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# Metric spaces are Ramsey

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**Abstract**

We prove that the class of all ordered finite metric spaces is a Ramsey class. This solves a problem of Kechris, Pestov and Todorćevic.


## 1. Ramsey classes

This paper contains one result which is formulated in the title. However the proof is a complex interplay of various structures and thus it is convenient to formulate the result more generally.

Let $\mathcal{K}$ be a class of objects which is isomorphism closed and endowed with subobjects. Given two objects $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ we denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all subobjects $\mathbf{A}'$ of $\mathbf{B}$ which are isomorphic to $\mathbf{A}$. (Thus in this notation the rôle of $\mathcal{K}$ is suppressed. It should always be clear from the context.) We say that the class $\mathcal{K}$ has the **A**-*Ramsey property* if the following statement holds:

For every positive integer $k$ and for every $\mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_k$. Here the last symbol (the *Erdős–Rado partition arrow*) has the following meaning.

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For every partition $\binom{\mathbf{C}}{\mathbf{A}} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_k$ there exists $\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}$ and an $i$, $1 \le i \le k$, such that $\binom{\mathbf{B}'}{\mathbf{A}} \subset \mathcal{A}_i$.

In the extremal case where a class $\mathcal{K}$ has the $\mathbf{A}$-Ramsey property for all its objects $\mathbf{A}$, we say that $\mathcal{K}$ is a *Ramsey class*.

These notions crystallized in the early 1970's; see e.g. [9,19,3]. This formalism and the natural questions it motivated essentially contributed to establishing Ramsey theory as a "theory" (as nicely put in the introduction to [4]). The notion of a Ramsey class is highly structured and in a sense it is the top of the line of the Ramsey notions ("one can partition everything in any number of classes to get anything homogeneous"). Consequently there are not many (essentially different) examples of Ramsey classes known. Examples of Ramsey classes include:

(i) The class of all finite ordered graphs.
(ii) The class of all finite partially ordered sets (with a fixed linear extension).
(iii) The class of all finite vector spaces (over a fixed field $F$).
(iv) The class of all (labeled) finite partitions.

For these results see [3,4,16,17,12]. We formulate explicitly one of the general results (for binary relational structures) which is tailored suit our proof below.

Let $I$ be a finite set of positive real numbers. $I$ is called the *type* (or *signature*). We consider objects, *ordered binary relational structures*, of the form $\mathbf{A} = (X, (R_i; i \in I))$ where $X$ is a non-empty ordered set and $R_i \subseteq X^2$ is a binary symmetric anti-reflexive relation for every $i \in I$. (That is, we assume $(x, y) \in R_i \Rightarrow (y, x) \in R_i$ while $(x, x) \notin R_i$ for every $x \in \underline{\mathbf{A}}$.) We assume that relations $R_i, i \in I$, are mutually disjoint: $R_i \cap R_j = \emptyset, i \ne j \in I$. The ordering of the set $X$ will be denoted by $\le_{\mathbf{A}}$ and called *standard* ordering. We also denote the type of $\mathbf{A}$ by $I(\mathbf{A}) = I$, the underlying set (vertices) of $\mathbf{A}$ by $\underline{\mathbf{A}} = X$ (sometimes we simply denote the set of vertices as $\mathbf{A}$) and the relations by $R_i(\mathbf{A}) = R_i$. Thus the type $I$ is just an index set of the system of relations. Later the actual values of $I$ will play a role.

We denote by *Rel* the class of all such ordered binary relational structures $\mathbf{A}$ of all possible (finite) types $I$. The class *Rel* will be considered with homomorphisms and embeddings (corresponding to induced substructures): For relational structures $\mathbf{A} = (X, (R_i; i \in I))$ and $\mathbf{A}' = (X', (R'_{i'}; i' \in I'))$ of types $I$ and $I'$ (note that the types $I$ and $I'$ may be different) a mapping $f : X \longrightarrow X'$ is called a *monotone homomorphism* of $\mathbf{A}$ into $\mathbf{A}'$ if $I \subset I'$ and if $f$ is a monotone mapping of $(X, \le_{\mathbf{A}})$ into $(X', \le_{\mathbf{A}'})$ satisfying $(f(x_1), f(x_2)) \in R'_i$ for all $(x_1, x_2) \in R_i$ for all $i \in I$. $f$ is called an *embedding* of $\mathbf{A}$ into $\mathbf{A}'$ if moreover $f$ is a monotone injection of $X$ into $X'$ satisfying $(f(x_1), f(x_2)) \in R'_i$ if and only if $(x_1, x_2) \in R_i$ for all $i \in I$.

As usual, an inclusion (resp. bijective) embedding is called a *substructure* (resp. *isomorphism*). Given two ordered relational structures $\mathbf{A}, \mathbf{B}$ we denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the class of all substructures $\mathbf{A}'$ of $\mathbf{B}$ which are isomorphic to $\mathbf{A}$. Any such $\mathbf{A}'$ is called a *copy of* $\mathbf{A}$ in $\mathbf{B}$. One more definition: For real numbers $d, D, 0 < d < D$, we denote by $Rel(d, D)$ the subclass of *Rel* induced by all systems $\mathbf{A} = (X, (R_i; i \in I))$ where $I$ is a subset of the interval $[d, D]$. We have the following:

**Theorem 1.1** *([15]). For every choice of reals $d, D, 0 < d < D$ the class $Rel(d, D)$ is a Ramsey class.*

*Explicitly: For every choice of a natural number $k$ and of structures $\mathbf{A}, \mathbf{B} \in Rel(d, D)$ there exists a structure $\mathbf{C} \in Rel(d, D)$ with the following property: For every partition $\binom{\mathbf{C}}{\mathbf{A}} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_k$ there exists $i, 1 \leq i \leq k$, and a substructure $\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\mathbf{B}'}{\mathbf{A}} \subset \mathcal{A}_i$.*

In [15,14,12] (and elsewhere) Theorem 1.1 is stated in a more general form as a Ramsey theorem for classes $Rel(\Delta)$ of all ordered relational structures of a fixed type $\Delta$. We shall make use of Theorem 1.1 in our proof of the following theorem which is the main result of this paper:

**Theorem 1.2.** *The class of all finite ordered metric spaces is a Ramsey class.*

Let us formulate Theorem 1.2 explicitly.

