Cartan invariants for the restricted toral rank two contact Lie algebra

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ABSTRACT

Restricted modules for the restricted toral rank two contact Lie algebra are considered. Contra-
gredients of the simple modules, Cartan invariants, and dimensions of the simple modules and their
projective covers are determined.

Let $L$ be a finite dimensional restricted Lie algebra. All $L$-modules in this
paper are assumed to be left, restricted and finite dimensional over the defining
field. Each simple $L$-module has a projective cover; the multiplicities of the
composition factors of the various projective covers are called Cartan invariants.

Here, we use the method of [3] to compute the Cartan invariants for the re-
stricted toral rank two contact Lie algebra $K(3, 1)$. To carry out the computa-
tion, one needs to know the simple modules and their multiplicities as composi-
contact Lie algebras of arbitrary toral rank – it was shown that these multi-
plcities are generically one, that is, the induced modules are, with a few excep-
tions, simple (see 1.1 below). Although it is not known at the time of this writing,
it is expected that (for arbitrary toral rank) the few exceptional induced modules
will not be simple. At least this is the case for the algebra $K(3, 1)$ as will be shown
in this paper (see 6.1).

In addition to the Cartan invariants for $K(3, 1)$ we will compute the dimen-

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sions of the simple modules which will give in turn the dimensions of their projective covers. Also, we determine the contragredient of each simple module.

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1. STATEMENT OF MAIN RESULTS

Let $F$ be an algebraically closed field of characteristic $p > 2$ and let $n = 2r + 1$ with $r \in \mathbb{N}$. For $1 \leq k \leq n$ let $\varepsilon_k$ be the $n$-tuple with $j$th component $\delta_{jk}$ (Kronecker delta). Set $A = \{a = \sum_{k=1}^{n} a_k \varepsilon_k \mid 0 \leq a_k < p\} \subset \mathbb{Z}^n$. For $a, b \in A$, define $\binom{a}{b} := \prod_{k} \binom{a_k}{b_k}$. The factors on the right are the usual binomial coefficients with the convention that $\binom{j}{j} = 0$ unless $0 \leq j \leq i$. The vector space $\mathcal{A}$ with $F$-basis $\{x^{(a)} \mid a \in A\}$ becomes an associative $F$-algebra by defining

$$x^{(a)} x^{(b)} = \binom{a + b}{a} x^{(a+b)}$$

(where $x^{(c)} = 0$ if $c \notin A$) and by extending this product linearly to $\mathcal{A}$. $\mathcal{A}$ is called the divided power algebra. It is a graded algebra with $i$th homogeneous component $\mathcal{A}_i$ defined to be the $F$-span of $\{x^{(a)} \mid |a| + a_n = i\}$ where $|a| := \sum_{k=1}^{n} a_k$.

If $n + 3 \equiv 0 \pmod{p}$ let $K$ be the $F$-span of $\{x^{(a)} \mid a \neq \sum (p-1)\varepsilon_k\}$ and otherwise let $K = \mathcal{A}$. Section 3 describes a way to give $K$ the structure of a restricted Lie algebra. With this structure, $K$ (denoted $K(3,1)$ in [9]) is called the (toral rank $r+1$) restricted contact Lie algebra. The toral rank refers to the dimension of a maximal torus of $K$. Also, $K$ has a restricted grading with $i$th homogeneous component $K_i$ defined to be the $F$-span of $\{x^{(a)} \mid \|a\| = i\}$ where $\|a\| = |a| + a_n - 2$. Note that $K_i \subseteq \mathcal{A}_{i+2}$.

For $1 \leq k \leq 2r$, set

$$\sigma(k) = \begin{cases} 1 & 1 \leq k \leq r, \\ -1 & r < k \leq 2r, \end{cases}$$

and $k' = k + \sigma(k)r$.

Let $A = \{\lambda = \sum_{i=1}^{r+1} \lambda_i \varepsilon_i \mid \lambda_i \in F_p\} = (F_p)^{r+1}$ (viewing $\varepsilon_i$ as an $r + 1$-tuple). For a $K_0$-module $V$ and $\lambda \in A$ define $V_\lambda = \{v \in V \mid x^{(\varepsilon_i + \varepsilon_j)} v = \lambda_i v \ (1 \leq i \leq r) \text{ and } x^{(\varepsilon_r)} v = \lambda_{r+1} v\}$. Any vector in $V_\lambda$ is said to have weight $\lambda$. A nonzero vector $m \in V_\lambda$ is a maximal vector (of weight $\lambda$) if $x^{(\varepsilon_i + \varepsilon_j)} m = 0$ for all $(i, j) \in I := \{(i, j) \mid 1 \leq i, j \leq r \text{ or } 1 \leq i \leq r, \ i' < j \leq 2r\}$.

For each $\lambda \in A$ there exists a simple (restricted) $K_0$-module $L_0(\lambda)$ possessing a unique (up to scalar multiple) maximal vector of weight $\lambda$. Moreover, $\{L_0(\lambda) \mid \lambda \in A\}$ is a complete set of representatives for the isomorphism classes of simple $K_0$-modules. In fact, $K_0$ is the direct sum of its $p$-ideals $\sum_{1 \leq i, j \leq 2r} Fx^{(\varepsilon_i + \varepsilon_j)} \cong \tilde{S}p(2r)$ and $Fx^{(\varepsilon_r)} \cong F$ and it is easy to see that $L_0(\lambda)$ is a simple $\tilde{S}p(2r)$-module on which $x^{(\varepsilon_r)}$ acts as multiplication by $\lambda_{r+1}$, so classical theory applies.

If $L$ is a restricted Lie algebra, its restricted universal enveloping algebra (u-algebra) is denoted $u(L)$.
Set \( N^+ = \sum_{i>0} K_i \). Then \( N^+ \cap N^+ =: K^+ \) and \( K^+/N^+ \cong K_0 \). In particular, any \( K_0 \)-module becomes a \( K^+ \)-module in the natural way.

In [2] it was shown that for \( \lambda \in \mathcal{A} \), the \( K \)-module \( Z(\lambda) = u(K) \otimes_{u(K^+)} L_0(\lambda) \) possesses a unique maximal submodule which is graded with respect to the natural grading of \( Z(\lambda) \) (cf. paragraph before 3.3 below). The quotient \( L(\lambda) \) of \( Z(\lambda) \) by this maximal submodule is simple and graded with homogeneous component of greatest degree \( K_0 \)-isomorphic to \( L_0(\lambda) \). \( \{ L(\lambda) \mid \lambda \in \mathcal{A} \} \) is a complete set of representatives for the isomorphism classes of simple \( K \)-modules. Moreover, denoting by \( L(\lambda)[i] \) the \( i \)th suspension of \( L(\lambda) \) (so that \( L(\lambda)[i] \) is the graded \( K \)-module with \( j \)th homogeneous component \( L(\lambda)_{i+j} \)), \( \{ L(\lambda)[i] \mid \lambda \in \mathcal{A}, i \in \mathbb{Z} \} \) is a complete set of representatives for the isomorphism classes of simple graded \( K \)-modules. Consequently, if \( V \) is a simple graded \( K \)-module with homogeneous component of highest degree \( K_0 \)-isomorphic to \( L_0(\lambda) \), then \( V \cong L(\lambda) \) as \( K \)-modules.

