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Unipotent variety in the group compactification

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Abstract

The unipotent variety of a reductive algebraic group G plays an important role in the representation theory. In this paper, we will consider the closure $\bar{\mathcal{U}}$ of the unipotent variety in the De Concini–Procesi compactification $\bar{G}$ of a connected simple algebraic group G. We will prove that $\bar{\mathcal{U}} - \mathcal{U}$ is a union of some G-stable pieces introduced by Lusztig in [Moscow Math. J 4 (2004) 869–896]. This was first conjectured by Lusztig. We will also give an explicit description of $\bar{\mathcal{U}}$. It turns out that similar results hold for the closure of any Steinberg fiber in $\bar{G}$.

MSC: 20G15

Keywords: Unipotent varieties; Steinberg fibers; Group compactifications

0. Introduction

A connected simple algebraic group G has a "wonderful" compactification $\bar{G}$, introduced by De Concini and Procesi. The variety $\bar{G}$ is a smooth, projective variety with $G \times G$ action on it. The $G \times G$-orbits of $\bar{G}$ are indexed by the subsets of the simple roots.

The group G acts diagonally on $\bar{G}$. Lusztig introduced a partition of $\bar{G}$ into finitely many G-stable pieces. The G-orbits on each piece are in one-to-one correspondence to

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the conjugacy classes of a certain reductive group. Based on the partition, he developed the theory of "Parabolic Character Sheaves" on $\bar{G}$.

In this paper, we study the closure $\bar{\mathcal{U}}$ of the unipotent variety $\mathcal{U}$ of G in $\bar{G}$, partially based on the previous work of [Spr2]. The main result is that the boundary of the closure is a union of some G-stable pieces. (see Theorem 4.3.)

The unipotent variety plays an important role in the representation theory. One would expect that $\bar{\mathcal{U}}$, the subvariety of $\bar{G}$, which is analogous to the subvariety $\mathcal{U}$ of G, also plays an important role in the theory of "Parabolic Character Sheaves". Our result is a step toward this direction.

The arrangement of this paper is as follows. In Section 1, we briefly recall some results on the $B \times B$-orbits of $\bar{G}$ (where B is a Borel subgroup of G) and results on $\bar{\mathcal{U}}$, which were proved by Springer in [Spr1] and [Spr2]. In Section 2, we first recall the definition of the G-stable pieces and then in 2.6, we show that any G-stable piece is the minimal G-stable subset of $\bar{G}$ that contains a particular $B \times B$-orbit. In the remaining part of Section 2, we establish some basic facts about the Coxeter elements, which will be used in Section 4 to prove our main theorem. In Section 3, we show case-by-case that certain G-stable pieces are contained in $\bar{\mathcal{U}}$. Hence a lower bound of $\bar{\mathcal{U}}$ is established.

A naive thought about $\bar{\mathcal{U}}$ is that the boundary of the "unipotent elements" are "nilpotent cone". In fact, it is true. A precise statement is given and proved in 4.3. Thus we obtain an upper bound of $\bar{\mathcal{U}}$. We also show in 4.3 that the lower bound is actually equal to the upper bound. Therefore, our main theorem is proved. In Section 4, we also consider the closure of arbitrary Steinberg fiber of G in $\bar{G}$. (An example of Steinberg fiber is $\mathcal{U}$.) The results are similar. In the end of Section 4, we calculate the number of points of $\bar{\mathcal{U}}$ over a finite field. The formula bears some resemblance to the formula for $\bar{G}$.

1. Preliminaries

1.1. Let G be a connected, simple algebraic group over an algebraically closed field k. Let B be a Borel subgroup of G, B^- be the opposite Borel subgroup and $T = B \cap B^-$. Let $(\alpha_i)_{i \in I}$ be the set of simple roots. For $i \in I$, we denote by $\alpha_i^\vee$, ω_i, $\omega_i^\vee$ and s_i the simple coroot, the fundamental weight, the fundamental coweight and the simple reflection corresponding to α_i. We denote by $<,>$ the standard pairing between the weight lattice and the root lattice. For any element w in the Weyl group $W = N(T)/T$, we will choose a representative $\dot{w}$ in $N(T)$ in the same way as in [L1, 1.1].