Denote by $\mathcal{M}$ the class of all finite ordered metric spaces (i.e. metric spaces where the set of points is linearly ordered). The ordering is again called *standard*. $\mathcal{M}$ will be considered with a mapping which is both an isometry and a monotone mapping with respect to standard orderings.

Theorem 1.2 asserts that $\mathcal{M}$ is a Ramsey class: For every choice of ordered metric spaces $(X, \rho), (Y, \sigma)$ (standard orderings are not indicated) and for every $k \geq 2$ there exists a metric space $(Z, \lambda)$ such that the following statement holds:

For every partition $\binom{(Z,\lambda)}{(X,\rho)} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_k$ there exists $(Y', \sigma') \in \binom{(Z,\lambda)}{(Y,\sigma)}$ and an $i, 1 \leq i \leq k$ such that $\binom{(Y',\sigma')}{(X,\rho)} \subset \mathcal{A}_i$.

This will be again denoted by

$$(Z, \lambda) \longrightarrow (Y, \sigma)_k^{(X,\rho)}.$$

Theorem 1.2 solves a problem of Kechris et al. see [6]. The paper [6] lists several consequences of Theorem 1.2 to dynamical systems and topological groups (extremal amenable groups, minimal flows). This also implies a remarkable property of the *Urysohn space* which is defined as a completion of the homogeneous universal rational metric space; see e.g. [22,23]. (The author, himself a student of Katětov, cannot resist mentioning that this construction was one of the last results of Urysohn [21] as well as of Katětov [5].) Theorem 1.2 also generalizes the Ramsey theorem for pairs in metric spaces stated in [14].

In this paper we concentrate solely on Theorem 1.2. The other applications of our proof will appear in [13]. Theorem 1.2 will be proved as a consequence of a more technical form stated in Sections 2 and 3. Here is the outline of the proof:

We view any metric space $(X, \rho)$ as a complete graph $K_X$ with weight-labelling $w : E(K_X) \longrightarrow \mathbb{R}$ where $w(x, y) = \rho(x, y)$. Of course we shall mostly denote the weight $w$ by $\rho$. But sometimes we denote the weight of edges by $w(x, y)$ when we want to stress that we deal with weights which are not known to satisfy the triangle inequality. Every metric space $(X, \rho)$ and every labelled complete graph $K_X$, $w : E(K_X) \longrightarrow \mathbb{R}$ may be viewed as a binary relational system of type $I$: we put $(x, y) \in R_i$ iff $\rho(x, y) = i$. (Thus $I$ is the set of all possible distances in $(X, \rho)$, and $i \in I$ may be viewed as the weight of an edge $(x, y) \in R_i$.) Clearly not every binary relational system corresponds to a metric space (we need symmetry and the triangle inequality). But every binary relational system $\mathbf{A} = (X, (R_i; i \in I))$ may be converted to a metric space $(X, \rho_{\mathbf{A}})$ by defining $\rho_{\mathbf{A}}(x, y)$ as the minimal weight

$$w(P) = i_1 + i_2 + \cdots + i_t$$

of a path $P = (x = x_0, x_1, \ldots, x_k = y)$ from $x$ to $y$ in $\mathbf{A}$ such that $(x_{j-1}, x_j) \in R_{i_j}$ for $j = 1, \ldots, t$. $\rho_{\mathbf{A}}$ is the *free (path)* metric generated by $\mathbf{A}$.

We denote by $F(\mathbf{A})$ the binary relational system corresponding to the metric space $(\underline{\mathbf{A}}, \rho_{\mathbf{A}})$. Clearly it can (only too often) happen that for a binary relational system $\mathbf{B}$ there is an embedding $\mathbf{B} \to \mathbf{A}$ while $\mathbf{B} \not\to F(\mathbf{A})$. That may of course happen regardless of whether $\mathbf{B}$ is metric space or not. Thus we shall introduce the notion of *$\ell$-approximative system*. We then prove by induction on $\ell$ a Ramsey type theorem for ordered $\ell$-approximative systems (Theorem 2.1). On the other hand, for each (fixed) metric system $\mathbf{B}$ there exists $\ell$ such that every $\ell$-approximative embedding of $\mathbf{B}$ into $\mathbf{A}$ is an isometry $\mathbf{B} \to F(\mathbf{A})$. This will be then used to prove that the Ramsey theorem for $\ell$-approximative systems implies Theorem 1.2.

We use the following convention: The *length* of a path $P$ is the number of edges it contains (i.e. $k$ above), while $w(P)$ will be called the *length* of $P$.

The paper is organized as follows. In Section 2 we state the Theorem 1.2 in a more technical form and introduce classes $Rel_{(\ell)}(d, D)$ of $\ell$-approximative systems (and a given range of edge lengths). In Sections 3 and 4 we further refine the classes $Rel_{(\ell)}(d, D)$ to classes $PartiRel_{(\ell)}(d, D)$ and prove the $\mathbf{A}$-Ramsey property by a variant of *amalgamation technique* (known also as *Partite Construction*); see [20,18,17,12]. This then implies Theorem 1.2. Section 5 contains concluding remarks and some related results.

## 2. Metric approximation

Let $d < D$ be positive real numbers, $\ell$ a positive integer. Before defining objects and morphisms of our classes we take time out for a definition: Given $\mathbf{A} = (X, (R_i; i \in I))$ we refer to a pair $(x, y)$ as an *edge of* $\mathbf{A}$ if $(x, y) \in R_i$ for some $i \in I$. We say that $(x, y) \in R_i$ is an *$\ell$-metric edge* in $\mathbf{A} = (X, (R_i; i \in I))$ if for any path $P$ in $\mathbf{A}$ from $x$ to $y$ in $x = x_0, x_1, \ldots, x_t, t \leq \ell$, with weights of edges $i_1, i_2, \ldots, i_t$ (i.e. we assume $\rho(x_{j-1}, x_j) = i_j$) has weight $i \leq i_1 + i_2 + \cdots + i_t$.

