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Hardy type derivations on fields of exponential logarithmic series

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ABSTRACT

We consider the valued field $\mathbb{K} := \mathbb{R}((\Gamma))$ of formal series (with real coefficients and monomials in a totally ordered multiplicative group Γ). We investigate how to endow $\mathbb{K}$ with a logarithm l, which satisfies some natural properties such as commuting with infinite products of monomials. We studied derivations on $\mathbb{K}$ (Kuhlmann and Matusinski, in press [KM10]). Here, we investigate compatibility conditions between the logarithm and the derivation, i.e. when the logarithmic derivative is the derivative of the logarithm. We analyze sufficient conditions on a given derivation to construct a compatible logarithm via integration of logarithmic derivatives. In Kuhlmann (2000) [Kuh00], the first author described the exponential closure $\mathbb{K}^{\mathrm{EL}}$ of $(\mathbb{K}, l)$. Here we show how to extend such a log-compatible derivation on $\mathbb{K}$ to $\mathbb{K}^{\mathrm{EL}}$.

1. Introduction

Consider the valued field $\mathbb{K} := \mathbb{R}((\Gamma))$ of generalized series, with real coefficients and monomials in a totally ordered multiplicative group Γ. We undertook the investigation of these fields in a series of publications [KKS97,Kuh00,FKK10,KM10]. We endeavor to endow these formal algebraic objects with the analogous of classical analytic structures, such as exponential and logarithmic maps, derivation, integration and difference operators. Hardy fields, extensively studied by M. Rosenlicht, are the natural domain for asymptotic analysis. Our investigations thus lead us to analyze the relationship between Hardy fields and generalized series fields. This paper is a further step in this direction. In particular, we interpret here some key ideas of [Ros83] in the formal setting of generalized series.

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In [KKS97], we proved that if $\Gamma \neq 0$, then $\mathbb{K}$ cannot be endowed with a logarithm (i.e. an isomorphism of ordered groups from its multiplicative group of positive elements onto its additive group). We established however that $\mathbb{K}$ always admits a pre-logarithm, i.e. a non-surjective logarithm. In this paper, we take a closer look at this aspect. We investigate how to endow $\mathbb{K}$ with a (non-surjective) logarithm l, which moreover satisfies some natural properties such as commuting with infinite products of monomials.

In [KM10], we studied derivations on $\mathbb{K}$ and introduced in particular Hardy type derivations (that is, derivations that behaves like derivations in a Hardy field). For the analysis of the derivations on $\mathbb{K}$, we worked with the chain of fundamental monomials $(\Phi, \preccurlyeq)$ of Γ (see Section 2). We gave a necessary and sufficient condition for a map $d : \Phi \to \mathbb{K}$ to extend naturally to such a derivation. Here, we investigate compatibility conditions between the logarithm and the derivation, i.e. when the logarithmic derivative is the derivative of the logarithm.

In [Kuh00], the first author described the exponential closure $\mathbb{K}^{\mathrm{EL}}$ of $(\mathbb{K}, l)$. Here we show how to extend such a log-compatible derivation on $\mathbb{K}$ to $\mathbb{K}^{\mathrm{EL}}$. This exponential closure $\mathbb{K}^{\mathrm{EL}}$ is an infinite towering extension, starting with a pre-logarithmic series field, i.e. a generalized series field endowed with a pre-logarithm (see Definition 2.7). Thus we begin in Section 3 by proving a criterion for a derivation on a pre-logarithmic series field to be compatible (see Proposition 3.8). This result is applied in Section 4. There, the main Theorem 4.10 deals with a Hardy type series derivation d, and gives sufficient conditions on d to define a d-compatible pre-logarithm. This pre-logarithm is constructed by a process of "iterated asymptotic integration" of the logarithmic derivatives (Corollary 4.13). This process is based on the computation of specific asymptotic integrals, which we do in Section 4.1. This allows us to provide many examples in Section 5. In Section 6, given some pre-logarithmic series field endowed with a Hardy type derivation, we show how to extend it to the corresponding exponential closure. Note that this has been considered for fields of transseries in [Sch01, Chapter 4.1.4]. However, our pre-logarithmic field $(\mathbb{K}, l)$ does not necessarily satisfy Axiom (T4) of [Sch01, Definition 2.2.1]. The last Section 7 is devoted to the questions of asymptotic integration and integration on EL-series fields.

In forthcoming papers, we extend our investigations to study Hardy type derivations on the field of Surreal Numbers [Con01], and investigate difference operators on generalized series fields.

2. Preliminaries

We summarize notation and terminology from [KM10]. Recall the following corollary to Ramsey's theorem [Ros82]:

Lemma 2.1. *Let Γ be a totally ordered set. Every sequence $(\gamma_n)_{n\in\mathbb{N}} \subset \Gamma$ has an infinite sub-sequence which is either constant, or strictly increasing, or strictly decreasing.*

2.1. Hahn groups

Definition 2.2. Let $(\Phi, \preccurlyeq)$ be a totally ordered set, the set of **fundamental monomials**. Consider the set $\mathbf{H}(\Phi)$ of formal products γ of the form

$$\gamma = \prod_{\phi \in \operatorname{supp}\gamma} \phi^{\gamma_\phi}$$

where $\gamma_\phi \in \mathbb{R}$, and **support** of γ, $\operatorname{supp}\gamma := \{\phi \in \Phi \mid \gamma_\phi \neq 0\}$, is an anti-well-ordered subset of Φ. We will refer to γ_ϕ as the **exponent** of ϕ. Multiplication of formal products is pointwise, and $\mathbf{H}(\Phi)$ is an abelian group with identity 1. We endow $\mathbf{H}(\Phi)$ with the anti lexicographic ordering $\preccurlyeq$ which extends $\preccurlyeq$ of Φ. Note that $\phi \succ 1$ for all $\phi \in \Phi$. The totally ordered abelian group $\mathbf{H}(\Phi)$ is the **Hahn group** over Φ, which elements are called the **(generalized) monomials**. The set Φ is the **rank**. By Hahn's embedding theorem [Hah07], every ordered abelian group Γ with rank Φ can be seen as a subgroup of $\mathbf{H}(\Phi)$.

From now on, we fix a totally ordered set $(\Phi, \preccurlyeq)$ and a subgroup Γ of $\mathbf{H}(\Phi)$.

Definition 2.3. The **leading fundamental monomial** of $1 \neq \gamma \in \Gamma$ is $\mathrm{LF}(\gamma) := \max(\operatorname{supp}\gamma)$, and $\mathrm{LF}(1) := 1$. This map verifies the **ultrametric triangular inequality**:

$$\forall \alpha, \beta \in \Gamma, \quad \mathrm{LF}(\alpha\beta) \preccurlyeq \max\{\mathrm{LF}(\alpha), \mathrm{LF}(\beta)\}.$$

The **leading exponent** of $1 \neq \gamma \in \Gamma$ is the exponent of $\mathrm{LF}(\gamma)$. We denote it by $\mathrm{LE}(\gamma)$. For $\alpha \in \Gamma$ set $|\alpha| := \max\{\alpha, \alpha^{-1}\}$.

2.2. Generalized series fields

Below, we adopt our notation as in [KM10].

Definition 2.4. Throughout this paper, $\mathbb{K} = \mathbb{R}((\Gamma))$ will denote the **generalized series field**. As usual, we write these series $a = \sum_{\alpha \in \operatorname{Supp} a} a_\alpha \alpha$, and denote by 0 the series with empty support. Here $\operatorname{Supp} a = \{\alpha \in \Gamma \mid a_\alpha \neq 0\}$ is anti-well-ordered in Γ.

For $a \in \mathbb{K}^*$, its **leading monomial** is: $\mathrm{LM}(a) := \max(\operatorname{Supp} a) \in \Gamma$. The map $\mathrm{LM} : \mathbb{K}^* \to \Gamma$ is the **canonical valuation** on $\mathbb{K}$. The **leading coefficient** of a is $\mathrm{LC}(a) := a_{\mathrm{LM}(a)} \in \mathbb{R}$. For nonzero $a \in \mathbb{K}$, the term $\mathrm{LC}(a)\,\mathrm{LM}(a)$ is called the **leading term** of a, that we denote by $\mathrm{LT}(a)$. We extend the notions of **leading fundamental monomial** and of **leading exponent** to $\mathbb{K}^*$ by setting $\mathrm{LF}(a) := \mathrm{LF}(\mathrm{LM}(a))$, respectively $\mathrm{LE}(a) := \mathrm{LE}(\mathrm{LM}(a))$.

We extend the ordering $\preccurlyeq$ on Γ to a **dominance relation** on $\mathbb{K}$ by setting $a \preccurlyeq b \Leftrightarrow \mathrm{LM}(a) \preccurlyeq \mathrm{LM}(b)$. We write: $a \asymp b \Leftrightarrow \mathrm{LM}(a) = \mathrm{LM}(b)$, and: $a \sim b \Leftrightarrow \mathrm{LT}(a) = \mathrm{LT}(b)$. Let $a \succ 1$, $b \succ 1$ be two elements of $\mathbb{K}$. a and b are **comparable** if and only if $\mathrm{LF}(a) = \mathrm{LF}(b)$. We also set $|a| := |\mathrm{LM}(a)|$.

The **anti lexicographic ordering** on $\mathbb{K}$ is defined as follows: $\forall a \in \mathbb{K}$, $a \leqslant 0 \Leftrightarrow \mathrm{LC}(a) \leqslant 0$. We denote as usual $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$, and $\mathbb{K}_{>0} := \{a \in \mathbb{K} \mid a > 0\}$. Note that $(\mathbb{K}_{>0}, \cdot)$ is an ordered abelian group.

Remark 2.5. The results in this paper hold for the generalized series field with coefficients in an arbitrary ordered exponential field $\mathcal{C}$ [Kuh00] containing $\mathbb{R}$ (instead of $\mathbb{R}$).

2.3. Pre-logarithmic sections

Definition 2.6. We denote by $\mathbb{K}^{\preccurlyeq 1} := \{a \in \mathbb{K} \mid a \preccurlyeq 1\}$ the **valuation ring** of $\mathbb{K}$. Similarly, we denote by $\mathbb{K}^{\prec 1} := \{a \in \mathbb{K} \mid a \prec 1\}$ the **maximal ideal** of $\mathbb{K}^{\preccurlyeq 1}$. We have $\mathbb{K}^{\preccurlyeq 1} = \mathbb{R} \oplus \mathbb{K}^{\prec 1}$. We denote by $\mathbb{K}^{\succ 1} := \mathbb{R}((\Gamma^{\succ 1}))$, the **subring of purely infinite series**.

We will use repeatedly the following direct sum, respectively direct product, decompositions of the ordered abelian groups $(\mathbb{K}, +, \leqslant)$, respectively $(\mathbb{K}_{>0}, \cdot, \leqslant)$ [Kuh00, Chapter 1]:

$$\mathbb{K} = \mathbb{K}^{\succ 1} \oplus \mathbb{R} \oplus \mathbb{K}^{\prec 1},$$

$$\mathbb{K}_{>0} = \Gamma.\mathbb{R}_{>0}.(1 + \mathbb{K}^{\prec 1})$$

Definition 2.7. Let $\mathbb{K}$ be a field of generalized series.

- The natural **logarithm on 1-units** is the following isomorphism of ordered groups [All62]:

$$\begin{aligned} l_1 : 1 + \mathbb{K}^{\prec 1} &\to \mathbb{K}^{\prec 1} \\ 1 + \epsilon &\mapsto \sum_{n \geqslant 1} (-1)^{n-1} \frac{\epsilon^n}{n}. \end{aligned} \tag{1}$$

- A **pre-logarithmic section** l of $\mathbb{K}$ is an embedding of ordered groups

$$l : (\Gamma, \cdot, \preccurlyeq) \to \left(\mathbb{K}^{\succ 1}, +, \leqslant\right).$$

- The **pre-logarithm** on $\mathbb{K}$ induced by a pre-logarithmic section l is the embedding of ordered groups defined by:

$$l : \left(\mathbb{K}^{>0}, \cdot, \leqslant\right) \to (\mathbb{K}, +, \leqslant)$$
$$a = a_\alpha \alpha(1 + \epsilon_a) \mapsto l(a) := \log(a_\alpha) + l(\alpha) + l_1(1 + \epsilon_a)$$

where log is the usual logarithm on positive real numbers and l_1 the logarithm on 1-units. The pair $(\mathbb{K}, l)$ is then called a **pre-logarithmic series field**.

