L$, then $l_1 (cl_2) = (l_1 c) l_2$, and $l_1 (cl_2) \in L^2 \subseteq L$, while $l_1 c \in C$ so $S^1 L_0 C S^1 = S^1 L S^1$ and $(l_1 c) \in L$ by the definition of $J = L$, since $L \cap C = \phi$, a contradiction. Now let $e \in L$ be idempotent. Then $C \cup \{e\}$ is a subsemigroup of $T$, hence equals $T$, so $T$ is combinatorial, a contradiction.

This completes the proof.
2. $\mathcal{J} = \mathcal{D}$ for Torsion Semigroups

The following results are proved by Green in [9]. We state them here for the convenience of the reader.

In this section only, $T$ denotes a torsion semigroup, i.e., every $t \in T$ generates a finite cyclic subsemigroup of $T$. Clearly, a semigroup is a torsion semigroup iff some power of each element is idempotent.

**Proposition 2.1.** (Green) Let $T$ be a torsion semigroup. Then $\mathcal{J} = \mathcal{D}$, i.e., for $a, b \in T$, $T^1aT^1 = T^1bT^1$ iff $T^1a \cap bT^1 \neq \emptyset$.

**Proof.** Clearly $T^1a \cap bT^1 \neq \emptyset$ implies $T^1aT^1 = T^1bT^1$. Conversely, suppose $T^1aT^1 = T^1bT^1$. Then for some $x_1, x_2, y_1, y_2 \in T^1$,

$$x_1ay_1 = b \quad \text{and} \quad x_2by_2 = a. \quad (2.1)$$

Thus $x_2x_1ay_1y_2 = a$, and we obtain inductively

$$(x_2x_1)^n(a(y_1y_2)^n) = a \quad n \geq 1. \quad (2.2)$$

Since $T$ is a torsion semigroup, there exists an integer $m$ such that $e_1 = (x_2x_1)^m$ and $e_2 = (y_1y_2)^m$ are idempotents. Thus by (2.2) $e_1ae_2 = a$, so $a = e_1a = ((x_2x_1)^{m-1}x_1)a$, and $a = ae_2 = ay_1(y_1y_2)^{m-1}$. Thus

$$T^1x_1a = T^1a \quad \text{and} \quad aT^1 = ay_1T^1. \quad (2.3)$$

From (2.1) and (2.3) we conclude that $bT^1 = x_1ay_1T^1 = x_1aT^1$, which together with the first equation of (2.3) says that $x_1a \in T^1a \cap bT^1$. This completes the proof.

We note that from Eq. (2.1) we can conclude that all of $a, ay_1, x_1a, b, by_2$, and $x_2b$ are $\mathcal{D}$-equivalent since (2.3) and the corresponding equations for $b$ imply that they are $\mathcal{J}$-equivalent. Further, (2.3) implies $x_1aLa, x_2bLb, ay_1Ra, \text{and by}_2Rb$. Thus we have

**Proposition 2.2.** (Green) Let $T$ be a torsion semigroup.

(a) Let $a, b \in T$ and suppose there exist $x_1, x_2, y_1, y_2 \in T^1$ such that (2.1) holds. Then $a, ay_1, x_1a, b, by_2$, and $x_2b$ are all $\mathcal{D}$-equivalent, and $x_1aLa, x_2bLb, ay_1Ra, \text{and by}_2Rb$.

(b) Let $h, x \in T^1$. Then $h \mathcal{J}hx \iff h \mathcal{R}hx$, and $h \mathcal{J}hx \iff h \mathcal{L}hx$.

**Remark 2.1.** (a) (Green) Using Proposition 2.1, it is possible to give a rather elementary proof of the Rees structure theorem for 0-simple torsion semigroups (see Section 3.2 of [1]).

(b) Let $S$ be a finite semigroup and let $I$ be a $0$-minimal ideal. Let $\equiv$ be the equivalence relation on $S$ with $s_1 \equiv s_2$ iff $s_1 = s_2 \in S - I$ or $s_1, s_2 \in I$. 

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and \( s_1\mathcal{H}s_2 \). Then Proposition 2.2(b) implies that \( I - \{0\} \) is a \( \mathcal{D} \)-class of \( S \) and \( = \) is a congruence on \( S \). If \( \phi : S \to S/\equiv \) is the natural homomorphism, then \( \phi(I) \) is a \( \mathcal{O} \)-minimal ideal of \( S/\equiv \) and each \( \mathcal{H} \)-class of \( S/\equiv \) contained in \( \phi(I) \) contains exactly one element.

3. **Local Properties of Homomorphisms**

If \( R \) is an equivalence relation on the set \( A \), then \( R_a \) will denote the equivalence class containing \( a \in A \). In this section we will use the ordering \( \leq \) on the relations \( \mathcal{J} \), \( \mathcal{L} \), and \( \mathcal{R} \) defined by \( \mathcal{J}_a \leq \mathcal{J}_b \) (respectively \( \mathcal{L}_a \leq \mathcal{L}_b \)) (respectively \( \mathcal{R}_a \leq \mathcal{R}_b \)) if there are idempotents \( a_1 \) in \( \mathcal{J}_a \) (respectively \( \mathcal{L}_a \)) (respectively \( \mathcal{R}_a \)) and \( b_1 \) in \( \mathcal{J}_b \) (respectively \( \mathcal{L}_b \)) (respectively \( \mathcal{R}_b \)) such that \( a_1b_1 = a_1 = b_1a_1 \) (respectively \( a_1b_1 = a_1 \) (respectively \( b_1a_1 = a_1 \)).

The following argument in its final form was given by Roger Renne.

**Proposition 3.1.** Let \( S \) be a semigroup, and let \( a \) and \( b \) be regular elements of \( S \). Then \( \mathcal{J}_a \leq \mathcal{J}_b \) (respectively \( \mathcal{L}_a \leq \mathcal{L}_b \)) (respectively \( \mathcal{R}_a \leq \mathcal{R}_b \)) if there are idempotents \( a_1 \) in \( \mathcal{J}_a \) (respectively \( \mathcal{L}_a \)) (respectively \( \mathcal{R}_a \)) and \( b_1 \) in \( \mathcal{J}_b \) (respectively \( \mathcal{L}_b \)) (respectively \( \mathcal{R}_b \)) such that \( a_1b_1 = a_1 = b_1a_1 \) (respectively \( a_1b_1 = a_1 \) (respectively \( b_1a_1 = a_1 \)).

The following argument in its final form was given by Roger Renne.

**Proof.** Suppose \( \mathcal{L}_a \leq \mathcal{L}_b \), i.e., \( S^1aS^1 \subseteq S^1bS^1 \). Since \( a \) and \( b \) are regular, there are elements \( x \) and \( y \) in \( S \) such that \( a = axa \) and \( b = byb \). Set \( a_1 = xa \) and \( b_1 = yb \). Then \( a_1^2 = a_1 \) and \( b_1^2 = b_1 \). Also, \( a_1 \in S^1a_1 = S^1a \subseteq S^1b = S^1b_1 \), so that there exists \( x \in S^1 \) such that \( a_1 = xb_1 \) and thus \( a_1b_1 = a_1 \) since \( b_1^2 = b_1 \).

When \( \mathcal{R}_a \leq \mathcal{R}_b \) the corresponding assertion is established dually.

If \( \mathcal{J}_a \leq \mathcal{J}_b \) with \( a = axa \) and \( b = byb \), we set \( b_1 = yb \) and \( a_2 = xa \). Then \( a_2^2 = a_2 \) and \( b_1^2 = b_1 \). Also, \( a_2 \in S^1a_2S^1 = S^1a^1 \subseteq S^1bS^1 = S^1b_1^1 \). Thus \( a_2 = s_1b_1s_2 \) for some \( s_1, s_2 \in S^1 \). Now set \( a_1 = b_1s_1a_2s_2b_1 \). Then \( a_1^2 = a_1a_1 = b_1s_1a_2a_2s_1a_2b_1 = b_1s_1a_2s_1b_1 = b_1s_1a_2s_2b_1 = a_1 \). So \( a_1 \) is idempotent. Also, \( S^1aS^1 = S^1a^1 \subseteq S^1bS^1 = S^1b_1^1 \), and \( a_2 = a_2s_1a_2s_2 \) for some \( s_1, s_2 \in S^1 \). Finally, \( a_1b_1 = a_1 = b_1a_1 \).

This establishes the "only if" part of the assertions. The converse is trivial.

The following construction is standard.

Since \( \mathcal{J} = \mathcal{D} \) for finite semigroups, we can introduce a coordinate system in every \( \mathcal{J} \)-class as follows. Let \( J \) be a \( \mathcal{J} \)-class of \( S \), and let \( H \subseteq J \) be an \( \mathcal{H} \)-class of \( S \). Let \( M_L : LJ_0(H) \to F_L(H) \) be the homomorphism given by \( (M_L(x))(h) = xh \). Let \( M_L(LJ_0(H)) = S_L \subseteq F_L(H) \). Then it is well known (see Section 2.4 of [1]) that \( S_L \) is a group and that there is a 1:1 correspondence \( s \leftrightarrow s^# \) between \( S_L \) and \( H \) such that for \( s_1, s_2 \in S_L \), \( s_1(s_2^#) = (s_1s_2)^# \). Dually,
we speak of $M_R : RI_S(H) \rightarrow F_S(H)$ with image $S_R$ and the correspondence $r \mapsto r^*$ between $S_R$ and $H$ with $(r^*)^* r_1 = (r^* r_1)^*$. We remark that we can choose the above correspondence so that $1^* = 1^*$. Further, the actions of $S_L$ and of $S_R$ on $H$ commute. Occasionally, for $s \in S_L$, we write $s^\ast$ for any element of $LI_S(H)$ for which $M_L(s^\ast) = s$ and $s^\ast$ for any element of $RI_S(H)$ for which $M_R(s^\ast) = s$. Also, $s \leftrightarrow s^\ast$ denotes the anti-isomorphism between $S_L$ and $S_R$ for which $s(1^*) = (1^*)s^\ast$.

Next, let $H_{11} = H$, and let $(H_{11}, \ldots, H_{m1})$ and $(H_{11}, \ldots, H_{m1})$ be the sets of $H$-classes respectively $H$- and $L$-equivalent with $H_{11}$ in $S$. Let $1 = r_1, \ldots, r_n$ and $1 = l_1, \ldots, l_m$ be chosen in $S^1$ so that $H_{11} r_k = H_{1k}$ and $l_1 H_{11} = H_{j1}$. Choose $1^* = 1^* = h_{11} \in H_{11}$. Then $S_L h_{11} = H_{11}$, since $S_L$ acts by its left regular representation, and for $h \in H_{11}$, the representation $h = s_h h_{11}$, $s_h \in S_L$, is unique. Thus, since $J = D$, every element of $J$ has a unique expression of the form $l_s h_{11} r_k$, so that $l_s h_{11} r_k \mapsto (j, s, k)$ is a well defined 1:1 mapping of $J$ onto $\{1, \ldots, n\} \times S_L \times \{1, \ldots, m\}$. Any such mapping is a coordinate map for $J$. The coordinate maps are extended to $J^0$ by sending zero into a zero added to the range.

**Proposition 3.2.** Let $\phi : S_1 \rightarrow S_2$ be an epimorphism and $\alpha(S_h)$ any of the relations $L$, $R$, $J$, or $H$ on $S_h$, $h = 1, 2$. Then

(a) $s_1 \alpha(S_1) s_1^\ast$ implies $\phi(s_1) \alpha(\phi(S_h))^\ast$.

(b) Let $J_2$ be a $J$-class of $S_2$. Then $\phi^{-1}(J_2)$ is a union of $J$-classes of $S_1$.

Let $J_1$ be a minimal (in the order $\subseteq$) $J$-class of $S_1$ contained in $\phi^{-1}(J_2)$. Then $\phi(J_1) = J_2$, and $\phi$ induces a homomorphism $\phi^* : J_1^0 \rightarrow J_2^0$. Further, each $H$-class (respectively $L$-class) of $S_1$ contained in $J_1$ maps under $\phi$ onto a $H$-class (respectively $L$-class) of $S_2$ contained in $J_2$. Each $K$-class (respectively $L$-class) of $S_2$ contained in $J_2$ is obtained in this manner. However, an $H$-class of $S_1$ contained in $J_1$ need not map under $\phi$ onto an $H$-class of $S_2$ if $J_2$ is not regular (see Example 3.1).

(c) There exist coordinate maps $C_1$ for $J_1$ and $C_2$ for $J_2$ such that $\psi = C_2 \psi^0 C_1^{-1} : (A_1 \times G_1 \times B_1)^0 \rightarrow (A_2 \times G_2 \times B_2)^0$ is given by $\psi(0) = 0$, $\psi(a_1, g_1, b_1) = (a(a_1), \lambda(a_1) \omega(g_1) \delta(b_1), \beta(b_1))$ where $\omega : G_1 \rightarrow G_2$ is a homomorphism (not necessarily onto) and $a : A_1 \rightarrow A_2$, $\beta : B_1 \rightarrow B_2$, $\lambda : A_1 \rightarrow G_2$, and $\delta : B_1 \rightarrow G_2$ are functions (but the range of $\lambda$ and $\delta$ need not be contained in $\omega(G_1)$). Further, we can assume that $1 \in (A_1 \cap B_1) \cap (A_2 \cap B_2)$, $a(1) = 1$, $\beta(1) = 1$, $\lambda(1) = 1$, and $\delta(1) = 1$. Also, for each $a_2 \in A_2$, $\lambda(a_2) \omega(G_1) = G_2$, and for each $b_2 \in B_2$, $\omega(G_1) \delta(\beta^{-1}(b_2)) = G_2$. In fact, let $H$ be an $H$-class of $S_2$ contained in $J_2$. Then for each $H$-class $R$ (respectively $L$-class) of $S_1$ contained in $J_1$ such that $X = R \cap \phi^{-1}(H) \neq \phi$ (respectively $X = L \cap \phi^{-1}(H) \neq \phi$) we have $\phi(X) = H$.

(d) $J_1$ is regular iff $J_2$ is regular. When $J_2$ is regular $\phi^{-1}(J_2)$ contains a
unique \( \leq \)-minimal \( \mathcal{J} \)-class, denoted by \( \phi^{-1}(J_2)' \), such that \( \phi^{-1}(J_2)' \) is regular and \( \phi(\phi^{-1}(J_2)') = J_2 \). In this case, as is well known, \( \phi' : \phi^{-1}(J_2)^0 \rightarrow J_2^0 \) carries each \( \mathcal{H} \)-class of \( \phi^{-1}(J_2)' \) onto an \( \mathcal{H} \)-class of \( J_2^0 \), so in (c), the choices can be made so that \( \omega(G_1) = G_2 \).