We shall define the class $Rel_{(\ell)}(d, D)$ as follows:

Objects of $Rel_{(\ell)}(d, D)$ (called *$\ell$-approximative systems* and usually denoted by $\mathbf{A}, \mathbf{B}, \ldots$) are those objects $\mathbf{A} = (X, (R_i; i \in I))$ of the class $Rel(d, D)$ which moreover satisfy the additional property that every edge of $\mathbf{A}$ is $\ell$-metric.

Thus the objects are binary relational structures where the relations are indexed by a set $I$ of positive real numbers which we may interpret as weights of edges; these weights are denoted by $\rho$: if $(x, y) \in R_i$ we also write $\rho(x, y) = i$.

Embeddings of $Rel_{(\ell)}(d, D)$ are just embeddings of $Rel(d, D)$.

An edge $(x, y)$ which is $\ell$-metric for every $\ell$ is called a *metric edge*. If all pairs of vertices of a system $\mathbf{A}$ are edges and they are all metric (and in this case it suffices that they are 2-metric), then of course $\mathbf{A}$ corresponds to a metric space $(\underline{\mathbf{A}}, \rho)$. In that case we also say that $\mathbf{A}$ is a *metric* system.

Note that the objects $\mathbf{A}, \mathbf{A}'$ of $Rel_{(\ell)}(d, D)$ need not correspond to metric spaces. However lengths of edges of an $\ell$-approximative systems cannot be "shortened" by paths of length $\leq \ell$. Thus for larger $\ell$ we get the better approximation of a metric space and of an isometry.

Given objects $\mathbf{A}, \mathbf{B}$ of $Rel_{(\ell)}(d, D)$ we again denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the class of all subobjects of $\mathbf{B}$ which are isomorphic to $\mathbf{A}$. Note also that for $\ell = 1$ the notion of $\ell$-approximative systems (and their $\ell$-approximative embeddings) coincides with the notion of relational structures (and their embeddings) – we have $Rel_{(1)}(d, D) = Rel(d, D)$. Thus the following generalizes Theorem 1.1:

**Theorem 2.1.** *Let $0 < d < D$ be real numbers, $\ell$ a positive integer. Then for all metric systems $\mathbf{A}$ and $\mathbf{B}$ in $Rel(d, D)$ there exists $\mathbf{C} \in Rel_{(\ell)}(d, D)$ such that we have (in the class $Rel_{(\ell)}(d, D)$)*

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

We postpone the proof of Theorem 2.1 to Section 4. Here we show that Theorem 2.1 implies Theorem 1.2.

**Proof.** Let $(X, \rho), (Y, \sigma)$ be finite ordered metric spaces. We may assume that $(Y, \sigma)$ contains an isometric copy of $(X, \rho)$. Put $d = \min\{\sigma(x, y)\}$ and $D = \max\{\sigma(x, y)\}$. Let $\ell \geq D/d$. Let $\mathbf{A} = (X, (R_i; i \in I))$ and $\mathbf{B} = (Y, (S_j; j \in J))$ be binary relational systems corresponding to the metric spaces $(X, \rho)$ and $(Y, \sigma)$ (thus both systems are metric). By Theorem 2.1 there exists a binary relational system $\mathbf{C} = (Z, (T_k; k \in K))$ which is Ramsey for $\mathbf{A}$ and $\mathbf{B}$ in the class $Rel_{(\ell)}(d, D)$. Let us write this explicitly:

For every partition $\binom{\mathbf{C}}{\mathbf{A}} = \mathcal{A}_1 \cup \mathcal{A}_2$ there exists an $\ell$-approximative embedding $g : \mathbf{B} \longrightarrow \mathbf{C}$ and $\iota \in \{1, 2\}$ such that for all $\ell$-approximative embeddings $f : \mathbf{A} \longrightarrow \mathbf{B}$ we have $g \circ f \in \mathcal{A}_\iota$.

In this situation, consider the metric space $(Z, \theta)$ freely generated by the binary relational system $\mathbf{C}$. Recall: we put $\theta(x, y) = \min\{i_1 + i_2 + \cdots + i_t\}$ where the second minimum is taken over all paths $x = x_0, x_1, \ldots, x_t$ where $(x_{r-1}, x_r) \in R_{i_r}$. Finally define the metric space $(Z, \vartheta)$ as $\vartheta(x, y) = \min\{D, \theta(x, y)\}$. We note that all the $\vartheta$ distances are in the interval $[d, D]$ and that the corresponding binary system $F(Z, \vartheta)$ is metric. As $\ell \geq D/d$ we have that for every edge $(x, y)$ of $\mathbf{C}$ it holds that $(x, y) \in R_i$ if and only if $\vartheta(x, y) = i$. It further follows that $f : \mathbf{A} \longrightarrow \mathbf{C}$ is an $\ell$-approximative embedding if and only if $f : \mathbf{A} \longrightarrow F(\mathbf{C})$ is an embedding, which in turn is equivalent to the fact that $f : (X, \rho) \longrightarrow (Z, \vartheta)$ is an isometry.

Similarly, $f : \mathbf{B} \longrightarrow \mathbf{C}$ is an $\ell$-approximative embedding iff $g : \mathbf{B} \longrightarrow F(\mathbf{C})$ is an embedding iff $g : (Y, \sigma) \longrightarrow (Z, \vartheta)$ is an isometry.