In particular, we are interested in pre-logarithmic sections that verify the **Growth Axiom Scheme**:

Definition 2.8. Let $(\mathbb{K}, l)$ be a pre-logarithmic series field. We say that the Growth Axiom Scheme holds if and only if we have:

(GA) $\forall \alpha \in \Gamma^{\succ 1}$, $l(\alpha) \prec \alpha$.

Axiom (GA) is satisfied by non-archimedean models of the theory $\mathrm{Th}(\mathbb{R}, \exp)$ of the ordered field of real numbers with the exponential function (see [Kuh00, Chapter 3] for more details).

3. Pre-logarithms and derivations

3.1. Defining pre-logarithms on generalized series fields

We consider *pre-logarithmic sections* on a generalized series field $\mathbb{K}$ (see Definition 2.7), which satisfy the following property:

Definition 3.1. A map $l : \Gamma \to \mathbb{K}^{\succ 1}$ is a **series morphism** if it satisfies the following axiom:

(L) $\forall \alpha = \prod_{\phi \in \operatorname{supp} \alpha} \phi^{\alpha_\phi} \in \Gamma$, $l(\alpha) = \sum_{\phi \in \operatorname{supp} \alpha} \alpha_\phi l(\phi)$.

Note that a series morphism l is in particular a group homomorphism. Moreover, a series morphism l is a pre-logarithmic section if and only if it is order preserving (i.e. for any $\alpha \prec \beta$ in Γ, we have $l(\alpha) < l(\beta)$).

We are interested in the following setting: given a map $l_\Phi : \Phi \to \mathbb{K}^{\succ 1} \setminus \{0\}$, we study necessary and sufficient conditions so that l_Φ extends to a series morphism $l_\Gamma : \Gamma \to \mathbb{K}^{\succ 1}$.

Recall the following definition from [KM10]:

Definition 3.2. Let I be an infinite index set and $\mathcal{F} = (a_i)_{i \in I}$ be a family of series in $\mathbb{K}$. Then $\mathcal{F}$ is said to be **summable** if the two following properties hold:

(SF1) $\operatorname{Supp} \mathcal{F} := \bigcup_{i \in I} \operatorname{Supp} a_i$ (the support of the family) is an anti-well-ordered subset of Γ.
(SF2) For any $\alpha \in \operatorname{Supp} \mathcal{F}$, the set $S_\alpha := \{i \in I \mid \alpha \in \operatorname{Supp} a_i\} \subseteq I$ is finite.

Write $a_i = \sum_{\alpha \in \Gamma} a_{i,\alpha} \alpha$, and assume that $\mathcal{F} = (a_i)_{i \in I}$ is summable. Then

$$\sum_{i \in I} a_i := \sum_{\alpha \in \operatorname{Supp} \mathcal{F}} \left(\sum_{i \in S_\alpha} a_{i,\alpha} \right) \alpha \in \mathbb{K}$$

is a well-defined element of $\mathbb{K}$ that we call the **sum** of $\mathcal{F}$.

Definition 3.3. Let

$$l_\Phi : \Phi \to \mathbb{K}^{\succ 1}\backslash\{0\}$$
$$\phi \mapsto l_\Phi(\phi)$$

be a map. We say that l_Φ **extends to a series morphism on Γ** if the following property holds:

(SL) For any anti-well-ordered subset $E \subset \Phi$, the family $(l_\Phi(\phi))_{\phi \in E}$ is summable.

Then the series morphism l_Γ on Γ **induced** by l_Φ is defined to be the map

$$l_\Gamma : \Gamma \to \mathbb{K}^{\succ 1}$$

obtained through the axiom (L) (which clearly makes sense by (SL)).

Note that, if the series morphism l_Γ is a pre-logarithmic section, then it extends to a pre-logarithm l on $\mathbb{K}_{>0}$ as in Definition 2.7. We are interested moreover in pre-logarithms which verify (GA) (see Definition 2.8).

In the next Proposition 3.4, we provide a necessary and sufficient condition on a map $l_\Phi : \Phi \to \mathbb{K}$ so that the properties (SL) and (GA) hold. (In the sequel, we drop the subscripts Φ and Γ of l_Φ and l_Γ to relax the notation.)

Proposition 3.4. *A map $l : \Phi \to \mathbb{K}^{\succ 1} \setminus \{0\}$ extends to a series morphism on Γ if and only if the following condition fails*:

(HL1) *there exist a strictly decreasing sequence $(\phi_n)_{n\in\mathbb{N}} \subset \Phi$ and an increasing sequence $(\lambda^{(n)})_{n\in\mathbb{N}} \subset \Gamma$ such that for any n, $\lambda^{(n)} \in \operatorname{Supp} l(\phi_n)$.*

Moreover, such an extension l is a pre-logarithmic section if and only if we have:

(HL2) *l is an embedding of ordered sets, i.e. for any $\phi \prec \psi \in \Phi$, $0 < l(\phi) < l(\psi)$.*

Moreover, such a pre-logarithmic section l satisfies (GA) *if and only if we have*:

(HL3) *for any $\phi \in \Phi$, $\mathrm{LF}(l(\phi)) \prec \phi$.*

Proof. Note that (HL1) is the exact analogue of (H1) in [KM10], replacing ϕ'/ϕ by $l(\phi)$. The proof of the first statement is the exact analogue of the proof of [KM10, Lemma 3.9], replacing ϕ'/ϕ by $l(\phi)$.

Let $\alpha = \prod_{\phi\in\operatorname{supp}\alpha} \phi^{\alpha_\phi} \in \Gamma$. Assume that $\alpha \succ 1$, thus $\mathrm{LE}(\alpha) > 0$. By the first statement, $l(\alpha) = \sum_{\phi\in\operatorname{supp}\alpha} \alpha_\phi l(\phi)$. Since by hypothesis l is order preserving on Φ, we have $\mathrm{LC}(l(\alpha)) = \mathrm{LE}(\alpha)\,\mathrm{LC}(\phi_0) > 0$ where $\phi_0 = \mathrm{LF}(\alpha)$, so $l(\alpha) > 0$.

For (HL3), we consider some monomial $\alpha = \prod_{\phi\in\operatorname{supp}\alpha} \phi^{\alpha_\phi}$ in $\Gamma \setminus \{1\}$. Then $\mathrm{LM}(l(\alpha)) = \mathrm{LM}(\sum_{\phi\in\operatorname{supp}\alpha} \alpha_\phi l(\phi)) = \mathrm{LM}(\alpha_{\phi_0} l(\phi_0))$ where $\phi_0 = \mathrm{LF}(\alpha)$ and $\alpha_{\phi_0} > 0$. So $l(\alpha) \prec \alpha$ for any α if and only if, for any ϕ_0 and $\alpha_{\phi_0} > 0$, $l(\phi_0) \prec \phi_0^{\alpha_{\phi_0}}$. This is equivalent to (HL3). □

Remark 3.5. As in [KM10, Corollaries 3.12 and 3.13], one can give analogous particular cases of (HL1).

3.2. Compatibility of pre-logarithms and derivations

We recall the following definition from [KM10]:

Definition 3.6. A map $d:\mathbb{K}\to\mathbb{K}$, $a\mapsto a'$, verifying the following axioms is called a **series derivation**:

(D0) $1'=0$;
(D1) Strong Leibniz rule: $\forall\alpha=\prod_{\phi\in\operatorname{supp}\alpha}\phi^{\alpha_\phi}\in\Gamma$, $(\alpha)'=\alpha\sum_{\phi\in\operatorname{supp}\alpha}\alpha_\phi\frac{\phi'}{\phi}$;
(D2) Strong linearity: $\forall a=\sum_{\alpha\in\operatorname{Supp}a}a_\alpha\alpha\in\mathbb{K}$, $a'=\sum_{\alpha\in\operatorname{Supp}a}a_\alpha\alpha'$.

Here we provide a criterion on the derivation to be compatible with the pre-logarithm:

Definition 3.7. Let $(\mathbb{K},l)$ be a pre-logarithmic field endowed with a derivation d. Then d is **log-compatible** if for all $a\in\mathbb{K}^{>0}$, we have $l(a)'=\frac{a'}{a}$. In this case, we shall say the pre-logarithm l is compatible with the derivation d or that d and l are compatible.

In the case of a series morphism and a series derivation, it is sufficient to verify the compatibility condition for the fundamental monomials:

Proposition 3.8. *Let $(\mathbb{K},l,d)$ be a generalized series field endowed with a series morphism l and a series derivation d. Then d is log-compatible if and only if the following property holds*:

(HL4) $\forall\phi\in\Phi$, $l(\phi)'=\frac{\phi'}{\phi}$.

Proof. Let $a=\alpha a_\alpha(1+\epsilon_a)\in\mathbb{K}_{>0}$ where $\alpha=\prod_{\phi\in\operatorname{supp}\alpha}\phi^{\alpha_\phi}$ and $a_\alpha\in\mathbb{R}_{>0}$. Using (L), (D1), (D2), we compute:

$$l(a)'=\left(l(\alpha)+\log(a_\alpha)+\sum_{k=1}^{+\infty}(-1)^{k-1}\epsilon_a^k\right)'$$
$$=\sum_{\phi\in\operatorname{supp}\alpha}\alpha_\phi l(\phi)'+0+\left(\sum_{k=1}^{+\infty}(-1)^{k-1}\epsilon_a^{k-1}\right)\epsilon_a'.$$

On the other hand, we compute: $a'=(\alpha a_\alpha(1+\epsilon_a))'=\alpha' a_\alpha(1+\epsilon_a)+\alpha a_\alpha\epsilon_a'$. Therefore:

$$\frac{a'}{a}=\frac{\alpha'}{\alpha}+\frac{\epsilon_a'}{1+\epsilon_a}.$$

Now, by (D1):

$$\alpha'=\alpha\sum_{\phi\in\operatorname{supp}\alpha}\alpha_\phi\frac{\phi'}{\phi}.$$

So:

$$\frac{a'}{a}=\sum_{\phi\in\operatorname{supp}\alpha}\alpha_\phi\frac{\phi'}{\phi}+\frac{\epsilon_a'}{1+\epsilon_a}=\sum_{\phi\in\operatorname{supp}\alpha}\alpha_\phi\frac{\phi'}{\phi}+\left(\sum_{k=1}^{+\infty}(-1)^{k-1}\epsilon_a^{k-1}\right)\epsilon_a'.$$

Consequently: $l(a)'=\frac{a'}{a}$ if and only if $\frac{\phi'}{\phi}=l(\phi)'$ for all $\phi\in\operatorname{supp}\alpha$. $\square$

4. Pre-logarithms and integration for Hardy type derivations

We recall the following definition from [KM10]:

Definition 4.1. A derivation d on $\mathbb{K}$ is a **Hardy type derivation** if:

(HD1) the **sub-field of constants** of $\mathbb{K}$ is $\mathbb{R}$;
(HD2) d verifies **l'Hospital's rule**: $\forall a, b \in \mathbb{K}^*$ with $a, b \not\asymp 1$ we have $a \preccurlyeq b \Leftrightarrow a' \preccurlyeq b'$;
(HD3) the logarithmic derivation is **compatible with the dominance relation**: $\forall a, b \in \mathbb{K}$ with $|a| \succ |b| \succ 1$, we have $\frac{a'}{a} \succcurlyeq \frac{b'}{b}$. Moreover, $\frac{a'}{a} \asymp \frac{b'}{b}$ if and only if a and b are comparable.

4.1. The monomial asymptotic integral

For the rest of this section, we assume that d is a Hardy type series derivation on $\mathbb{K}$.

Notation 4.2. Set

$$\forall \phi \in \Phi, \quad \theta^{(\phi)} := \mathrm{LM}\left(\frac{\phi'}{\phi}\right), \qquad \Theta := \left\{\theta^{(\phi)},\ \phi \in \Phi\right\}, \quad \text{and} \quad \hat{\theta} := \text{g.l.b.}_{\preccurlyeq}\Theta$$

if it exists in Γ.

Adopting the notation of [Ros83], we write below: $\Psi := \{\mathrm{LM}(\frac{a'}{a});\ a \in \mathbb{K}^*,\ a \not\asymp 1\}$.

We will make use of the following result [KM10, Theorem 4.3; Corollary 4.4]:
A series derivation on $\mathbb{K}$ is of Hardy type if and only if the following condition holds:

(H3′) $\forall \phi \prec \psi \in \Phi, \theta^{(\phi)} \prec \theta^{(\psi)}$ *and* $\mathrm{LF}(\frac{\theta^{(\phi)}}{\theta^{(\psi)}}) \prec \psi$.