\textbf{Proof.} The proof of (a) is trivial, and (a) implies that \( \phi^{-1}(J_2) \) is a union of \( \mathcal{J}(S_1) \)-classes, proving the first assertion of (b). Now if \( J_1 \) is as in the statement of (b), then \( \phi(S_1)J_1S_1') = I \) is an ideal of \( S_2 \) which meets, hence contains \( J_2 \). Further, \( I_1 = S_1J_1S_1' - J_1 \) is an ideal of \( S_1 \), and \( \phi(I_1) \cap J_2 = \phi \) by the minimality of \( J_1 \). Thus \( \phi(J_1) = J_2 \). Now if \( R \) is an \( \mathcal{R} \)-class of \( S_1 \) contained in \( J_1 \), then \( R \) is a \( \leq \)-minimal \( \mathcal{R} \)-class of \( \phi^{-1}(J_2) \) by minimality of \( J_1 \) and since, by Proposition 2.2(b), no two distinct \( \mathcal{R} \)-classes of a \( \mathcal{J} \)-class are comparable. Thus, by an argument similar to the above \( \phi(R) \) is an \( \mathcal{R} \)-class of \( S_2 \) contained in \( J_2 \). The dual argument proves the corresponding fact about \( \mathcal{L} \)-classes. (b) now follows.

Let \( C_1 : J_1 \rightarrow A_1 \times G_1 \times B_1 \) be a coordinate map given by \( C_1(l, sh_{11}, k) = (l, s, r_k) \), and let \( C_2 : J_2 \rightarrow A_2 \times G_2 \times B_2 \) be a coordinate map where \( C_2(x, t, e_{11}, y_b) = (a, t, b) \), and \( e_{11} = \phi(h_{11}) \). Let \( a : A_1 \rightarrow A_2 \) and \( \beta : B_1 \rightarrow B_2 \) be defined by \( \phi(l_j) \mathcal{X}_{\alpha(j)} \) and \( \phi(r_k) \mathcal{Y}_{\beta(k)} \). Let \( \lambda : A_1 \rightarrow G_2 \) and \( \delta : A_1 \rightarrow G_2 \) be defined by \( \phi(l_j)h_{11} = \mathcal{X}_{\alpha(j)}h_{11} \) and \( \phi(h_{11}r_k) = \delta(k)e_{11}y_{\beta(k)} \). Let \( \omega : G_1 \rightarrow G_2 \) be the homomorphism defined by \( \phi(h_{11}) = \omega(s)e_{11} \). Then \( \phi(l_jsh_{11}r_k) = \phi(l_j)\phi(h_{11})\phi(r_k) = \phi(l_j)(\phi(s)e_{11})\phi(r_k) = (\phi(l_j)\omega(s)e_{11})(\delta(k)e_{11}y_{\beta(k)}) = (\phi(l_j)\omega(s)e_{11})(\delta(k)e_{11}y_{\beta(k)}) = \mathcal{X}_{\alpha(j)}(\lambda(j)\omega(s)\delta(k))e_{11}y_{\beta(k)} = x_{\alpha(j)}(\lambda(j)\omega(s)\delta(k))e_{11}y_{\beta(k)} \).

Thus, \( \psi \) has the form required in (c), and \( a(1) = \beta(1) = 1 \), and \( \lambda(1) = 1 \).

Now assume \( H, L, X \) are as in the statement of (c). If \( h' \in H \) and \( h'' \in \phi(X) \), then \( h' = sh'' = s'h'' \). Let \( r \in S_1 \) such that \( \phi(r) = s' \), and let \( h \in X \subseteq J_1 \) such that \( \phi(h) = h'' \). Then \( \phi(\mathfrak{r}h) = s'h'' = h' \in H \subseteq J_2 \). However, by the minimal property of \( J_1 \) we have \( \mathfrak{r}h \in J_1 \) and thus \( \mathfrak{r}h \mathcal{J}h \). Thus by Proposition 2.2(b) \( \mathfrak{r}h \mathcal{J}h \). But \( \phi(\mathfrak{r}h) = h' \in H \) so \( \mathfrak{r}h \in X \). Thus \( \phi(X) = H \). Now all the assertions of (c) follow easily.

We now prove (d). A \( \mathcal{J} \)-class \( J \) is regular iff \( J \subseteq J^2 \) (by Rees Theorem) and not regular iff \( J^2 \cap J \) is empty. Thus \( J_1 \) is regular iff \( J_2 \) is regular since from (b) \( \phi(J_1) = J_2 \). Now suppose \( J_2 \) is regular and \( J_1, J_1' \) are two distinct \( \leq \)-minimal \( \mathcal{J} \)-classes of \( S_1 \) contained in \( \phi^{-1}(J_2) \). Then by (b) \( J_2 = \phi(J_1)' = \phi(J_1) \). But \( J_2 \subseteq J_2J_2 = \phi(J_1)\phi(J_1') = \phi(J_1J_1') \). Also \( J_1J_1' \cap \phi^{-1}(J_2) = \phi \) since \( J_1, J_1' \) are two distinct minimal \( \mathcal{J} \)-classes of \( \phi^{-1}(J_2) \). Thus \( J_2 \subseteq \phi(J_1J_1') \).
and $J_2 \cap \phi(J_1 J'_1) = \phi$. This is a contradiction. Thus $\phi^{-1}(J_2)$ contains a unique \(-\)-minimal \(\mathcal{J}\)-class, $\phi^{-1}(J_2)'$.

If $J_2$ is regular, then $J_2$ contains some idempotent. Thus $e^2 = e \in H$ with $H$ an $\mathcal{H}$-class of $S_2$ contained in $J_2$. Thus by Green's relations (Section 2.1 of [1]) $H$ is a group. Now $(\phi^{-1}(H) \cap \phi^{-1}(J_2)')^0 = T$ is a subsemigroup of $J_2^0$. Let $G$ be a subsemigroup of $T$ of smallest order such that $\phi'(G) = H$. Then $G$ is a group. Let $H'$ be the $\mathcal{H}$-class of $J_1$ containing $G$. Then $\phi'(H') = H$, so $\mathcal{H}$-classes map onto $\mathcal{H}$-classes under $\phi'$ by (c).

This proves Proposition 3.2.

**Remark 3.1.** (a) Let $\phi: S_1 \rightarrow S_2$ Then from the previous proof we obtain the following result. Let $J_1$ be a $\mathcal{J}$-class of $S_1$ and let $J_2$ be the $\mathcal{J}$-class of $S_2$ such that $\phi(J_1) \subseteq J_2$. Then there exist coordinate maps $C_k$ for $J_k$, $K = 1, 2$, such that $\psi = C_2gC_1^{-1}: A_1 \times G_1 \times B_1 \rightarrow A_2 \times G_2 \times B_2$ is given by $\psi(a_1, \tilde{x}_1, b_1) = (a(a_1), \lambda(a_1) \omega(\tilde{x}_1) \delta(b_1), \beta(b_1))$, where $a: A_1 \rightarrow A_2$, $\beta: B_1 \rightarrow B_2$, $\lambda: A_1 \rightarrow G_1$, $\delta: A_1 \rightarrow G_2$ are functions and $\omega: G_1 \rightarrow G_2$ is a homomorphism.

(b) Let $S$ be a regular semigroup, and let $Q$ be a congruence on $S$. Then $\alpha(S) \lor Q = Q \cdot \alpha(S) \cdot Q = \alpha(S/Q)$, where $\alpha(S/Q)$ is regarded as a partition on $S$, $\alpha$ denotes any of $\mathcal{J}$, $\mathcal{L}$, $\mathcal{H}$, or $\mathcal{R}$, $\cdot$ is composition, and $\lor$ is the join operation of the lattice of equivalence relations on $S$.

**Example 3.1.** Let $G$ be a group, $H$ a subgroup, $\tilde{x}_1, \ldots, \tilde{x}_n$ a set of representatives for the cosets $\{gH : g \in G\}$, and $\tilde{y}_1, \ldots, \tilde{y}_n$ a set of representatives for $\{Hg : g \in G\}$. Let $A$ and $B$ be finite sets. Let $R$ denote the set $(0) \cup (A \times \{\tilde{x}_1, \ldots, \tilde{x}_n\} \times H \times \{\tilde{y}_1, \ldots, \tilde{y}_n\} \times B)$ and let $V$ denote the set $(A \times G \times B) \cup (0)$. Let $T$ denote the group $SYM(A) \times G \times SYM(B)$. Let $S_1$ be the semigroup with elements $T \cup R$ (disjoint union) where $T$ is a subgroup, $R$ is a null subsemigroup, $0$ is the zero of $S_1$, and for $(f_1, g, f_2) \in T$, $(a, \tilde{x}_k, h, \tilde{y}_j, b) \in R - \{0\}$, we have

$$(f_1, g, f_2) \cdot (a, \tilde{x}_k, h, \tilde{y}_j, b) = (f_1(a), \tilde{x}_k', h'h, \tilde{y}_j, b)$$

and

$$(a, \tilde{x}_k, h, \tilde{y}_j, b) \cdot (f_1, g, f_2) = (a, \tilde{x}_k, hh', \tilde{y}_j', f_2(b)),$$

where $h', h'^*, \tilde{x}_k', \tilde{y}_j'$ are defined by $g\tilde{x}_k = \tilde{x}_k, h'$ and $\tilde{y}_j g = h^* \tilde{y}_j$. Let $S_2$ be the semigroup with elements $T \cup V$ (disjoint union) where $T$ is a subgroup, $V$ is a null subsemigroup, $0$ is the zero of $S_2$, and if $(f_1, g, f_2) \in T$ and $(a, g', b) \in V - \{0\}$ then

$$(f_1, g, f_2) \cdot (a, g', b) = (f_1(a), gg', b)$$

and

$$(a, g', b) \cdot (f_1, g, f_2) = (a, g'g, f_2(b)).$$
Two elements of \( R \) (respectively \( V \)) are \( \mathcal{H} \)-equivalent in \( S_1 \) (respectively \( S_2 \)) iff all but their center, i.e., \( H \) (respectively \( G \)) coordinates agree. Let \( \phi : S_1 \to S_2 \) be defined as follows: \( \phi \) is the identity on \( T \), while \( \phi(0) = 0 \) and \( \phi(a, \bar{x}_k, h, \bar{y}_j, b) = (a, \bar{x}_k \bar{y}_j, b) \). Then \( \phi \) is an epimorphism, but \( \phi \) carries each \( \mathcal{H} \)-class of \( S_1 \) onto an \( \mathcal{H} \)-class of \( S_2 \) iff \( H = G \).

4. Nilpotent Elements and Irreducible Characters

Let \( C \) denote the complex numbers, and if \( M \) is a square matrix, let \( \text{tr}(M) \) denote its trace. The ideas of this section are related to those of Munn ([1], Theorem 5.33).

**Proposition 4.1.** Let \( S \) be a semigroup with zero. Then

(a) For each idempotent \( 0 \neq e \in S \), there exists an irreducible complex character \( \chi \) of \( S \) such that \( \chi(e) = 1 \) and \( \chi(0) = 0 \).

(b) Let \( x \in S \). Then the following assertions are equivalent.

\[
x^n = 0 \quad \text{for some} \quad n \geq 1
\]

\( \chi \) an irreducible complex character of \( S \) and \( \chi(0) = 0 \) implies \( \chi(x) = 0 \).

**Proof.** Let \( e \in S \), \( 0 \neq e = e^2 \). Let \( J \) be the non-zero regular \( J \)-class of \( S \) containing \( e \). Consider the standard (Schützenberger) representation of \( S \) by row-monomial matrices acting on the right of \( J \) (see Chapter 3 of [1]) which gives a homomorphism \( \phi \) of \( S \) into the semigroup of row-monomial matrices over \( G \) where \( G \) is a maximal subgroup of \( J \). Let \( \psi \) be the homomorphism of \( S \) into \( n \times n \) complex matrices with \( \psi(s)_{ij} = \phi(s)_{ij} \), where \( x'_{ij} = 0 \) if \( x_{ij} = 0 \) and \( x_{ij} = 1 \) if \( x_{ij} \in G \). Then it is trivial to verify that \( \psi(e) \) has rank one. Since \( e^2 = e \), we have \( (\psi(e))^2 = \psi(e) \), so that \( \psi(e) \) is similar to the \( n \times n \) complex matrix \( M \) with \( M_{11} = 1 \) and \( M_{ij} = 0 \) for \( (i, j) \neq (1, 1) \). Thus \( \text{tr}(\psi(e)) = 1 \). Let \( \chi \) be the character of \( \psi \). Then \( \chi(e) = 1 \) and \( \chi(0) = 0 \). Let \( \chi_1, \ldots, \chi_k \) be the irreducible constituents of \( \psi \) (the diagonal components when \( \psi \) is trianglized), and let \( \chi_1, \ldots, \chi_k \) be the corresponding irreducible characters. Then \( \chi = \chi_1 + \cdots + \chi_k \). Since \( e^2 = e \), \( \psi(e) \) is idempotent, so \( \chi_i(e) = \text{rank of} \ \psi_i(e) \). Thus, each \( \chi_i(e) \) is a non-negative integer, and \( \chi_1(e) + \cdots + \chi_k(e) = 1 \), so (by reordering if necessary) \( \chi_1(e) = 1 \), \( \chi_2(e) = \cdots = \chi_k(e) = 0 \). Now \( \chi_i \) is irreducible, so either \( \chi_i(0) = 0 \) or \( \chi_i(s) = 1 \) for all \( s \in S \). But \( 0 = \chi(0) = \chi_1(0) + \cdots + \chi_k(0) \), so \( \chi_i(0) = \cdots = \chi_k(0) = 0 \). Thus, \( \chi_1(e) = 1 \) and \( \chi_1(0) = 0 \), so (a) is proved.

We next show that (4.1) implies (4.2). Let \( \chi \) be an irreducible character.
with \( \chi(0) = 0 \). If \( \phi \) is the corresponding irreducible representation, then \( \phi(0) = 0 \), the zero matrix, and \((\phi(x))^n = \phi(x^n) = \phi(0) = 0\), so \( \phi(x) \) is nilpotent. Thus, as is well-known, \( \chi(x) = \text{tr}(\phi(x)) = 0 \).

Finally we show that (4.2) implies (4.1). Let \( x \in S \) and suppose (4.1) does not hold for \( x \). Thus for some \( n \), \( x^n = e = e^2 \neq 0 \). By (a), we can choose an irreducible character \( \chi \) such that \( \chi(e) = 1 \) and \( \chi(0) = 0 \). Let \( \phi \) be the corresponding irreducible representation. Let \( \phi(x) = M \) and by Fitting’s lemma write \( M' = A \oplus N \), the direct sum of an invertible matrix \( A \) and a nilpotent matrix \( N \), where \( M' \) is similar to \( M \), so that \( \chi(x) = \text{tr}(M') \). Then \( \chi(e) = \chi(x^n) = \text{tr}((\phi(x))^n) = \text{tr}(A^n \oplus N^n) \). But \( \phi(e) \) is an idempotent matrix, so \( A^n \) and \( N^n \) are idempotents, and thus \( N^n = 0 \). Hence \( 1 = \chi(e) = \text{tr}(M^n) = \text{rank of } A^n = \text{rank of } A, \) so \( A = (\omega) \), a \( 1 \times 1 \) matrix, where \( \omega^n = 1 \). Thus \( \chi(x) = \omega + \text{tr}(N) = \omega \neq 0 \), so (4.2) does not hold for \( x \).