It follows that $\binom{\mathbf{C}}{\mathbf{A}} \subseteq \binom{F(Z,\vartheta)}{\mathbf{A}}$ while $\mathbf{B} = F(\mathbf{B})$ and $\mathbf{A} = F(\mathbf{A})$. Combining all this we get $F(\mathbf{C}) \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ (in the class $Rel_{(\ell)}(d, D)$) and also $(Z, \vartheta) \longrightarrow (Y, \sigma)^{(X,\rho)}_2$. This proves Theorem 1.2. $\square$

## 3. Partite approximative classes

Our proof proceeds by a double induction and towards this end we introduce a version of Partite Construction (see Introduction).

We define the class $\mathit{PartiRel}_{(\ell)}(d, D)$ of structures as follows:

An object is a triple $(\mathbf{B}, \mathbf{A}, \iota)$ where:

(i) $\mathbf{A}$, $B$ are ordered binary relational structures $\mathbf{B} \in \mathit{Rel}_{(\ell)}(d, D)$, $\mathbf{A} \in \mathit{Rel}_{(\ell-1)}(d, D)$. Put explicitly, $\mathbf{A} = (X, (R_i; i \in I))$, $\mathbf{B} = (Y, (S_j; j \in J))$, $I, J$ are finite set of reals $I, J \subset [d, D]$.
(ii) $\iota : \mathbf{B} \to \mathbf{A}$; $\iota$ is a monotone homomorphism. Let us define explicitly the properties of $\iota$:
(iia) $\iota : X \longrightarrow Y$ is a monotone mapping with respect to standard ordering of $\leq_{\mathbf{A}}$ and $\leq_{\mathbf{B}}$ (note that $\iota$ need not be injective);
(iib) if $(x, y) \in S_j$, then $(\iota(x), \iota(y)) \in R_j$ (thus $J \subset I$).

We also call $\mathbf{B}$ an $\mathbf{A}$-*partite* (binary relational) system. This looks like a small change. But considering partite ("levelled") systems (sets of the form $\iota^{-1}(x)$ are sometimes called *parts* of $\mathbf{B}$) allows us to derive more complex Ramsey type statements from simpler ones and to start the induction procedure in our case. And for this the key is the definition of morphisms which is as follows:

Let $(\mathbf{B}, \mathbf{A}, \iota)$ and $(\mathbf{B}', \mathbf{A}', \iota')$ be objects of $\mathit{PartiRel}_{(\ell)}(d, D)$. An *embedding* $(\mathbf{B}, \mathbf{A}, \iota) \longrightarrow (\mathbf{B}', \mathbf{A}', \iota')$ is a pair $(f, \alpha)$ with the following properties:

(i) $\alpha : \mathbf{A} \to \mathbf{A}'$ is an embedding (in the class $\mathit{Rel}_{(\ell-1)}(d, D)$);
(ii) $f : \mathbf{B} \to \mathbf{B}'$ is an embedding (in the class $\mathit{Rel}_{(\ell)}(d, D)$);
(iii) $\iota' \circ f = \alpha \circ \iota$.

This means that the mappings $f$ and $\alpha$ commute with $\iota$'s as indicated by the following diagram.

$$\begin{tikzcd} \mathbf{B} \arrow[r, "\iota"] \arrow[d, "f"'] & \mathbf{A} \arrow[d, "\alpha"] \\ \mathbf{B}' \arrow[r, "\iota'"] & \mathbf{A}' \end{tikzcd}$$

(Thus an embedding has to preserve parts of $\mathbf{B}$ and $\mathbf{B}'$.)

Consider an object $(\mathbf{B}, \mathbf{A}, \iota) \in \mathit{PartiRel}_{(\ell)}(d, D)$, $\iota : \mathbf{B} \to \mathbf{A}$. If $\iota$ is an injective mapping then we say that $\mathbf{B}$ is a *transversal* system. Clearly any $\mathbf{B} \in \mathit{Rel}_{(\ell)}(d, D)$ can be regarded as a *transversal system* $(\mathbf{B}, \mathbf{B}, \mathbf{1}) \in \mathit{PartiRel}_{(\ell)}(d, D)$ where $\mathbf{1} : \mathbf{B} \longrightarrow \mathbf{B}$ is the identity mapping. This is a functorial correspondence: $f : (\mathbf{B}, \mathbf{B}, \mathbf{1}) \longrightarrow (\mathbf{B}', \mathbf{B}', \mathbf{1})$ is an embedding in $\mathit{PartiRel}_{(\ell)}(d, D)$ iff $f : \mathbf{B} \longrightarrow \mathbf{B}'$ is an embedding in $\mathit{Rel}_{(\ell)}(d, D)$. Thus we may regard $\mathit{Rel}_{(\ell)}(d, D)$ as a subcategory of $\mathit{PartiRel}_{(\ell)}(d, D)$.

We shall prove the following technical result:

**Theorem 3.1.** *Let $\ell$ be a positive integer; let $\mathbf{A}$ and $\mathbf{B}$ be metric systems in $\mathit{Rel}_{\ell}(d, D)$ (considered as transversal systems). Then there exists $\mathbf{C} \in \mathit{PartiRel}_{(\ell)}(d, D)$ such that we* *have (in the class $PartiRel_{(\ell)}(d, D)$)*

$$\mathbf{C} \longrightarrow (\mathbf{B})_2^{\mathbf{A}}.$$

We could also prove that the classes $Rel_{(\ell)}(d, D)$ and $PartiRel_{(\ell)}(d, D)$ are Ramsey classes. (We want to keep generalities at the minimum and concentrate on the proof of Theorem 1.2 only; we shall publish generalizations elsewhere.)

## 4. Proofs

As stated above, we apply Partite Construction at the heart of which lies the amalgamation property.

The amalgamation property now takes the following technical form. (To simplify the notation, the symbol **1** will denote an inclusion embedding or identity mapping between sets.)