For \( 1 \leq k \leq r+1 \) set \( \zeta_k = -\sum_{i=1}^{r-k+1} \varepsilon_i \) (the empty sum being zero). A weight \( \lambda \in A \) is exceptional if \( \lambda = \zeta_k + (\pm k - r - 1)\varepsilon_{r+1} \) for some \( k \) (\( 1 \leq k \leq r+1 \)). The following theorem was proved in [4].

**1.1. Theorem.** If \( \lambda \in A \) is not exceptional, then \( L(\lambda) \cong Z(\lambda) \).

In order to state the main results of this paper, we assume \( n = 3 \) so that \( K \) is the toral rank two restricted contact Lie algebra. Note that in this case, the exceptional weights are \( (0,0), (0,-4), (-1,-1) \) and \( (-1,-3) \).

In the following theorem \( V^* \) denotes the contragredient of the \( K \)-module \( V \).

**Theorem** (see 5.3).

1. If \( \lambda \in \mathcal{A} \) is not exceptional, then \( L(\lambda)^* \cong L(\lambda_1, -\lambda_2 - 4) \).
2. \( L(0,0)^* \cong L(0,0) \).
3. \( L(0,-4)^* \cong L(-1,-1) \).
4. \( L(-1,-3)^* \cong L(-1,-3) \).

If \( \lambda = (\lambda_1, \lambda_2) \in \mathcal{A} \), then view \( \lambda_i \in \mathbb{Z} \) with \( -p < \lambda_i \leq 0 \) by identifying \( \mathbb{F}_p \) with \( \mathbb{Z}/p\mathbb{Z} \) and using the coset representative in the indicated range.

**Theorem** (see 5.5).

1. If \( \lambda \in \mathcal{A} \) is not exceptional, then \( \dim_F L(\lambda) = p^3(1 - \lambda_1) \).
2. \( \dim_F L(0,0) = 1 \).
3. \( \dim_F L(0,-4) = \dim_F L(-1,-1) = p^3 - 1 \).
4. \( \dim_F L(-1,-3) = p^3 - 2 \).

Let \( P(\lambda) \) denote the projective cover of \( L(\lambda) \) and let \( C_{\lambda\mu} \) be the multiplicity of \( L(\mu) \) as a composition factor of \( P(\lambda) \). \( C_{\lambda\mu} \) is called a *Cartan invariant* and the matrix \( C = [C_{\lambda\mu}] \) is called the *Cartan matrix* of \( K \). (All indexing by \( \Lambda \) is assumed to be relative to a fixed ordering.)
Theorem (see 6.3 and 6.4). Assume $p > 3$. Let $C = p^2 X'X$ where $X = [X_\lambda]$ is the $p^2 \times 1$-matrix with

$$X_\lambda = \begin{cases} 16 & \lambda = (0,0), \\ 4 & \lambda = (0,-4), (-1,-1) \text{ or } (-1,-3), \\ 1 & \lambda_1 = 1 - p, \\ 2 & \text{otherwise}, \end{cases}$$

and $X'$ denotes the transpose of $X$.

Theorem (see 6.5). Assume $p > 3$. For $\lambda \in \Lambda$, $\dimFP(\lambda) = p^\alpha X_\lambda$ where $\alpha = p^3 - 6$ and $X_\lambda$ is as above.

2. MODULES FOR THE WITT ALGEBRA

Let $n$ be arbitrary once again and for each $k$ ($1 \leq k \leq n$) let $D_k$ be the derivation of $\mathfrak{U}$ given by $D_kx^{(a)} - x^{(a-e_k)}$ with the convention that $x^{(b)} = 0$ if $b \notin A$. Then $W = \sum_k \mathfrak{U}D_k = \text{Der}_F \mathfrak{U}$ is a restricted Lie algebra called the Witt algebra. It has as $F$-basis $\{x^{(a)}D_k \mid a \in A, 1 \leq k \leq n\}$. The bracket product in $W$ satisfies

$$[x^{(a)}D_k, x^{(b)}D_l] = \left( \begin{array}{c} a+b-\xi_k \\ a \end{array} \right) x^{(a+b-\xi_k)}D_l - \left( \begin{array}{c} a+b-\xi_l \\ b \end{array} \right) x^{(a+b-\xi_l)}D_k.$$

View $W$ as an $\mathfrak{U}$-module in the natural way and set $M = \text{Hom}_\mathfrak{U}(W, \mathfrak{U})$. $M$ is a free $\mathfrak{U}$-module with base $\{dx_1, \ldots, dx_n\}$ where $x_k := x^{(e_k)}$ and $d : \mathfrak{U} \to M$ is given by $dx : D \mapsto Dx$ ($x \in \mathfrak{U}$, $D \in W$). $M$ becomes a $W$-module by defining $D \cdot \phi = D \circ \phi - \phi \circ (ad D)$ ($D \in W$, $\phi \in M$) (cf. [1, p. 125]).

The following lemma is easy to verify.

2.1. Lemma.

$$x^{(a)}D_k \cdot dx_j = \begin{cases} \sum_k x^{(a-e_k)}dx_k & i = j, \\ 0 & i \neq j. \end{cases}$$

Let $\Omega$ denote the exterior algebra of $M$ over $\mathfrak{U}$. For $1 \leq k \leq n$, set $\Gamma_k = \{ \gamma = \sum_{i=1}^k \gamma_i e_i \mid 1 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_k \leq n \}$ and for $\gamma \in \Gamma_k$ set $e_\gamma = dx_{\gamma_1} \wedge \cdots \wedge dx_{\gamma_k} \in \Omega$. Also, let $\Gamma_0 = \{ \zeta \}$ where $\zeta$ denotes the empty tuple, $\zeta = ()$, and set $e_\zeta = 1 \in \mathfrak{U}$. Then $\Omega = \sum_k \Omega_k$ where $\Omega_k := \sum_{\gamma \in \Gamma_k} \mathfrak{U}e_\gamma$ if $0 \leq k \leq n$ (and, for convenience, $\Omega_k = 0$ if $k < 0$ or $k > n$). For $0 \leq k \leq n$, $\Omega_k$ has $F$-basis $\{x^{(a)}e_\gamma \mid a \in A, \gamma \in \Gamma_k\}$.

The action of $W$ on $M$ extends uniquely to an action on $\Omega$ subject to the rules

$$D \cdot (v \wedge w) = D \cdot v \wedge w + v \wedge D \cdot w$$
$$D \cdot (xv) = (Dx) v + x(D \cdot v)$$

for $D \in W$, $x \in \mathfrak{U}$ and $v, w \in \Omega$. It is clear from 2.1 that each $\Omega_k$ is a $W$-submodule of $\Omega$. 294
Define an $F$-linear map $\delta_j : \Omega_j \to \Omega_{j+1}$ by setting
\[
\delta_j(x^{(a)}e_\gamma) = \sum_{k=1}^n x^{(a-\varepsilon_k)}e_\gamma \wedge dx_k
\]
if $0 \leq j \leq n$ and by setting $\delta_j = 0$ if $j < 0$ or $j > n$. It is straightforward to check that $\delta_j$ is a $W$-homomorphism and that $\delta_j \delta_{j-1} = 0$.

For $0 \leq j, k \leq n$ set
\[
B_{jk} = \{ x^{(a)}e_\gamma \in \Omega_j \mid k \notin \gamma, \ a_k \neq 0 \text{ and for } i < k, \ a_i = 0 \text{ if } i \notin \gamma \text{ and } a_i = p - 1 \text{ if } i \in \gamma \}
\]
where $k \in \gamma$ means $k = \gamma_i$ for some $i$. Let $B_j = \bigcup_k B_{jk}$.