Definition 4.3. We say that $b \in \mathbb{K}$ is an **asymptotic integral** of $a \in \mathbb{K}$ if $b' \sim a$, equivalently if $b' \sim \mathrm{LT}(a)$. We say that b is an **integral** of a if $b' = a$.

Theorem 4.4. *A series $a \in \mathbb{K}^*$ has an asymptotic integral if and only if $a \not\asymp \text{g.l.b.}_{\preccurlyeq}\Psi$.*

This theorem is proved for Hardy fields in [Ros83, Proposition 2 and Theorem 1]. As noted in [KM10], it suffices to observe that Rosenlicht's proof only uses the properties of what we call a Hardy type derivation in Definition 4.1. If d is moreover a series derivation, it suffices to consider fundamental monomials as we establish below.

Proposition 4.5. *Assume that d is a Hardy type series derivation on $\mathbb{K}$. Let $a \in \mathbb{K}^*$ with $a \not\asymp 1$. Then*

$$\mathrm{LT}\left(\frac{a'}{a}\right) = \mathrm{LE}(a)\,\mathrm{LT}\left(\frac{\mathrm{LF}(a)'}{\mathrm{LF}(a)}\right).$$

More precisely $\mathrm{LM}(\frac{a'}{a}) = \theta^{(\mathrm{LF}(a))}$ *and* $\mathrm{LC}(\frac{a'}{a}) = \mathrm{LE}(a)\,\mathrm{LC}(\frac{\mathrm{LF}(a)'}{\mathrm{LF}(a)})$.

In particular, $\hat{\theta} = \text{g.l.b.}_{\preccurlyeq}\Psi$.

Proof. Let $1 \not\asymp a = a_\alpha\alpha + \cdots \in \mathbb{K}^*$ with $\alpha = \prod_{\phi \in \operatorname{supp}\alpha} \phi^{\alpha_\phi}$. Set $\phi_0 = \mathrm{LF}(a) = \mathrm{LF}(\alpha)$ and $\alpha_{\phi_0} = \mathrm{LE}(a) = \mathrm{LE}(\alpha)$. We compute:

$$a' = a_\alpha\alpha' + \cdots = a_\alpha\alpha\left(\alpha_{\phi_0}\frac{\phi_0'}{\phi_0} + \cdots\right) + \cdots = (a_\alpha\alpha_{\phi_0})\alpha\frac{\phi_0'}{\phi_0} + \cdots.$$

Therefore:

$$\mathrm{LT}\left(\frac{a'}{a}\right)=\frac{\mathrm{LT}(a')}{\mathrm{LT}(a)}=\frac{(a_\alpha\alpha_{\phi_0})\alpha\,\mathrm{LT}(\frac{\phi_0'}{\phi_0})}{a_\alpha\alpha}=\alpha_{\phi_0}\,\mathrm{LT}\left(\frac{\phi_0'}{\phi_0}\right). \qquad \square$$

[Ros83, Theorem 1] gives a parametrized family of asymptotic integrals of an (asymptotically integrable) element a. For a Hardy type series derivations, we compute in Proposition 4.8 below a specific asymptotic integral, which turns out to be a non-monic monomial (i.e. of the form $r\alpha$ with $r\in\mathbb{R}$ and $\alpha\in\Gamma$), uniquely determined by a.

Notation 4.6. We call the asymptotic integral computed in Proposition 4.8 below the **monomial asymptotic integral** of a, and denote it by a.i.(a).

Lemma 4.7. *Let $\alpha\in\Gamma$ with $\alpha\neq\hat{\theta}$. There exists a uniquely determined fundamental monomial $\psi_\alpha\in\Phi$ which satisfies* $\mathrm{LF}(\frac{\alpha}{\theta^{(\psi_\alpha)}})=\psi_\alpha$.

Proof. First, suppose that $\alpha\succ\hat{\theta}$. Take a monomial $\beta\succ 1$ with $\alpha\succ\frac{\beta'}{\beta}$. Set $\phi:=\mathrm{LF}(\beta)$, so $\frac{\beta'}{\beta}\asymp\theta^{(\phi)}$ by Proposition 4.5. Set $\beta_0:=\min\{\beta,\frac{\alpha}{\theta^{(\phi)}}\}$ and $\phi_0:=\mathrm{LF}(\beta_0)$. Since $\beta\succcurlyeq\beta_0\succ 1$, we have $\phi\succcurlyeq\phi_0$, so $\theta^{(\phi)}\succcurlyeq\theta^{(\phi_0)}$. We deduce that $\alpha\succ\theta^{(\phi_0)}$ and $\frac{\alpha}{\theta^{(\phi_0)}}\succcurlyeq\frac{\alpha}{\theta^{(\phi)}}\succcurlyeq\beta_0\succ 1$. If we set $\phi_1:=\mathrm{LF}(\frac{\alpha}{\theta^{(\phi_0)}})$, then $\phi_1\succcurlyeq\phi_0$. We compute: $\mathrm{LF}(\frac{\alpha}{\theta^{(\phi_1)}})=\mathrm{LF}(\frac{\alpha}{\theta^{(\phi_0)}}.\frac{\theta^{(\phi_0)}}{\theta^{(\phi_1)}})$. By (H3′): $\mathrm{LF}(\frac{\theta^{(\phi_0)}}{\theta^{(\phi_1)}})\prec\phi_1$. We obtain: $\mathrm{LF}(\frac{\alpha}{\theta^{(\phi_1)}})=\max\{\mathrm{LF}(\frac{\alpha}{\theta^{(\phi_0)}});\mathrm{LF}(\frac{\theta^{(\phi_0)}}{\theta^{(\phi_1)}})\}=\phi_1$. Set $\psi_\alpha:=\phi_1$.

Now suppose that $\alpha\prec\hat{\theta}$. Let $\alpha_1\in\Gamma$ be such that $\alpha\prec\alpha_1\preccurlyeq\hat{\theta}$. Set $\phi_0:=\mathrm{LF}(\frac{\alpha}{\alpha_1})$, then $\frac{\alpha}{\theta^{(\phi_0)}}=\frac{\alpha}{\alpha_1}.\frac{\alpha_1}{\theta^{(\phi_0)}}\preccurlyeq\frac{\alpha}{\alpha_1}\prec 1$. Set $\phi_1:=\mathrm{LF}(\frac{\alpha}{\theta^{(\phi_0)}})$. We deduce that $\phi_1\succcurlyeq\phi_0$, and compute $\mathrm{LF}(\frac{\alpha}{\theta^{(\phi_1)}})=\phi_1$ as above. Set $\psi_\alpha:=\phi_1$. This concludes the proof of the existence of ψ_α.

Consider now a monomial $\alpha\not\asymp\hat{\theta}$, and denote by ψ_1 and ψ_2 two fundamental monomials such that $\mathrm{LF}(\frac{\alpha}{\theta^{(\psi_i)}})=\psi_i$ for $i=1,2$. Assume for instance that $\psi_1\prec\psi_2$. We would have $\mathrm{LF}(\frac{\alpha}{\theta^{(\psi_2)}})=\mathrm{LF}(\frac{\alpha}{\theta^{(\psi_1)}}.\frac{\theta^{(\psi_1)}}{\theta^{(\psi_2)}})=\psi_2$. Since $\mathrm{LF}(\frac{\alpha}{\theta^{(\psi_1)}})=\psi_1$, we would have $\mathrm{LF}(\frac{\theta^{(\psi_1)}}{\theta^{(\psi_2)}})=\psi_2$, which contradicts (H3′). $\square$

Proposition 4.8. *Let $a\in\mathbb{K}^*$ with $a\not\asymp\hat{\theta}$, and set $\alpha:=\mathrm{LM}(a)$. Then*:

$$\text{a.i.}(\alpha)=\frac{\alpha}{\mathrm{LE}(\frac{\alpha}{\theta^{(\psi_\alpha)}})\,\mathrm{LT}(\frac{\psi_\alpha'}{\psi_\alpha})}\quad \textit{and}\quad \text{a.i.}(a)=\mathrm{LC}(a)\text{a.i.}(\alpha).$$

Proof. Below, set $m:=\text{a.i.}(\alpha)=\frac{\alpha}{\mathrm{LE}(\frac{\alpha}{\theta^{(\psi_\alpha)}})\,\mathrm{LC}(\frac{\psi_\alpha'}{\psi_\alpha})\theta^{(\psi_\alpha)}}$.

Since $\mathrm{LF}(m)=\mathrm{LF}(\frac{\alpha}{\theta^{(\psi_\alpha)}})=\psi_\alpha$, using Proposition 4.5, we compute:

$$\mathrm{LT}\left(\frac{m'}{m}\right)=\mathrm{LE}(m)\,\mathrm{LT}\left(\frac{\psi_\alpha'}{\psi_\alpha}\right).$$

Since $\mathrm{LE}(m)=\mathrm{LE}(\frac{\alpha}{\theta^{(\psi_\alpha)}})$, we compute:

$$\mathrm{LT}\big(m'\big)=m\,\mathrm{LE}(m)\,\mathrm{LT}\left(\frac{\psi_\alpha'}{\psi_\alpha}\right)=\frac{\alpha}{\mathrm{LE}(\frac{\alpha}{\theta^{(\psi_\alpha)}})\,\mathrm{LT}(\frac{\psi_\alpha'}{\psi_\alpha})}.\mathrm{LE}\left(\frac{\alpha}{\theta^{(\psi_\alpha)}}\right)\mathrm{LT}\left(\frac{\psi_\alpha'}{\psi_\alpha}\right)=\alpha,$$

as desired.

Denote $b:=\text{a.i.}(a)$. We have: $\mathrm{LT}(b')=\mathrm{LT}(\mathrm{LC}(a)m')=\mathrm{LC}(a)\,\mathrm{LT}(m')=\mathrm{LC}(a)\alpha=\mathrm{LT}(a)$, as desired. $\square$

Notation 4.9. In the sequel, to simplify the notations, we will write ψ instead of ψ_α (of Lemma 4.7) if the context is clear.

4.2. Constructing pre-logarithms as anti-derivatives

In the following theorem, we give a criterion for $(\mathbb{K}, d)$ to carry a pre-logarithm, compatible with the derivation. Moreover, we will require this pre-logarithm to be induced by a pre-logarithmic section which is a series morphism. The construction relies on the computation of the anti-derivatives of $\frac{\phi'}{\phi}$, $\phi \in \Phi$.

Theorem 4.10. *Let d be a Hardy type series derivation on $\mathbb{K}$. There exists a unique pre-logarithmic section l on $\mathbb{K}$ which is a series morphism, for which the induced pre-logarithm is compatible with the derivation, if and only if the following two conditions hold*:

1. $\hat{\theta} \notin \bigcup_{\phi \in \Phi} \operatorname{Supp} \frac{\phi'}{\phi}$;
2. $\forall \phi \in \Phi$, $\forall \tau^{(\phi)} \in \operatorname{Supp} \frac{\phi'}{\phi}$, $\text{a.i.}(\tau^{(\phi)}) \succ 1$.

Moreover, this pre-logarithm verifies (GA).

Proof. To define a pre-logarithm l on $\mathbb{K}_{>0}$, it suffices to define a pre-logarithmic section l on Γ. We set $l(1) := 0$. By (D1), for any $\alpha = \prod_{\phi \in \operatorname{supp}\alpha} \phi^{\alpha_\phi} \in \Gamma \backslash \{1\}$, we have $\frac{\alpha'}{\alpha} = \sum_{\phi \in \operatorname{supp}\alpha} \alpha_\phi \frac{\phi'}{\phi}$. Assume that for any $\phi \in \Phi$, there exists $l(\phi) \in \mathbb{K}^{\succ 1}$ such that (HL4) holds, i.e. $l(\phi)' = \frac{\phi'}{\phi}$. (The proof of the existence of such $l(\phi) \in \mathbb{K}^{\succ 1}$ will be established below.) We apply Proposition 3.4 to extend l to a series morphism on Γ. Suppose, as in (HL1), that there exist a strictly decreasing sequence $(\phi_n)_{n \in \mathbb{N}} \subset \Phi$ and an increasing sequence $(\lambda^{(n)})_{n \in \mathbb{N}} \subset \Gamma$ such that for any n, $\lambda^{(n)} \in \operatorname{Supp} l(\phi_n)$. By (HD2), $\tau^{(n)} := \operatorname{LM}((\lambda^{(n)})')$ defines an increasing sequence in Γ such that for any n, $\tau^{(n)} \in \operatorname{Supp} \frac{\phi_n'}{\phi_n}$. This implies that [KM10, (H1)] holds, contradicting the fact that d is a series derivation. Therefore, for any $\alpha \in \Gamma$, we can indeed define $l(\alpha) := \sum_{\phi \in \operatorname{supp}\alpha} \alpha_\phi l(\phi)$.