This proves Proposition 4.1.

Let \( \chi \) be an irreducible complex character of \( S \) for which \( \chi(0) = 0 \) and \( \chi(S) \neq \{0\} \). Then apex (\( \chi \)) is the unique \( \leq \)-minimal \( \mathcal{J} \)-class of \( S \) in \( \{J : J \text{ is a } \mathcal{J} \text{-class of } S \text{ and } \chi(J) \neq \{0\} \} \). (See Section 5.3 of [1], i.e., use the technique of Section 3 of this paper). Apex (\( \chi \)) is regular.

We say \( x \in S \) is eventually in the \( \mathcal{J} \)-class \( J \) iff there exists an integer \( n \) such that \( x^{n+k} \in J \) for \( k = 0, 1, \ldots \). Clearly, every \( x \in S \) is eventually in a unique \( \mathcal{J} \)-class \( J \) of \( S \), and \( J \) is regular. In fact \( J \) is the regular \( \mathcal{J} \)-class of \( S \) containing the unique idempotent among the powers of \( x \).

**Corollary 4.1.** Let \( 0 \neq x \in S \). Then \( x \) is eventually in \( J \) iff \( J \) is the unique \( \leq \)-maximal \( \mathcal{J} \)-class of \( S \) in \( \{\text{Apex (} \chi \text{)} : x \text{ is an irreducible complex character of } S, \chi(x) \neq 0, \text{ and } \chi(0) = 0\} \).

**Proof.** If \( I \) is an ideal of \( S \), then Proposition 4.1 applied to \( S/I \) implies \( x^k \in I \) for some \( k \) iff all irreducible characters \( \chi \) of \( S \) with \( \chi(0) = 0, \chi(S) \neq \{0\}, \) and apex disjoint from \( I \), vanish at \( x \). Now the corollary follows easily by letting \( I = F(s) = \{J' : J' \text{ is a } \mathcal{J} \text{-class of } S \text{ and } \mathcal{J}_s \leq \mathcal{J}' \text{ is false} \} \) and letting \( s \) vary over \( S \).

5. **Complexity of** \( F_R(X_n) \)

Let \( X_n = \{1, \ldots, n\} \) \( n \geq 2 \). Then \( F_R(X_n) \) denotes the semigroup of all mappings of \( X_n \) into \( X_n \) under the multiplication \((f \cdot g)(x) = g(f(x)) \).

The complexity of a semigroup was first defined in [3] and [4]. We first give an equivalent definition of complexity.

**Notation 5.1.** The 4-tuple \( \mathcal{T} = (x, X, S, \phi) \) denotes a transformation
semigroup, where $X$ is a finite non-empty set, $x \in X$, $\phi : S \rightarrow F_R(X)$ is a homomorphism, and $X = \{\phi(s)(x) : s \in S\} \cup \{x\} = \{xs : s \in S\} \cup \{x\} = xS$. We say $\mathcal{I}$ is faithful if $\phi$ is 1:1. $F(\mathcal{I}) = (x, X, S/F, \psi)$ where $\equiv$ is the congruence induced on $S$ by $\phi$ and $\psi([s]) = \phi(s). F(\mathcal{I})$ is clearly faithful.

We write $(y, Y, T, \psi) \subseteq (x, X, S, \phi)$ iff $T$ is a subsemigroup of $S$, $Y \subseteq X$, $y = x$, and $\psi$ is $\phi$ restricted to $T$. We write

$$(y, Y, T, \psi) \leftrightarrow (x, X, S, \phi)$$

iff there exists $\theta : X \rightarrow Y$ with $\theta(x) = y$ and a homomorphism $\alpha : S \rightarrow T$ such that $\theta(xs) = \theta(x)\alpha(s)$ for $x \in X, s \in S$. $\mathcal{I}_1 \approx \mathcal{I}_2$ iff $\mathcal{I}_1 \leftrightarrow \mathcal{I}_2$ and $\mathcal{I}_2 \leftrightarrow \mathcal{I}_1$.

We write $\mathcal{I}_1 = (y, Y, T, \psi) \parallel (x, X, S, \phi) = \mathcal{I}_2$ iff there exists $(x, Z, U, \beta) = \mathcal{I}_2$ such that

$$\mathcal{I}_1 \leftrightarrow \mathcal{I}_2 \subseteq \mathcal{I}_3.$$

We write $\mathcal{I}_1 \mid \mathcal{I}_2$ read $\mathcal{I}_1$ divides $\mathcal{I}_2$ iff $F(\mathcal{I}_1) \mid F(\mathcal{I}_2)$.

We write $(x, X, S)$ for $(x, X, S, \phi)$ when $S$ is a subsemigroup of $F_R(X)$ and $\phi$ is the inclusion map.

Let $\mathcal{I}_k = (x_k, X_k, S_k)$ be given for $k = 1, 2, ..., n$. Then the wreath product $\mathcal{I}_2 \setminus \mathcal{I}_1 = ((x_2, x_1), X_2 \times X_1, S_2 \circ S_1)$ where $S_2 \circ S_1 = \{(w(f_2, f_1) \in F_R(X_2 \times X_1) : f_1 \in S_1, f_2 \in S_2, w(f_2, f_1) = (w(f_2, f_1)(y_2, y_1) = (f_2(y_2)(y_1), f_1(y_1))) \text{ for } (y_2, y_1) \in X_2 \times X_1 \}$. $\mathcal{I}_2 \setminus \mathcal{I}_1 = \mathcal{I}_3 \setminus \mathcal{I}_2 \setminus \mathcal{I}_1$, so $\mathcal{I}_n \setminus \cdots \setminus \mathcal{I}_1$ is well-defined (up to $\approx$).

Let $(x, X, S)$ be given. We are interested in solutions of

$$(x, X, S) \mid (x_n, X_n, S_n) \setminus \cdots \setminus (x_1, X_1, S_1). \quad (5.1)$$

**Definition 5.1.** Let $\mathcal{I} = (x, X, S)$ be given. Then $\#(\mathcal{I})$, the complexity number of $\mathcal{I}$, is the smallest positive integer $n$ such that Eq. (5.1) holds where either

(a) $S_1, S_2, S_3, ...$ are groups and $S_4, S_5, S_6, ...$ combinatorial semigroups or

(b) $S_1, S_2, S_3, ...$ are combinatorial semigroups and $S_4, S_5, S_6, ...$ are groups.

**Notation 5.2.** We define $C(\mathcal{I}) = (n, G)$ iff (a) holds for $n = \#(\mathcal{I})$ but (b) never holds with $n = \#(\mathcal{I})$. Similarly, $C(\mathcal{I}) = (n, C)$ iff (b) can hold with $n = \#(\mathcal{I})$, but (a) cannot. Finally, $C(\mathcal{I}) = (n, C \lor G)$ iff either (a) or (b) can hold with $n = \#(\mathcal{I})$. That $C(\mathcal{I})$ is well defined follows easily from Corollary 3.2 (a) of [2].

The set of all complexities, $\mathcal{C} = \{1, 2, ..., \} \times \{C, G, C \lor G\}$. Let $\# : \mathcal{C} \rightarrow$
{1, 2,...} with \( \#(n, a) = n \). Then \( \#(\mathcal{F}) = \#(C(\mathcal{F})) \). \( \mathcal{C} \) is partially ordered by \( \leq \), where \( C_1 \leq C_2 \) iff

(a) \( C_1 = C_2 \) or
(b) \( \#(C_1) < \#(C_2) \) or
(c) \( \#(C_1) = \#(C_2) = n \) and \( C_1 = (n, C \vee G) \).

Then \( (\mathcal{C}, \leq) \) is a lattice with minimal element \((1, C \vee G)\).

Let \( S' \) denote the semigroup obtained from \( S \) by adjoining an identity \( I \) regardless of whether or not \( S \) already has an identity. Let \( \mathcal{R}(S) = (I, S', \mathbb{R}(S)) \) where \( \mathbb{R} : S \rightarrow F_{\mathcal{R}}(S') \) with \( (\mathbb{R}(s))(s') = s' s \). Clearly \( \mathbb{R} \) is a 1:1 homomorphism.

**Definition 5.2.** If \( S \) is a semigroup, \( C(S) = C(\mathcal{R}(S)) \).

We next show that \( C((x, X, S)) \) depends only on \( S \).

**Proposition 5.1.** (a) \( C((x, X, S)) = C(S) \).
(b) if \( C_4(S) \) denotes the complexity of \( S \) as defined in [4], then \( C(S) = C_4(S) \).

Before giving the proof, we recall an equivalent definition of \( C_4(S) \) phrased in our present terminology (see Section 1 of [4]). Consider the equation

\[
\mathcal{R}(S) \text{ "divides" } \mathcal{R}(S_1) \setminus \cdots \setminus \mathcal{R}(S_n)
\]  

where "divides" here means that the relation " | " of Notation 1.1 holds between the abstract semigroups defined by each side. (5.2) does not mean division in the sense of Notation 5.1. Now \( C_4(S) \) is defined by replacing (5.1) by (5.2) in Definition 5.1.

We now prove Proposition 5.1.

We first observe that \((x, X, S) \preceq \mathcal{R}(S) \) with \( \theta : S' \rightarrow X \) given by \( \theta(s) = xs \) \((xI = x)\), and \( \alpha : S \rightarrow S \) the identity map. Thus \( C(x, X, S) \leq C(S) \).

By a similar technique it is easy to verify that (5.1) implies (5.2). Thus \( C_4(S) \leq C(x, X, S) \), so \( C_4(S) \leq C(x, X, S) \leq C(S) \), and it is sufficient to prove

\[
C(S) \leq C_4(S).
\]  

To prove (5.3) we appeal to the techniques of [2]. The following lemma is a slight variation of Proposition 1.2 of [2]. See Notation 1.1. If \( h : A \rightarrow B \), then \( h^\tau : \sum A \rightarrow \sum B \) is the unique extension of \( h \) to a homomorphism, so \( h^\tau(a_1, ..., a_n) = (h(a_1), ..., h(a_n)) \).

**Lemma 5.1.** Let

\[
S' = h_3 S_2 h_2^\tau S_1^\tau h_1^\tau.
\]

Then

\[
\mathcal{R}(S) \setminus (\mathcal{R}(S_1) \setminus \mathcal{R}(S_2)).
\]
Proof. Let \( \theta : S_2^I \times S_1^I \rightarrow S^I \) be defined by \( \theta(I, s_1) = I \) and \( \theta(s_2, s_1) = h_2(s_2) \) for \( s_2 \in S_2 \), \( s_1 \in S_1^I \). For \( s \in S \) let \( \bar{s} = (\bar{s}(I), \bar{s}(s)) \in \mathcal{R}(S_2) \cap \mathcal{R}(S_1) \) with \( \bar{s}(I) = R(h_1(s)) \) and \( \bar{s}(s) : S_1^I \rightarrow R(S_2) \) defined by \( \bar{s}^{(s)}(s_1) = R(h_2(s_1^{(s)})) \). Let \( T \) be the subsemigroup of \( S_2 \times S_1 \) generated by \( \{ t : s \in S \} \). Let \( X = (I, I)T \cup \{(I, I)\} \). Then for \( u_1, \ldots, u_n \in S \), \( \theta(u_1 \cdots u_n) = u_1 \cdots u_n \) is a well-defined homomorphism of \( T \) onto \( S \) such that \( \theta(xu_1 \cdots u_n) = \theta(x)\psi(u_1 \cdots u_n) = \theta(x)u_1 \cdots u_n \). This proves Lemma 5.1.

Now Lemma 3.1 of [2] almost proves a converse to the above lemma. In fact, Eq. (5.2) and the first line of the proof of Lemma 3.1 of [2] imply that

\[
S' = h_2T_{2n-1}^{I}h_2^{I} \cdots h_2 \cdot T^{I} \cdot h_1^{I},
\]

where for \( 1 \leq k \leq n - 1 \), \( T_{2k} = C_k \), a combinatorial semigroup, and for \( 1 \leq k \leq n \), \( T_{2k - 1} = V_k \times A_k^r \) where \( A_k \) is a finite non-empty set and \( V_k \) is the direct product of \( S_k \) with itself \( a(k) \) times. Now \( S^I \) is a subsemigroup of \( S \times \{0\}^I \), so \( T_{2k+1} \) can be taken to be either \( U_k \times W_k \times A_k^r \) or \( H_k \times A_k^r \times U_k \) where \( U_k \) (respectively \( W_k \)) denotes the direct product of \( S_k \) (respectively \( \{0\}^I \)) with itself \( a(k) \) times. Further, for semigroups \( T, T' \),

\[
\mathcal{R}(T \times T') \cap \mathcal{R}(T) \times \mathcal{R}(T'),
\]

and \( C_1, \ldots, C_j \) combinatorial implies

\[
\mathcal{R}(C_j) \cap \cdots \cap \mathcal{R}(C_1) \cap \mathcal{R}(C)
\]

for some combinatorial semigroup \( C \) (see [2]). Thus, (5.4), Lemma 5.1 (extended inductively to \( 2n - 1 \) terms) and the associativity of "\( \cap \)" imply that (5.1) holds for some \( S_1', \ldots, S_n' \), where case (a) or case (b) of Definition 5.2 holds depending respectively on whether case (a) or case (b) holds for \( S_1, \ldots, S_n \) in (5.2).

This completes the proof of Proposition 5.1.

It is convenient to introduce the following notation. For \( n \geq 1 \), let \( (C \vee G, n) = (n, C \vee G), (G, 2n) = (2n, C), (C, 2n) = (2n, G), (G, 2n - 1) = (2n - 1, G), \) and \( (C, 2n - 1) = (2n - 1, C) \).

**Proposition 5.2.** \( C(F_R(X_{n})) = (2(n - 1), G) = (C, 2(n - 1)) \) for \( n \geq 2 \).

The proof will follow from the following lemmas.