2.2. Lemma. For $0 \leq j < n$, $\delta_j(B_j)$ is an $F$-basis for im $\delta_j$.

**Proof.** Fix $j$ with $0 \leq j < n$. It will first be shown that $\delta_j(B_j)$ spans im $\delta_j$. Now im $\delta_j$ is spanned by $\delta_j(\bigcup_k Y_k)$ where $Y_k = \{ x^{(a)}e_\gamma \mid k \notin \gamma, \ a_k \neq 0 \}$ since any standard basis element $x^{(a)}e_\gamma$ of $\Omega_j$ not in $Y_k$ is in ker $\delta_j$. Therefore, it is enough to prove that for each $k$, $\delta_j(Y_k)$ is contained in the $F$-span of $\delta_j(B_j)$, and this will be done by induction on $k$. Since $Y_1 = B_j$, the first step is trivial. Now assume $k > 1$ and let $y = x^{(a)}e_\gamma$ be an arbitrary element of $Y_k$. If $a_i \neq 0$ for some $1 \leq i < k$ with $i \notin \gamma$, then $y \in Y_1$ and the induction hypothesis implies $\delta_j(y)$ is in the $F$-span of $\delta_j(B_j)$. So assume that $a_i = 0$ for each $1 \leq i < k$ with $i \notin \gamma$. Similarly, if $a_i = p - 1$ for each $1 \leq i < k$ with $i \in \gamma$, then $y \in B_{jk}$ so that $\delta_j(y) \in \delta_j(B_j)$. So assume otherwise and let $i$ be the least index for which $i \in \gamma$ and $a_i \neq p - 1$. Define $\eta = (\gamma_1, \gamma_2, \ldots, \hat{i}, \ldots, \gamma_l)$ ("delete i") and $b = a + \varepsilon_i$. Then $\delta_{j-1}(x^{(b)}e_\eta) = \sum_l x^{(b-\varepsilon_l)}e_\eta \wedge dx_l = \pm y + \sum_{l > i} x^{(b)}e_\eta \wedge dx_l$. Each term in the sum on the right is either zero or contained in $B_{ji}$. Therefore, applying $\delta_j$ and using the fact that $\delta_j \delta_{j-1} = 0$ it follows that $\delta_j(y)$ is in the $F$-span of $\delta_j(B_j)$. This completes the proof that $\delta_j(B_j)$ spans im $\delta_j$.

Next, let $1 \leq k \leq l \leq n$ and pick $y = x^{(a)}e_\gamma \in B_{jk}$ and $z = x^{(b)}e_\theta \in B_{ji}$. Assume $x^{(b - \varepsilon_i)}e_\theta \wedge dx_i = x^{(a - \varepsilon_k)}e_\gamma \wedge dx_k$ for some $i$. It will be shown that, under this assumption, $l = k$ and $z = y$. This, and the easily verified fact that no element of $\delta_j(B_j)$ is zero, will imply linear independence of $\delta_j(B_j)$. Since $x^{(a - \varepsilon_k)}e_\gamma \wedge dx_k \neq 0$, it follows that $i \geq l$. If $k \in \theta$, then $k < l$ which yields the contradiction $(b - \varepsilon_i)_k = b_k = p - 1 \neq a_k - 1 = (a - \varepsilon_k)_k$. Therefore, $k \notin \theta$. Since $e_\theta \wedge dx_i = e_\gamma \wedge dx_k \neq 0$ it must be the case that $i = k$. It now easily follows that $k = l$ and $z = y$, as desired. □

2.3. Corollary (cf. [8], Theorem 2.1). For $-1 \leq j < n$,

(1) $\dim_F \text{im } \delta_j = (^{n-1}_j)(p^n - 1)$,

(2) $\dim_F \ker \delta_{j+1}/\text{im } \delta_j = (^{n}_j+1)$ and $W$ acts trivially on ker $\delta_{j+1}/\text{im } \delta_j$.

**Proof.** Part (1) follows directly from 2.2 by a counting of the elements of $B_j$.
For (2), first note that \( \dim_F \ker \delta_{j+1}/\im \delta_j = \dim_F \Omega_{j+1} - \dim_F \Omega_{j+1}/\ker \delta_{j+1} - \dim_F \im \delta_j = (p^{n+1} - (p-1) - (p^n - 1)) = (p^n - 1) \) by part (1).

Therefore, all that remains to be proved is the second claim of (2). For \( 1 \leq k \leq n \), set \( f_k = x^{(p-1)\varepsilon_k} \) and for \( \gamma \in I_{j+1} \), set \( f_\gamma = f_{\gamma_1} \wedge f_{\gamma_2} \wedge \cdots \wedge f_{\gamma_{j+1}} \) \( (f \in \Omega_{j+1}) \). Then \( f_\gamma \in \ker \delta_{j+1} \). Moreover, \( \{ f_\gamma + \im \delta_j \gamma \in I_{j+1} \} \) is clearly linearly independent and, since the cardinality of this set is \( \binom{n+1}{j+1} \), it must be a basis for \( \ker \delta_{j+1}/\im \delta_j \) by the previous paragraph. Let \( x^{(a)} D_i \in W \). It suffices to show that \( (x^{(a)} D_i) \cdot f_\gamma \in \im \delta_j \) for each \( \gamma \in I_{j+1} \). If \( j = -1 \), then \( f_\gamma = 1 \) and so \( (x^{(a)} D_i) \cdot f_\gamma = 0 \in \im \delta_j \). Therefore, assume \( j > -1 \). Now,

\[
(x^{(a)} D_i) \cdot f_k = \left\{
\begin{array}{ll}
\sum_{l=0}^{i} x^{(a+(p-1)\varepsilon_l)} \varepsilon_l \wedge dx_l & \text{if } i = k \text{ and } a_i = 0,
0 & \text{otherwise}.
\end{array}
\right.
\]

So \( (x^{(a)} D_i) \cdot f_\gamma = 0 \) unless \( i \in \gamma \) and \( a_i = 0 \) in which case

\[
(x^{(a)} D_i) \cdot f_\gamma = \pm \sum_{l=0}^{i} x^{(b-\varepsilon_l)} \varepsilon_l \wedge dx_l = \delta_j(\pm x^{(b)} e_\eta) \in \im \delta_j,
\]

where \( b = a + \sum_k (p-1) \varepsilon_k \) and \( \eta = (\gamma_1, \gamma_2, \ldots, i, \ldots, \gamma_{j+1}) \) ("delete \( i \)"). This completes the proof. □

Let \( \omega_K := dx_n + \sum \sigma(k) x_k dx_k \) and define \( \tau_j : \Omega_j \to \Omega_{j+1} \) by \( \tau_j(v) = \omega_K \wedge v. \) Set \( C_j = \{ x^{(a)} e_\gamma \mid a \in A, \gamma \in I_j, n \not\in \gamma \} \) \((0 \leq j < n)\).

**2.4. Lemma.** For \( 0 \leq j < n \),

1. \( \Omega_{j-1} \cap \Omega_j \cap \Omega_{j+1} \) is exact,
2. \( \tau_j(C_j) \) is an \( F \)-basis for \( \im \tau_j \),
3. \( \dim_F \im \tau_j = p^n. \)

**Proof.** First observe that \( \tau_j(C_j) \) is linearly independent, for if \( v = x^{(a)} e_\gamma \in C_j \), then \( \tau_j(v) \) is the sum of \( x^{(a)} dx_n \wedge e_\gamma \) and scalar multiples of terms of the form \( x^{(b)} e_\rho \) with \( n \not\in \rho \).