Note that by (HD2), (HL2) holds. Thus l would be the pre-logarithmic section induced by the given l on Φ. Furthermore, this series morphism l is compatible with the derivation (Proposition 3.8).

It remains to prove the existence of such $l(\phi) \in \mathbb{K}^{\succ 1}$. We adapt to our context [Kuh, Theorem 1], with the "spherically complete" ultrametric space $(\mathbb{K}, u)$ where $u(a, b) := \operatorname{LM}(a - b)$, and the map $f := d$.

Lemma 4.11. *(See [Kuh, Theorem 1].) Let $\phi \in \Phi$. We suppose that for any $a \in \mathbb{K}$ with $a' \neq \frac{\phi'}{\phi}$, there exists $b \in \mathbb{K}$ such that*:

(AT1) $\operatorname{LM}(b' - \frac{\phi'}{\phi}) \succ \operatorname{LM}(a' - \frac{\phi'}{\phi})$;
(AT2) $\forall c \in \mathbb{K}$, *if* $\operatorname{LM}(a - c) \succ \operatorname{LM}(a - b)$, *then* $\operatorname{LM}(a' - c') \succ \operatorname{LM}(a' - \frac{\phi'}{\phi})$.

Then there exists $l(\phi) \in \mathbb{K}$ such that $l(\phi)' = \frac{\phi'}{\phi}$.

Proof. Let $a \in \mathbb{K}$. By (D1) and (D2), we can denote $\operatorname{LT}(a' - \frac{\phi'}{\phi}) = c_0 \alpha \tau^{(\tilde{\phi})}$ for some $c_0 \in \mathbb{R}$, $\alpha \in \operatorname{Supp} a \cup \{1\}$, $\tilde{\phi} \in \operatorname{supp}\alpha \cup \{\phi\}$ and $\tau^{(\tilde{\phi})} \in \operatorname{Supp} \frac{\tilde{\phi}'}{\tilde{\phi}}$.

Claim 4.12. *Provided Hypotheses* 1 *and* 2 *of Theorem* 4.10, *we consider $\alpha \in \Gamma$, $\alpha \not\asymp 1$. Then any monomial $\beta = \alpha \tau^{(\tilde{\phi})} \in \operatorname{Supp}(\alpha')$ (where $\tilde{\phi} \in \operatorname{supp}\alpha$ and $\tau^{(\tilde{\phi})} \in \operatorname{Supp} \frac{\tilde{\phi}'}{\tilde{\phi}}$ by* (D1)) *admits an asymptotic integral. Moreover, $\psi_\beta = \operatorname{LF}(\alpha)$ and $\operatorname{LE}(\beta) = \operatorname{LE}(\alpha)$.*

Indeed, by Lemma 4.7 and Proposition 4.8, we show that $\mathrm{LF}(\frac{\alpha\tau^{(\tilde\phi)}}{\theta^{(\psi)}}) = \psi$. Set $\psi := \mathrm{LF}(\alpha)$, therefore $\psi \succcurlyeq \tilde\phi$. Denote by $\tilde\psi$ the unique fundamental monomial such that $\mathrm{LF}(\frac{\tau^{(\tilde\phi)}}{\theta^{(\tilde\psi)}}) = \tilde\psi$ (which exists since $\tau^{(\tilde\phi)} \not\asymp \hat\theta$ by Hypothesis 1). Since $\frac{\tau^{(\tilde\phi)}}{\theta^{(\tilde\psi)}} \succ 1$ by Hypothesis 2, we have $\tilde\psi \prec \tilde\phi$. Consequently, $\tilde\psi \prec \psi$, so $\mathrm{LF}(\frac{\theta^{(\tilde\psi)}}{\theta^{(\psi)}}) \prec \psi$ by (H3′). Then, using the ultrametric triangular inequality for LF, we compute:

$$\mathrm{LF}\left(\frac{\alpha\tau^{(\tilde\phi)}}{\theta^{(\psi)}}\right) = \mathrm{LF}\left(\alpha\frac{\tau^{(\tilde\phi)}}{\theta^{(\tilde\psi)}}\frac{\theta^{(\tilde\psi)}}{\theta^{(\psi)}}\right) = \mathrm{LF}(\alpha) = \psi \quad \text{and} \quad \mathrm{LE}(\beta) = \mathrm{LE}(\alpha).$$

Consequently, $c_0\alpha\tau^{(\tilde\phi)}$ admits an asymptotic integral monomial. To conclude the proof of (AT1), it suffices to set $b := a - \mathrm{a.i.}(c_0\alpha\tau^{(\tilde\phi)})$.

Concerning (AT2), we consider $c \in \mathbb{K}$ such that

$$\mathrm{LM}(a-c) \succ \mathrm{LM}(a-b) = \mathrm{LM}\big(\mathrm{a.i.}\big(c_0\alpha\tau^{(\tilde\phi)}\big)\big) = \frac{\alpha\tau^{(\tilde\phi)}}{\theta^{(\psi)}}.$$

By (HD2), we have:

$$\mathrm{LM}\big(a'-c'\big) \succ \mathrm{LM}\left[\left(\frac{\alpha\tau^{(\tilde\phi)}}{\theta^{(\psi)}}\right)'\right] = \alpha\tau^{(\tilde\phi)} = \mathrm{LM}\left(a' - \frac{\phi'}{\phi}\right). \qquad \square$$

Note that $l(\phi)$ is defined up to addition by a real constant. *We choose the $l(phi)$'s so that this real constant is zero, i.e.* $1 \notin \mathrm{Supp}\, l(\phi)$.

We prove now that $l(\phi) \in \mathbb{K}^{\succ 1}$ for any $\phi \in \Phi$. Suppose not, and denote by $\lambda^{(\phi)}$ the greatest monomial in $\mathrm{Supp}\, l(\phi)$ such that $\lambda \prec 1$. Then, $\mathrm{LM}(\lambda') = \lambda\theta^{(\psi)}$, where $\psi = \mathrm{LF}(\lambda)$. We consider two cases. Either $\lambda\theta^{(\psi)} = \tau \in \mathrm{Supp}\,\frac{\phi'}{\phi}$, which is impossible since $\mathrm{a.i.}(\tau) \succ 1$ by Hypothesis 2. Or $\lambda\theta^{(\psi)} = \tilde\lambda\tilde\tau$ for some $\tilde\lambda \succ 1$, $\tilde\phi \in \mathrm{supp}\,\tilde\lambda$ and $\tilde\tau \in \mathrm{Supp}\,\frac{\tilde\phi'}{\tilde\phi}$, meaning that, up to multiplication by a real coefficient, λ is the asymptotic integral monomial of $\tilde\lambda\tilde\tau$. But, computing $\mathrm{a.i.}(\tilde\lambda\tilde\tau)$ as in the proof of (AT1) in the preceding lemma, we obtain:

$$\mathrm{LM}\big[\mathrm{a.i.}(\tilde\lambda\tilde\tau)\big] = \frac{\tilde\lambda\tau^{(\tilde\phi)}}{\theta^{(\psi)}}$$

with $\psi := \mathrm{LF}(\tilde\lambda) = \mathrm{LF}(\frac{\tilde\lambda\tau^{(\tilde\phi)}}{\theta^{(\psi)}})$ and $\mathrm{LE}(\frac{\tilde\lambda\tau^{(\tilde\phi)}}{\theta^{(\psi)}}) = \mathrm{LE}(\tilde\lambda) > 0$. This means that $\mathrm{a.i.}(\tilde\lambda\tilde\tau) \succ 1$: contradiction.

To conclude the proof of the theorem, we show that the pre-logarithm is uniquely determined, and that it verifies (GA). Indeed, let l_1 and l_2 be two pre-logarithms compatible with d, and $a \in \mathbb{K}_{>0}$. So $l_1(a)' = \frac{a'}{a} = l_2(a)'$, which means that $l_1(a) = l_2(a) + c$ for some $c \in \mathbb{R}$. But if we take $a = 1$, then $l_1 = l_2 = \log$, so $l_1(1) = l_2(1) = 0$ which implies that $c = 0$.

Concerning (GA), since the derivation verifies l'Hospital's rule (HD2), we observe that, for any ϕ, the leading monomial of $l(\phi)$ is $\frac{\theta^{(\phi)}}{\theta^{(\psi)}}$ where ψ is the fundamental monomial such that $\mathrm{LF}(\frac{\theta^{(\phi)}}{\theta^{(\psi)}}) = \psi$ (exists by Hypothesis 1). By Hypothesis 2, we have moreover that $\frac{\theta^{(\phi)}}{\theta^{(\psi)}} \succ 1$, so $\phi \succ \psi$. Thus, we obtain $\mathrm{LF}(l(\phi)) = \psi \prec \phi$, as desired. $\square$

In the next result and in its proof, we give a description of the $l(\phi)$'s via a method that we may call an **iterated asymptotic integration**.

Corollary 4.13. *With the same hypothesis as in Theorem* 4.10*, for any $\phi \in \Phi$, if we denote $l(\phi) = \sum_{\lambda \in \mathrm{Supp}\, l(\phi)} d_\lambda \lambda \in \mathbb{K}^{\succ 1}$, then for any $\lambda \in \mathrm{Supp}\, l(\phi)$, there is $n \in \mathbb{N}$ such that*:

$$\lambda = \prod_{i=1}^{n} \frac{\tau^{(\phi_i)}}{\theta^{(\psi)}} \quad \textit{and} \quad d_\lambda = \frac{\prod_{i=1}^{n} c_{\tau^{(\phi_i)}}}{(\beta_0 c_{0,\psi})^n}$$

where:

a) $\tau^{(\phi_1)} = \tau^{(\phi)} \in \operatorname{Supp} \frac{\phi'}{\phi}$ *and* $\psi = \psi_{\tau^{(\phi)}}$ *(Lemma 4.7: i.e.* ψ *verifies* $\mathrm{LF}(\frac{\tau^{(\phi)}}{\theta^{(\psi)}}) = \psi$*);*

b) *for any* $i = 2, \ldots, n$, $c_{\tau^{(\phi_i)}} \tau^{(\phi_i)}$ *is a monomial of* $\frac{\phi_i'}{\phi_i}$ *for some* $\phi_i \preccurlyeq \psi$ *with* $\tau^{(\phi_i)} \prec \theta^{(\psi)}$;

c) $\beta_0 = \mathrm{LE}(\frac{\tau^{(\phi)}}{\theta^{(\psi)}}) > 0$ *and* $c_{0,\psi} = \mathrm{LC}(\frac{\psi'}{\psi})$;

d) *for any* $k = 1, \ldots, n$, $\mathrm{LF}(\prod_{i=1}^{k} \frac{\tau^{(\phi_i)}}{\theta^{(\psi)}}) = \psi$ *and* $\mathrm{LE}(\prod_{i=1}^{k} \frac{\tau^{(\phi_i)}}{\theta^{(\psi)}}) = \beta_0 > 0$.

Proof. Let $\phi \in \Phi$. We set the iterated asymptotic integration of $\frac{\phi'}{\phi}$ as being the fixed point of the following map f (we prove below that such a fixed point is well defined, unique and equal to $l(\phi)$). Given a series $l = \sum_{\lambda \in S} d_\lambda \lambda$ (which can be thought as an approximation of $l(\phi)$), by (D1) and (D2) we have:

$$\big[l(\phi) - l\big]' = \frac{\phi'}{\phi} - l' = \sum_{\lambda \in \operatorname{Supp} l(\phi) \setminus S} \; \sum_{\tilde{\phi} \in \operatorname{supp} \lambda} \; \sum_{\tau^{(\tilde{\phi})} \in \operatorname{Supp} \phi'/\phi} (d_\lambda \tilde{c}_{\tilde{\phi}}).\lambda \tau^{(\tilde{\phi})}.$$

Since any of the terms $(d_\lambda \tilde{c}_{\tilde{\phi}}).\lambda \tau^{(\tilde{\phi})}$ admits an asymptotic integral monomial (Claim 4.12), we set:

$$\mathrm{A.I.}\big(\big[l(\phi) - l\big]'\big) := \sum_{\lambda \in \operatorname{Supp} l(\phi) \setminus S} \; \sum_{\tilde{\phi} \in \operatorname{supp} \lambda} \; \sum_{\tau^{(\tilde{\phi})} \in \operatorname{Supp} \phi'/\phi} \mathrm{a.i.}\big[(d_\lambda \tilde{c}_{\tilde{\phi}}).\lambda \tau^{(\tilde{\phi})}\big] \quad \text{and} \quad \mathrm{A.I.}(0) := 0$$

and

$$f(l) := l + \mathrm{A.I.}\big(\big[l(\phi) - l\big]'\big).$$

Note that $l(\phi)$ is a fixed point for f. We adapt to our context [PCR93, Theorem 1] for the ultrametric $u(a, b) := \mathrm{LM}(a - b)$, provided the fact that $(\mathbb{K}, u)$ is spherically complete:

Lemma 4.14. *(See [PCR93, Theorem 1].) Since* $\mathbb{K}$ *is spherically complete, any contracting map* $f : \mathbb{K} \to \mathbb{K}$ *has exactly one fixed point.*

Our map f is contracting. Indeed, given $l_1, l_2 \in \mathbb{K}$, $l_1 \neq l_2$, we compute:

$$f(l_1) - f(l_2) = l_1 - \mathrm{A.I.}\left(\frac{\phi'}{\phi} - l_1'\right) - l_2 + \mathrm{A.I.}\left(\frac{\phi'}{\phi} - l_2'\right)$$
$$= (l_1 - l_2) - \mathrm{A.I.}\big[(l_1 - l_2)'\big].$$

Therefore: $u[f(l_1) - f(l_2)] = \mathrm{LM}[(l_1 - l_2) - \mathrm{A.I.}[(l_1 - l_2)']] < \mathrm{LM}(l_1 - l_2) = u(l_1, l_2)$. Consequently, $l(\phi)$ is the unique fixed point of f.