**Lemma 5.2.** \( C(F_R(X_n)) \leq (2(n - 1), G) \) for \( n \geq 2 \).

**Proof.** This immediately follows from the method of Zeiger [8]. We give a brief outline of the details.

Let \( PR(X_p) \) denote the subsemigroup of \( F_R(X_p) \) consisting of all functions \( f \) whose range contains either \( p \) elements or 1 element, i.e., the permutations
and the constants (resets) on $X_n$. Then for $n \geq 2$ $F_R(X_n)$ divides (in the sense of Notation 1.1) the semigroup $W_n$ determined by

$$W_n = (1, X_{n-1}, F_R(X_{n-1})) \cap (1, X_n, PR(X_n)).$$

To see this, let $f \in F_R(X_n)$. Let $A_1, \ldots, A_n$ be the subsets of $X_n$ of order $n - 1$ where $A_j = \{a_{j1}, \ldots, a_{j(n-1)}\}$. If $f$ is a permutation on $X_n$ let $f_1 \in PR(X_n)$ be determined by $f(A_j) = A_{f(j)}$, so $f_1$ is a permutation on $X_n$. In this case let $(f_2(j))(l) = m$, where

$$f_2(j) = \begin{cases} a_{j1}, \ldots, a_{j(n-1)} & \text{if } f \text{ is a permutation on } X_n, \\ a_{j1}, \ldots, a_{j(m-1)}, a_{q(m+1)}, \ldots, a_{q(n-1)} & \text{if } f \text{ is not a permutation on } X_n \end{cases}$$

so $f_2(j)$ is a permutation on $X_{n-1}$. If $f$ is not a permutation on $X_n$ choose $A_j'$ so that $f(X_n) \subseteq A_j'$, and let $f_2(p) = j'$ for all $p \in X_n$ so $f_1$ is a constant mapping on $X_n$. Let $(f_2(j))(l) = m$ where

$$f_2(j) = \begin{cases} a_{j1}, \ldots, a_{j(n-1)} & \text{if } f \text{ is a permutation on } X_n, \\ a_{j1}, \ldots, a_{j(m-1)}, a_{j'(m+1)}, \ldots, a_{j'(n-1)} & \text{if } f \text{ is not a permutation on } X_n \end{cases}$$

Let $\psi(f) = (f_2, f_1) \in W_n$. Let $T$ be the subsemigroup of $W_n$ generated by $\{\psi(f) : f \in F_R(X_n)\}$. Then it is easy to verify that $\phi : T \to F_R(X_n)$ given by $\phi(\psi(h_1, \ldots, h_k)) = h_1 \cdots h_k$, $h_i \in F_R(X_n)$, is a well-defined homomorphism.

Now it is not difficult to show that $C(PR(X_n)) \leq (C, 2)$ (see Lemma 3.6 of [2]). Now Lemma 5.2 follows easily.

**Lemma 5.3.** (a) $C(F_R(X_n)) = (k, G)$ for some $k \geq 1$ if $n \geq 2$.

(b) Let $IG(S)$ denote the subsemigroup of $S$ generated by $I(S) = \{e \in S : e^2 = e\}$. If $S - IG(S) \neq \{1\}$, then $C(S) = (k, G)$ for some $k \leq 1$.

Assume for a moment that Lemma 5.3 has been proved. Then we will derive Proposition 5.2 by induction on $n$. Let $n = 2$. Then from Lemma 5.2, $C(F_R(X_2)) \leq (2, G)$. If equality does not hold, then by Lemma 5.3 (a) $C(F_R(X_2)) = (1, G)$, which is absurd. Thus $C(F_R(X_2)) = (2, G)$.

Now assume Proposition 5.2 is true for $n = k \geq 2$. Then by Lemma 5.2 $C(F_R(X_{k+1})) \leq (2k, G)$. Also, $F_R(X_n) \mid F_R(X_{n+1})$ (as in Notation 1.1), so by induction and by the above, $$(2(n - 1), G) \leq C(F_R(X_{n+1})) \leq (2n, G).$$

Then by Lemma 5.3(a) $C(F_R(X_{n+1})) = (2n, G)$ as desired or $C(F_R(X_{n+1})) = (2n - 1, G)$ or $(2(n - 1), G)$. Further, $IG(F_R(X_{n+1}))$ is easily seen to be $F_R(X_{n+1}) - H_{n+1}$, where $H_{n+1}$ is the set of all permutations on $X_{n+1}$. Hence $F_R(X_n) \mid IG(F_R(X_{n+1}))$ so if either of the latter two alternatives holds, then

$$(2(n - 1), G) = C(F_R(X_n)) \leq C(IG(F_R(X_{n+1}))) \leq (2(n - 1), C)$$

a contradiction. Thus, $C(F_R(X_{n+1})) = (2n, G)$. 

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We now prove Lemma 5.3. $F_R(X_n)$ is generated by $H_n$ and any subsemigroup $\{e_n, e_{n-1}\}$ in which $e_n$ is the identity of $H_n$, and $e_{n-1}$ is an idempotent with $n - 1$ elements in its range. Thus, we can apply the following lemmas to $F_R(X_n)$. In the following we write $S_2 \setminus S_1$ for $\mathcal{R}(S_2) \setminus \mathcal{R}(S_1)$ and $S \setminus S' \setminus S''$ for $\mathcal{R}(S) \setminus \mathcal{R}(S') \setminus \mathcal{R}(S'')$.

**Lemma 5.4.** Let $S \setminus S_2 \setminus C$ where $C$ is a combinatorial semigroup. Let $e_n \geq e_{n-1} \geq \cdots \geq e_1$ be a chain of idempotents of $S$, i.e., $e_i e_j = e_j e_i = e_i$ for $n \geq i \geq j \geq 1$. Let $H_k$ be the maximal subgroup of $S$ containing $e_k$. Let $S'$ be the subgroup of $S$ generated by $H_n \cup \cdots \cup H_1$. Then $S' \setminus S_2 \setminus B$, where $B$ is a chain of idempotents.

**Proof.** First notice that $e_k$ is the identity of the subsemigroup of $S$ generated by $H_k \cup \cdots \cup H_1$. Let $T \subseteq S_2 \setminus C$ be a subsemigroup and $\phi: T \to S$ a homomorphism. Let $G_n$ be a subgroup of $T$ such that $\phi(G_n) = H_n$. (Any subsemigroup of $T$ minimal with respect to being mapped by $\phi$ onto $H_n$ is a subgroup of $T$). Let $f_n$ be the identity of $G_n$. Then $\phi(f_n) = e_n$, and $\phi(f_n f_{n-1})$ contains $S'$. Let $f_{n-1}$ be the identity of $G_{n-1}$. By continuing in this fashion we can choose subgroups $G_n, ..., G_1$ of $T$ with identities $f_n, ..., f_1$ respectively such that $\phi(G_k) = H_k$ and $f_n \geq f_{n-1} \geq \cdots \geq f_1$. Then $\phi$ maps the subsemigroup $T'$ of $T$ generated by $G_n \cup \cdots \cup G_1$ onto $S'$. Let $p_1$ be the projection homomorphism of $S_2 \setminus C$ onto $C$. Then $p_1(G_k) = \{h_k\}$ since $C$ is combinatorial and $G_k$ is a group. Further, $h_k^2 = h_k$, and $h_n \geq \cdots \geq h_1$, so $p_1(T') = \{h_n, ..., h_1\} = B$. Now Lemma 5.4 follows easily.

**Lemma 5.5.** Let $U_2 = \{0, 1\}^I$. Then

(a) $B$ is a commutative band iff $B$ is a subsemigroup of the direct product $U_2 \times \cdots \times U_2$ of $U_2$ with itself $n$ times for some $n$.

(b) Let $G$ be a group. Then $G \wr U_2 = F(U_2, G) \times F(U_2, G')$ divides $C \wr G'$, where $C$ is some combinatorial semigroup and $G' = F(U_2, G)$.

**Proof.** (a) $U_2 \times \cdots \times U_2$ and all its subsemigroups are commutative bands. Conversely, if $B$ is a commutative band it can be represented with respect to each $\mathcal{J}$-class by $1 \times 1$ matrices over $U_2$, and the direct product of these representations is $1 : 1$ (see [1], Chapter 3).

(b) $(G \wr U_2)'$ has the decomposition

$$(G \wr U_2)' = h_5(X \times Y) h_4 \gamma_2 x^o (U_2 \times Y)^{\phi_0 (\{0, 1\}^r \times F(U_2, G))^{\phi_1} h_1}$$

where $Y = F(U_2, G), X = \{0, 1\} \times Y, 2_x: \Sigma X \to (X \cup \{0\}) \times X$ is given by $2_x(x_1) = (\{0\}, x_1)$ and $2_x(x_1, \ldots, x_n) = (x_{n-1}, x_n)$ for $n \geq 2$. $(2_x)^s$ is combinatorial, since $2_x$ divides $D_x \times X^r$. (See [2]). $h_5: G \wr U_2 \to X$, $h_4: G \wr U_2 \to Y$, $h_3: G \wr U_2 \to X^r$, $h_2: G \wr U_2 \to Y^{\phi_0}$, $h_1: G \wr U_2 \to X^r$. (See [2]).
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$h_4 : (X \cup \{x\}) \times X \to X \times (Y \cup \{1\})$, and $h_5 : X \times (Y \cup \{1\}) \to G \wr U_2$

are defined by $h_4(f, x) = (x, f)$; $h_4(1, f) = (1, f, 1)$, $h_5(x, 0, f) = (0, f, 1)$,

$h_4((0, f), (0, g)) = (0, g, 1)$, $h_4((1, f), (1, g)) = (1, g, 1)$, $h_4((1, f), (1, g)) = (1, g, 1)$, and $h_4((1, f), (0, g)) = (0, g, f)$; $h_4((x, f), 1) = (x, f)$ and $h_5((x, f), g) = (x, g \cdot C_{x-1(\cdot)}(\cdot))$ where $C_x$ denotes the constant function to $x$, and $g^{-1}(x) = g(x)^{-1}$. Now Lemma 5.5 follows from Lemma 5.1.

Corollary 5.1. (of Lemmas 5.4 and 5.5). Let $S \mid S \subseteq G \setminus C$ where $C$ is combinatorial and $G$ is a group. Let $S'$ be as in Lemma 5.4. Then $S' \mid S \subseteq C' \setminus G'$ where $C'$ is combinatorial and $G'$ is a group.

Now since $F_{\mathbb{R}}(X_n)$ is generated by $H_n$ and $e_{n-1}$, Corollary 5.1 implies Lemma 5.3(a). (The case $n = 2$ can be argued similarly.) Now Lemma 5.3(b) follows from Lemma 2.1 of [4].

This proves Proposition 5.2.

Remark 5.1. (a) Let $F_L(X_n)$ be the reverse semigroup of $F_{\mathbb{R}}(X_n)$. Then $C(F_L(X_n)) \geq (2(n - 1), G)$ since the proof that $C(F_{\mathbb{R}}(X_n)) \geq (2(n - 1), G)$ by Lemmas 5.3–5.5 holds also for $F_L(X_n)$.

(b) At another time we will show $C(F_L(X_n)) = (2(n - 1), G)$ and investigate the relation between $C(S)$ and $C(r(S))$ where $r(S)$ denotes the reverse semigroup.

(2) Let the width $w(S)$ of a semigroup $S$ be the maximum number of $\mathcal{L}$- or $\mathcal{R}$-classes contained in any $\mathcal{J}$-class of $S$. Then assuming that $C(F_L(X_n)) = (2(n - 1), G) = C(F_{\mathbb{R}}(X_n))$ and that $S$ is a regular semigroup, we have

$$\#(S) \leq 2w(S) + 1.$$ 

This follows from the wreath product expressions for the Preston-Schützenberger representations of $S$ by row-monomial and column-monomial matrices over groups. See the first paragraph of the proof of Proposition 6.9.

6. Decompositions of Homomorphisms

All semigroups are assumed to be of finite order unless the contrary is explicitly stated.

Definition 6.1. Let $S$ be a fixed semigroup. A property of the homomorphisms of $S$ is a collection $\mathcal{P}$ of pairs $(\phi, T)$, where $\phi : S \to T$ such that whenever $(\phi_1, T_1) \in \mathcal{P}$ and $j : T_1 \to T_2$ is an isomorphism then $(j \phi_1, T_2) \in \mathcal{P}$.

We write $(\phi_1, T_1) \leq (\phi_2, T_2)$ iff there exists a homomorphism $f : T_2 \to T_1$ such that $\phi_1 = f \phi_2$. $(\phi, T)$ is a minimal (respectively maximal) homomorphic
image of $S$ with respect to $\mathcal{P}$ iff $(\phi, T)$ is a minimal (respectively maximal) element of $P$ under $\leq$. To say that $(\phi, T)$ is minimal means that if $(\phi_1, T_1) \in \mathcal{P}$, then there exists a homomorphism $j$ to make the following diagram commutative:

$$
\begin{array}{ccc}
S & \xrightarrow{\phi_1} & T_1 \\
& j & \\
& \phi & T
\end{array}
$$

**Remark 6.1.** (a) Let $(\phi_1, T_1)$ and $(\phi_2, T_2)$ be minimal homomorphic images of the semigroup $S$ with respect to $\mathcal{P}$. Then there is an isomorphism $j : T_1 \rightarrow T_2$ such that $\phi_2 = j\phi_1$ and $\phi_1 = j^{-1}\phi_2$.

(b) Let $P$ be a partition on $S$. Let $\mathcal{P}(S, P)$ be the collection of $(\phi, T)$ such that $\phi(s_1) = \phi(s_2)$ implies $s_1 \equiv s_2$ (mod $P$). Let $Q$ be the congruence generated by $P$, i.e., $s_1 \equiv s_2$ (mod $Q$) iff $\alpha_1\beta \equiv \alpha_2\beta$ (mod $P$) for all $\alpha, \beta \in S^1$. Let $\eta : S \rightarrow S/Q$ be the natural homomorphism. Then $\eta, S/Q$ is the minimal homomorphic image of $S$ with respect to $\mathcal{P}(S, P)$. We denote this minimal homomorphic image by $S^\mathcal{P}$. We write $\phi : S \rightarrow T$ iff $\phi(s_1) = \phi(s_2)$ implies $s_1 \equiv s_2$ (mod $P$).

(c) Let $\mathcal{P}$ be a property of the homomorphisms of $S$. If $Q$ is a congruence on $S$, let $(\eta_Q) : S \rightarrow S/Q$ be the natural homomorphism. Let $Q' = \text{lub}\{Q : Q$ is a congruence on $S$ and $((\eta_Q), S/Q) \in \mathcal{P}\}$. Then $S$ has a minimal homomorphic image with respect to $\mathcal{P}$ iff $((\eta_{Q'}), S/Q') \in \mathcal{P}$. In this case, $((\eta_{Q'}), S/Q')$ is the minimal homomorphic image. Thus, if $\mathcal{P}$ is non-empty, then $S$ has a minimal homomorphic image with respect to $\mathcal{P}$ if $((\eta_Q), S/Q), (\eta_{Q_2}), S/Q_2) \in P$ implies $((\eta_Q), S/Q) \in \mathcal{P}$, where $Q = \text{lub}\{Q_1, Q_2\}$.

(d) If $|A| \geq 3$, then $A^r$ has no minimal homomorphic image with respect to $\mathcal{P}$ if $\mathcal{P}$ is the collection of $(\phi, T)$ such that $\phi : S \rightarrow T$ and $|T| = 2$.

**Notation 6.1.** (see Definition 5.1 of [4]). Let $M$ be a semigroup. For $m \in M$, let $\bar{m}$ denote the $J$-class of $M$ containing $m$ and let $F(m) = F(\bar{m})$ be the ideal $\cup\{\bar{m}_1 : \bar{m} \leq \bar{m}_1 \text{ is false}\}$. Let $(\eta_m) : M \rightarrow M/F(m)$ be the natural homomorphism. $GM_m^z(M) = \left(M/F(\bar{m})\right)/\equiv$ where for $s_1, s_2 \in M/F(\bar{m})$, $s_1 \equiv s_2$ if $x_1s_1x_2 = x_1s_2x_2$ in $M/F(\bar{m})$ for all $x_1, x_2 \in \bar{m}$. Let $(\eta \equiv (\eta_m)) = H_m^z : M \rightarrow GM_m^z(M)$. Let $GM_m(M) = GM_m^z(M)$ if $\bar{m}$ is not combinatorial, and let $GM_m(M) = \{0\}$ if $\bar{m}$ is combinatorial. Let $H_m : M \rightarrow GM_m(M)$ be respectively $H_m^z$ or the trivial map.