Next, since \( \omega_K \wedge \omega_K = 0 \), it follows that \( \im \tau_{j-1} \subseteq \ker \tau_j \).

Therefore, \( (p^{n+1} - (p-1) - (p^n - 1)) = (p^n - 1) \). The next to the last equality from the induction hypothesis (or the fact that \( \im \tau_{j-1} = 0 \) if \( j = 0 \)) Hence, both inequalities are, in fact, equalities and all three statements are established. □

The element \( \omega_K \) defined above can be used to give an alternate definition of \( K \) (see [1]).

**3. MODULES FOR \( K \) (n ARBITRARY)**

In Section 1 an \( F \)-basis was given for the underlying vector space of the contact algebra \( K \). Now the Witt algebra \( W \) will be used to introduce the bracket product on \( K \). (For details, see [9].)
Define an $F$-linear map $D_K : K \to W$ by means of $D_K(f) = \sum_{k=1}^{n} f_k D_k$, where

$$f_k = x_k D_n(f) + \sigma(k') D_{k'}(f), \quad k \leq 2r,$$

$$f_n = 2f - \sum_{k=1}^{2r} \sigma(k) x_k f_k.$$

(These formulas are taken from [9]. They differ from those in [1] by some signs which explains why our definition of $w_K$ above is different from that given in [1].) $D_K$ is injective and its image is closed under both the bracket product and the $p$-mapping of $W$. Therefore, identifying $K$ with its image under $D_K$, $K$ is a restricted Lie algebra and $K < W$. In fact, $K$ is a simple algebra.

To avoid confusion with the trivial bracket product on $\mathfrak{g}$, denote the induced bracket product on $K$ by $\langle f, g \rangle$ ($f, g \in K$). The following list of formulas will be useful.

3.1. Lemma ([9, p. 173]),

1. $\langle x^{(0)}, x^{(a)} \rangle = 2x^{(a - e_\epsilon)}$.
2. $\langle x^{(e_\epsilon)}, x^{(a)} \rangle = \sigma(k) x^{(a - e_\epsilon)} + (a_k + 1) x^{(a + e_\epsilon - e_\epsilon)}$, $1 \leq k \leq 2r$.
3. $\langle x^{(e_\epsilon)}, x^{(a)} \rangle = ||a|| x^{(a)}$.
4. $\langle x^{(e_\epsilon + e_\epsilon' + e_\epsilon)}, x^{(a)} \rangle = \sigma(k)(a_k - a_k) x^{(a)}$, $1 \leq k \leq 2r$.

For any nonzero finite dimensional graded vector space $V = \sum_i V_i$ define the length of $V$ to be $M - m$ where $M$ (resp., $m$) is the maximum (resp., minimum) of the set $\{i \mid V_i \neq 0\}$.

3.2. Lemma (cf. [7, Lemma 2.2]). Let $V = \bigoplus_i V_i$ be a nonzero graded $K$-module and assume that the length of $V$ is less than

$$l := \begin{cases} (n + 1)(p - 1) - 3 & p = 3, \\ (n + 1)(p - 1) - 2 & p \neq 3. \end{cases}$$

Then $K$ acts trivially on $V$.

Proof. The annihilator of $V$ is an ideal of $K$ which contains the nonzero component $K_I$. Since $K$ is simple, the result follows. □

As in [4], set

$$T_k = \begin{cases} x_k & 1 \leq k < n, \\ x^{(0)} & k = n, \end{cases}$$

and for $a = \sum_k a_k e_k \in \mathbb{Z}^n$ define $T^a = \prod_{k=1}^{n} T^a_k \in u(K)$, where $T^i_k := 0$ if $i < 0$. It is a consequence of the $p$-mapping defined on $K$ that $T^i_k = 0$ if $i \geq p$, so that $T^a = 0$ if and only if $a \not\in A$.

The $K$-module $Z(\lambda) := u(K) \otimes_{u(K^+)} L_0(\lambda)$ is graded with $i$th homogeneous component $Z(\lambda)_i = \sum_{|a| + a_0 = -i} T^a \otimes L_0(\lambda)$.

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3.3. Lemma. If \( N \) is a nonzero submodule of the \( K \)-module \( Z(\lambda) \) \((\lambda \in \Lambda)\), then \( N \supseteq T^\omega \otimes L_0(\lambda) \) where \( \omega = \sum_{k=1}^n (p-1)\epsilon_k \).

Proof. Let \( 0 \neq v \in N \). It follows from the PBW theorem that \( v \) can be written (uniquely) in the form
\[
v = \sum_{a \in A} T^a \otimes s_a
\]
with \( s_a \in L_0(\lambda) \). Order \( A \) by setting \( a < a' \) if for some \( k \) \((1 \leq k \leq n)\) \( a_i = a_i' \) for all \( i > k \) and \( a_k < a_k' \). Let \( \eta \) be the least element for which \( s_\eta \neq 0 \) and set \( y = \prod_{i=1}^n \mathcal{P}_i^{\eta_i} - \eta_i \). Then, using 3.1, \( T^\omega \otimes s_\eta = yv \in N \). Now \( T^\omega \otimes L_0(\lambda) \) is a \( \mathcal{K}_0 \)-submodule of \( Z(\lambda) \), namely, the homogeneous component of \( Z(\lambda) \) of least degree. Moreover, \( T^\omega \otimes L_0(\lambda) \) is simple so, since it intersects \( N \) nontrivially, it must be contained in \( N \). \( \square \)

For each \( 0 < j < n \), the \( K \)-module \( \Omega_j \) is graded with \( i \)th homogeneous component being the \( F \)-span of all \( x^{(a)}d\gamma \) \((a \in A, \gamma \in \Gamma_j)\) for which
\[
i = |a| + a_n +  \begin{cases} 
    j + 1 & n \in \gamma, \\
    j & n \notin \gamma.
\end{cases}
\]

3.4. Lemma. For each \( 0 < j < n \), im \( \tau_j \) is a graded \( K \)-submodule of \( \Omega_{j+1} \).

Proof. It is routine to check that for each \( f \in K \), \( D_K(f) \cdot \omega_K = 2D_n(f) \omega_K \). Therefore, if \( v \in \Omega_j \) and \( f \in K \), then
\[
D_K(f) \cdot \tau_j(v) = D_K(f) \cdot (w_K \land v)
\]
\[
= (D_K(f) \cdot w_K) \land v + w_K \land (D_K(f) \cdot v)
\]
\[
= w_K \land (2D_n(f)v + D_K(f) \cdot v) \in \text{im} \tau_j.
\]
Since \( w_K \) is a homogeneous element of \( \Omega_1 \) (of degree 2) it follows that the image under \( \tau_j \) of a homogeneous element is homogeneous, so that \( \text{im} \tau_j \) is graded. \( \square \)

It should be pointed out that \( \tau_j \) is not a \( K \)-homomorphism, in general.

4. SIMPLE MODULES FOR \( K \) \((n = 3)\)

For the rest of the paper, we assume \( n = 3 \) so that \( K \) is the toral rank two contact algebra.

It will be convenient to have explicit formulas for the actions of certain elements of \( K \) on the standard basis vectors of \( \Omega \).