For any subset J of I, let W_J be the subgroup of W generated by $\{s_j \mid j \in J\}$ and W^J (resp. ${}^J W$) be the set of minimal length coset representatives of W/W_J (resp. $W_J \backslash W$). Let w_0^J be the unique element of maximal length in W_J. (We will simply write w_0^I as w_0.) For $J, K \subset I$, we write ${}^J W^K$ for ${}^J W \cap W^K$.

1.2. For $J \subset I$, let $P_J \supset B$ be the standard parabolic subgroup defined by J and $P_J^- \supset B^-$ be the opposite of P_J. Set $L_J = P_J \cap P_J^-$. Then L_J is a Levi subgroup of

P_J and P_J^-. Let Z_J be the center of L_J and $G_J = L_J/Z_J$ be its adjoint group. We denote by π_{P_J} (resp. $\pi_{P_J^-}$) the projection of P_J (resp. P_J^-) onto G_J.

Let $\bar{G}$ be the wonderful compactification of G ([DP] deals with the case $k = \mathbf{C}$. The generalization to arbitrary k was given in [Str]). It is an irreducible, projective smooth $G \times G$-variety. The $G \times G$-orbits Z_J of $\bar{G}$ are indexed by the subsets J of I. Moreover, $Z_J = (G \times G) \times_{P_J^- \times P_J} G_J$, where $P_J^- \times P_J$ acts on the right on $G \times G$ and on the left on G_J by $(q, p) \cdot z = \pi_{P_J^-}(q) z \pi_{P_J}(p)^{-1}$. Let h_J be the image of $(1, 1, 1)$ in Z_J.

We will identify Z_I with G and the $G \times G$-action on it is given by $(g, h) \cdot x = gxh^{-1}$.

For any subvariety X of $\bar{G}$, we denote by $\bar{X}$ the closure of X in $\bar{G}$.

For any finite set A, we will write $|A|$ for the cardinality of A.

1.3. For any closed subgroup H of G, we denote by H_{diag} the image of the diagonal embedding of H in $G \times G$ and by $\text{Lie}(H)$ the corresponding Lie subalgebra of H. For $g \in G$, we write ${}^g H$ for gHg^{-1}.

For any parabolic subgroup P, we denote by U_P its unipotent radical. We will simply write U for U_B and U^- for U_{B^-}. For $J \subset I$, set $U_J = U \cap L_J$ and $U_J^- = U^- \cap L_J$.

For parabolic subgroups P and Q, define

$$P^Q = (P \cap Q) U_P.$$

It is easy to see that for $J, K \subset I$ and $u \in {}^J W^K$, $P_J^{({}^{\dot{u}} P_K)} = P_{J \cap \text{Ad}(u) K}$.

Let $\mathcal{U}$ be the unipotent variety of G. Then $\mathcal{U}$ is stable under the action of G_{diag} and U is stable under the action of $U \times U$ and T_{diag}. Moreover, $\mathcal{U} = G_{\text{diag}} \cdot U$. Similarly, $\bar{\mathcal{U}} = G_{\text{diag}} \cdot \bar{U}$ (see [Spr2, 1.4]).

1.4. Now consider the $B \times B$-orbits on $\bar{G}$. We use the same notation as in [Spr1]. For any $J \subset I$, $u, v \in W$, set $[J, u, v] = (B \times B)(\dot{u}, \dot{v}) \cdot h_J$. It is easy to see that $[J, u, v] = [J, x, vz^{-1}]$, where $u = xz$ with $x \in W^J$ and $z \in W_J$. Moreover, $\bar{G} = \bigsqcup_{J \subset I} \bigsqcup_{x \in W^J, w \in W} [J, x, w]$. Springer proved the following result in [Spr1, 2.4].

Theorem. *Let $x \in W^J$, $x' \in W^K$, $w, w' \in W$. Then $[K, x', w']$ is contained in $\overline{[J, x, w]}$ if and only if $K \subset J$ and there exists $u \in W_K$, $v \in W_J \cap W^K$ with $xvu^{-1} \leqslant x'$, $w'u \leqslant wv$ and $l(wv) = l(w) + l(v)$.*

As a consequence of the theorem, we have the following properties which will be used later.

(1) For any $K \subset J$, $w \in W^J$ and $v \in W_J$, $[K, wv, v] \subset \overline{[J, w, 1]}$.
(2) For any $J \subset I$, $w, w' \in W^J$ with $w \leqslant w'$, then $[J, w', 1] \subset \overline{[J, w, 1]}$.