**Proposition 6.1.** Let $S$ be a semigroup. Let $\mathcal{P}(S, \gamma)$ be the collection of $(\phi, T)$ such that $\phi : S \rightarrow T$ and $\phi$ restricted to any subgroup of $S$ is 1:1. Then $S$ has a minimal homomorphic image with respect to $\mathcal{P}(S, \gamma)$. 
Proof. If $S$ is combinatorial, then $S \rightarrow \{1\}$ is the minimal homomorphic image with respect to $\mathcal{P}(S, \gamma)$. Assume $S$ is not combinatorial, and let $J_1, \ldots, J_k$ be the $k$ distinct non-combinatorial $\mathcal{J}$-classes of $S$, ordered as they appear in some principal ideal series, i.e., so ordered that if $i < j$, then $J_j \leq J_i$ is false. Let $m_i \subset J_i$ for $1 \leq i \leq k$. We now define homomorphisms $\psi_1, \ldots, \psi_k$ of $S$ inductively as follows: $\psi_1 = H_{m_1} : S \rightarrow GM_{m_1}(S). \psi_1$ is $1:1$ on subgroups of $S$ contained in $J_1$. Now assume $1 \leq i < k$ and $\psi_i$ has been defined. If $\psi_i$ is $1:1$ on subgroups of $J_{i+1}$, let $\psi_{i+1} = \psi_i$. Otherwise, let $\psi_{i+1} = (\psi_i \times H_{m_{i+1}})\Delta : S \rightarrow \psi_i(S) \times GM_{m_{i+1}}(S)$, where $\Delta : S \rightarrow S \times S$ is given by $\Delta(s) = (s, s)$, so $\psi_{i+1}(s) = (\psi_i(s), H_{m_{i+1}}(s))$. Let $\psi = \psi_k$. We prove that $(\psi, \psi(S))$ is the minimal homomorphic image of $S$ with respect to $\mathcal{P}(S, \gamma)$.

We write $\phi : S \rightarrow T$ iff $\phi$ is $1:1$ on subgroups. It is trivial to verify that $(\psi, \psi(S)) \in \mathcal{P}(S, \gamma)$. Now if $\phi : S \rightarrow T$, we must show that $\phi(s_1) = \phi(s_2)$ implies $\psi(s_1) = \psi(s_2)$. Equivalently, we will show by induction that $\phi(s_1) = \phi(s_2)$ implies $\psi(s_1) = \psi(s_2)$ for $1 \leq i \leq k$. Suppose $\phi(s_1) = \phi(s_2)$. Let $x_1, x_2 \in J_1$. We first show that $x_1x_2 \in J_1$ iff $x_1x_2 \in J_1$. If $x_1, x_2 \in J_1$ and $x_1x_2 \notin J_1$, then $x_1x_2 \in B(J_1) = \cup\{J : J$ is a $\mathcal{J}$-class of $S$ and $J < J_1\}$, which is a combinatorial ideal or is empty. But $J_1$ is regular, so $(J_1 \cup B(J_1))/B(J_1)$ is a noncombinatorial 0-simple semigroup. Thus, there exist $v_1, v_2 \in J_1$ such that $v_1x_1x_2v_2$ generates a non-trivial cyclic group, while $v_1x_1x_2v_2$ lies in the combinatorial ideal $B(J_1)$. But $\phi(v_1x_1x_2v_2) = r = \phi(v_1x_1x_2v_2)$, so $r$ lies in the combinatorial ideal $B(J_1)$, and $r$ generates a non-trivial cyclic group, since $\phi$ is $1:1$ on subgroups of $S$. This is a contradiction, proving that $x_1x_2 \in J_1$ iff $x_1x_2 \in J_1$.

We next show that if $x_1x_2 \in J_1$ and $x_1x_2 \in J_1$, then $x_1x_2 = x_1x_2$. By Rees theorem and the well-known properties of translations of 0-simple semigroups, $x_1x_2 \in \mathcal{H}x_1x_2$, and there exist $z_1, z_2, a_1, a_2 \in J_1$ so that if $b_1 = x_1x_2z_2$ and $b_2 = x_1x_2z_2$, then $b_1$ and $b_2$ lie in the same maximal subgroup of $J_1$, and $a_1b_1a_2 = x_1x_2z_2$, $a_2b_2a_1 = x_1x_2z_2$. Now, since $\phi$ is $1:1$ on subgroups and $\phi(b_1) = \phi(b_2)$, we have $b_1 = b_2$, so $x_1x_2 = x_1x_2$. Thus, for all $x_1, x_2 \in J_1$, either $x_1x_2$ and $x_1x_2$ both lie in $B(J_1)$, or $x_1x_2 = x_1x_2 \in J_1$. Hence, $\psi(s_1) = \psi(s_2)$.

Now by induction assume that $\phi(s_1) = \phi(s_2)$ implies $\psi(s_1) = \psi(s_2)$ for all $i$ such that $1 \leq i < k$. We must show that $\phi(s_1) = \phi(s_2)$ implies $\psi(s_1) = \psi(s_2)$. If $\psi_i$ is $1:1$ on subgroups of $J_{i+1}$, then $\psi_i = \psi_{i+1}$, and the induction step proceeds trivially. Hence, assume there exists a maximal subgroup $G \subset J_{i+1}$, $G \neq \{1\}$, and elements $g_1, g_2 \in G$ such that $\psi_i(g_1) = \psi_i(g_2)$. Then $\psi_{i+1} = (\psi_i \times H_{m_{i+1}})\Delta$ where $\Delta : S \rightarrow S \times S$ with $\Delta(s) = (s, s)$. Let $x_1, x_2 \in J_{i+1}$. We now show that $x_1x_2 \in J_{i+1}$ iff $x_1x_2 \in J_{i+1}$, since we can then proceed as before to prove that $\psi_{i+1}(s_1) = \psi_{i+1}(s_2)$.

Thus suppose $\phi(s_1) = \phi(s_2)$. Assume $x_1x_2 \in J_{i+1}$ and $x_1x_2 \notin J_{i+1}$. Then $x_1x_2 \in B(J_{i+1})$, and $B(J_{i+1})$ is a maximal ideal of $B(J_{i+1}) \cup J_{i+1}$, so
\( \phi(B(J_{j+1})) = \phi(B(J_{j+1}) \cup J_{j+1}) \text{ or } \phi(J_{j+1}) \) is disjoint from \( \phi(B(J_{j+1})) \). The second alternative cannot occur, since \( \phi(x_1x_2) = \phi(x_1x_2) \in \phi(J_{j+1}) \cap \phi(B(J_{j+1})) \). Thus \( \phi(B(J_{j+1})) = \phi(B(J_{j+1}) \cup J_{j+1}) \).

Now let \( H = \phi(G) \cong G \). Let \( S_1 = \phi^{-1}(H) \cap (B(J_{j+1}) \cup J_{j+1}) \). Then \( \phi(S_1) - H \), and, since \( \phi(B(J_{j+1})) = \phi(B(J_{j+1}) \cup J_{j+1}) \), \( S_1 \cap B(J_{j+1}) \) is a nonempty ideal of \( S_1 \), so \( \phi(S_1 \cap B(J_{j+1})) = H \). Let \( K(S_1) \) denote the kernel of \( S_1 \). Then \( K(S_1) \subseteq B(J_{j+1}) \) and \( \phi(K(S_1)) = H \). Since \( \phi \) is 1:1 on subgroups of \( K(S_1) \), there exists an isomorphism \( \alpha : K(S_1) \rightarrow A^+ \times B^+ \times H_1 \), where \( H_1 \cong H_1 \). (See [1], Exercise for Section 3.4.) It then follows (see [1], Exercise 3 of Section 3.5) that for \( x \in S_1 \) and \( y \in K(S_1) \) with \( \alpha(y) = (a, b, h_1) \) we have

\[ \alpha(yx) = (u, f_\alpha(b), h_2) \text{ with } h_2 \text{ independent of } b, \]

for all \( a, a', b, b', h, h', x \). Moreover, \( x \rightarrow h_2 \) is a homomorphism of \( S_1 \) onto \( H_1 \), and the congruence \( x \rightarrow h_2 \) induces on \( S_1 \) equals the congruence induced by \( H_2 : S_1 \rightarrow GM_g(S_1) \), i.e., \( h_2 = h_2 \) iff \( H_2(x) = H_2(y) \), where \( h \in K(S_1) \) and \( GM_g(S_1) \cong H_1 \). Now, since \( \phi \) is 1:1 on subgroups, \( g \rightarrow h_2 \) is an isomorphism of \( G \) onto \( H_1 \), for if \( \phi_1 \) is \( \phi \) restricted to \( K(S_1) \), then \( h \rightarrow p_0 \phi_1^{-1}(h) \) (where \( p_0 \) is the projection \( p_0(a, b, h) = h \)) is a well-defined isomorphism of \( H \) onto \( H_1 \), so if \( \phi_2 \in H_1 \), then \( \phi_2 = p_0 \phi_1^{-1}(a, b, 1)g = p_0 \phi_1^{-1}(a, b, 1)g \), since \( \phi_1^{-1}(a, b, 1)g \).

Now \( K(S_1) \subseteq J_i \) for some \( i < j + 1 \), so \( \psi_i \) is 1:1 on the subgroups of \( K(S_1) \). Now let \( g_1, g_2 \in G \), \( g_1 \neq g_2 \cdot \) Then \( h_2 \neq h_2 \), so for \( x_1, x_2 \in K(S_1) \), \( x_1g_1x_2 \neq x_1g_2x_2 \), but both lie in the same maximal subgroup of \( K(S_1) \).

Thus \( \psi_i(x_1g_1x_2) \neq \psi_i(x_1g_2x_2) \), so \( \psi_i(g_1) \neq \psi_i(g_2) \), a contradiction. Hence, \( x_1g_1x_2 \in J_{j+1} \) iff \( x_1g_2x_2 \in J_{j+1} \), and the remainder of the proof proceeds as for \( J_i \).

This proves Proposition 6.1.

**Notation 6.2.** Let \( S \) be a semigroup. Then \( S \rightarrow S^\gamma \) denotes the minimal homomorphic image with respect to \( \mathcal{P}(S, \gamma) \). We write \( \psi : S \rightarrow T \) iff \( \psi \in \mathcal{P}(S, \gamma) \).

**Remark 6.2.** (a) \( S \rightarrow T \) implies \( S \rightarrow T \rightarrow S^\gamma \). \( S^\gamma = \{1\} \) iff \( S \) is combinatorial.

(b) Let \( R \) be a subsemigroup of \( S \), and let \( \psi : S \rightarrow T \). Then if \( \psi' \) is \( \psi \) restricted to \( R \) and \( R' = \psi(R) \subseteq T \), then \( \psi' : R \rightarrow R' \), so \( R \rightarrow R' \rightarrow R^\gamma \).

(c) Assume \( S \) is regular. Then \( \psi : S \rightarrow T \) iff \( \psi \) is 1:1 when restricted to any \( \mathcal{P}' \)-class of \( S \). This follows easily from Remark 3.1(a).

**Corollary 6.1** (of the proof of Proposition 6.1). Let \( I \) be a maximal ideal of \( S \). Suppose there exists \( \phi : S \rightarrow \phi(S) = \phi(I) \). Then
(a) Let $\psi : S \rightarrow T$. Then $\psi$ is a $\gamma$-homomorphism iff $\psi$ restricted to $I$ is a $\gamma$ homomorphism.

(b) Let be a maximal subgroup of $S$ contained in $S - I$. Then there exists a subsemigroup $S_1$ of $S$ such that $G \subseteq S_1$, $K(S_1) \subseteq I$, and for $a \in K(S_1)$, $H_a : S_1 \rightarrow GM_a(K(S_1)) = H_1$, and $H_a$ restricted to $G$ is an isomorphism of $G$ onto $H_1$.

**Corollary 6.2.** Let $Q_1, Q_2$ be congruences on $S$ such that $S \rightarrow S/\gamma Q_i$, $i = 1, 2$. Then $S \rightarrow S/(Q_1 \vee Q_2)$, where $Q_1 \vee Q_2 = \text{lub}(Q_1, Q_2)$.

**Proof.** $S \rightarrow S/\gamma Q_i \rightarrow S^\gamma$ for $i = 1, 2$. Hence $S \rightarrow S/(Q_1 \vee Q_2) \rightarrow S^\gamma$, so $S \rightarrow S/(Q_1 \vee Q_2)$ not a $\gamma$-map implies $S \rightarrow S^\gamma$ not a $\gamma$ map, a contradiction.

**Example 6.1.** Let $G \neq \{1\}$ be a group. Let $M_2 = \{0, 1\}$ be the multiplicative semigroup of integers modulo 2, and let $S = G \times M_2$. Then $S \rightarrow S^\gamma$ is given by $(g, x) \rightarrow (g, 0)$, so $S^\gamma \equiv G$. Also $G \times \{0\}$ is an ideal of $S$ and $S/(G \times \{0\}) \equiv G^0$. Thus $S \rightarrow S/(G \times \{0\})$, but $S^\gamma \equiv G$ does not map homomorphically onto $(S/(G \times \{0\}))^\gamma \equiv (G^0)^\gamma = G^0$. Thus it is false that $S \mid T$ implies $S^\gamma \mid T^\gamma$.

In the following proposition, the elementary assertion (c1) is very important and was first proved by Dennis Allen, Jr.

**Proposition 6.2.** Let $S$ be a semigroup with partition $P$ (see Remark 6.1(b)). Then

(a) $S \rightarrow T$ implies $S \rightarrow T \rightarrow S^p$, where $P'$ is the partition induced on $T$ by $P$, i.e., $\phi(x) \equiv \phi(y) \pmod{P'}$ iff $x \equiv y \pmod{P}$. Further, let $S \rightarrow T \rightarrow T'$ be given. Then $\phi' \phi$ is a $P$-homomorphism iff $\phi$ is a $P$-homomorphism and $\phi'$ is a $P'$ homomorphism.

(b) Let $\alpha(S)$ denote any of $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, or $\mathcal{J}(-\mathcal{D})$ on $S$. Let $\psi : S \rightarrow T$. Then $S^\alpha(S) \rightarrow T^\alpha(T)$.

(c) Let $T$ be a subsemigroup of $S$. Let $t_1, t_2 \in T$. Then:

(c1) If $t_1, t_2$ are regular elements of $T$ and $\alpha$ is any of $\mathcal{L}$, $\mathcal{R}$ or $\mathcal{H}$, then $t_1 \equiv t_2 \pmod{\alpha(T)}$ iff $t_1 \equiv t_2 \pmod{\alpha(S)}$.