4.1. Lemma. Let \( v = x^{(a)}e_\gamma \in \Omega \) and set
\[
\chi_k = \begin{cases} 
1 & k \in \gamma, \\
0 & k \notin \gamma.
\end{cases}
\]

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Proof. Recall that if \( f \in K \), then \( f \cdot v = D_K(f) \cdot v \), by definition. The lemma now follows from 2.1, the action of \( W \) on \( \Omega \), and the following formulas:

\[
D_K(x^{(0)}) = 2D_3, \quad D_K(x_1) = D_2 + x_1D_3, \quad D_K(x_2) = -D_1 + x_2D_3, \quad D_K(x^{(x_1 + x_2)}) = -x_1D_1 + x_2D_2, \quad D_K(x_3) = x_1D_1 + x_2D_2 + 2x_3D_3, \quad D_K(x^{(2x_1)}) = x_1D_2. \]

4.2. Lemma.

(1) \( \Omega_0 \cong Z(0, -4) \).

(2) \( \Omega_1/\ker \tau \cong Z(-1, -3) \).

Proof. As before, let \( \omega = \sum_k (p - 1)\varepsilon_k \).

(1) \( x^{(\omega)} \) is a maximal vector of weight \((0, -4)\) (using 4.1(4,5,6)) in the one-dimensional space \((\Omega_0)_{\mathfrak{q}p - q} \) which is the homogeneous component of \( \Omega_0 \) of greatest degree. Hence there is a \( K^+ \)-isomorphism \( L_0(0, -4) \rightarrow (\Omega_0)_{\mathfrak{q}p - q} \) (see 5.1 below) which induces a \( K \)-homomorphism \( \varphi : Z(0, -4) \rightarrow \Omega_0 \). Using 4.1(1,2,3), it can be seen that \( \varphi \) sends the space \( T^\omega \otimes L_0(0, -4) \) onto \( Fx^{(0)} \neq 0 \) so that \( \varphi \) is injective, by 3.3. Since the dimension of each space is \( p^3 \), it follows that \( \varphi \) is an isomorphism.

(2) Here, there is a \( K^+ \)-isomorphism \( L_0(-1, -3) \rightarrow (\Omega_1/\ker \tau)_{\mathfrak{q}p - q} \) since this last space has basis \( \{ x^{(\omega)}dx_1 + \im \tau, x^{(\omega)}dx_2 + \im \tau \} \) according to 2.4, and \( x^{(\omega)}dx_1 + \im \tau \) is a maximal vector of weight \((-1, -3)\) (see also 5.1). This yields a \( K \)-homomorphism \( \varphi : Z(-1, -3) \rightarrow \Omega_1/\ker \tau \). Notice that \( \varphi \) sends \( T^\omega \otimes L_0(-1, -3) \) onto the \( F \)-span of \( \{ x^{(0)}dx_1 + \im \tau, x^{(0)}dx_2 + \im \tau \} \) which is nonzero by 2.4. Hence 3.3 implies \( \varphi \) is injective. Since the dimension of each space is \( 2p^3 \) (see 2.4 and 5.1), \( \varphi \) is an isomorphism. \( \square \)

Set \( S_j = \im \tau_{j-1} + \ker \delta_j \) and \( \Omega_j = \Omega_j/S_j \) and let \( B_j \) be as in Section 2. Let \( E_1 = \{ x^{(a)}dx_3 \mid a_1 \neq 0 \text{ or } a_2 \neq 0 \} \) and \( E_2 = \{ x^{(a)}dx_2 \mid a = (1, 0, c) \text{ for some } c \} \) and set \( E = E_1 \cup E_2 \). Note that \( E \subseteq B_1 \). Define \( H_0 = B_0 \) and \( H_1 = B_1 \setminus E \) and set \( H_j = \{ h + S_j \mid h \in H_j \} \) (\( j = 0, 1 \)).

4.3. Lemma. \( H_j \) is an \( F \)-basis for \( \Omega_j \) (\( j = 0, 1 \)).

Proof. First, note that \( S_0 = \ker \delta_0 \), so that the case \( j = 0 \) is handled by 2.2.

Using 4.1, \( v := \tau_0(x^{(\omega)}) = x^{(\omega)}dx_3 \) (where \( \omega = \sum_k (p - 1)\varepsilon_k \)) is a maximal vector of weight \((0, -2)\). Also \( Fv \) is the homogeneous component of \( \im \tau_0 \) of greatest degree. Hence, there is a \( K^+ \)-isomorphism \( L_0(0, -2) \rightarrow Fv \) which extends to a \( K \)-homomorphism \( \varphi : Z(0, -2) \rightarrow \im \tau_0 \). Since \((0, -2)\) is not exceptional, \( Z(0, -2) \) is simple (see 1.1) so that \( \varphi \) is an isomorphism. Hence, \( \im \tau_0 \) is
simple and of dimension \( p^3 \). Also, \( \nu \in \im_0 \setminus \ker \delta_1 \) so \( \im_0 \cap \ker \delta_1 = \{0\} \). This gives \( \dim_F \Omega_1 = \dim_F \Omega_0 / \ker \delta_1 - \dim_F \im_0 = |B| - p^3 = |\widetilde{H}_1| \). Therefore, it suffices to show \( \widetilde{H}_1 \) spans \( \Omega_0 \) and for this it need only be shown that for each \( b \in E, b + S_1 \) is in the \( F \)-span of \( H_1 \) (cf. 2.2).

Let \( a \in A \) and consider the equation

\[
\tau_0(x(a)) = x(a) \, dx_3 - (a + 1) x(a + \varepsilon_2) \, dx_1 + (a_1 + 1) x(a + \varepsilon_1) \, dx_2.
\]

Assume \( x(a) \, dx_3 \in E_1 \). If \( a_1 = p - 1 \), then the equation implies that \( x(a) \, dx_3 \equiv (a + 1) x(a + \varepsilon_2) \, dx_1 \) (mod \( S_1 \)) and since \( x(a + \varepsilon_2) \, dx_1 \) is either zero or in \( H_1 \), this implies \( x(a) \, dx_3 + S_1 \) is in the \( F \)-span of \( H_1 \). The case \( a_2 = p - 1 \) is handled similarly. Now assume \( a_1 \neq p - 1 \neq a_2 \). Then

\[
\delta_0(x(a + \varepsilon_1 + \varepsilon_2)) = x(a + \varepsilon_2) \, dx_1 + x(a + \varepsilon_1) \, dx_2 + x(a + \varepsilon_1 + \varepsilon_2 - \varepsilon_1) \, dx_3.
\]

Since \( \delta_1 \delta_0 = 0 \), it follows that

\[
x(a + \varepsilon_2) \, dx_1 \equiv x(a + \varepsilon_1) \, dx_2 - x(a + \varepsilon_1 + \varepsilon_2 - \varepsilon_1) \, dx_3 \pmod{\ker \delta_1},
\]

whence, from (*),

\[
x(a) \, dx_3 \equiv -(a_2 + a_2 + 2) x(a + \varepsilon_1) \, dx_2 - (a_2 + 1) x(a + \varepsilon_1 + \varepsilon_2 - \varepsilon_1) \, dx_3 \pmod{S_1}.
\]

Since either \( a_1 \neq 0 \) or \( a_2 \neq 0 \) the first term on the right is in \( FH_1 \). By using reverse induction on \( a_1 \) it follows that \( x(a) \, dx_3 + S_1 \) is in the \( F \)-span of \( H_1 \).