1.5. In this subsection, we recall some results of [Spr2].

Let ε be an indeterminate. Put $o = k[[\varepsilon]]$ and $K = k((\varepsilon))$. An o-valued point of a k-variety Z is a k-morphism $\gamma : \mathrm{Spec}(o) \to Z$. We write $Z(o)$ for the set of all o-valued points of Z. Similarly, we write $Z(K)$ for the set of all K-valued points of Z. For $\gamma \in Z(o)$, we have that $\gamma(0) \in Z$, where 0 is the closed point of $\mathrm{Spec}(o)$.

By the valuative criterion of completeness (see [EGA, Ch II, 7.3.8 & 7.3.9]), for the complete k-variety $\bar{G}$, the inclusion $o \hookrightarrow K$ induces a bijective from $\bar{G}(o)$ onto $\bar{G}(K)$. Therefore, any $\gamma \in \bar{G}(K)$ defines a point $\gamma(0) \in \bar{G}$. In particular, any $\gamma \in U(K)$ defines a point $\gamma(0) \in \bar{G}$. Here we regard $U(K)$ as a subset of $\bar{G}(K)$ in the natural way.

We have that $x \in \bar{U}$ if and only if there exists $\gamma \in U(K)$ such that $\gamma(0) = x$ (see [Spr2, 2.2]).

Let Y be the cocharacter group of T. An element $\lambda \in Y$ defines a point in $T(k[\varepsilon, \varepsilon^{-1}])$, hence a point p_λ of $T(K)$. Let $H \subset G(o)$ be the subgroup consisting of elements γ with $\gamma(0) \in B$. Then for $\gamma \in U(K)$, there exists $\gamma_1, \gamma_2 \in H$, $w \in W$ and $\lambda \in Y$, such that $\gamma = \gamma_1 \dot{w} p_\lambda \gamma_2$. Moreover, w and λ are uniquely determined by γ (see [Spr2, 2.6]). In this case, we will call (w, λ) admissible. Springer showed that $(w, \lambda - w^{-1}\lambda)$ is admissible for any dominant regular coweight λ (see [Spr2, 3.1]).

For $\lambda \in Y$ and $x \in W$ with $x^{-1} \cdot \lambda$ dominant, we have that $p_\lambda(0) = (\dot{x}, \dot{x}) \cdot h_{I(x^{-1}\lambda)}$, where $I(x^{-1}\lambda)$ is the set of simple roots orthogonal to $x^{-1}\lambda$ (see [Spr2, 2.5]). If moreover, (w, λ) is admissible, then there exists some $t \in T$ such that $(U \times U)(\dot{w}\dot{x}t, \dot{x}) \cdot h_{I(x^{-1}\lambda)} \subset \bar{U}$.

2. The partition of Z_J

2.1. We will follow the set-up of [L4, 8.18].

For any $J \subset I$, let $\mathcal{P}^J$ be the set of parabolic subgroups conjugate to P_J. We will write $\mathcal{B}$ for $\mathcal{P}^{\varnothing}$. For $P \in \mathcal{P}^J$, $Q \in \mathcal{P}^K$ and $u \in {}^J W^K$, we write $\mathrm{pos}(P, Q) = u$ if there exists $g \in G$, such that ${}^g P = P_J, {}^g Q = {}^{\dot{u}} P_K$. For $J, J' \subset I$ and $y \in {}^{J'} W^J$ with $\mathrm{Ad}(y) J = J'$, define

$$\tilde{Z}_J^y = \{(P, P', \gamma) \mid P \in \mathcal{P}^J, P' \in \mathcal{P}^{J'}, \gamma = U_{P'} g U_P, \mathrm{pos}(P', {}^g P) = y\}$$

with the $G \times G$ action given by $(g_1, g_2) \cdot (P, Q, \gamma) = ({}^{g_1} P, {}^{g_2} P', g_2 \gamma g_1^{-1})$.