**Lemma 4.1** (*Amalgamation Lemma*). *Let* $\mathbf{C} \in Rel_{(\ell-1)}(d, D)$, *and let* $\mathbf{A}$ *be a metric subsystem of* $\mathbf{C}$. *Denote by* $\mathbf{1} : \mathbf{A} \longrightarrow \mathbf{C}$ *the inclusion map. Let for* $i = 1, 2$ *there be given systems* $(\mathbf{B}_i, \mathbf{C}, \iota_i : \mathbf{B}_i \longrightarrow \mathbf{C}) \in PartiRel_{(\ell)}(d, D)$. *Let* $(\mathbf{B}_0, \mathbf{A}, \iota_0 : \mathbf{B}_0 \longrightarrow \mathbf{A})$ *be a system with embeddings* $(f_i, \mathbf{1}) : (\mathbf{B}_0, \mathbf{A}, \iota_0) \longrightarrow (\mathbf{B}_i, \mathbf{C}, \iota_i), i = 1, 2,$ *in* $PartiRel_{(\ell)}(d, D)$. *Then there exists* $(\mathbf{B}_3, \mathbf{C}, \iota_3) \in PartiRel_{(\ell)}(d, D)$ *and embedding* $(g_i, \mathbf{1}) : (\mathbf{B}_i, \mathbf{C}, \iota_i) \longrightarrow (\mathbf{B}_3, \mathbf{C}, \iota_3) \in PartiRel_{(\ell)}(d, D)$ *such that* $(g_i, \mathbf{1})$ *is an amalgam of* $(f_i, \mathbf{1}), i = 1, 2$. *Explicitly, we have* $g_1 \circ f_1 = g_2 \circ f_2$ *while the embeddings* $g_i$ *commute with homomorphisms* $\iota_i$*; see the following scheme.*

$$\begin{tikzcd}
\mathbf{B}_0 \arrow[rr, "\iota_0"] \arrow[dr, "f_2"] \arrow[dd, "f_1"'] & & \mathbf{A} \arrow[dr, "\mathbf{1}"] \arrow[dd, "\mathbf{1}"'] & \\
& \mathbf{B}_2 \arrow[rr, "\iota_2"] \arrow[dd, "g_2"'] & & \mathbf{C} \arrow[dd, "\mathbf{1}"] \\
\mathbf{B}_1 \arrow[rr, "\iota_1"] \arrow[dr, "g_1"'] & & \mathbf{C} \arrow[dr, "\mathbf{1}"] & \\
& \mathbf{B}_3 \arrow[rr, "\iota_3"] & & \mathbf{C}
\end{tikzcd}$$

**Proof.** Without loss of generality let us assume that the mappings $f_1$ and $f_2$ and **1** are inclusions. Assume also that the sets $\underline{\mathbf{B}}_1 \setminus f_1(\underline{\mathbf{B}}_0)$ and $\underline{\mathbf{B}}_2 \setminus f_2(\underline{\mathbf{B}}_0)$ are disjoint. In this situation we define $\mathbf{B}_3$ simply as the union of systems $\mathbf{B}_1$ and $\mathbf{B}_2$: $\mathbf{B}_3 = (\underline{\mathbf{B}}_1 \cup \underline{\mathbf{B}}_2, (R_i(\mathbf{B}_1) \cup R_i(\mathbf{B}_2); i \in I))$ (this is sometimes called *free amalgamation*). We define the mapping $\iota_3$ as follows: $\iota_3(x) = \iota_i(x)$ for every $x \in \underline{\mathbf{B}}_i, i = 1, 2$. The standard ordering $\leq_{\mathbf{B}_3}$ can then be chosen arbitrarily so that the mapping $\iota_3$ is monotone.

Define the mappings $g_i : \underline{\mathbf{B}}_i \longrightarrow \underline{\mathbf{B}}_3$ by $g_i(x) = x$ for $x \in \underline{\mathbf{B}}_i, i = 1, 2$. These mappings induce embeddings $(g_i, \mathbf{1}) : (\mathbf{B}_i, \mathbf{C}, \iota_i) \longrightarrow (\mathbf{B}_3, \mathbf{C}, \iota_3)$.

It remains to justify the assertion that $(\mathbf{B}_3, \mathbf{C}, \iota_3)$ belongs to the class $PartiRel_{(\ell)}(d, D)$. Let $\{x, y\}$ be an edge of $\mathbf{B}_3$. By definition, $\mathbf{B}_3\{x, y\}$ is an edge of either $\mathbf{B}_1$ or $\mathbf{B}_2$; assume without loss of generality that $\{x, y\}$ is an edge of $\mathbf{B}_1$. Let $P = (x = x_0, x_1, \ldots, x_t = y)$ be a path in $\mathbf{B}_3$ from $x$ to $y$ of length $t \leq \ell$. Recall that the weight $w(x, y)$ of edge $(x, y)$ of $\mathbf{B}_3$ is defined by $w(x, y) = i$ iff $(x, y) \in R_i(\mathbf{B}_3)$. Thus all our mappings preserve weights. We have to prove that the weight $w(x, y)$ of the edge $\{x, y\}$ satisfies $w(x, y) \leq w(P) = \sum_{i=1}^{t} w(x_{i-1}, x_i)$. Towards this end consider the image $\iota_3(P) = (\iota_3(x_0), \iota_3(x_1), \ldots, \iota_3(x_t))$. Note that $w(x_i, x_{i+1}) = w(\iota_3(x_i), \iota_3(x_{i+1}))$ (as $\iota_3$ is a homomorphism of a binary relational system). The sequence $\iota_3(P) = (\iota_3(x_0), \iota_3(x_1), \ldots, \iota_3(x_t))$ induces a trail in $\mathbf{C}$ and some vertices and edges may be identified by $\iota_3$. However if this really happens then the length $w(P) = w(\iota_3(P))$ is bounded by $w(\bar{P})$ where $\bar{P}$ is a path (a subpath of $\iota(P)$) from $\iota(x)$ to $\iota(y)$ of length $\leq \ell - 1$ and thus (as $\mathbf{C} \in Rel_{(\ell)}(d, D)$) we have that $w(\iota(x), \iota(y)) = w(x, y) \leq w(\bar{P})$ is an $\ell$-metric edge. Thus we can assume that $\iota_3$ is injective. Consequently $\iota_3(P)$ is a path of length $t = \ell$ in $\mathbf{C}$. We distinguish two cases:

If $P$ is a subset of $\underline{\mathbf{B}}_1$, then $w(x, y) \leq w(P) = \sum_{i=1}^{t} w(x_{i-1}, x_i)$ (as $(\mathbf{B}_1, \mathbf{C}, \iota_1) \in PartiRel_{(\ell)}(d, D)$).