Now assume \( x(a) \, dx_2 \in E_2 \). The equations

\[
\tau_0(x(a - \varepsilon_1)) = x(a - \varepsilon_1) \, dx_3 - x(a - \varepsilon_1 + \varepsilon_2) \, dx_1 + x(a) \, dx_2
\]

and

\[
\delta_0(x(a + \varepsilon_2)) = x(a + \varepsilon_2 - \varepsilon_1) \, dx_1 + x(a) \, dx_2 + x(a + \varepsilon_2 - \varepsilon_1) \, dx_3
\]

along with the fact that \( x(a - \varepsilon_1) \, dx_3 \in \ker \delta_1 \) show that \( 2x(a) \, dx_2 \equiv -x(a + \varepsilon_1 - \varepsilon_2) \, dx_3 \pmod{S_1} \). The previous paragraph now shows that \( x(a) \, dx_2 + S_1 \) is in the \( F \)-span of \( H_1 \) (recalling that \( p \neq 2 \)).

4.4. Lemma. \( \Omega_j \) \( (j = 0,1) \) has no submodule isomorphic to the one-dimensional trivial module \( F \).

Proof. \((j = 0)\) Let \( \nu \in \Omega_0 \) be a vector on which \( K \) acts trivially. By 4.3 there are unique scalars \( c_a \in F \) for which \( \nu = \sum_{a \neq 0} c_a(x(a) + S_0) \). First note that 4.1(5) implies \( c_a = 0 \) if \( a = (0,0,1) \). Using 4.1(1), \( 0 = x(0) \cdot \nu - \sum_{a \neq 0} 2c_a(x(a - \varepsilon_1) + S_0) \). For \( a \neq 0 \), either \( c_a = 0 \), \( x(a - \varepsilon_1) = 0 \), or \( x(a - \varepsilon_1) + S_0 \) is in \( H_0 \). Moreover, the elements of \( H_0 \) appearing are all distinct. Therefore, by linear independence, \( c_a = 0 \) if \( a \neq 0 \). Next, 4.1(4), implies \( c_a = 0 \) if \( a_1 \neq a_2 \). So finally, using 4.1(2), \( c_a = 0 \) for all \( a \).

\( (j = 1) \) Proceeding as above, let \( \nu \in \Omega_1 \) be a vector on which \( K \) acts trivially. Write \( \nu \) as a linear combination of the basis vectors in \( \widetilde{H}_1 \) (see 4.3): \( \nu = \sum_{(a, \gamma)} c_{a, \gamma}(x(a) e_\gamma + S_1) \) where \( c_{a, \gamma} \in F \) and the sum is over all pairs \((a, \gamma)\) for
which $x^{(a)}e_\gamma \in H_1$. If $x^{(a)}e_\gamma \in B_{1,3}$ (definition before 2.2) and $a_1 = 1$, then 4.1(5) implies $x_3 \cdot x^{(a)}e_\gamma = 2x^{(a)}e_\gamma$ so that $c_{a,\gamma} = 0$. Now, using 4.1(1) as in the case $j = 0$ above, it follows that $c_{a,\gamma} = 0$ if $a_3 \neq 0$. Next, by using 4.1(4), $c_{a,\gamma} = 0$ if either $a = (p-1,1,0)$ and $\gamma = (1)$ or $a = (1,1,0)$ and $\gamma = (2)$. Therefore, 4.1(2) implies $c_{a,\gamma} = 0$ if $a_2 \neq 0$. Finally 4.1(4) implies each $c_{a,\gamma}$ is zero.

Since $\delta_j$ is a graded $K$-homomorphism, $\ker \delta_j$ is a graded $K$-submodule of $\Omega_j$. Hence, by 3.4, $S_j$ is a graded $K$-submodule of $\Omega_j$. Therefore, $\Omega_j$ is a graded $K$-module ($j = 0, 1$).

4.5. Theorem.

(1) $\mathcal{O}_0 \cong L(0, -4)$.

(2) $\mathcal{O}_1 \cong L(-1, -3)$.

Proof. (1) By 4.2, $\mathcal{O}_0$ is a homomorphic image of $Z(0, -4)$, so it suffices to show that $\mathcal{O}_0$ is simple. In fact, from the comments before 1.1, it is enough to show that $\mathcal{O}_0$ is simple as a graded module. Suppose $N$ is a simple proper graded submodule of $\mathcal{O}_0$. Note that, by 4.3, the homogeneous component $(\mathcal{O}_0)_i$ is zero if either $i < 1$ or $i > 4p - 4$. Assume $N_{4p-4} \neq 0$. Then $N_{4p-4} = (\mathcal{O}_0)_{4p-4} = F(x^{(c)} + S_0)$ where $\omega = \sum_k (p-1)e_k$. But, by 4.2(1) and its proof, this space generates $\mathcal{O}_0$ as $K$-module which leads to the contradiction $N = \mathcal{O}_0$. Therefore, $N_{4p-4} = 0$. Now $N$ has length at most $4p - 6$ and is hence, by 1.1, isomorphic to $L(\lambda)$ with $\lambda$ an exceptional weight other than $(0, 0)$ (see 4.4). Since $\lambda \in \{(0,-4),(1,-1),(1,-3)\}$ and $x_3 \cdot m = -5m$ for any $m \in N_{4p-5}$ (see 4.1(5)), it follows that $N_{4p-5} = 0$. In particular, $N$ has length at most $4p - 7$ so that, by 3.2, $p = 3$ and $N_{4p-6} = N_6 \neq 0$. Since $x_3 \cdot m = 6m$ for $m \in N_6$, this says $N_6 \cong L(-1,-3)$ as $K_0$-modules. A contradiction is obtained by observing that $(\mathcal{O}_0)_6$ (and hence $N_6$) contains no nonzero vector of weight $(-1,-3)$. Hence $\mathcal{O}_0$ is simple, as desired.

(2) As in part (1), it is enough to prove that $\mathcal{O}_1$ is simple as a graded $K$-module. Let $N$ be a simple proper graded submodule of $\mathcal{O}_1$. Here, $(\mathcal{O}_1)_i = 0$ if either $i < 3$ or $i > 4p - 3$ (see 4.3). Assume $N_{4p-3} \neq 0$. Then $N_{4p-3} = (\mathcal{O}_1)_{4p-3}$ since this latter space is isomorphic to the simple $K_0$-module $L_0(-1,-3)$. By 4.2(2) and its proof, $(\mathcal{O}_1)_{4p-3}$ generates $\mathcal{O}_1$ as $K$-module giving the contradiction $N = \mathcal{O}_1$. Hence $N_{4p-3} = 0$. Therefore, $N$ has length at most $4p - 7$ which implies $p = 3$ and $N_{4p-4} = N_8 \neq 0$. Once again, by reason of length, $N \cong L(\lambda)$ with $\lambda$ an exceptional weight other than $(0, 0)$. In particular, $\lambda \in \{(0,-4),(1,-1)\}$ since $x_3 \cdot m = 8m$ for $m \in N_8$. By part (1), $L(-4) \cong \mathcal{O}_0$ which has length 7. Hence $\lambda = (-1,-1)$. As in the proof of part (1), a contradiction is now obtained by observing that $N_8$ contains no nonzero vector of weight $(-1,-1)$. Thus, $\mathcal{O}_1$ is simple and the proof is complete.