(c2) The result of (c1) need not hold for $\alpha = \mathcal{J} = \mathcal{D}$ even if $S$ is regular and $T$ is a union of groups.

(c3) Let $S$ be a union of groups. Then $T$ is a union of groups so all of its elements are regular and the result of (c1) holds for $\alpha = \mathcal{J} = \mathcal{D}$.

(c4) Let $\phi : S \rightarrow S'$. Then $\phi$ restricted to $T$ is an $\alpha(T)$ homomorphism under either of the following assumptions:
(i) $S$ is a union of groups and $\alpha$ is any of $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{I}$, or $\mathcal{J}$.

(ii) $T$ is regular and $\alpha$ is any of $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$.

Proof. (a) is trivial. (b) follows from Proposition 3.2(a) and Remark 6.1(b). We now prove (c1). First, trivially, $t_1 \equiv t_2 \pmod{\alpha(T)}$ implies $t_1 \equiv t_2 \pmod{\alpha(S)}$. For the converse, any regular element is congruent mod $\mathcal{I}(T)$ with an idempotent. Hence, without loss of generality assume $t_1, t_2$ are idempotents. Now $t_1 \equiv t_2 \pmod{\mathcal{I}(S)}$ iff there exist $x_1, x_2 \in S$ such that $x_1 t_1 = t_2$ and $x_2 t_2 = t_1$. Then $t_1 t_2 = x_1 t_1^2 = x_1 t_1 = t_2$ and $t_1 t_2 = t_1$, so $t_1 \equiv t_2 \pmod{\mathcal{I}(T)}$. Similarly, $t_1 \equiv t_2 \pmod{\mathcal{H}(S)}$ implies $t_1 \equiv t_2 \pmod{\mathcal{I}(T)}$. The case $\alpha = \mathcal{H}$ follows from the fact that $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

To prove (c2), let $S = M^p(\{1, 2\})$, where $C$ is the $2 \times 2$ identity matrix. Then $S$ is the 0-simple inverse semigroup with maximal subgroups $\mathbb{Z}_2$ and order 9. Let $T$ be the union of the maximal subgroups of $S$, i.e., $T = \{(g)_{aa} : a \in \{1, 2\}\} \cup \{0\}$. Then $(1)_{11} \equiv (1)_{22} \pmod{\mathcal{J}(S)}$ but $(1)_{11} \not\equiv (1)_{22} \pmod{\mathcal{J}(T)}$.

We next prove (c3). If $t_1 \equiv t_2 \pmod{\alpha(S)}$, then $t_1$ and $t_2$ are in the same $\mathcal{J}$-class $J$ of $S$. Then $T \cap J$ is a subsemigroup of the simple semigroup $J$, hence by Rees Theorem is simple, so $t_1 \equiv t_2 \pmod{\alpha(J \cap T)}$. Thus $t_1 \equiv t_2 \pmod{\alpha(T)}$.

Now (c4) follows from (c1), (c2) and (c3).

Notation 6.3. $\mathcal{P}(S, \gamma, \mathcal{J})$ is the collection of $(\phi, T)$ such that $\phi : S \rightarrow T$ and $\phi : S \rightarrow T$.

Proposition 6.3. $S$ has a minimal homomorphic image with respect to $\mathcal{P}(S, \gamma, \mathcal{J})$, denoted $S \rightarrow S^\mathcal{J}$ or $S \rightarrow S^\mathcal{J}$. We have the following formulas for $S \rightarrow S^\mathcal{J}$:

(a) Let $\phi : S \rightarrow S$ and $\psi : S \rightarrow S$.

(b) Assume $S$ is regular. Let $\psi = \prod[H_m : m \in S] \Delta : S \rightarrow \prod[Gm_n : m \in S]$, where $\Delta : S \rightarrow S \times \cdots \times S$ is the diagonal map, $\Delta(s) = (s, \ldots, s)$. Then $\psi$ equals $S \rightarrow S^\mathcal{Y}$.

(c) Let $Q$ be the congruence on $S$ defined by $s_1 = s_2 \pmod{Q}$ iff for each $\mathcal{J}$-class $J$ of $S$ and for all $x_1, x_2 \in J$, $x_1 x_2 = x_1 x_2$ in $(J \cup F(J))/F(J)$. Then $S \rightarrow S/Q$ is $S \rightarrow S^\mathcal{Y}$ when $S$ is regular (see Notation 6.1 for the definition of $F(J)$).

Proof. (a) Clearly $((\phi \times \psi) \Delta, (\phi \times \psi) \Delta(S)) \in \mathcal{P}(S, \gamma, \mathcal{J})$. Let $(\theta, T) \in \mathcal{P}(S, \gamma, \mathcal{J}) = \mathcal{P}(S, \gamma) \cap \mathcal{P}(S, \mathcal{J})$. Then $(\phi, S^\gamma) \leq (\theta, T)$ and $(\psi, S^\mathcal{Y}) \leq (\theta, T)$. Thus, $(\text{GLB}(\phi, S^\gamma), (\psi, S^\mathcal{Y})) = ((\phi \times \psi) \Delta, (\phi \times \psi) \Delta(S)) \leq (\theta, T)$.

We now prove (b) and (c). Clearly $H_m$ is 1:1 on subgroups of $\mathcal{H}$, the
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\#-class containing \( m \), so \( \psi \) is a \( \gamma \)-map. That \( \psi \) is a \( \mathcal{J} \)-map follows from the fact that \( S \) is regular. For, suppose \( m_1, m_2 \in S, m_1 \equiv m_2 \pmod{\mathcal{J}} \). By renaming, assume \( m_1 \in F(\overline{m}_2) = \{ j : J \text{ is a } \mathcal{J}\text{-class of } S, \text{ and } J \geq \overline{m}_2 \text{ is false} \} \). Then \( H^*_{m_1}(m_2) = 0 \neq H^*_{m_1}(m_2) \), since \( \overline{m}_2 \) is regular.

Further, it is clear that \( \psi \) induces on \( S \) the congruence \( Q \) of \( (c) \). Now let \( \phi : S \rightarrow T \) be both a \( \gamma \)- and a \( \mathcal{J} \)-homomorphism. Then \( s_1 \equiv s_2 \pmod{Q} \) implies \( \phi(s_1) \neq \phi(s_2) \). Now \( s_1 \equiv s_2 \pmod{Q} \) iff there exists a \( \mathcal{J} \)-class \( J \) of \( S \) and \( x_1, x_2 \in J \) so that either (1) \( x_1s_1x_2 \in J \) and \( x_1s_2x_2 \notin J \) or (2) both \( x_1s_1x_2 \) and \( x_1s_2x_2 \) lie in \( J \) and \( x_1s_1x_2 \neq x_1s_2x_2 \). In case (1), since \( \phi \) is a \( J \)-homomorphism we have \( \phi(x_1s_1x_2) \neq \phi(x_1s_2x_2) \), so \( \phi(s_1) \neq \phi(s_2) \).

Further, it is clear that \( I + \gamma \) induces on \( S \) the congruence \( Q \) of \( (c) \). Now let \( \tau : S \rightarrow T \) be both a \( \gamma \)- and a \( \mathcal{J} \)-homomorphism. Then \( s_1 + s_2 \pmod{Q} \) implies \( \tau(s_1) \neq \tau(s_2) \).

**Proposition 6.4.** (a) \( S^\mathcal{J} \) is a commutative band iff \( S \) is a union of groups.

Assume that \( S \) is a union of groups. Then \( \mathcal{J} \) is a union of \( \mathcal{J} \)-homomorphisms.

(b) Let \( (\phi, T) \in \mathcal{P}(S, \gamma, \mathcal{J}) \). Then for any subsemigroup \( S_1 \) of \( S \), \( \phi \) restricted to \( S_1 \) is in \( \mathcal{P}(S_1, \gamma, \mathcal{J}) \). Thus \( S_1^\mathcal{J} \uparrow S^\mathcal{J} \).

(c) Let \( \phi : S \rightarrow T \). Then \( S^\mathcal{J} \rightarrow T^\mathcal{J} \), see Example 6.1 for the falsity of (c) if \( B(S, \gamma, \mathcal{J}) \) is replaced by \( B(S, \gamma) \).

(d) \( T \uparrow S \) implies \( T \) is a union of groups and \( T^\mathcal{J} \uparrow S^\mathcal{J} \).

**Proof.** The statement and proof of (a) is standard ([I], Section 4.1). (b) follows from Proposition 6.2 (c4), and (c) follows from Proposition 6.3(c) and Proposition 3.2(b). Finally, (d) follows from (b) and (c).

The statement and proof of (c) hold for all regular semigroups \( S \).

**Notation 6.4.** Let \( M \) be a semigroup. For \( m \in M \), let \( RLM_m(M) = (M/F(\overline{m})) \equiv R \) (respectively \( LLM_m(M) = (M/F(\overline{m})) \equiv L \)) (see Notation 6.1) where \( s_1 \equiv s_2 \) (respectively \( s_1 \equiv Ls_2 \)) for \( s_1, s_2 \in M/F(\overline{m}) \) iff for all \( x_1 \in \overline{m} \), \( x_1s_1Lx_1s_2 \) (respectively \( s_1x_1Rx_1x_2 \)). Let \( (\eta \equiv_R)\eta_m = R_m : M \rightarrow RLM_m(M) \) and \( (\eta \equiv_L)\eta_m = L_m : M \rightarrow LLM_m(M) \).

Let \( I \) be a left ideal of \( M \). Then \( M^L : M \rightarrow F_L(I) \) is the homomorphism defined by \( (M^L(m))(x) = mx \) for \( m \in M, x \in I \). \( M^R : M \rightarrow F_R(J) \) for \( J \) a right ideal of \( M \) is defined dually.

\( M \) is a left letter mapping (LLM) semigroup iff \( M \) has a minimal or 0-minimal combinatorial (two-sided) ideal \( I \) for which \( M^L \) is 1:1. \( M \) is a right letter mapping (RLM) semigroup iff \( r(M) \) is a LLM semigroup.

Sometimes say \( M \) is a LM semigroup in either of the above cases.

\( M \) is a group mapping (GM) semigroup iff \( M = \{ 1 \} \) or \( M \) has a minimal
or zero-minimal (two-sided) noncombinatorial ideal \( I \) so that both \( M_I^L \) and \( M_I^R \) are 1:1. \( M \) is a \( GM^z \) semigroup iff \( M \) has a minimal or zero minimal (two-sided) ideal \( I \) so that both \( M_I^L \) and \( M_I^R \) are 1:1. Note that if \( I \) is non-regular, then \( M = \{0\} \).

\( M^\# \) denotes the subset \( M - \{0\} \subseteq M \).

It is easy to verify that \( GM_m(M), GM^z_m(M), RLM_m(M), \) and \( LLM_m(M) \) are respectively \( GM, GM^z, RLM, \) and \( LLM \) semigroups.

**Proposition 6.5.** Let \( \phi : S \rightarrow T \) and suppose that \( T \) is a \( RLM, LLM, \) \( GM^z, \) or \( GM \) semigroup with respect to the ideal \( I \subseteq T \). Let \( J \) be the minimal \( J \)-class of \( S \) such that \( \phi(J) = I^\# \) (see Proposition 3.2). Let \( j \in J \). Then either \( RLM(j) \rightarrow T, \) \( LLM(j) \rightarrow T, \) \( GM^z(j) \rightarrow T, \) or \( GM(j) \rightarrow T, \) respectively.

**Proof.** Suppose \( T \) is a \( GM \) semigroup. We may assume \( T \neq \{1\} \) so that \( I \) is noncombinatorial and thus contains a non-trivial subgroup. Let \( s_1, s_2 \in S, \) and assume \( H(s_1) = H(s_2) \). That is, for all \( x_1, x_2 \in J, \) either \( x_1 s_1 x_2 = x_1 s_2 x_2 \in J \) or \( x_1 s_1 x_2 \) and \( x_1 s_2 x_2 \) do not lie in \( J \). We must show that \( \phi(s_1) = \phi(s_2) \).

Suppose \( x_1 s_1 x_2 \notin J \). Then \( x_1 s_1 x_2 \notin B(J) \), so by the minimality of \( J \), \( \phi(x_1 s_1 x_2) = 0 \in I. \) Thus \( \phi(x_1 s_1 x_2) = 0 \) iff \( \phi(x_1 s_1 x_2) = 0 \) for all \( x_1, x_2 \in J, \) i.e., \( i_1 \phi(s_1) i_2 = i_2 \phi(s_2) i_2 \) for all \( i_1, i_2 \notin I^\# \). Since \( T \) is a \( GM \) semigroup with respect to \( I \), this implies that \( \phi(s_1) = \phi(s_2) \). The other cases are similar.

**Notation 6.5.** Let \( Q(S, GM) = \text{glb}\{Q' : Q' \text{ is a congruence on } S \text{ and } S/Q' \text{ is a } GM \text{ semigroup}\}. \) Let \( S^{GM} = S/Q(S, GM) \), and let \( S \rightarrow S^{GM} \) denote the canonical map.

**Proposition 6.6.** (a) Let \( s_1, s_2 \in S \). Then \( s_1 \equiv s_2 \pmod{Q(S, GM)} \) iff for each non-combinatorial \( J \)-class \( j \) of \( S \) and \( x_1, x_2 \in j \), either \( x_1 s_1 x_2 = x_1 s_2 x_2 \in j \) or \( x_1 s_1 x_2, x_1 s_2 x_2 \in S - j \). \( Q(S, GM) \) is induced by \( \prod \{H_m : \overline{m} \text{ is a } J \)-class of \( S\} : S \rightarrow \prod \{GM_m(S) : \overline{m} \text{ is a } J \)-class of \( S\} \).

(b) Let \( S \rightarrow T \). Then \( S^{GM} \rightarrow T^{GM} \).

(c) Let \( T \) be a subsemigroup of \( S \), and let \( Q' \) be \( Q \) restricted to \( T \). Then \( T/Q' \rightarrow T^{GM} \).

(d) \( T/S \) implies \( T^{GM}/S^{GM} \).

(e) If \( S \) is regular, \( S \rightarrow S^{\mathcal{F}} \rightarrow S^{GM} \rightarrow S^{\mathcal{F}}. \)

**Proof.** (a) follows from Proposition 6.5. (b) follows from Proposition 6.5 or from (a) and Proposition 3.2.

To prove (c) it suffices to show that \( t_1, t_2 \in T, t_1 \neq t_2 \pmod{Q(T, GM)} \) implies \( t_1 \neq t_2 \pmod{Q(S, GM)} \). Using (a) we may assume there exists a
$J$-class $J$ of $T$ and $x_1, x_2 \in J$ such that either (i) $x_1x_1x_2, x_1x_2x_2 \in J$ and $x_1x_1x_2 \neq x_1x_2x_2$; (ii) $x_1x_1x_2 \in J$ and $x_1x_2x_2 \notin J'$, the $J$-class of $S$ containing $J$; or (iii) $x_1x_1x_2 \in J$ and $x_1x_2x_2 \in J'$ and $x_1x_1x_2 \neq x_1x_2x_2$, so $s_1 \neq s_2 \pmod{Q(S, GM)}$. In case (ii), $s_1s_2x_1 \in J'$, and $s_1s_2x_2 \notin J'$, so again $s_1 \neq s_2 \pmod{Q(S, GM)}$.