To $z = (P, P', \gamma) \in \tilde{Z}_J^y$, we associate a sequence $(J_k, J'_k, u_k, y_k, P_k, P'_k, \gamma_k)_{k \geqslant 0}$ with $J_k, J'_k \subset I$, $u_k \in W$, $y_k \in {}^{J'_k} W^{K_k}$, $\mathrm{Ad}(y_k) J_k = J'_k$, $P_k \in \mathcal{P}_{J_k}, P'_k \in \mathcal{P}_{J'_k}, \gamma_k = U_{P'_k} g U_{P_k}$ for some $g \in G$ satisfies $\mathrm{pos}(P'_k, {}^g P_k) = u_k$. The sequence is defined as follows.

$$P_0 = P, P'_0 = P', \gamma_0 = \gamma, J_0 = J, J'_0 = J', u_0 = \mathrm{pos}(P'_0, P_0), y_0 = y.$$

Assume that $k \geqslant 1$, that $P_m, P'_m, \gamma_m, J_m, J'_m, u_m, y_m$ are already defined for $m < k$ and that $u_m = \text{pos}(P'_m, P_m)$, $P_m \in \mathcal{P}_{J_m}$, $P'_m \in \mathcal{P}_{J'_m}$ for $m < k$. Let

$$J_k = J_{k-1} \cap \text{Ad}(y_{k-1}^{-1} u_{k-1}) J_{k-1},\ J'_k = J_{k-1} \cap \text{Ad}(u_{k-1}^{-1} y_{k-1}) J_{k-1},$$

$$P_k = g_{k-1}^{-1} ({}^{g_{k-1}} P_{k-1})^{(P'_{k-1}{}^{P_{k-1}})} g_{k-1} \in \mathcal{P}_{J_k},\ P'_k = P_{k-1}^{P'_{k-1}} \in \mathcal{P}_{J'_k},$$

where

$$g_{k-1} \in \gamma_{k-1} \text{ is such that } {}^{g_{k-1}} P_{k-1} \text{ contains some Levi of } P_{k-1} \cap P'_{k-1},$$

$$u_k = \text{pos}(P'_k, P_k),\ y_k = u_{k-1}^{-1} y_{k-1},\ \gamma_k = U_{P'_k} g_{k-1} U_{P_k}.$$

It is known that the sequence is well defined. Moreover, for sufficient large n, we have that $J_n = J'_n = J_{n+1} = J'_{n+1} = \cdots$ and $u_n = u_{n+1} = \cdots = 1$. Now we set $\beta(z) = u_0 u_1 \cdots u_n$, $n \gg 0$. Then we have that $\beta(z) \in {}^{J'}W$. By [L4, 8.18], [L3, 2.5], the sequence $(J_k, J'_k, u_k, y_k)_{k \geqslant 0}$ is uniquely determined by $(J, \beta(z), y)$.

The map $w \mapsto yw^{-1}$ is a bijection between W^J and ${}^{J'}W$. For $w \in W^J$, set

$$\tilde{Z}^y_{J,w} = \{z \in \tilde{Z}^y_J \mid \beta(z) = yw^{-1}\}.$$

Then $(\tilde{Z}^y_{J,w})_{w \in W^J}$ is a partition of $\tilde{Z}^y_J$ into locally closed G-stable subvarieties. For $w \in W^J$, let $(J_k, J'_k, u_k, y_k)_{k \geqslant 0}$ be the sequence uniquely determined by (J, yw^{-1}, y). Then $(P, P', \gamma) \mapsto (P_1, P'_1, \gamma_1)$ define a G-equivariant map $\vartheta : \tilde{Z}^y_{J,w} \to \tilde{Z}^{y_1}_{J_1, u_0^{-1} w}$.

2.2. Let $J \subset I$. Set $\tilde{Z}_J = \tilde{Z}_J^{w_0 w_0^J}$ and $J^* = \text{Ad}(w_0 w_0^J) J$. For $w \in W^J$, set $w_J = w_0 w_0^J w^{-1}$. The map $w \mapsto w_J$ is a bijection between W^J and ${}^{J^*}W$. For any $w \in W^J$, let