5. CONTRAGREIDENT MODULES AND DIMENSIONS OF SIMPLE MODULES

Let $L$ be a Lie algebra and let $V$ be an $L$-module. The contragredient of $V$ is the space $V^* := \text{Hom}_F(V, F)$ on which $L$ acts according to the rule
(x \cdot \varphi) = -\varphi(x \cdot v) (x \in L, \varphi \in V^*, v \in V). If L is graded and V is a graded L-module, then V^* is also graded by setting \((V^*)_i = \{ \varphi \in V^* | \varphi(V_j) = 0 \text{ for all } j \neq -i \}. In particular, \((V^*)_i \neq 0 \) if and only if \(V_{-i} \neq 0 \).

Recall the convention stated before that if \( \lambda \in \Lambda \), then \( \lambda_i \) is viewed as an integer with \(-p < \lambda_i \leq 0 \) by identifying \( F_p \) with \( \mathbb{Z}/p\mathbb{Z} \) and using the coset representative in the indicated range.

5.1. Lemma ([5, Lemma 2]). For \( \lambda \in \Lambda \), \( \dim_F L_0(\lambda) = 1 - \lambda_1 \).

5.2. Lemma. \( L_0(\lambda)^* \cong L_0(\lambda_1, -\lambda_2) \) as \( K_0 \)-modules.

Proof. By 5.1, \( \dim_F L_0(\lambda) = 1 - \lambda_1 \) so it follows that \( L_0(\lambda)^* \cong L_0(\lambda_1, c) \) for some \( c \in \mathbb{F}_p \). Let \( \varphi \in L_0(\lambda)^* \). For any \( v \in L_0(\lambda) \), \( (x_3 \cdot \varphi)(v) = -\varphi(x_3 \cdot v) = -\varphi(\lambda_2 v) = (-\lambda_2 \varphi)(v) \) so that \( x_3 \cdot \varphi = -\lambda_2 \varphi \). This says \( c = -\lambda_2 \) which completes the proof.

5.3. Theorem.

(1) If \( \lambda \in \Lambda \) is not exceptional, then \( L(\lambda)^* \cong L(\lambda_1, -\lambda_2 - 4) \).
(2) \( L(0, 0)^* \cong L(0, 0) \).
(3) \( L(0, -4)^* \cong L(-1, -1) \).
(4) \( L(-1, -3)^* \cong L(-1, -3) \).

Proof. (1) Assume \( \lambda \in \Lambda \) is not exceptional. Then 1.1 says \( L(\lambda) \cong Z(\lambda) \). The homogeneous component of \( Z(\lambda) \) of least degree is \( T^\omega \otimes L_0(\lambda) \) where \( \omega = \sum_k (p - 1) \varepsilon_k \). This is a simple \( K_0 \)-module and is therefore, using 5.1, isomorphic to \( L_0(\lambda_1, c) \) for some \( c \in \mathbb{F}_p \). By 3.1(3), \( x_3 \) acts on this space by the scalar \( \lambda_2 + 4 \). Hence \( c = \lambda_2 + 4 \). It follows from 5.2 that the homogeneous component of \( Z(\lambda)^* \) of greatest degree is \( K_0 \)-isomorphic to \( L_0(\lambda_1, -\lambda_2 - 4) \). The result now follows from the remarks before 1.1.

(2) Again, using the remarks before 1.1, \( L(0, 0) \) is the one-dimensional trivial module, so the claim is obvious.

(3) Using 4.5 and 4.3 it is easy to see that the homogeneous component of \( L(0, -4) \) of least degree is \( L(0, -4)_1 \) which is \( K_0 \)-isomorphic to \( L_0(-1, 1) \). The result now follows from 5.2 as in the proof of part (1).

(4) Here, the homogeneous component of \( L(-1, -3) \) of least degree is \( L(-1, -3)_3 \) which is \( K_0 \)-isomorphic to \( L_0(-1, 3) \). (This part also follows from the previous parts using the process of elimination.)

5.4. Lemma. \( Z(\lambda)^* \cong Z(\lambda_1, -\lambda_2 - 4) \) (\( \lambda \in \Lambda \)).

Proof. If \( \lambda \) is not exceptional, the statement follows from 5.3 and 1.1.

Assume \( \lambda = (0, -4) \). By 4.2(1) it is enough to prove \( \Omega_0^* \cong Z(0, 0) \). The homogeneous component of \( \Omega_0 \) of least degree is \( (\Omega_0)_0 \) which is \( K_0 \)-isomorphic to \( L_0(0, 0) \). Hence, the homogeneous component of \( \Omega_0^* \) of greatest degree is \( (\Omega_0^*)_0 \approx ((\Omega_0)_0)^* \cong L_0(0, 0) \). Therefore, a \( K^+ \)-isomorphism \( L_0(0, 0) \rightarrow (\Omega_0^*)_0 \)
induces a $K$-homomorphism $\varphi : Z(0, 0) \to \Omega_0^*$ which sends $T^\omega \otimes L_0(0, 0)$ ($\omega = \sum_k (p - 1) \varepsilon_k$) onto $(\Omega_0^*)_{-4p+4} \neq 0$ (using 4.1). By 3.3, $\varphi$ is injective. Since both spaces have dimension $p^3$, $\varphi$ is an isomorphism.

Next, assume $\lambda = (-1, -3)$. By 4.2(2), it is enough to prove that $V^* \cong Z(-1, -1)$ where $V := \Omega_1/\text{im } \tau_0$. The homogeneous component of $V$ of least degree is the two-dimensional space $V_1$ (see 2.4) which is $K_0$-isomorphic to $L_0(-1, 1)$. This yields a $K$-homomorphism $\varphi : Z(-1, -1) \to V^*$ which sends $T^\omega \otimes L_0(-1, -1)$ onto $(V^*)_{-4p+3} \neq 0$. By 3.3, $\varphi$ is injective and, since both spaces have dimension $2p^3$, $\varphi$ is an isomorphism. \(\square\)

5.5. Theorem.

(1) If $\lambda \in A$ is not exceptional, then $\dim_F L(\lambda) = p^3(1 - \lambda_1)$.
(2) $\dim_F L(0, 0) = 1$.
(3) $\dim_F L(0, -4) = \dim_F L(-1, -1) = p^3 - 1$.
(4) $\dim_F L(-1, -3) = p^3 - 2$.

Proof. (1) Use 1.1 and 5.1.
(2) $L(0, 0)$ is the trivial one-dimensional module.
(3) Use 4.5(1), 4.3 and 5.3(3).
(4) Use 4.5(2) and 4.3. \(\square\)

6. CARTAN INVARIANTS

Let $V$ be a $K$-module and let $[V]$ denote the element of the Grothendieck group of $u(K)$ corresponding to $V$. Then $[V]$ can be written uniquely in the form $[V] = \sum_{\lambda \in A} [V : L(\lambda)] [L(\lambda)]$ with $[V : L(\lambda)] \in \mathbb{Z}$. In fact, $[V : L(\lambda)]$ is just the multiplicity of $L(\lambda)$ as a composition factor of $V$.

6.1. Lemma.

(1) If $\lambda \in A$ is not exceptional, then $[Z(\lambda)] = [L(\lambda)]$.
(2) $[Z(0, -4)] = [L(0, -4)] + [L(0, 0)]$.
(3) $[Z(-1, -3)] = [L(-1, -3)] + 3[L(0, 0)] + [L(0, -4)]$.
(4) $[Z(-1, -1)] = [L(-1, -1)] + 3[L(0, 0)] + [L(-1, -3)]$.
(5) $[Z(0, 0)] = [L(0, 0)] + [L(-1, -1)]$.