Thus assume that there exists $x_j \notin \underline{\mathbf{B}}_1$. Let $r$ be the smallest index such that all elements $x_{r+1}, \ldots x_j$ do not belong to $\underline{\mathbf{B}}_1$; similarly let $s$ be the maximal index such that all elements $x_j, x_{j+1}, \ldots, x_{s-1}$ do not belong to $\underline{\mathbf{B}}_1$. Note that then necessarily $x_r, x_s \in \underline{\mathbf{A}}$ and $r + 1 < s$. Let $P', P'', P'''$ be subpaths of $P$ with vertices $(x_0, x_1, \ldots x_r)$, $(x_k, x_{k+1}, \ldots, x_s)$, and $(x_s, x_{s+1}, \ldots, x_t)$. We have $w(P') = w(\iota_3(P'))$, $w(P'') = w(\iota_3(P''))$, $w(P''') = w(\iota_3(P'''))$ and also $w(P) = w(P') + w(P'') + w(P''')$. However $w(P'') = w(\iota_3(P''))$ and $w(\iota_3(P'')) \geq w(\iota_3(x_r), \iota_3(x_s))$ as $(\iota_3(x_r), \iota_3(x_s))$ is an $\ell - 1$-metric edge of $\mathbf{C}$. Thus $w(P) \geq w(\iota_3(P')) + w(\iota_3(x_r), \iota_3(x_s)) + w(\iota_3(P'''))$. $P', P'''$ together with edge $(w(\iota_3(x_r), \iota_3(x_s)))$ forms a path in $\mathbf{C}$ from $\iota_3(x)$ to $\iota_3(y)$ of length $\leq \ell - 1$ of weight $\leq w(P)$ and thus $w(x, y) \leq w(P)$ as $(x, y)$ is an $\ell - 1$-metric edge. □

Let us formulate the following as a step towards the construction of Ramsey structures. We introduce here notation which is tailored for the proof of Theorem 3.1.

**Lemma 4.2** (*Multiple Amalgamation Lemma*). *Let* $\mathbf{R} \in Rel_{(\ell-1)}(d, D)$*, and let* $\mathbf{A}$ *be a metric subsystem of system* $\mathbf{C}$ *(in* $Rel_{(\ell-1)}(d, D)$*). Denote by* $\mathbf{1} : \mathbf{A} \longrightarrow \mathbf{R}$ *the inclusion embedding. Let the system* $(\mathbf{B}_i, \mathbf{R}, \iota_{\mathbf{B}} : \mathbf{B}_i \longrightarrow \mathbf{R}) \in PartiRel_{(\ell)}(d, D)$ *be given. Let there be given systems* $(\mathbf{D}_i, \mathbf{A}, \iota_{\mathbf{D}} : \mathbf{D}_i \longrightarrow \mathbf{A}) \in PartiRel_{(\ell)}(d, D)$ *and* $(\mathbf{E}_i, \mathbf{A}, \iota_{\mathbf{E}} : \mathbf{E}_i \longrightarrow \mathbf{A}) \in PartiRel_{(\ell)}(d, D)$*. Let* $\mathcal{F} = \{f_1, f_2, \ldots, f_t\}$ *be inclusion embeddings* $(\mathbf{D}_i, \mathbf{A}, \iota_{\mathbf{D}}) \longrightarrow (\mathbf{E}_i, \mathbf{A}, \iota_{\mathbf{E}})$*,* $i = 1, \ldots, t$*. Then there exists a system* $(\mathbf{P}, \mathbf{R}, \iota_{\mathbf{P}} : \mathbf{P} \longrightarrow \mathbf{R}) \in PartiRel_{(\ell)}(d, D)$ *such that for every embedding* $f_i$ *there exists an embedding* $g_i : (\mathbf{E}_i, \mathbf{A}, \iota_{\mathbf{E}}) \longrightarrow (\mathbf{P}_i, \mathbf{R}, \iota_{\mathbf{P}})$ *such that* $g_i$ *restricted to the set* $\underline{\mathbf{D}}_i$ *coincides with* $f_i$ *for all* $i = 1, \ldots, t$*.*

**Proof.** Despite its formal complexity this is easy to prove by induction on $t$: at each step we use Lemma 4.1. □

We are now in position to prove Theorem 3.1.

**Proof.** We shall proceed by induction on $\ell$. As explained above, for $\ell = 1$ Theorem 3.1 reduces to Theorem 1.1.

In the induction step ($\ell - 1 \Rightarrow \ell$) we assume that Theorem 3.1 holds for $\ell - 1$. By the above remark ($Rel_{(\ell-1)}(d, D)$ is a subcategory of $PartiRel_{(\ell-1)}(d, D)$) we know that Theorem 2.1 holds for $\ell - 1$. Let $\mathbf{A}, \mathbf{B}$ be metric binary systems considered as transversal systems in $PartiRel_{(\ell)}(d, D)$. Let $\mathbf{R} \in Rel_{(\ell-1)}(d, D)$ be a system satisfying $\mathbf{R} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ in the class $Rel_{(\ell-1)}(d, D)$. $\mathbf{R}$ will be fixed from now on and it will be considered as a transversal system (in $PartiRel_{(\ell-1)}(d, D)$). We shall construct $\mathbf{R}$-partite systems $\mathbf{P}^0, \mathbf{P}^1, \ldots, \mathbf{P}^a$ where $a = \left|\binom{\mathbf{R}}{\mathbf{A}}\right|$. The system $\mathbf{C} = \mathbf{P}^b$ will satisfy (as we shall show below) all the required properties of Theorem 2.1.