To obtain the desired properties for $l(\phi)$, we proceed by induction along the iterated asymptotic integration. We begin with $l = 0$. Thus, we compute the asymptotic integral of any monomial $c_{\tau^{(\phi)}} \tau^{(\phi)}$ of $\frac{\phi'}{\phi}$. By Proposition 4.8 and Hypothesis 1, its asymptotic integral exists and is of the form:

$$d_\lambda \lambda := \frac{c_{\tau^{(\phi)}}}{\beta_0 c_{0,\psi}} \frac{\tau^{(\phi)}}{\theta^{(\psi)}}$$

where $\psi := \psi_{\tau^{(\phi)}}$, $\beta_0 := \mathrm{LE}(\frac{\alpha}{\theta^{(\psi)}})$ and $c_{0,\psi} := \mathrm{LC}(\frac{\psi'}{\psi})$. Moreover, by Hypothesis 2, $\lambda \succ 1$ as desired.

We consider now $f^n(0)$ for some $n \in \mathbb{N}$ which we denote by the series $l = \sum d_\lambda \lambda$, supposing that properties a)–d) hold for it. Then any term in $[l(\phi) - l]'$ is of the form

$$(d_\lambda \tilde{c}_{\tilde{\phi}}).\lambda \tau^{(\tilde{\phi})} = \frac{(\prod_{i=1}^n c_{\tau^{(\phi_i)}})\tilde{c}_{\tilde{\phi}}}{(\beta_0 c_{0,\psi})^n} \left(\prod_{i=1}^n \frac{\tau^{(\phi_i)}}{\theta^{(\psi)}} \right) \tau^{(\tilde{\phi})}$$

where $\tilde{\phi} \in \operatorname{supp} \lambda$, so $\tilde{\phi} \preccurlyeq \psi$, and $\tilde{c}_{\tilde{\phi}} \tau^{(\tilde{\phi})}$ is a monomial of $\operatorname{Supp} \frac{\tilde{\phi}'}{\tilde{\phi}}$ with $\tau^{(\tilde{\phi})} \prec \theta^{(\psi)}$. By Proposition 4.8, Claim 4.12 and the induction hypothesis, its asymptotic integral is:

$$d_{\tilde{\lambda}} \tilde{\lambda} := \frac{d_\lambda \tilde{c}_{\tilde{\phi}}}{\beta_0 c_{0,\psi}} \frac{\lambda}{\theta^{(\psi)}} = \frac{(\prod_{i=1}^{n+1} c_{\tau^{(\phi_i)}})}{(\beta_0 c_{0,\psi})^{n+1}} \prod_{i=1}^{n+1} \frac{\tau^{(\phi_i)}}{\theta^{(\psi)}}$$

where $\phi_{n+1} := \tilde{\phi}$, $\tau^{(\phi_{n+1})} := \tau^{(\tilde{\phi})}$ and $c_{\tau^{(\phi_{n+1})}} := \tilde{c}_{\tilde{\phi}}$. Note that $\mathrm{LF}(d_{\tilde{\lambda}} \tilde{\lambda}) = \psi$ and $\mathrm{LF}(d_{\tilde{\lambda}} \tilde{\lambda}) = \beta_0 > 0$, which implies that $d_{\tilde{\lambda}} \tilde{\lambda} \succ 1$ as desired. $\square$

5. Pre-logarithms and derivations induced by decreasing automorphisms

5.1. Decreasing automorphisms and monomial series morphisms

Definition 5.1. Let $(\Phi, \preccurlyeq)$ be a chain. A **decreasing endomorphism** σ of Φ is an order preserving map $\sigma : \Phi \to \Phi$, such that for all $\phi \in \Phi$, $\sigma(\phi) \prec \phi$. If this map is surjective, we call it a **decreasing automorphism**.

Remark 5.2. Note that, if Φ has a decreasing endomorphism, then it has necessarily no least element. It would be interesting to characterize linear orderings which admit a decreasing endomorphism.

Definition 5.3. A pre-logarithm on $\mathbb{K}$ is **monomial** if its restriction to the fundamental monomials has its image in the monomials:

$$l : \Phi \to \mathbb{R}^*.\Gamma.$$

In [KM10, Proposition 5.2], we study derivations on $\mathbb{K}$ that are also called **monomial** (i.e. such that their restrictions to the fundamental monomials have their image in the monomials), and we prove that:

Proposition 5.4. *A monomial derivation d extends to a Hardy type series derivation on $\mathbb{K}$ if and only if the condition* (H3$'$) *holds.*

Here we prove that:

Proposition 5.5. *Let d be a monomial Hardy type series derivation on $\mathbb{K}$. Assume that the set $\Theta = \{\theta^{(\phi)}, \phi \in \Phi\}$ has no least element. Then there exists a unique pre-logarithmic section l on $\mathbb{K}$ which is a series morphism, for which the induced pre-logarithm is compatible with the derivation. Moreover, this pre-logarithm verifies* (GA).

Proof. We just need to check the hypotheses of Theorem 4.10. Indeed, for any ϕ, $\theta^{(\phi)} \neq \hat{\theta}$, which implies that assumption 1 of Theorem 4.10 holds. We compute now:

$$\text{a.i.}\big(\theta^{(\phi)}\big) = \frac{\theta^{(\phi)}}{\text{LE}(\frac{\theta^{(\phi)}}{\theta^{(\psi)}})\,\text{LT}(\frac{\psi'}{\psi})} = \frac{1}{\text{LE}(\frac{\theta^{(\phi)}}{\theta^{(\psi)}})\,\text{LC}(\theta^{(\psi)})}\frac{\theta^{(\phi)}}{\theta^{(\psi)}}$$

with $\text{LF}(\frac{\theta^{(\phi)}}{\theta^{(\psi)}}) = \psi$ (as in Lemma 4.7 with $\alpha = \theta^{(\phi)}$). Since d is a Hardy type derivation, by (H3′) we have: $\text{LF}(\frac{\theta^{(\phi)}}{\theta^{(\psi)}}) \prec \max\{\phi, \psi\}$ for any $\phi \neq \psi$. Consequently, $\phi = \max\{\phi, \psi\} \succ \psi$, which implies also that $\theta^{(\phi)} \succ \theta^{(\psi)}$. Assumption 2 of Theorem 4.10 holds, as desired. □

Example 5.6. We define the basic pre-logarithmic section on $\mathbb{K}$ by:

$$l\bigg(\prod_{\phi\in\Phi}\phi^{\gamma_\phi}\bigg) = \sum_{\phi\in\Phi}\gamma_\phi\phi.$$

Here (SL) is readily verified. The basic pre-logarithmic section l does *not* satisfy (GA) (e.g. $l(\phi) = \phi$). To remedy to this problem, we fix a decreasing endomorphism

$$\sigma : \Phi \to \Phi.$$

We define the **pre-logarithmic section** l_σ **induced by** σ as follows:

$$l_\sigma\bigg(\prod_{\phi\in\Phi}\phi^{\gamma_\phi}\bigg) = \sum_{\phi\in\Phi}\gamma_\phi\sigma(\phi).$$

The **induced pre-logarithm** (given in Definition 2.7) is denoted by l_σ. We leave it to the reader to verify that l_σ satisfies (GA) (see [Kuh00, Chapter 3] for more details).

As an elementary but important illustration, take the following chain of infinitely increasing real germs at infinity (applying the usual comparison relations of germs):

$$\Phi := \big\{\exp^n(x);\ n \in \mathbb{Z}\big\}$$

where $\exp^n$ denotes for positive n, the nth iteration of the real exponential function, for negative n, the $|n|$'s iteration of the logarithmic function, and for $n = 0$ the identical map. The restriction of the (germ of the) natural logarithmic function log to Φ is such an embedding σ. We leave it to the reader to verify that its lifting as a pre-logarithm on $\mathbb{K}$, extends the logarithmic function on the rational functions field $\mathbb{R}(\exp^n(x),\ n \in \mathbb{Z})$.

5.2. Defining a compatible monomial derivation from a series morphism

We study now the converse situation of Proposition 5.5. We consider the chain $(\Phi, \preccurlyeq)$ endowed with a decreasing automorphism σ, and the induced pre-logarithm l_σ. We want to know when we can define a log-compatible Hardy type series derivation on $\mathbb{K}$, and describe it.

Definition 5.7. Given an ordered chain $(\Phi, \preccurlyeq)$, an element $\phi \in \Phi$ and a decreasing endomorphism $\sigma : \Phi \mapsto \Phi$, we call:

- the $\mathbb{Z}$**-orbit** of ϕ: $\mathcal{O}(\phi) = \{\sigma^k(\phi) \mid k \in \mathbb{Z}\}$;
- the **convex orbit** of ϕ: $\mathcal{C}(\phi) = \{\psi \in \Phi \mid \exists k \in \mathbb{N},\ \sigma^k(\phi) \preccurlyeq \psi \preccurlyeq \sigma^{-k}(\phi)\}$;
- For any $\alpha = \prod_{\phi\in\operatorname{supp}\alpha}\phi^{\alpha_\phi} \in \Gamma$, any $\psi \in \Phi$ and any binary relation $\mathcal{R} \in \{\prec, \preccurlyeq, \succ, \succcurlyeq\}$, we denote $S_\psi = \{\phi \in \operatorname{supp}\alpha \mid \phi\mathcal{R}\psi\}$, and define the corresponding **truncation** of α as $\text{Tr}_{\mathcal{R}\psi}(\alpha) := \prod_{\phi\in S_\psi}\phi^{\gamma_\phi}$.

Notation 5.8. Given a family $\mathcal{F} \subset \Phi$ of representatives of the convex orbits of Φ, given $\phi \in \mathcal{F}$, we denote $\mathcal{S}_{\mathcal{F},\phi} := \{\psi \in \Phi \mid \phi \preccurlyeq \psi \prec \sigma^{-1}(\phi)\}$, and $\mathcal{S}_{\mathcal{F}} := \bigcup_{\phi \in \mathcal{F}} \mathcal{S}_{\mathcal{F},\phi}$.

Proposition 5.9. *Let σ be a decreasing automorphism on Φ, and l_σ the induced pre-logarithm. There exists a log-compatible monomial Hardy type series derivation on $\mathbb{K}$ if and only if there exists a map $\Phi \to \Gamma$, $\phi \mapsto \theta^{(\phi)}$, such that*:

(M) *for any $\phi \prec \psi \in \Phi$, $\mathrm{Tr}_{\succcurlyeq \mathcal{C}_\psi}(\frac{\theta^{(\psi)}}{\theta^{(\phi)}}) = \mathrm{Tr}_{\succcurlyeq \mathcal{C}_\psi}(\prod_{j=1}^{\infty} \frac{\sigma^j(\psi)}{\sigma^j(\phi)})$, with in particular, for any $k \in \mathbb{N}$, $\theta^{(\sigma^k(\phi))} = \frac{\theta^{(\phi)}}{\prod_{j=1}^{k} \sigma^j(\phi)}$.*

Moreover, given a family $\mathcal{F}$ of representatives of the various convex orbits of Φ, such a derivation d is unique up to the definition of the corresponding map $\mathcal{S}_{\mathcal{F}} \to \mathbb{R}^* \cdot \Gamma$, $\psi \mapsto t_\psi \theta^{(\psi)}$ (for arbitrary $t_\psi \in \mathbb{R}^*$). In particular, when Φ admits only one convex orbit, say $\mathcal{C}_\phi$, then d is unique up to the definition of $\theta^{(\phi)} \in \Gamma$, and $t_\psi \in \mathbb{R}^*$ for $\psi \in \Phi$ with $\phi \preccurlyeq \psi \prec \sigma^{-1}(\phi)$. More precisely, we have $\theta^{(\psi)} = \theta^{(\phi)} \prod_{k=1}^{\infty} \frac{\sigma^k(\psi)}{\sigma^k(\phi)}$.