Proof. (1) Use 1.1.
(2) By 4.2(1), $Z(0, -4) \cong \Omega_0$. According to 2.3(2), $L(0, 0) \cong \ker \delta_0 \leq \Omega_0$ and, by 4.5(1), $\Omega_0/\ker \delta_0 \cong L(0, -4)$.
(3) By 4.2(2), $Z(-1, -3) \cong \Omega_1/\text{im } \tau_0$. Now $\text{im } \delta_0 \leq \ker \delta_1 \leq \Omega_1$. Therefore, if $\pi : \Omega_1 \to \Omega_1/\text{im } \tau_0$ denotes the canonical epimorphism, then $\pi(\text{im } \delta_0) \leq \pi(\ker \delta_1) \leq \pi(\Omega_1)$. As in the proof of 4.3, $\text{im } \tau_0 \cap \ker \delta_1 = \{0\}$, so $\pi(\text{im } \delta_0) \cong \text{im } \delta_0 \cong L(0, -4)$ (by 4.5(1)), $\pi(\ker \delta_1)/\pi(\text{im } \delta_0) \cong \ker \delta_1/\text{im } \delta_0$ which has three composition factors, each isomorphic to $L(0, 0)$ (by 2.3(2)), and $\pi(\Omega_1)/\pi(\ker \delta_1) \cong \Omega_1 \cong L(-1, -3)$ (by 4.5(2)).
The Lie algebra $K_0$ has a $p$-grading: $K_0 = (K_0)_{-1} + (K_0)_0 + (K_0)_1$ where $(K_0)_{-1} = Fx^{(2a_2)}$, $(K_0)_0 = Fx^{(e_1 + e_2)} + Fx_3$ and $(K_0)_1 = Fx^{(e_1)}$. The component $(K_0)_0$ is a maximal torus for $K$. Also, $(K_0)_1 \triangleleft T + (K_0)_1 := K_0^+$ and $K_0^+/(K_0)_1 \cong T$ so that any $T$-module becomes a $K_0^+$-module in the natural way.

For $\lambda \in \Lambda$, let $F_\lambda$ denote the one-dimensional $T$-module on which $x^{(e_1 + e_2)}$ (resp., $x_3$) acts as multiplication by $\lambda_1$ (resp., $\lambda_2$). Then $Z_0(\lambda) := u(K_0) \otimes_{u(K_0^+)} F_\lambda$ has unique simple quotient $L_0(\lambda)$.

The following lemma gives the composition factors of $Z_0(\lambda)$ (by way of formulas in the Grothendieck group for $u(K_0)$). It follows easily from [6, Theorem 1] and [2, Theorem 5.1].

**6.2. Lemma.**

$$[Z_0(\lambda)] = \begin{cases} [L_0(\lambda)] + [L_0(2 - p - \lambda_1, \lambda_2)] & \lambda_1 \neq 1 - p, \\
[L_0(\lambda)] & \lambda_1 = 1 - p. \end{cases}$$

In the following discussion, all parametrizations by $\Lambda$ are assumed to be relative to a fixed ordering.

Let $X$ be the $p^2 \times 1$-matrix with $\lambda$-entry

$$X_\lambda = \sum_{\mu, \nu} [Z_0(\nu) : L_0(\mu)][Z(\mu) : L(\lambda)].$$

**6.3. Lemma.**

$$X_\lambda = \begin{cases} 16 & \lambda = (0, 0), \\
4 & \lambda = (0, -4), (-1, -1) \text{ or } (-1, -3), \\
1 & \lambda_1 = 1 - p, \\
2 & \text{otherwise}. \end{cases}$$

**Proof.** Assume $\lambda_1 \neq 1 - p$. Then 6.1 implies $[Z(1 - p, c) : L(\lambda)] = 0$ for all $c \in F_p$. Hence, $X_\lambda = \sum_{\mu \neq (1 - p, c)} [Z(\mu) : L(\lambda)]$ by 6.2. Using 6.1 the result follows.

Now assume $\lambda_1 = 1 - p$. If $\mu_1 \neq 1 - p$, then $[Z(\mu) : L(\lambda)] = 0$. Therefore, $X_\lambda = \sum_{\mu_1 = 1 - p} [Z(\mu) : L(\lambda)] = 1$, again by 6.2 and 6.1. \(\square\)

Let $C$ be the Cartan matrix of $K$. Hence, $C$ is the $p^2 \times p^2$-matrix with $(\lambda, \mu)$-entry $C_{\lambda \mu} := [P(\lambda) : L(\mu)]$ where $P(\lambda)$ is the projective cover of $L(\lambda)$.

**6.4. Theorem.** Assume $p > 3$. $C = p^\beta X^t X$ where $\beta = p^3 - 12$ and $^tX$ denotes the transpose of $X$.

**Proof.** Note that for $\lambda \in \Lambda$, $F_\lambda^* \cong F_{-\lambda}$. According to [3], $C - p^\beta R^t Q^t U Q R$ where $Q_{\lambda \mu} = [Z_0(\lambda) : L_0(\mu)]$. 304
\[ \begin{align*}
Q^i_{\lambda \mu} &= [Z_0(-\lambda) : I_0(\mu)], \\
R^i_{\lambda \mu} &= [Z(\lambda) : L(\mu)], \\
R^i_{\lambda \mu} &= [Z(\lambda) : L(\mu)^*], \\
U_{\lambda \mu} &= 1.
\end{align*} \]

Evidently, \( R' = RP_1 \) and \( Q' = P_2Q \) for some permutation matrices \( P_1 \) and \( P_2 \). Since \( P_2U = U \), it follows that \( C = p^{\beta} P_1 U R' Q U R P_1 X 'X \). By 5.3, \( P_1 \) is obtained from the identity matrix by transposing the \((0, -4)\)- and \((-1, -1)\)-columns and transposing the \((\lambda_1, \lambda_2)\)- and \((\lambda_1, -\lambda_2 - 4)\)-columns for \( \lambda \) not exceptional. Hence, 6.3 implies \( P_1 X = X \).  

6.5. **Theorem.** Assume \( p > 3 \). For \( \lambda \in \Lambda \), \( \dim_F P(\lambda) = p^\alpha X_\lambda \) where \( \alpha = p^3 - 6 \) and \( X_\lambda \) is as in 6.4.

**Proof.** \( \dim_F Z(\mu) = p^3 \dim_F L_0(\mu) \) and \( \dim_F Z(\nu) = p \), so that
\[
\sum_{\lambda} X_\lambda \dim_F L(\lambda) = \sum_{\nu, \mu, \lambda} [Z_0(\nu) : L_0(\mu)] [Z(\mu) : L(\lambda)] \dim_F L(\lambda)
= p^3 \sum_{\nu, \mu} [Z_0(\nu) : L_0(\mu)] \dim_F L_0(\mu)
= p^4 \sum_{\nu} 1 = p^6.
\]

Therefore, denoting by \( \dim P \) (resp., \( \dim L \)) the \( p^2 \times 1 \)-matrix with \( \lambda \)-entry \( \dim_F P(\lambda) \) (resp., \( \dim_F L(\lambda) \)), 6.4 implies \( \dim P = C \dim L = p^\beta X 'X \) \( \dim L = p^{\alpha} X \).  

**REFERENCES**