Put explicitly, $\binom{\mathbf{R}}{\mathbf{A}} = \{\mathbf{A}^1, \mathbf{A}^2, \ldots, \mathbf{A}^a\}$ and also $\binom{\mathbf{R}}{\mathbf{B}} = \{\mathbf{B}^1, \mathbf{B}^2, \ldots, \mathbf{B}^b\}$. (This is shorthand notation for systems in $PartiRel_{(\ell)}(d, D)$ but this suffices as both systems $\mathbf{A}$ and $\mathbf{B}$ are transversal.) Let the system $(\mathbf{P}^0, \mathbf{R}, \iota^0 : \mathbf{P}^0 \longrightarrow \mathbf{R})$ be any system in the class $PartiRel_{(\ell)}(d, D)$ for which the mapping $\iota^0$ satisfies:

For any $i = 1, \ldots, b$ the set $(\iota^0)^{-1}(\mathbf{B}^i)$ contains a subsystem isomorphic to $\underline{\mathbf{B}}^i$ (in $PartiRel_{(\ell+1)}(d, D)$).

The system $(\mathbf{P}^0, \mathbf{R}, \iota^0 : \mathbf{P}^0 \longrightarrow \mathbf{R})$ is easy to construct: we can take the disjoint union of $b$ copies of $\mathbf{B}$ and define mapping $\iota^0$ such that the above condition holds.

In the induction step ($i - 1 \Rightarrow i$) let an $\mathbf{R}$-partite system $(\mathbf{P}^{i-1}, \mathbf{R}, \iota^{i-1}) \in PartiRel_{(\ell)}(d, D)$ be given. Consider the system $\mathbf{A}$ and denote by $(\mathbf{D}^i, \mathbf{A}, \iota^{i-1})$ the subsystem of $(\mathbf{P}^{i-1}, \mathbf{R}, \iota^{i-1})$ induced by the set $(\iota^{i-1})^{-1}(\mathbf{A}^i)$ (we denoted the restriction of $\iota^{i-1}$ to the subset $(\iota^{i-1})^{-1}(\mathbf{A}^i)$ by the same symbol $\iota^{i-1}$). We have $(\mathbf{D}^i, \mathbf{A}, \iota^i) \in PartiRel_{(\ell)}(d, D)$; thus by the induction hypothesis there exits a system $(\mathbf{E}^i, \mathbf{A}, \lambda^i : \mathbf{E}^i \longrightarrow \mathbf{A})$ such that

$$\mathbf{E}^i \longrightarrow (\mathbf{D}^i)^{\mathbf{A}}_k$$

(in the class $PartiRel_{(\ell-1)}(d, D)$).

But in fact we can assume that $(\mathbf{E}^i, \mathbf{A}, \lambda^i : \mathbf{E}^i \longrightarrow \mathbf{A}) \in PartiRel_{(\ell)}(d, D)$. This needs an explanation: Recall that $\mathbf{A}$ is a transversal system and $\lambda^i : \mathbf{E}^i \longrightarrow \mathbf{A}$. Thus we may assume that $\lambda^i : \mathbf{E}^i \longrightarrow \mathbf{A}$ (as the elements of the system $\mathbf{E}^i$ which do not map to $\mathbf{A}$) are irrelevant for $\mathbf{E}^i \longrightarrow (\mathbf{D}^i)^{\mathbf{A}}_k$. Thus we may assume $(\mathbf{E}^i, \mathbf{A}, \lambda^i : \mathbf{E}^i \longrightarrow \mathbf{A}) \in PartiRel_{(\ell)-1}(d, D)$. That may be improved: Let $(v, y)$ be an edge of $\mathbf{E}^i$ and let $P$ be a path in $\mathbf{E}^i$ from $x$ to $y$ of length $\ell$. If $\lambda^i$ restricted to $P$ is not injective then we now that the weight of $P$ is at least the weight of $(x, y)$. But if $\lambda^i$ is injective on $P$ then $\lambda^i(P)$ is a path of length $\ell$ in $\mathbf{A}$ and thus $w(x, y) = w(\lambda^i(x), \lambda^i(y)) \leq w(\lambda^i(P))$ as $\mathbf{A}$ is metric. Thus we have $(\mathbf{E}^i, \mathbf{A}, \lambda^i : \mathbf{E}^i \longrightarrow \mathbf{A}) \in PartiRel_{(\ell)}(d, D)$.

The assumptions Lemma 4.2 are satisfied; let $(\mathbf{P}^i, \mathbf{R}, \iota^i)$ be a multiple amalgamation of copies of $(\mathbf{P}^{i-1}, \mathbf{R}, \iota^{i-1})$ such that every copy of $(\mathbf{D}^i, \mathbf{A}, \iota^{i-1})$ in $(\mathbf{E}^i, \mathbf{A}, \lambda^i)$ is extended to the unique copy of $(\mathbf{P}^i, \mathbf{R}, \iota^i)$. According to Lemma 4.2 we know that $(\mathbf{P}^i, \mathbf{R}, \iota^i) \in PartiRel_{(\ell)}(d, D)$.

Put $(\mathbf{C}, \mathbf{R}, \iota) = (\mathbf{P}^a, \mathbf{R}, \iota^a) \in PartiRel_{(\ell)}(d, D)$. We have to show that

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

However this is the underlying idea of the Partite Construction and this follows by backwards induction for $i = a, a-1, \ldots, 1, 0$. Let $\binom{\mathbf{C}}{\mathbf{A}} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$ be an arbitrary partition (coloring). By induction for $i = a, a-1, \ldots, 1, 0$ we prove that there exists a subsystem $(\widetilde{\mathbf{P}}^i, \mathbf{R}, \tilde{\iota}^i)$ (in $PartiRel_{(\ell)}(d, D)$) isomorphic to $\mathbf{P}^i$ such that for all $j > i$ all copies $\tilde{\mathbf{A}} \in \binom{\mathbf{C}}{\mathbf{A}}$ for which $\tilde{\iota}(\tilde{\mathbf{A}}) = \mathbf{A}^j$ get the same color, say $c(j)$.