Proof. By Proposition 5.4, the existence of a monomial Hardy type series derivation on $\mathbb{K}$ reduces to the existence of a map $d : \Phi \to \mathbb{R}^*\Gamma$ such that (H3′) holds. By Proposition 3.8, such a series derivation is log-compatible if and only if (HL4) holds, which means, in the monomial case, that for any $\phi \in \Phi$, $(\sigma(\phi))' = \frac{\phi'}{\phi} = t_\phi.\theta^{(\phi)}$. But, $(\sigma(\phi))' = t_{\sigma(\phi)}.\theta^{(\sigma(\phi))}\sigma(\phi)$ by definition. Therefore, we obtain $t_{\sigma(\phi)}.\theta^{(\sigma(\phi))} = t_\phi.\frac{\theta^{(\phi)}}{\sigma(\phi)}$, and by induction, for any $k \in \mathbb{N}^*$, $t_{\sigma^k(\phi)}.\theta^{(\sigma^k(\phi))} = t_\phi.\frac{\theta^{(\phi)}}{\prod_{j=1}^{k}\sigma^j(\phi)}$, and $t_{\sigma^{-k}(\phi)}.\theta^{(\sigma^{-k}(\phi))} = t_\phi.\theta^{(\phi)} \prod_{j=0}^{k-1} \sigma^{-j}(\phi)$. Now, consider $\psi \in \Phi$ such that $\phi \preccurlyeq \psi \prec \sigma^{-1}(\phi)$, so $\sigma^k(\phi) \preccurlyeq \sigma^k(\psi) \prec \sigma^{k-1}(\phi)$ for any $k \in \mathbb{N}$. We deduce that $\frac{\theta^{(\phi)}}{\prod_{j=1}^{k}\sigma^j(\phi)} \preccurlyeq \frac{\theta^{(\psi)}}{\prod_{j=1}^{k}\sigma^j(\psi)} \prec \frac{\theta^{(\phi)}}{\prod_{j=1}^{k-1}\sigma^j(\phi)}$, and equivalently $1 \preccurlyeq \frac{\theta^{(\psi)}}{\theta^{(\phi)}} \prod_{j=1}^{k} \frac{\sigma^j(\phi)}{\sigma^j(\psi)} \prec \sigma^k(\phi)$. By letting k tends to $+\infty$, we deduce that $1 \preccurlyeq \frac{\theta^{(\psi)}}{\theta^{(\phi)}} \prod_{j=1}^{+\infty} \frac{\sigma^j(\phi)}{\sigma^j(\psi)} \prec \chi$ for all $\chi \in \mathcal{C}_\phi$. For $\psi \in \Phi$ such that $\sigma^{-k}(\phi) \preccurlyeq \psi \prec \sigma^{-k-1}(\phi)$, we set $\tilde{\psi} := \sigma^k(\psi)$. Then, $\frac{\theta^{(\psi)}}{\theta^{(\phi)}} = \frac{\theta^{(\psi)}}{\theta^{(\tilde{\psi})}} \frac{\theta^{(\tilde{\psi})}}{\theta^{(\phi)}} = \prod_{j=1}^{k} \sigma^j(\psi) \frac{\theta^{(\tilde{\psi})}}{\theta^{(\phi)}}$. We are reduced to the preceding case. Finally, assume that $\mathcal{C}_\phi \prec \mathcal{C}_\psi$, i.e. for any $k, l \in \mathbb{N}$, $\sigma^{-k}(\phi) \prec \sigma^l(\psi)$. By (H3′), we have $\mathrm{LF}(\frac{\theta^{(\sigma^l(\psi))}}{\theta^{(\phi)}}) = \mathrm{LF}(\frac{\theta^{(\psi)}}{\theta^{(\phi)}\prod_{j=1}^{l}\sigma^j(\psi)}) \prec \sigma^l(\psi)$, which implies that $\mathrm{LF}(\frac{\theta^{(\psi)}}{\theta^{(\phi)}\prod_{j=1}^{\infty}\sigma^j(\psi)}) \prec \mathcal{C}_\psi$. To conclude, it suffices to note that, in the present case, $\mathrm{Tr}_{\succcurlyeq \mathcal{C}_\psi}(\prod_{j=1}^{\infty} \frac{\sigma^j(\psi)}{\sigma^j(\phi)}) = \prod_{j=1}^{\infty} \sigma^j(\psi)$.

Conversely, suppose now that there is a map $\Phi \to \Gamma$, $\phi \mapsto \theta^{(\phi)}$, such that condition (M) holds. We set $t_\phi := 1$ for any $\phi \in \Phi$. It remains to verify that (H3′) and (HL4) hold for such a map $d : \Phi \to \Gamma$, $\phi \mapsto \phi' = \theta^{(\phi)}\phi$. Condition (HL4) holds since, for any $\phi \in \Phi$, $\sigma(\phi)' = \theta^{(\sigma(\phi))}\sigma(\phi) = \frac{\theta^{(\phi)}}{\sigma(\phi)}\sigma(\phi) = \frac{\phi'}{\phi}$. For (H3′), we consider $\phi \prec \psi \in \Phi$, and deduce from (M) that: $\mathrm{LF}(\frac{\theta^{(\psi)}}{\theta^{(\phi)}}) = \sigma(\psi) \prec \psi$, and $\mathrm{LE}(\frac{\theta^{(\psi)}}{\theta^{(\phi)}}) = 1 > 0$.

Concerning the second part of the statement of the proposition, we observe from the preceding proof that, whenever we fix $\frac{\phi'}{\phi} := t_\phi.\theta^{(\phi)}$, this determines the values of $\frac{\psi'}{\psi}$ for any $\psi \in \mathcal{O}(\phi)$. Then note that $\mathcal{S}_{\mathcal{F},\phi}$ is a family of representatives of the $\mathbb{Z}$-orbits included in $\mathcal{C}(\phi)$. Therefore, $(\mathcal{S}_{\mathcal{F},\phi})_{\phi \in \mathcal{F}}$ is a partition of Φ, and $\mathcal{S}_{\mathcal{F}}$ is a family of representatives of the $\mathbb{Z}$-orbits of Φ. □

5.3. Examples

1. Our purpose is to illustrate Proposition 5.9, in particular when the chain $\Phi = \{\phi_i \mid i \in \mathbb{Z}\}$ is isomorphic to $\mathbb{Z}$. Let $n \in \mathbb{N}^*$ be given. We consider the corresponding automorphism σ of Φ defined by $\phi_i \mapsto \phi_{i-n}$. For instance, we set $\theta^{(\phi_0)} := 1$. In order to build a log-compatible monomial Hardy type series derivation on $\mathbb{K}$, we have to set $\theta^{(\sigma^{-k}(\phi_0))} := \theta^{(\phi_{kn})} = \prod_{l=0}^{k-1} \phi_{ln}$, and $\theta^{(\sigma^k(\phi_0))} = \theta^{(\phi_{-kn})} := \frac{1}{\prod_{l=1}^{k}\phi_{-ln}}$,

for any $k \in \mathbb{N}$. Furthermore, for any $j \in \{1, \ldots, n-1\}$, we have to set $\theta^{(\phi_j)} := \prod_{l=1}^{+\infty} \frac{\phi_{j-ln}}{\phi_{-ln}}$. Then, for any $k \in \mathbb{N}$, $\theta^{(\sigma^{-k}(\phi_j))} = \theta^{(\phi_{j+kn})} := \prod_{l=1}^{+\infty} \frac{\phi_{j-ln}}{\phi_{-ln}} \prod_{l=0}^{k-1} \phi_{j+ln} = \frac{\prod_{l=-k+1}^{+\infty} \phi_{j-ln}}{\prod_{l=1}^{+\infty} \phi_{-ln}}$, and $\theta^{(\sigma^k(\phi_j))} = \theta^{(\phi_{j-kn})} := \prod_{l=k+1}^{+\infty} \frac{\phi_{j-ln}}{\phi_{-ln}} \frac{1}{\prod_{l=1}^{k} \phi_{-ln}} = \frac{\prod_{l=k+1}^{+\infty} \phi_{j-ln}}{\prod_{l=1}^{+\infty} \phi_{-ln}}$.

As an illustration with germs of real functions at $+\infty$, consider for any $i \in \mathbb{Z}$, $\phi_{2i} := \log^{-i+1}(x)$ (with $\log^0(x) := x$), and $\phi_{2i+1} := \log^{-i+1} \circ g(x)$, where g is an (ultimately positive and differentiable) half compositional iterate of exp (i.e. $g \circ g(x) = \exp(x)$: see [Bos86, Section 6]). The automorphism of the chain Φ is the usual real logarithmic function. We have: $\sigma(\phi_i) = \phi_{i+2}$. By applying the usual derivation with respect to x, for any $k \in \mathbb{N}^*$, we compute: $\frac{\phi'_{2k}}{\phi_{2k}} = \exp(x)\exp^2(x)\cdots\exp^{k-1}(x) = \prod_{l=0}^{k-1} \phi_{2l}$, and $\frac{\phi'_{-2k}}{\phi_{-2k}} = \frac{1}{\log^k(x)\cdots\log(x)x} = \prod_{l=1}^{k} \phi_{-l}$. Concerning the fundamental monomial with odd indexes, following Proposition 5.9, we have to set: $\frac{\phi'_{2k+1}}{\phi_{2k+1}} := \frac{\prod_{l=-k+1}^{+\infty} \phi_{1-2l}}{\prod_{l=1}^{+\infty} \phi_{-2l}} = \frac{\prod_{l=-k+1}^{+\infty} \log^{l+1} \circ g(x)}{\prod_{l=1}^{+\infty} \log^{l+1}}$, and $\frac{\phi'_{-2k+1}}{\phi_{-2k+1}} := \frac{\prod_{l=k+1}^{+\infty} \phi_{1-2l}}{\prod_{l=1}^{+\infty} \phi_{-2l}} = \frac{\prod_{l=k+1}^{+\infty} \log^{l+1} \circ g(x)}{\prod_{l=1}^{+\infty} \log^{l+1}}$. In particular, $\frac{g'(x)}{g(x)} = \frac{\phi'_{-1}}{\phi_{-1}} := \frac{\prod_{l=2}^{+\infty} \phi_{1-2l}}{\prod_{l=1}^{+\infty} \phi_{-2l}} = \prod_{l=1}^{+\infty} \frac{\log^{l+1} \circ g(x)}{\log^l(x)}$.

It would be interesting to investigate the possible analytic meaning of such a formal definition for the derivative of g.

2. The purpose now is to provide a general example illustrating Proposition 5.9, with a uniform definition for the $\theta^{(\phi)}$'s. Let $(\Phi, \preccurlyeq)$ be a chain endowed with a decreasing automorphism $\sigma : \Phi \to \Phi$. Set $\theta^{(\phi)} := \prod_{k=1}^{+\infty} \sigma^k(\phi)$ and $t_\phi = 1$ for any $\phi \in \Phi$. These monomials verify (M), since for any $\phi \prec \psi \in \Phi$, we have $\frac{\theta^{(\psi)}}{\theta^{(\phi)}} = \prod_{k=1}^{+\infty} \frac{\sigma^k(\psi)}{\sigma^k(\phi)}$.

In the case of germs of real functions described at the end of Example 5.6 ($\Phi \approx \mathbb{Z}$), the present one can be seen as a limit case. Indeed, instead of differentiating with respect to the variable x, one may differentiate with respect to ϕ_i, with $i \to -\infty$. This can be viewed as a differentiation with respect to a variable ρ dominated by all the ϕ's in Φ: $\rho \prec \Phi$. In other words, differentiation with respect to a *translogarithm* (i.e. the compositional inverse of a transexponential: see [Bos86]).