(d) follows from (b) and (c). To prove (e), notice that $s_1 \equiv s_2 \pmod{Q(S, GM)}$ implies $s_1$ and $s_2$ have different images under $S \rightarrow S$ by (a) and Proposition 6.3. Thus, $S \rightarrow S \rightarrow S \rightarrow S \rightarrow S \rightarrow S$. Now clearly, $S \rightarrow S \rightarrow S \rightarrow S$, so by Remark 6.2, we have $S \rightarrow S \rightarrow S \rightarrow S$.

We now proceed to generalize Lemma 4.1 of [4].

**Definition 6.2.** Let $\phi : S \rightarrow T$, and let $\alpha$ be one of $\mathcal{L}, \mathcal{R}, \mathcal{H},$ or $\mathcal{J}$. We write $\phi : S \rightarrow T$ iff $s_1, s_2$ regular elements of $S$ and $\phi(s_1) = \phi(s_2)$ imply $s_1 = s_2$.

Let $\mathcal{P}(S, \alpha')$ be the collection of $(\phi, T)$ such that $\phi : S \rightarrow T$. Let $S^{RLM}$ be the image of $\prod \{R_m : \bar{m}$ is a regular $\mathcal{J}$-class of $S \}$ $A : S \rightarrow \prod \{R_{LM}(S) : \bar{m}$ is a regular $\mathcal{J}$-class of $S \}$. Let $S^{LM}$ be defined dually (see Notation 6.3 and Remark 6.1(b)).

**Proposition 6.7.** (a) Let $G$ be left simple (e.g., a group). Then the projection map $\phi : GX \rightarrow S$ for any homomorphism $Y : S \rightarrow \text{Endo}_G(G)$. (See Notation 1.1).

(b) $\phi : S \rightarrow T$ implies $\phi : S \rightarrow T$ for $\alpha = \mathcal{L}, \mathcal{R}, \mathcal{H},$ or $\mathcal{J}$. $\phi : S \rightarrow T$, and $S_1$ a subsemigroup of $S$ implies $\phi$ restricted to $S_1$ is an $\alpha'$-map for $\alpha = \mathcal{L}, \mathcal{R},$ or $\mathcal{H}$.

(c) Let $\phi : S \rightarrow T$. Let $J$ be a regular $\mathcal{J}$-class of $S$. Then $\phi(J)$ is a regular $\mathcal{J}$-class of $T$ and $X = \phi^{-1}(\phi(J))$ is an union of $\mathcal{J}$-classes of $S$ containing $J$ and as the unique minimal $\mathcal{J}$-class of $S$ contained in $X$.

(d) $S$ has a minimal homomorphic image, denoted $S \rightarrow S^\alpha$, with respect to $\mathcal{P}(S, \alpha')$, for $\alpha = \mathcal{L}, \mathcal{R},$ or $\mathcal{H}$. In fact, $S \rightarrow S^{\mathcal{J}}$ equals $S \rightarrow S^{RLM}$, $S \rightarrow S^{\mathcal{J}}$ equals $S \rightarrow S^{LM}$ and $S^{\mathcal{J}}$ is induced by $S \rightarrow S^{RLM} \times S^{LM}$. $S^{RLM} = \{(\prod \{\phi : \phi(S)$ is a $RLM$ semigroup$\})A(S)$. A similar formula holds for $S^{LM}$. Further, $(S^{\alpha'})^{\alpha} = S^{\alpha}$ for $\alpha = \mathcal{L}, \mathcal{R},$ or $\mathcal{H}$.

(e) $\phi : S \rightarrow T$ iff $s_1, s_2$ regular elements of $S$ and $\phi(s_1)\phi(s_2)$ imply $s_1s_2$, where $\alpha = \mathcal{L}, \mathcal{H}, \mathcal{H},$ or $\mathcal{J}$. Hence, $\phi\psi$ is an $\alpha'$-homomorphism iff $\phi$ and $\psi$ are $\alpha'$-homomorphisms.

(f) $S \rightarrow T$ implies $S^{\alpha} \rightarrow T^{\alpha}$ for $\alpha = \mathcal{L}, \mathcal{R},$ or $\mathcal{H}$.

(g) $S^{\mathcal{J}} \neq \{1\}$ implies $C(S^{\mathcal{J}}) = (C, n)$. See Definitions 5.1 and 5.2 and Notation 5.2.

(h) $S \neq \{1\}$ a $RLM$ semigroup implies $C(S) = (C, n)$. 

Proof. (a) Let \((g_1, s), (g_2, s) \in GX_p S^1\). Then we must show that \((g_1, s) \in L (g_2, s)\). Let \(h_1, h_2 \in G\) such that \(g_1 = h_1(V(1))(g_2)\) and \(g_2 = h_2(V(1))(g_1)\). These choices are possible since \(G\) is left simple. Then \((g_1, s) = (h_1, 1) (g_2, s)\), and \((g_2, s) = (h_2, 1)(g_1, s)\), so that \((g_1, s) \in L (g_2, s)\).

(b) follows from Definition 6.2 and Proposition 6.2(c1).

(c) Since \(\phi\) is a homomorphism, there is a \(J\)-class \(J_1 \subseteq T\) such that \(\phi(J) \subseteq J_1\), and \(J_1\) is regular since \(J\) is regular. Then by Proposition 3.2 \(\phi^{-1}(J_1)\) is a union of \(J\)-classes of \(S\) with a unique minimal class \(J'\), and \(\phi(J') = J_1\). \(J'\) is regular, since \(J_1\) is regular, and hence there exist \(x \in J\), \(x' \in J'\), such that \(\phi(x) = \phi(x')\). Since \(\phi\) is an \(\alpha'\)-map, this implies that \(xax'\) which in turn implies \(x\not\in x'\). Thus \(J = J'\). If \(J'\) is any regular \(J\)-class of \(S\) contained in \(\phi^{-1}(\phi(J))\), then the same argument shows that \(J = J'\). Thus \(J\) is the unique regular \(J\)-class in \(\phi^{-1}(\phi(J))\).

(d) By Proposition 6.5, the formula for \(S^{RLM} \) is valid. Also, clearly, \(S \rightarrow S^{RLM}\). Now let \(\phi : S \rightarrow T\), and suppose \(\phi(s_1) = \phi(s_2)\). We must show that \(R_{\alpha}(s_1) = R_{\alpha}(s_2)\) for every regular \(J\)-class \(m\) of \(S\), i.e., for all such \(m\) and all \(x \in m\), either (i) \(x_1, x_2 \in m\) and \(x_1 \not\in m\) or \(x_1, x_2 \in S - m\). If \(x_1, x_2 \in m\), then since \(\phi\) is an \(L\)-map, \(x_1 \not\in m\), \(x_1 \not\in m\) \(S - m\). Assume \(x_1, x_2 \in S - m\), \(J'\) be the unique minimal \(J\)-class of \(\phi(J)\). \(J'\) is regular by (c). Now \(x_1, x_2 \not\in J\), \(J' \subseteq x_1, x_2 \not\in \bar{m}\). However, by (c), \(\bar{m} = J'\), a contradiction. The cases \(\alpha = R, \mathcal{K}\) are similar. (d) now follows.

(e) Let \(\phi : S \rightarrow T\) denote a homomorphism satisfying the condition in (e). Then clearly \(\phi : S \rightarrow T\). Conversely, suppose that \(\phi : S \rightarrow T\). We show that \(\phi : S \rightarrow T\). Let \(s_1, s_2\) be regular elements of \(S\) such that \(\phi(s_1) = \phi(s_2)\). Then \(\phi(s_1)\) and \(\phi(s_2)\) both lie in some \(J\)-class \(J\) of \(T\). Then since \(s_1\) and \(s_2\) are regular, we have by (c) that \(\phi(s_1) = \phi(s_2) = J\) and \(s_1 = s_2\) is the unique regular \(J\)-class of \(\phi^{-1}(J)\). Then by Proposition 3.2, \(\phi\) maps \(\alpha\)-classes of \(s_1\) onto \(\alpha\)-classes of \(J\). Thus \(\phi(s_1) = \phi(s_2)\).

The remaining assertion is now easy to verify.

(f) Let \(S \rightarrow S\) denote a homomorphism satisfying the condition in (e). Then clearly \(\phi : S \rightarrow T\). Conversely, suppose that \(\phi : S \rightarrow T\). We show that \(\phi : S \rightarrow T\). Let \(s_1, s_2\) be regular elements of \(S\) such that \(\phi(s_1) = \phi(s_2)\).

(g) If \(S \rightarrow G \wedge T\), \(G\) a group, then by (d), (f), and (a), \(S \rightarrow (S \rightarrow T) \rightarrow T\).

(h) follows from (g) since \(S\) a RLM semigroup implies \(S = S^{RLM}\).

Remark 6.3. If \(S\) is regular then \(\phi : S \rightarrow T\) is an \(\alpha\) homomorphism iff \(\phi\) is an \(\alpha'\) homomorphism. Thus \(S^{\alpha} = S^{\alpha'}\) for \(\alpha = L, R, \mathcal{K}\) or \(J\). Hence, if \(S\) and \(T\) are regular semigroups, then by Proposition 6.7 (f), \(S \rightarrow T\) implies \(S^{\alpha} \rightarrow T^{\alpha}\).
Lemma 6.1. (a) Let $\phi_i : S_i \rightarrow T_i$ for $i = 1, 2$, be a $\gamma$-(respectively $\alpha'$-) homomorphism, where $\alpha = \mathcal{L}, \mathcal{R}$, or $\mathcal{H}$. Then $\phi = (\phi_1 \times \phi_2) : S_1 \times S_2 \rightarrow T_1 \times T_2$, where $\phi(s_1, s_2) = (\phi_1(s_1), \phi_2(s_2))$, is a $\gamma$- (respectively $\alpha'$-) homomorphism.

(b) Let $S = S^\gamma$ and $S = S^{\alpha'}$ where $\alpha = \mathcal{L}, \mathcal{R}$ or $\mathcal{H}$. Then $S = \{1\}$.

Proof. (a) The statement for $\alpha'$-homomorphisms is trivial. Let $\phi_i$ be a $\gamma$-homomorphism for $i = 1, 2$, and $G$ be a subgroup of $S_1 \times S_2$ such that $\phi(G) = \{e\}$, where $e = (e_1, e_2)$ is an idempotent of $T_1 \times T_2$. Then $S_1' = \phi_1^{-1}(e_1)$ and $S_2' = \phi_2^{-1}(e_2)$ are combinatorial subsemigroups of $S_1$ and $S_2$ respectively (since $\phi_i$ is a $\gamma$-map), and $G \subseteq S_1' \times S_2'$. Hence $|G| = 1$.

(b) $S = S^\gamma$ implies $S = \{1\}$ or $S$ has no combinatorial ideals, for if $I$ were a combinatorial ideal, $S \rightarrow S/I$ would be a $\gamma$-map. Hence, let $I$ be a 0-minimal ideal of $S$. Then $I - \{0\}$ is a regular $J$-class of $S$ whose $H$-classes have order $\geq 2$. But now the homomorphism of Remark 2.1(b) is an $\alpha'$-map which is not $1:1$, contradicting $S = S^{\alpha'}$. Thus, $S = \{1\}$. This completes the proof of Lemma 6.1.

The following proposition is very important.

Proposition 6.8. Let $\phi : S \rightarrow T$, and let $\alpha$ be one of $\mathcal{L}, \mathcal{R}$, or $\mathcal{H}$. Then $\phi = \phi_\alpha \cdots \phi_1$, where $\phi_1$, $\phi_\alpha$, ... are $\gamma$-homomorphisms and $\phi_\alpha$, $\phi_\beta$, ... are $\alpha'$-homomorphisms.

Proof. Consider

$$S \rightarrow S^\gamma \rightarrow S^{\alpha'} \rightarrow S^{\alpha''} \rightarrow \cdots \rightarrow \{1\}. \quad (6.1)$$

Lemma 6.1(b) assures that this series reaches $\{1\}$. Let $\bar{A} : S \rightarrow S \times S$ be given by $\bar{A}(s) = (s, \phi(s))$, and consider

$$S \xrightarrow{\bar{A}} S \times T \rightarrow S^\gamma \times T \rightarrow S^{\alpha'} \times T \rightarrow \cdots \rightarrow \{1\} \times T = T,$$

where the maps on $T$ are the identity maps, and the maps on the first factor are given by the series (6.1). That these homomorphisms are alternately $\gamma$- and $\alpha'$-homomorphisms follows from Lemma 6.1(a). Then the restriction of this series to the images of $S$ is

$$S \rightarrow S \rightarrow S_1 \rightarrow S_2 \rightarrow \cdots \rightarrow T,$$

since the restriction of a $\gamma$-map is a $\gamma$-map and (by Proposition 6.7(b)) the restriction of an $\alpha'$-map is an $\alpha'$-map. The composed homomorphism is $\phi : S \rightarrow T$. The result is proved.
**COROLLARY 6.3.** Let \((\phi, T)\) be a maximal proper homomorphic image of \(S\), i.e., \(\phi = \psi_{1}\psi_{2} : S \rightarrow T\) implies that exactly one of \(\psi_{1}, \psi_{2}\) is \(1:1\). Then \(\phi\) is either a \(\gamma\)-map or an \(\mathcal{H}^e\)-map.

**DEFINITION 6.3.** If \(N\) is a subgroup of \(S\), we write \(N \triangleleft S\) iff \(N\) is a normal subgroup of a maximal subgroup of \(G \subseteq S\). Notice that every normal subgroup of \(S\) is contained in a regular \(\mathcal{J}\)-class of \(S\). If \(N \triangleleft S\), we define \((\eta N) : S \rightarrow S/N\) by \((\eta N) = (\eta =)(\eta n)\) (See Notation 6.1), where \(n \in N\) and \(=\) is the congruence on \(S/F(n)\) defined as follows: Let \(P\) be a partition on \(n^0\) defined by \(x_1 \equiv x_2 (\text{mod } P)\) for \(x_1, x_2 \in n^0\) iff \(x_1, x_2 \in n\), and for all \(z_1, z_2 \in n, (z_1, z_2)(z_1, z_2)^{-1} \in N\) whenever \(z_1x_1z_2\) lies in \(G\), where \(G\) is the maximal subgroup of \(S\) containing \(N\). Then \(s_1 \equiv s_2\), for \(s_1, s_2 \in F(n)\), iff for all \(n_1, n_2 \in n, s_1n_1 \equiv s_2n_1 \) (mod \(P\)) and \(n_1s_1 \equiv n_2s_2 \) (mod \(P\)).