$$\tilde{Z}_{J,w} = \{z \in \tilde{Z}_J \mid \beta(z) = w_J\}.$$

Then $(\tilde{Z}_{J,w})_{w \in W^J}$ is a partition of $\tilde{Z}_J$ into locally closed G-stable subvarieties. Let $(J_k, J'_k, u_k, y_k)_{k \geqslant 0}$ be the sequence determined by $(J, w_J, w_0 w_0^J)$ (see 2.1). Assume that $J_n = J'_n = J_{n+1} = J'_{n+1} = \cdots$ and $u_n = u_{n+1} = \cdots = 1$. Set $v_0 = w_J$ and $v_k = u_{k-1}^{-1} v_{k-1}$ for $k \in \mathbf{N}$. By [L4, 8.18], [L3, 2.3], we have $u_k \in {}^{J'_k}W^{J_k}$ and $u_{k+1} \in W_{J_k}$ for all $k \geqslant 0$. Hence $v_{k+1} \in W_{J_k}$ for all $k \geqslant 0$. Moreover, it is easy to see by induction on k that $y_k = v_k w$. In particular, $w = y_n \in {}^{J_n}W^{J_n}$, $\mathrm{Ad}(w)J_n = J_n$ and $\dot{w}$ normalizes $B \cap L_{J_n}$. We have the following result.

2.3. Lemma. *Keep the notation of* 2.2. *Let* $z = (P_J, {}^{\dot{w}_J^{-1}}P_{J^*}, \dot{w}_J^{-1} U_{P_{J^*}} \dot{w}_J \dot{w} b U_{P_J})$, *where* $b \in {}^{\dot{w}^{n-1}\dot{v}_n^{-1}}(U_{P_{J'_n}} \cap U_{J_{n-1}})^{\dot{w}^{n-2}\dot{v}_{n-1}^{-1}}(U_{P_{J'_{n-1}}} \cap U_{J_{n-2}}) \cdots {}^{\dot{v}_1^{-1}}(U_{P_{J'_1}} \cap U_{J_0})T$ *or* $b \in B$. *Then* $z \in \tilde{Z}_{J,w}$.

Proof. For any k, set $P_k = P_{J_k}$, $P'_k = {}^{\dot{v}_k^{-1}}P_{J'_k}$. Then

$$P_k \cap P'_k = P_{J_k} \cap {}^{\dot{v}_{k+1}^{-1}\dot{u}_k^{-1}}P_{J'_k} = {}^{\dot{v}_{k+1}^{-1}}(P_{J_k} \cap {}^{\dot{u}_k^{-1}}P_{J'_k}).$$

Note that $u_k^{-1} \in {}^{J_k}W^{J'_k}$. Then $L_{J_k} \cap {}^{\dot{u}_k^{-1}}L_{J'_k} = L_{J_k \cap \mathrm{Ad}(\dot{u}_k^{-1})J'_k} = L_{J'_{k+1}}$. Thus ${}^{\dot{v}_{k+1}^{-1}}L_{J'_{k+1}} = {}^{\dot{v}_{k+1}^{-1}}(L_{J_k} \cap {}^{\dot{u}_k^{-1}}L_{J'_k})$ is a Levi factor of $P_k \cap P'_k$. Moreover, we have

$$P_k^{P'_k} = P_{J_k}^{({}^{\dot{v}_k^{-1}}P_{J'_k})} = {}^{\dot{v}_{k+1}^{-1}}(P_{J_k}^{({}^{\dot{u}_k^{-1}}P_{J'_k})}) = {}^{\dot{v}_{k+1}^{-1}}P_{J_k \cap \mathrm{Ad}(\dot{u}_k^{-1})J'_k} = {}^{\dot{v}_{k+1}^{-1}}P_{J'_{k+1}}$$

$$\begin{aligned} P_k'^{P_k} &= {}^{\dot{v}_k^{-1}}(P_{J'_k}^{({}^{\dot{v}_k}P_{J_k})}) = {}^{\dot{v}_k^{-1}}(P_{J'_k}^{({}^{\dot{u}_k}P_{J_k})}) = {}^{\dot{v}_k^{-1}}P_{J'_k \cap \mathrm{Ad}(\dot{u}_k)J_k} \\ &= {}^{\dot{v}_k^{-1}}P_{\mathrm{Ad}(\dot{y}_k)(J_k \cap \mathrm{Ad}(\dot{y}_k^{-1}\dot{u}_k)J_k)} = {}^{\dot{v}_k^{-1}}P_{\mathrm{Ad}(\dot{y}_k)J_{k+1}}. \end{aligned}$$