6. Derivation on EL-series field

We consider $\mathbb{K}$ endowed with a pre-logarithm l. The **exponential-logarithmic series** (EL-series for short) field $(\mathbb{K}^{\mathrm{EL}}, \log)$ corresponding to the pre-logarithmic series field $(\mathbb{K}, l)$, is built as an infinite towering extension of $\mathbb{K}$, namely its exponential closure (see below, [Kuh00] and [KT] for details). Given a log-compatible series derivation d on $\mathbb{K}$, the purpose of this section is to show how to extend d to a log-compatible series derivation (also denoted by d) on $\mathbb{K}^{\mathrm{EL}}$. If we assume moreover that d is of Hardy type, then so will be its extension.

6.1. *The exponential closure of a pre-logarithmic series field*

Recall that the pre-logarithmic section $l : \Gamma \to \mathbb{K}^{\succ 1}$ is an embedding of ordered groups. We denote by $\hat{\Gamma} = \mathbb{K}^{\succ 1} \setminus l(\Gamma)$ the set complement of $l(\Gamma)$ in $\mathbb{K}^{\succ 1}$, and by $\tilde{\Gamma} = e^{\hat{\Gamma}}$ a multiplicative copy of it (the choice of e as abstract variable will result obvious from the definition of the new pre-logarithm $l^\sharp$ below). We endow the later with an ordering $\preccurlyeq$: $\forall e^a, e^b \in \tilde{\Gamma}$, $e^a \prec e^b \Leftrightarrow a < b$. Then we define a new group $\Gamma^\sharp = \Gamma \cup \tilde{\Gamma}$ with the following multiplicative rule: if $\alpha^\sharp, \beta^\sharp \in \Gamma^\sharp$ both belong to Γ, multiply them there; similarly if they both belong to $\tilde{\Gamma}$. If $\alpha^\sharp = \alpha \in \Gamma$ and $\beta^\sharp = e^a \in \tilde{\Gamma}$ (i.e. $a \in \hat{\Gamma}$), then set $\alpha^\sharp.\beta^\sharp := e^{l(\alpha)+a}$. Therefore $\Gamma^\sharp$ is a group extension of Γ.

We extend also l to the following isomorphism:

$$l^\sharp : (\Gamma^\sharp, .) \to (\mathbb{K}^{\succ 1}, +)$$

$$\alpha^\sharp \mapsto l^\sharp(\alpha^\sharp)$$

by defining $l^\sharp_{|\Gamma} := l$, and for any $\alpha^\sharp = e^a \in \tilde{\Gamma}$, $l^\sharp(\alpha^\sharp) := l^\sharp(e^a) = a$. Subsequently, we endow $\Gamma^\sharp$ with the ordering $\preccurlyeq$ defined as the transfer of the ordering $\leqslant$ on $\mathbb{K}^{\succ 1}$. Hence it extends the ordering $\preccurlyeq$ on Γ.

We set $\mathbb{K}^\sharp := \mathbb{R}((\Gamma^\sharp))$, and the corresponding $(\mathbb{K}^\sharp)^{\prec 1}$, $(\mathbb{K}^\sharp)^{\preccurlyeq 1}$, $(\mathbb{K}^\sharp)^{\succ 1}$ as before. Note that $\mathbb{K}^{\succ 1} \subset (\mathbb{K}^\sharp)^{\succ 1}$, so $l^\sharp : \Gamma^\sharp \to (\mathbb{K}^\sharp)^{\succ 1}$ is a pre-logarithmic section. We extend it to a pre-logarithm $l^\sharp$ on $\mathbb{K}^\sharp$ as in Definition 2.7.

Repeating this process, we obtain inductively the nth extension of $(\mathbb{K}, l)$, denoted by $(\mathbb{K}^{\sharp n}, l^{\sharp n})$, $n \in \mathbb{N}$. The corresponding EL-series field is defined as follows:

Definition 6.1. Set $\mathbb{K}^{\mathrm{EL}} = \bigcup_{n\in\mathbb{N}} \mathbb{K}^{\sharp n}$ and $\log = \bigcup_{n\in\mathbb{N}} l^{\sharp n}$. We call $(\mathbb{K}^{\mathrm{EL}}, \log)$ the EL-series field over the pre-logarithmic field $(\mathbb{K}, l)$.

Note that $\log : ((\mathbb{K}^{\mathrm{EL}})^{>0}, \cdot) \to (\mathbb{K}^{\mathrm{EL}}, +)$ is then an *order preserving isomorphism*. We denote by $\exp = \log^{-1}$ its inverse map.

6.2. Extending derivations to the exponential closure

Consider a strongly linear (i.e. which verifies (D2)) and log-compatible derivation d on $\mathbb{K}$. We show how to extend d to the corresponding EL-series field $\mathbb{K}^{\mathrm{EL}}$. Note that this has been considered for fields of transseries in [Sch01, Chapter 4.1.4]. However, our pre-logarithmic field $(\mathbb{K}, l)$ does not necessarily satisfy Axiom (T4) of [Sch01, Definition 2.2.1].

Theorem 6.2. *The strongly linear and log-compatible derivation d on $\mathbb{K}$ extends to a strongly linear and log-compatible derivation on $\mathbb{K}^{\mathrm{EL}}$, and this extension is uniquely determined. Moreover, if d is of Hardy type, then so is its extension to $\mathbb{K}^{\mathrm{EL}}$.*

To prove the theorem, we proceed by induction along the towering extension process. *Hence $(\mathbb{K}, l)$ represents from now until the end of this section, for simplicity of the notations, $(\mathbb{K}^{\sharp n}, l^{\sharp n})$ for some $n \in \mathbb{N}$.* We suppose $\mathbb{K}$ endowed with a strongly linear and log-compatible derivation d, and require that its extension to $\mathbb{K}^\sharp$ (also denoted by d) is also strongly linear and log-compatible:

Lemma 6.3. *For any $a^\sharp = \sum_{\alpha^\sharp \in \operatorname{Supp} a^\sharp} a_{\alpha^\sharp} \alpha^\sharp \in \mathbb{K}^\sharp$, if we set*

$$d(a^\sharp) = (a^\sharp)' = \sum_{\alpha^\sharp \in \operatorname{Supp} a^\sharp} a_{\alpha^\sharp} \alpha^\sharp \big(l^\sharp(\alpha^\sharp)\big)',$$

then d is well defined. Moreover, d is the unique strongly linear and log-compatible derivation on $\mathbb{K}^\sharp$ that extends d.

Furthermore, if d is a Hardy type derivation on $\mathbb{K}$, then so it is on $\mathbb{K}^\sharp$.

Proof. Consider $a^\sharp = \sum_{\alpha^\sharp \in \operatorname{Supp} a^\sharp} a^\sharp_{\alpha^\sharp} \alpha^\sharp \in \mathbb{K}^\sharp$. For any $\alpha^\sharp \in \operatorname{Supp} a^\sharp$, we denote $\alpha^\sharp = \alpha$ if $\alpha^\sharp \in \Gamma$, and $\alpha^\sharp = e^a$ for some $a \in \hat{\Gamma}$ if $\alpha^\sharp \in \tilde{\Gamma}$. Then, by definition, we have:

$$\big(a^\sharp\big)' = \sum_{\alpha \in (\operatorname{Supp} a^\sharp)\cap\Gamma} a^\sharp_\alpha \alpha' + \sum_{e^a \in (\operatorname{Supp} a^\sharp)\cap\tilde{\Gamma}} a^\sharp_{e^a} a' e^a.$$

If we denote $a = \sum_{\alpha \in \operatorname{Supp} a} a_\alpha \alpha \in \mathbb{K}^{\succ 1}$, and $a' = \sum_{\beta \in \operatorname{Supp} a'} b_\beta \beta \in \mathbb{K}$, then:

$$\big(a^\sharp\big)' = \sum_{\alpha \in (\operatorname{Supp} a^\sharp)\cap\Gamma} a^\sharp_\alpha \alpha' + \sum_{e^a \in (\operatorname{Supp} a^\sharp)\cap\tilde{\Gamma}} \sum_{\beta \in \operatorname{Supp} a'} a^\sharp_{e^a} b_\beta e^{a+l(\beta)}.$$

First, we verify that $(a^\sharp)'$ is well defined. We set $S := (\operatorname{Supp} a^\sharp) \cap \Gamma$, and $\tilde{S} := (\operatorname{Supp} a^\sharp) \cap \tilde{\Gamma}$. Observe that S and $\tilde{S}$ are anti-well-ordered subsets of Γ and $\tilde{\Gamma}$ respectively. Hence, if we set $\hat{S} := l^\sharp(\tilde{S})$, then $\hat{S}$ is anti-well-ordered in $(\mathbb{K}^{\succ 1}, \leqslant)$. The first sum is the derivative of $\sum_{\alpha \in S} a^\sharp_\alpha \alpha$, which is an element of $\mathbb{K}$. By the induction hypothesis, it is well defined. For the second sum, we have to show that the family $(a'e^a)_{a \in \hat{S}}$ is summable (see Definition 3.2). As noted above, the elements of the support of this family are of the form $e^{a+l(\beta)}$, where $a \in \hat{S}$ and $\beta \in \operatorname{Supp} a'$. Hence, to proceed by contradiction, we suppose that there is an increasing sequence $c_0 \leqslant c_1 \leqslant c_2 \leqslant \cdots$ of elements of $\mathbb{K}$ with $c_n := a_n + l(\beta^{(n)})$, $a_n \in \hat{S}$, $\beta^{(n)} \in \operatorname{Supp} a'_n$ for any n. Consider the corresponding sequence $(a_n)_{n \in \mathbb{N}}$. Since $\hat{S}$ is anti-well-ordered in $\mathbb{K}^{\succ 1}$, it cannot have an increasing subsequence. Moreover, if it had a stationary subsequence, we would have a corresponding increasing subsequence of $l(\beta^{(n)})$'s. Since l is order preserving, we would have an increasing subsequence of β_n's, all of them belonging to the support of a same a'_{n_0}, contradicting the induction hypothesis. Therefore, by Lemma 2.1, there is a strictly decreasing subsequence $a_{i_0} > a_{i_1} > a_{i_2} > \cdots$. Since the corresponding sequence $c_{i_0} < c_{i_1} < \cdots$ is increasing, we must have $l(\beta^{(i_0)}) < l(\beta^{(i_1)}) < \cdots$, and equivalently $\beta^{(i_0)} \prec \beta^{(i_1)} \prec \cdots$. Subsequently, we observe that:

$$\forall k < l \in \mathbb{N}, \quad 0 < a_{i_k} - a_{i_l} < l\big(\beta^{(i_l)}\big) - l\big(\beta^{(i_k)}\big) = l\left(\frac{\beta^{(i_l)}}{\beta^{(i_k)}}\right) \tag{2}$$

with $\beta^{(i_k)} \in \operatorname{Supp} a'_{i_k}$ and $\beta^{(i_l)} \in \operatorname{Supp} a'_{i_l}$. By the induction hypothesis, we denote $\beta^{(i_n)} = \alpha^{(i_n)}\gamma^{(i_n)}$, where $\alpha^{(i_n)} \in \operatorname{Supp} a_{i_n}$ and $\gamma^{(i_n)} \in \operatorname{Supp} \frac{(\alpha^{(i_n)})'}{\alpha^{(i_n)}} = \operatorname{Supp} l(\alpha^{(i_n)})'$ for any $n \in \mathbb{N}$. We observe that $\alpha^{(i_n)} \in \Gamma^{\succ 1}$ since $a_{i_n} \in \mathbb{K}^{\succ 1}$, and $l(\frac{\beta^{(i_l)}}{\beta^{(i_k)}}) = l(\frac{\alpha^{(i_l)}}{\alpha^{(i_k)}}) + l(\frac{\gamma^{(i_l)}}{\gamma^{(i_k)}})$ for any k, l. Consider the sequence $(\alpha^{(i_n)})_{n \in \mathbb{N}}$. If it had a strictly decreasing subsequence, we could define the series $\tilde{a} = \sum_{n \in \mathbb{N}} \alpha^{(i_n)}$. But the corresponding increasing subsequence of $(\beta^{(i_n)})_{n \in \mathbb{N}}$ would be included in $\operatorname{Supp} \tilde{a}'$, contradiction. Neither can $(\alpha^{(i_n)})_{n \in \mathbb{N}}$ have any stationary subsequence. Indeed, the corresponding increasing subsequence of $(\beta^{(i_n)})_{n \in \mathbb{N}}$ would be included in $\operatorname{Supp}(\alpha^{(i_{n_0})})'$ for a fixed n_0. Hence, by Lemma 2.1, $(\alpha^{(i_n)})_{n \in \mathbb{N}}$ has a strictly increasing subsequence, say $(\alpha^{(j_n)})_{n \in \mathbb{N}}$. Observe now that:

$$\forall k < l \in \mathbb{N}, \quad 1 \prec \frac{\gamma^{(j_k)}}{\gamma^{(j_l)}} \preccurlyeq \frac{\alpha^{(j_l)}}{\alpha^{(j_k)}}. \tag{3}$$

It implies that $0 < l(\frac{\gamma^{(j_k)}}{\gamma^{(j_l)}}) \leqslant l(\frac{\alpha^{(j_l)}}{\alpha^{(j_k)}}) < l(\alpha^{(j_l)})$ (since $\alpha^{(j_k)} \in \Gamma^{\succ 1}$). But, since l verifies (GA), we also have $l(\alpha^{(j_l)}) \prec \alpha^{(j_l)}$. Therefore, $l(\frac{\alpha^{(j_l)}}{\alpha^{(j_k)}}) \prec \alpha^{(j_l)}$, which implies that $l(\frac{\gamma^{(j_k)}}{\gamma^{(j_l)}}) \prec \alpha^{(j_l)}$. We obtain that $l(\frac{\beta^{(i_l)}}{\beta^{(i_k)}}) \prec \alpha^{(j_l)}$. But, we deduce from (2) that $a_{i_k} - a_{i_l} \prec \alpha^{(j_l)}$. It implies that the term with monomial $\alpha^{(j_l)}$ in a_{i_l} has been canceled by a term in a_{i_k}. Therefore, $\alpha^{(j_l)} \in \operatorname{Supp} a_{i_k}$, so $\beta^{(j_l)} \in \operatorname{Supp} a'_{i_k}$. By a straightforward induction, we obtain that the strictly increasing sequence $(\beta^{(j_n)})_{n \in \mathbb{N}}$ is included in $\operatorname{Supp} a'_{i_0}$, contradiction. The extension of d to $\mathbb{K}^\sharp$ is well defined.