In the induction step (for $i = a$ the statement clearly holds) we consider a copy $\widetilde{\mathbf{P}}^i$ of $\mathbf{P}^i$ with the stated properties. In the set $\binom{\widetilde{\mathbf{P}}^i}{\mathbf{A}}$ consider those $\tilde{\mathbf{A}}$ for which $\tilde{\iota}^i(\tilde{\mathbf{A}}) = \mathbf{A}^i$. These copies of $\mathbf{A}$ all lie in a copy of $\mathbf{A}$-partite system which is isomorphic to $\mathbf{E}^i$ and thus by $\mathbf{E}^i \longrightarrow (\mathbf{D}^i)^{\mathbf{A}}_k$ we get that there exists a subsystem $(\widetilde{\mathbf{P}^{i-1}}, \mathbf{R}, \tilde{\iota}^{i-1})$ of $(\widetilde{\mathbf{P}}^i, \mathbf{R}, \tilde{\iota}^i)$ which is isomorphic to $(\mathbf{P}^{i-1}, \mathbf{R}, \iota^{i-1})$ with the stated properties.

Finally, we obtain a copy $(\widetilde{\mathbf{P}}^0, \mathbf{R}, \tilde{\iota}^0)$ of $(\mathbf{P}^0, \mathbf{R}, \iota^0)$ such that for every $\tilde{\mathbf{A}} \in \binom{\widetilde{\mathbf{P}}^0}{\mathbf{A}}$ its color depends only on $\tilde{\iota}^0(\tilde{\mathbf{A}})$. But this in turn induces a coloring $\widetilde{\mathcal{A}}_1 \cup \cdots \cup \widetilde{\mathcal{A}}_k$ of the set $\binom{\mathbf{R}}{\mathbf{A}}$ defined by $\mathbf{A}_j \in \widetilde{\mathcal{A}}_i$ iff $c(j) = i$. Thus there exists $\widetilde{\mathbf{B}} \in \binom{\mathbf{R}}{\mathbf{B}}$ and $i(0)$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}} \subset \widetilde{\mathcal{A}}_{i(0)}$. Consequently by the construction of $\mathbf{P}^0$ any $\mathbf{B}' \in \binom{\widetilde{\mathbf{P}}^0}{\mathbf{B}}$ with $\tilde{\iota}(\mathbf{B}') = \widetilde{\mathbf{B}}$ satisfies $\binom{\mathbf{B}'}{\mathbf{A}} \subset \mathcal{A}_{i(0)}$ which we wanted to prove. $\square$

## 5. Remarks and open problems

**1**. Theorem 1.2 also implies the following (*ordering property of finite metric spaces*):

**Theorem 5.1.** *For every metric space $(X, \rho)$ there exists a metric space $(Z, \delta)$ such that for any linear orderings $\preceq_X$ and $\preceq_Z$ of $X$ and $Z$ there exists a monotone isometry $(X, \rho) \longrightarrow (Z, \delta)$.*

**Proof** (*Sketch*). Given $(X, \rho)$, find an ordered metric space $(Y, \sigma)$ (with standard ordering $\leq_Y$):

1. for every ordering of $X$ there is a monotone isometry of $(X, \rho)$ into $(Y, \sigma)$;
2. the elements of $Y$ are ordered $x_1, x_2, \ldots, x_n$ (in the standard ordering $\leq$) such that $d = \rho'(x_{i-1}, x_i)$ for all $1 \leq i \leq n$ for some $d > 0$.

(We consider the disjoint union of all possible orderings of $(X, \rho)$ and eventually add some more elements.) Let now $(Z, \delta)$ (with the standard ordering $\leq_Z$) satisfy $(Z, \delta) \longrightarrow (Y, \sigma)^{\mathbf{2}}_2$ where we denoted by $\mathbf{2}$ the metric space with elements $\{0, 1\}$ and the distance of 0 and 1 equal to $d$. We claim that $(Z, \delta)$ has the desired properties. Towards this end let $\preceq_Y, \preceq_Z$ be an arbitrary ordering of $Y$ and $Z$. Define a partition $\binom{(Z,\delta)}{(\mathbf{2})} = \mathcal{A}_1 \cup \mathcal{A}_2$ as follows:

$\{x, y\} \in \mathcal{A}_1$ iff $\delta(x, y) = d$ and $x \leq_Y y$ and $x \preceq_Y y$.

Let $(Y', \sigma')$ be a homogeneous copy of $(Y, \sigma)$ in $(Z, \delta)$. It is then easy to see that $(Y', \sigma')$ with the ordering $\preceq'_Y$ (which is the restriction of $\preceq_Z$ to the set $Y'$) is either monotone isomorphic to $(Y, \sigma)$ with the standard ordering or monotone isomorphic to

$(Y, \sigma)$ with the reversed standard ordering. As $(Y, \sigma)$ contains all possible orderings of $(X, \rho)$ the result follows. $\square$

(See [11] for another proof of Theorem 5.1.)

**2**. One can prove results analogous to Theorem 1.2 for other classes of metric spaces: for example one can consider metric spaces where the metric attains only rational, or integer values. One can also consider only those metric spaces which correspond to graphs. All these classes are again Ramsey. We only have to check that the amalgamation property holds for these classes. Rational metrics then apply to the Urysohn space.

**3**. Perhaps in the spirit of [10,11] one could ask for a characterization of all Ramsey classes of metric spaces. However this seems to be beyond reach as the corresponding characterization of (ultra)homogeneous metric spaces (and thus equivalently Fraissé classes) seems not be known; compare [2,7,8,10,11].

**4**. It would be interesting to investigate the "simple" Ramsey properties (such as the vertex- and edge-partitions) of the (countable rational) Urysohn space by analogy with similar results for the random graph [1].

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