If $b \in B$, then set $g_k = \dot{w}b$, $\gamma_k = U_{P'_k} g_k U_{P_k}$ and $z_k = (P_k, P'_k, \gamma_k)$ for all k. In this case, ${}^{\dot{v}_{k+1}^{-1}}L_{J'_{k+1}} = {}^{\dot{w}\dot{y}_{k+1}^{-1}}L_{J'_{k+1}} = {}^{\dot{w}}L_{J_{k+1}} \subset {}^{\dot{w}}P_k = {}^{g_k}P_k$. Thus ${}^{g_k}P_k$ contains some Levi of $P_k \cap P'_k$. Moreover,

$$\begin{aligned} g_k^{-1}({}^{g_k}P_k)^{({}^{\dot{v}_k^{-1}}P_{\mathrm{Ad}(\dot{y}_k)J_{k+1}})}g_k &= P_k^{({}^{b^{-1}\dot{w}^{-1}\dot{v}_k^{-1}}P_{\mathrm{Ad}(\dot{y}_k)J_{k+1}})} = {}^{b^{-1}}(P_k^{{}^{\dot{y}_k^{-1}}P_{\mathrm{Ad}(\dot{y}_k)J_{k+1}}}) \\ &= {}^{b^{-1}}P_{J_k \cap \mathrm{Ad}(\dot{y}_k^{-1})\mathrm{Ad}(\dot{y}_k)J_{k+1}} = {}^{b^{-1}}P_{J_{k+1}} = P_{J_{k+1}}. \end{aligned}$$

Therefore, $\vartheta(z_k) = z_{k+1}$.

If $b = (\dot{w}^{n-1}\dot{v}_n^{-1}b_n\dot{v}_n\dot{w}^{-n+1}) \cdots (\dot{v}_1^{-1}b_1\dot{v}_1)(\dot{w}^n t \dot{w}^{-n})$, where $b_j \in U_{P_{J'_j}} \cap U_{J_{j-1}}$ for $1 \leqslant j \leqslant n$ and $t \in T$, then set

$$a_k = (\dot{w}^{n-k}\dot{v}_n^{-1}b_n\dot{v}_n\dot{w}^{-n+k}) \cdots (\dot{v}_k^{-1}b_k\dot{v}_k)(\dot{w}^{n+1-k}t\dot{w}^{-n-1+k}).$$

In this case, set $g_k = \dot{w}a_{k+1}$, $\gamma_k = U_{P'_k} g_k U_{P_k}$ and $z_k = (P_k, P'_k, \gamma_k)$. For $j \geqslant 0$, $J_{j+1} = J_j \cap \mathrm{Ad}(\dot{y}_{j+1}^{-1}) J_j$ and $v_{j+1} \in W_{J_j}$. Thus ${}^{\dot{w}} L_{J_{j+1}} = {}^{\dot{v}_{j+1}^{-1} \dot{y}_{j+1}} L_{J_{j+1}} \subset {}^{\dot{v}_{j+1}^{-1}} L_{J_j} = L_{J_j}$. Then ${}^{\dot{w}^j \dot{v}_{k+j+1}^{-1}} U_{J_{k+j}} \subset {}^{\dot{w}^j} L_{J_{k+j}} \subset L_{J_k}$. So $a_{k+1} \in P_k$. Thus ${}^{g_k} P_k = {}^{\dot{w}} P_k$ contains some Levi of $P_{J_k} \bigcap {}^{\dot{v}_k^{-1}} P_{J'_k}$. Moreover,

$$g_k^{-1} ({}^{g_k} P_k)^{({}^{\dot{v}_k^{-1}} P_{\mathrm{Ad}(\dot{y}_k) J_{k+1}})} g_k = {}^{a_{k+1}^{-1}} P_{J_{k+1}}.$$

Thus $\vartheta(z_k) = (Q, Q', \gamma')$, where $Q = {}^{a_{k+1}^{-1}} P_{J_{k+1}}$, $Q' = {}^{\dot{v}_{k+1}^{-1}} P_{J'_{k+1}}$ and $\gamma' = U_{Q'} g_k U_Q$. Note that ${}^{\dot{v}_{k+1}^{-1}} U_{P_{J'_{k+1}}} \subset Q'$ and $T \subset Q'$. Moreover, for $j \geqslant 1$, ${}^{\dot{w}^j \dot{v}_{k+j+1}^{-1}} U_{J_{k+j}} \subset {}^{\dot{w}^j} L_{J_{k+j}} \subset {}^{\dot{w}} L_{J_{k+1}} = {}^{\dot{v}_{k+1}^{-1} \dot{y}_{k+1}} L_{J_{k+1}} = {}^{\dot{v}_{k+1}^{-1}} L_{J'_{k+1}} \subset Q'$. Thus $a_{k+1} \in Q'$. Hence, $z_{k+1} = (a_{k+1}, a_{k+1}) \cdot \vartheta(z_k)$.