The proofs that such an extension is a log-compatible derivation follow straightly from the definitions and are left to the reader.

Moreover, we observe that the extension of d to $\mathbb{K}^\sharp$ is uniquely determined by its definition, since we suppose that it is strongly linear and log-compatible:

$$d\big(a^\sharp\big) = \big(a^\sharp\big)' = \sum_{\alpha^\sharp \in \operatorname{Supp} a^\sharp} a^\sharp_{\alpha^\sharp} \alpha^\sharp \big(l^\sharp\big(\alpha^\sharp\big)\big)'.$$

Suppose now additionally that d is a Hardy type derivation on $\mathbb{K}$. To prove that d verifies l'Hospital's rule (HD2) on $\mathbb{K}^\sharp$, it suffices to prove it for its monomials. Hence, we consider $\alpha^\sharp, \beta^\sharp \in \Gamma^\sharp \backslash \{1\}$ with

$\alpha^\sharp \prec \beta^\sharp$. It means that $l^\sharp(\alpha^\sharp) < l^\sharp(\beta^\sharp)$ in $\mathbb{K}^{\succ 1}$, which is equivalent, by the induction hypothesis, to: $l^\sharp(\alpha^\sharp)' < l^\sharp(\beta^\sharp)'$. Therefore: $(\alpha^\sharp)' = \alpha^\sharp l^\sharp(\alpha^\sharp)' \prec \beta^\sharp l^\sharp(\beta^\sharp)' = (\beta^\sharp)'$.

To determine the subfield of constants of $\mathbb{K}^\sharp$, suppose now that there exists $a^\sharp = \sum_{\alpha^\sharp \in \operatorname{Supp} a^\sharp} a^\sharp_{\alpha^\sharp}\alpha^\sharp \in \mathbb{K}^\sharp \setminus \mathbb{R}$ such that $(a^\sharp)' = 0$. We denote as before:

$$\begin{aligned}(a^\sharp)' &= \sum_{\alpha^\sharp \in \operatorname{Supp} a^\sharp} a^\sharp_{\alpha^\sharp}\alpha^\sharp\big(l^\sharp(\alpha^\sharp)\big)' \\ &= \sum_{\alpha\in S} a^\sharp_\alpha \alpha' + \sum_{e^a\in\tilde{S}} a^\sharp_{e^a} a' e^a \\ &= \sum_{\alpha\in S} a^\sharp_\alpha \alpha' + \sum_{a\in\hat{S}}\sum_{\beta\in\operatorname{Supp} a'} a^\sharp_{e^a} b_\beta e^{a+l(\beta)},\end{aligned}$$

where $S := (\operatorname{Supp} a^\sharp)\cap\Gamma$, $\tilde{S} := (\operatorname{Supp} a^\sharp)\cap\tilde{\Gamma}$, and $\hat{S} := l^\sharp(\tilde{S})$. Set $\alpha_0^\sharp := \max(\operatorname{Supp} a\backslash\{1\})$. There are two possibilities. Either $\alpha_0^\sharp = \alpha_0 \in \Gamma$. By the induction hypothesis, it implies that there is $\beta_0 \in \operatorname{Supp}\alpha_0'$. So the corresponding term in $(a^\sharp)'$ must have been canceled by the leading term of the second sum. Or $\alpha_0^\sharp = e^{a_0}$ for some $a_0 \in \hat{\Gamma}$. Then there exist $\hat{\alpha}_0 := \operatorname{LM}(a_0) \succ 1$ and $\hat{\beta}_0 := \operatorname{LM}(\hat{\alpha}_0') \neq 0$ (by the induction hypothesis). So $e^{a_0+l(\hat{\beta}_0)}$ is the leading monomial of the second sum. The corresponding term in $(a^\sharp)'$ must have been canceled by the leading term of the first sum, say β_0. Therefore, in the two cases, there must be an equality $e^{a_0+l(\hat{\beta}_0)} = \beta_0$, which is equivalent to: $\frac{e^{a_0+l(\hat{\beta}_0)}}{\beta_0} = 1$. But $\frac{e^{a_0+l(\hat{\beta}_0)}}{\beta_0} = e^{a_0+l(\hat{\beta}_0)-l(\beta_0)}$. So we should have $a_0 + l(\hat{\beta}_0) - l(\beta_0) = a_0 + l(\frac{\hat{\beta}_0}{\beta_0}) = 0$, which is absurd since $a_0 \in \hat{\Gamma} = \mathbb{K}^{\succ 1} \setminus l(\Gamma)$.

It remains to prove (HD3) on $\mathbb{K}^\sharp$. We consider $a^\sharp, b^\sharp \in \mathbb{K}_\sharp$ with $|a^\sharp| \succ |b^\sharp| \succ 1$. Note that, by replacing $a^\sharp$ and $b^\sharp$ by $-a^\sharp$, $\pm\frac{1}{a^\sharp}$, and $-b^\sharp$, $\pm\frac{1}{b^\sharp}$ respectively, we still have $|a^\sharp| \succ |b^\sharp| \succ 1$, and the leading monomials of $\frac{(a^\sharp)'}{a^\sharp}$ and $\frac{(b^\sharp)'}{b^\sharp}$ are preserved. So we can suppose without loss of generality that $a^\sharp \succ b^\sharp \succ 1$. Consequently, $l^\sharp(a^\sharp) > l^\sharp(b^\sharp) > 0$ in $\mathbb{K}$. This implies that $\frac{(a^\sharp)'}{a^\sharp} = l^\sharp(a^\sharp)' \succcurlyeq l^\sharp(b^\sharp)' = \frac{(b^\sharp)'}{b^\sharp}$. Moreover, $a^\sharp \succ b^\sharp \succ 1$ if and only if $l^\sharp(a^\sharp) \succ l^\sharp(b^\sharp)$ in $\mathbb{K}$, which means that $\frac{(a^\sharp)'}{a^\sharp} = l^\sharp(a^\sharp)' \succ l^\sharp(b^\sharp)' = \frac{(b^\sharp)'}{b^\sharp}$.

This concludes the proof of Lemma 6.3, and so the one of Theorem 6.2. □

7. Asymptotic integration and integration on EL-series

Let $(\mathbb{K}, l, d)$ be a pre-logarithmic series field endowed with a strongly linear and log-compatible Hardy type derivation d. Recall that $\hat{\theta} := \text{g.l.b.}_{\preccurlyeq}\{\theta^{(\phi)},\ \phi\in\Phi\}$, whenever it exists in Γ. In particular, in the case where d is a series derivation, $\hat{\theta} = \text{g.l.b.}_{\preccurlyeq}\{\operatorname{LM}(\frac{a'}{a});\ a\in\mathbb{K}^*,\ a \not\asymp 1\}$ (see Proposition 4.5).

Theorem 7.1. *Let* $(\mathbb{K}^{\mathrm{EL}}, \log, d)$ *be the induced differential EL-series field as in Theorem* 6.2. *A series* $a \in \mathbb{K}^{\mathrm{EL}}$ *admits an asymptotic integral if and only if* $a \not\asymp \hat{\theta}$.

Proof. Recall that the induced derivation d on $\mathbb{K}^{\mathrm{EL}}$ is itself strongly linear, log-compatible and of Hardy type (Theorem 6.2). We proceed by induction along the towering extension construction of $\mathbb{K}^{\mathrm{EL}}$. The initial step is given by Theorem 4.4. Consider $(\mathbb{K}, l, d)$ as in the statement of the theorem. We want to show that its extension $(\mathbb{K}^\sharp, l^\sharp)$ verifies $\text{g.l.b.}_{\preccurlyeq}\{\operatorname{LM}(\frac{(a^\sharp)'}{a^\sharp});\ a^\sharp\in\mathbb{K}^\sharp\backslash\{0\},\ a^\sharp \not\asymp 1\} = \hat{\theta}$ (indeed, we will then obtain the desired result by applying Theorem 4.4 to $\mathbb{K}^\sharp = \mathbb{R}((\Gamma^\sharp))$). Let $1 \not\asymp a^\sharp \in \mathbb{K}^\sharp\backslash\{0\}$. We denote $\alpha^\sharp := \operatorname{LM}(a^\sharp)$. By (HD2), $\operatorname{LM}(\frac{(a^\sharp)'}{a^\sharp}) = \operatorname{LM}(\frac{(\alpha^\sharp)'}{\alpha^\sharp})$. There are two possibilities. Either $\alpha^\sharp = \alpha \in \Gamma$, which implies that $\operatorname{LM}(\frac{(\alpha^\sharp)'}{\alpha^\sharp}) \succcurlyeq \hat{\theta}$. Or $\alpha^\sharp = e^a \in \tilde{\Gamma}$ for some $a \in \hat{\Gamma} = \mathbb{K}^{\succ 1}\backslash l(\Gamma)$. We denote $\alpha := \operatorname{LM}(a)$. Then $\operatorname{LM}(\frac{(\alpha^\sharp)'}{\alpha^\sharp}) = \operatorname{LM}(a') = \operatorname{LM}(\alpha') = \alpha\operatorname{LM}(\frac{(\alpha)'}{\alpha})$. Since $\operatorname{LM}(\frac{(\alpha)'}{\alpha}) \succcurlyeq \hat{\theta}$, and $\alpha \succ 1$, then $\mathrm{LM}(\frac{(\alpha^\sharp)'}{\alpha^\sharp}) \succ \theta$. Therefore, by induction we obtain that $\hat{\theta} = \text{g.l.b.}_{\preccurlyeq}\{\mathrm{LM}(\frac{(\hat{a})'}{\hat{a}});\ \hat{a} \in \mathbb{K}^{\sharp n}\backslash\{0\},\ \hat{a} \not\asymp 1\}$ for any $n \in \mathbb{N}$, so $\hat{\theta} = \text{g.l.b.}_{\preccurlyeq}\{\mathrm{LM}(\frac{(\hat{a})'}{\hat{a}});\ \hat{a} \in \mathbb{K}^{\mathrm{EL}}\backslash\{0\},\ \hat{a} \not\asymp 1\}$ as desired. □

Denote $\Gamma^{\mathrm{EL}} := \mathrm{LM}(\mathbb{K}^{\mathrm{EL}}) = \bigcup_{n\in\mathbb{N}} \Gamma^{\sharp n}$.

Corollary 7.2. *The EL-series field $\mathbb{K}^{\mathrm{EL}}$ is closed under integration if and only if $\hat{\theta} \notin \Gamma^{\mathrm{EL}}$.*

Proof. By [Kuh, Theorems 55 and 56], it suffices to prove that $\hat{\theta} \notin \Gamma^{\mathrm{EL}}$ implies that any element of $\mathbb{K}^{\mathrm{EL}}$ has an asymptotic integral, which was proved in the preceding theorem. □

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