If \(N \Delta G_j \subseteq S, G_j\) a maximal subgroup of \(S\) for \(j = 1, \ldots, m\), then

\[
(\eta(N_j : j = 1, \ldots, m)) : S \rightarrow S/N_j = S/S_j
\]
equals \((\prod(\eta N_j : j = 1, \ldots, m))\)\(\Delta'\), where \(\Delta' : S \rightarrow S \times \cdots \times S\) (\(m\) terms) is given by \(\Delta'(s) = (s, \ldots, s)\) and \(\prod\) denotes the direct product of homomorphisms.

Let \(J\) be a regular \(\mathcal{J}\)-class of \(S\) with maximal subgroup \(G\), and let \(m \in J\). Let \(\mathcal{RM}(n, G)\) (respectively \(\mathcal{CM}(n, G)\)) denote the semigroup of all row-monomial (respectively column-monomial) \(n \times n\) matrices with coefficients in \(G^0\). Let \(M^R_m : S \rightarrow M^R_m(S) \subseteq \mathcal{RM}(n, G)\) (respectively \(M^L_m : S \rightarrow M^L_m(S) \subseteq \mathcal{CM}(n, G)\)) denote the so-called right (respectively left) Schützenberger representations ([1], Chapter 3). Let \(\psi : G \rightarrow H\) be a homomorphism. Then \(\psi^H : \mathcal{RM}(n, G) \rightarrow \mathcal{RM}(n, H)\) is given by \(\psi^H(M)_ij = \psi(M)_ij\), where \(\psi' : G^0 \rightarrow H^0\) with \(\psi'(0) = 0\) and \(\psi(g) = \psi(g)\) for \(g \neq 0\).

**PROPOSITION 6.9.\(a\)** Let \(n \in \Delta G \subseteq S\) with \(G\) a maximal subgroup of \(S\), and let \(\gamma : G \rightarrow G/N\) be the canonical map. Then \((\eta N) : S \rightarrow S/N\) induces the same congruence on \(S\) as does \(\psi = (L_n \times \eta M^R_n)\Delta' : S \rightarrow \psi(S)\), where \(\Delta' : S \rightarrow S \times S\) with \(\Delta'(s) = (s, s)\), and the other maps are defined in Notation 6.4 and Definition 6.3. Let \(S_n \circ \cdots \circ S_1\) denote the abstract semigroup defined by \(\mathcal{R}(S_n) \cap \cdots \cap \mathcal{R}(S_1)\) (see Notation 5.1). Then

\[
S/N | (G/N)^0 \circ R_n(S) \times L_n(S).
\]

**PROPOSITION 6.9.\(b\)** Let \(S\) be a regular semigroup, and let \(\phi : S \rightarrow T\). Let \(J_1, \ldots, J_k\) be the distinct \(\mathcal{J}\)-classes of \(T\). Let \(Q_j = \phi^{-1}(J_i)\) for \(j = 1, \ldots, k\). Then by Proposition 6.7(c) and the fact that \(\phi\) is an \(\mathcal{H}^e\)-map, \(Q_j\) is a \(\mathcal{J}\)-class of \(S\), and \(Q_1, \ldots, Q_k\) are the \(k\) distinct \(\mathcal{J}\)-classes of \(S\). Let \(G_j\) be a maximal subgroup of \(Q_j\) for \(j = 1, \ldots, k\), and let \(N_j\) be the kernel of the restriction of \(\phi\) to \(G_j\). We say
that \( \{N_1, \ldots, N_k\} \) is the kernel of \( \phi \). Then \( \psi = (\eta N_1, \ldots, N_k) : S \twoheadrightarrow S/(N_1, \ldots, N_k) \) induces the same congruence as does \( \phi \), i.e., there exists an isomorphism \( j : S/(N_1, \ldots, N_k) \twoheadrightarrow T \) such that \( \phi = j\psi \).

(c) Let \( S \) be a regular semigroup. Let \( G_1, \ldots, G_m \) be a collection of maximal subgroups of \( S \), one from each \( \mathcal{J} \)-class of \( S \). Then

\[
S / ((G_1^0 \times \cdots \times G_m^0) \circ S^\mathcal{R}) \times S^\mathcal{R} / (\Gamma_{I} \circ S^\mathcal{R})\times S^\mathcal{R},
\]

where \( C \) is some combinatorial semigroup. In particular,

\[
C(S) \leq (G_1 \circ 1) \oplus C(S^\mathcal{R}).
\]

(See Notation 5.5 of [4] and Notation 6.6 below.)

**Proof.** Let \( n \in S \) be a regular element. Then \( \alpha_n = (M_n^R \times M_n^L)\Delta' \) is 1:1 when restricted to \( \bar{n} \). Further, since for \( n_1, n_2 \in \bar{n}, s \in S \), we must have \( (n_1 s) n_2 = n_1 (s n_2) \), it is easy to verify that \( \beta_n = (L_n \times M_n^R)\Delta' \) induces the same congruence on \( S \) as does \( \alpha \), so \( \beta \) is also 1:1 when restricted to \( \bar{n} \).

Then (using Section 3 or Chapter 3 of [1]) it is easy to verify that \( \eta N \) and \( \psi \) induce the same congruence on \( \bar{n} \). It then follows easily that \( (\eta N) \) and \( \psi \) induce the same congruence on \( S \). The final assertion follows from the well known relation between \( R.M(n, G^0) \) and wreath products [2].

(b) is established by proving by direct computation that \( \phi(s_1) = \phi(s_2) \) iff \( \psi(s_1) = \psi(s_2) \).

(c) can be proved from (a) and (b) with the aid of a technique used in Lemma 3.6 of [2] and in Lemma 5.4 of [4]. We give a more direct proof.

Let \( n_i \in G_i \) for \( i = 1, \ldots, m \). Then it is well known that \( \prod_{i=1}^{m} (M_{n_i}^R \times M_{n_i}^L)\Delta' \), the Preston-Schützenberger representation, is 1:1 on \( S \). Thus, in the notation of the proof of (a), using the fact that \( \alpha_n \) and \( \beta_n \) induce the same congruence on \( \bar{n} \), we have

\[
S / \prod_{i=1}^{m} (M_{n_i}^R(S) \times L_{n_i}(S)) \big| \prod_{i=1}^{m} ((G_i^0 \circ R_{n_i}(S)) \times L_{n_i}(S)) \big|
\]

\[
((G_1^0 \times \cdots \times G_m^0) \circ S^\mathcal{R}) \times S^\mathcal{R}
\]

with the last division following from Proposition 6.7(d). It is easy to verify that \( G_1^0 \times \cdots \times G_m^0 \circ (G_1 \times \cdots \times G_m) \circ C \) for some combinatorial semigroup \( C \). (In fact, \( C \) may be taken to be \( \{1 \}^0 \times \cdots \times \{1 \}^0 \) (\( m \) terms). By Proposition 6.7(d), \( S^\mathcal{R} = (\prod_{i=1}^{m} L_i)\Delta'(S) \) and \( S^\mathcal{R} = (\prod_{i=1}^{m} R_i)\Delta'(S) \). Clearly, \( S^\mathcal{R} \rightarrow S^\mathcal{R} \) and \( S^\mathcal{R} \rightarrow S^\mathcal{R} \), so the result now follows from Proposition 6.7(g) and (h). This completes the proof of Proposition 6.9.

**Remark 6.4.** In [5] Munn proved that the collection of all \( \mathcal{H}(-= \mathcal{H}') \)
homomorphisms of a regular semigroup form a modular lattice. In fact, he showed that if \( Q_1 \) and \( Q_2 \) are congruences on \( S \) such that \( S \twoheadrightarrow S/Q_i \) for \( i = 1, 2 \), then \( Q_1 \cdot Q_2 = Q_2 \cdot Q_1 = Q \), so \( Q = \text{Lub}(Q_1, Q_2) \), and \( \mathcal{H} S \twoheadrightarrow S/Q \) is an \( \mathcal{H} \)-map. We can prove this in our present notation. Let \( \phi_k : S \twoheadrightarrow T_k \) for \( k = 1, 2 \), and let \( Q_k \) be the congruence on \( S \) induced by \( \phi_k \). Suppose \( S \) is regular, and let \( \text{Ker}(\phi_k) = (N_1^{(k)}, \ldots, N_q^{(k)}) \) where \( N_j^{(1)} \) and \( N_j^{(2)} \) are chosen in the same maximal subgroup \( G_j \) for \( j = 1, \ldots, q \), and \( q \) is the number of \( J \)-classes of \( S \). Now it is easy to verify that \( \psi(Q_1 \cdot Q_2) \) induces the same congruence on \( S \) as \( \psi([N_1^{(1)}, N_1^{(2)}], \ldots, [N_q^{(1)}, N_q^{(2)}]) \). But since \( N_j^{(1)} \Delta G_j \) and \( N_j^{(2)} \Delta G_j \), we have \( N_j^{(1)} N_j^{(2)} = N_j^{(2)} N_j^{(1)} \) for \( j = 1, \ldots, q \), so \( Q_1 \cdot Q_2 = Q_2 \cdot Q_1 \).

See notation 6.6 below for the notation used in the following.

**Proposition 6.10.** Let \( S \) be a semigroup which is a union of groups. Let \( \phi : S \twoheadrightarrow T \) be a maximal proper homomorphic image of \( S \). Then by Corollary 6.3, \( \phi \) is either a \( y \)-map or an \( \mathcal{H} \)-map. In the first case

\[
(C(S) \leq (C, 1) \oplus C(T')) \leq (C, 1) \oplus C(T)
\]

and in the second case

\[
C(S) \leq (G, 1) \oplus C(T).
\]

In either case, it follows that \( \#(S) - 1 \leq \#(T') \leq \#(S) \).

Thus let \( \psi : S \twoheadrightarrow T \) be an arbitrary homomorphism of \( S \) onto \( T \) with \( \#(S) = n \gg k = \#(T) \). Then there exist \( T_n, T_{n-1}, \ldots, T_{k+1}, T_k = T, T_{k-1}, \ldots, T_1 \) such that

\[
S \twoheadrightarrow T_n \twoheadrightarrow T_{n-1} \rightarrow \cdots \rightarrow T_1
\]

and \( \#(T_j) = j \) for \( j = 1, \ldots, n \).

**Proof.** To prove Proposition 6.10 we require the main theorem of [4] which we now state for the convenience of the reader.

**Notation 6.6.** Let \( n \) be a non-negative integer. Then \( X \) equaling \( M = T_0 \rightarrow T_1 \rightarrow T_2 \rightarrow \cdots \rightarrow T_n \rightarrow T_{n+1} = \{1\} \) is a right composition string of \( M \) of length \( n \) iff \( n = 0 \) or \( n > 0, T_n \neq \{1\}, T_k \neq G^a \) for \( k \neq n, 0 \), and either

(a) \( T_1, T_3, T_5, \ldots \) are RLM semigroups and \( T_2, T_4, T_6, \ldots \) are GM semigroups, or

(b) \( T_2, T_4, T_6, \ldots \) are RLM semigroups and \( T_1, T_3, T_5, \ldots \) are GM semigroups.

We now wish to define \( C(X) \), the complexity of a composition string \( X \) of \( M \) (See Notation 5.2). However, we must first introduce the following notation.
Let \((C \lor G, n) = (n, C \lor G)\) for all \(n \geq 1\). Let \((C, 2n) = (2n, G)\) and \((G, 2n) = (2n, C)\) for \(n = 1, 2, 3, \ldots\). Let \((C, 2n + 1) = (2n + 1, C)\) and \((G, 2n + 1) = (2n + 1, G)\) for \(n = 0, 1, 2, \ldots\).

Let \(X\) be a composition string of \(M\). When \(X\) has length zero, let \(C(X) = (C \lor G, 1)\). When \(X\) has length \(n > 0\) let \(C(X) = (C, n)\) in case (a), and let \(C(X) = (G, n)\) in case (b).

Finally we introduce the following notation. Let \((C, 1) \oplus (C, n) = (C, 1) \oplus (C \lor G, n) = (C, n)\). Let \((C, 1) \oplus (G, n) = (C, n + 1)\). We notice that \((C, 1) \oplus \text{LUB}(X) = \text{LUB}((C, 1) \oplus a : a \in X)) \geq \text{LUB}(X)\) for any finite set \(X\) of complexities. Similarly we define \((G, 1) \oplus (a, n)\) as follows. Let \((G, 1) \oplus (G, n) = (G, 1) \oplus (C \lor G, n) = (G, n)\). Let \((G, 1) \oplus (C, n) = (G, n + 1)\). We notice that \((G, 1) \oplus \text{LUB}(X) = \text{LUB}((G, 1) \oplus X) \geq \text{LUB}(X)\) for any finite set \(X\) of complexities.

**Theorem A.** Let \(M\) be a finite semigroup which is a union of groups. Let \(a(M) = \text{LUB}((C(X) : X \text{ is a right composition string of } M))\). Then

\[
\alpha(M) \leq C(M) \leq (C, 1) \oplus a(M). 
\]

Now Theorem A immediately implies that \(C(S) \leq (C, 1) \oplus C(S^{GM})\) (see Notation 6.5 and Proposition 6.6). However, Proposition 6.4(a) and Proposition 6.6(e) imply \((C, 1) \oplus C(S') \geq C(S^{GM})\), and thus, \(C(S) \leq (C, 1) \oplus C(S')\). Now Proposition 6.8 and 6.9(c) prove Proposition 6.10.

**Remark 6.5.** Let \(S\) be a union of groups. Then Theorem A of [4] and Proposition 6.7(d) imply

\[
C(S) \leq (C, 1) \oplus (G, 1) \oplus C(S^{\mathcal{E}}). 
\]

Also Proposition 6.7(g) implies \(C(S^{\mathcal{E}}) = (C, \#(S^{\mathcal{E}}))\) when \(S^{\mathcal{E}} \neq \{1\}\). Further, Lemma 4.2 of [4] implies that \(C(T) = (G, \#(T))\) if \(T\) is a GM semigroup which is a union of groups and such that \(C(T) \neq (G, 1)\). Thus, the proof of Proposition 6.1 of this paper and Proposition 1.1 of [4] imply \(C(S') = (G, \#(S'))\) if \(C(S) \neq (C \lor G, 2)\) (or more specifically, if \(S\) is not inverse and union of groups, i.e., \(S\) does not divide \(G_1^n \times \cdots \times G_n^n\) for groups \(G_1, \ldots, G_n\)).

Now consider

\[
S \rightarrow S' \rightarrow S'^{\mathcal{E}} \rightarrow S'^{\mathcal{E}, \prime} \rightarrow \cdots \rightarrow \{1\}. 
\]

(See Lemma 6.1(b).) Then it follows that \(C(S') = (G, k)\), \(C(S'^{\mathcal{E}}) = (C, k - 1)\), \(C(S'^{\mathcal{E}, \prime}) = (G, k - 2)\),... with a correction possibly necessary on the last non-trivial term. Also \(C(G, k) \leq C(S) \leq (C, k + 1)\), and thus, if \(l\) is the length of the above series, we have \(l \leq \#(S) \leq l + 1\).
REFERENCES