In both cases, $\vartheta(z_k)$ is in the same G orbit as z_{k+1}. Thus

$$\beta(z) = \beta(z_0) = u_1 \beta(z_1) = \cdots = u_1 u_2 \cdots u_n = w_J. \qquad \square$$

Remark. 1. From the proof of the case where $b \in B$, we can see that

$$\vartheta^n(P_J, {}^{\dot{w}_J^{-1}} P_{J^*}, \dot{w}_J^{-1} U_{P_{J^*}} \dot{w}_J \dot{w} b U_{P_J}) = (P_{J_n}, P_{J_n}, U_{P_{J_n}} \dot{w} b U_{P_{J_n}}).$$

This result will be used to establish a relation between the G-stable pieces and the $B \times B$-orbits.

2. The fact that $(P_J, {}^{\dot{w}_J^{-1}} P_{J^*}, \dot{w}_J^{-1} U_{P_{J^*}} \dot{w}_J \dot{w} b U_{P_J})$ is contained in $\tilde{Z}_{J,w}$ for any $b \in {}^{\dot{w}^{n-1} \dot{v}_n^{-1}} (U_{P_{J'_n}} \cap U_{J_{n-1}})^{\dot{w}^{n-2} \dot{v}_{n-1}^{-1}} (U_{P_{J'_{n-1}}} \cap U_{J_{n-2}}) \cdots {}^{\dot{v}_1^{-1}} (U_{P_{J'_1}} \cap U_{J_0}) T$ plays an important role in Section 3. We will discuss about it in more detail in 3.1.

2.4. Let $(J_n, J'_n, u_n, y_n)_{n \geqslant 0}$ be the sequence that is determined by w_J and $w_0 w_0^J$. Assume that $J_n = J'_n = J_{n+1} = J'_{n+1} = \cdots$ and $u_n = u_{n+1} = \cdots = 1$. Then $z \mapsto \vartheta^n(z)$ is a G-equivariant morphism from $\tilde{Z}_{J,w}$ to $\tilde{Z}^w_{J_n,1}$ and induces a bijection from the set of G-orbits on $\tilde{Z}_{J,w}$ to the set of G-orbits on $\tilde{Z}^w_{J_n,1}$.

Set $\tilde{L}_{J,w} = L_{J_n}$ and $\tilde{C}_{J,w} = \dot{w} \tilde{L}_{J,w}$. Let $N_G(\tilde{L}_{J,w})$ be the normalizer of $\tilde{L}_{J,w}$ in G. Then $\tilde{C}_{J,w}$ is a connected component of $N_G(\tilde{L}_{J,w})$ and $\tilde{Z}^w_{J_n,1}$ is a fibre bundle over $\mathcal{P}^{J_n}$ with fibres isomorphic to $\tilde{C}_{J,w}$. There is a natural bijection between $\tilde{C}_{J,w}$ and $F = \{z = (P_{J_n}, P_{J_n}, \gamma_n) \mid z \in \tilde{Z}^w_{J_n,1}\}$ under which the action of $\tilde{L}_{J,w}$ on $\tilde{C}_{J,w}$ by conjugation corresponds to the action of $P_{J_n}/U_{P_{J_n}}$ on F by conjugation. Therefore, we obtain a canonical bijection the set of G-stable subvarieties of $\tilde{Z}_{J,w}$ and the set of $\tilde{L}_{J,w}$-stable subvarieties of $\tilde{C}_{J,w}$ (see [L4, 8.21]). Moreover, a G-stable subvariety of $\tilde{Z}_{J,w}$ is closed if and only if the corresponding $\tilde{L}_{J,w}$-stable subvariety of $\tilde{C}_{J,w}$ is closed. By the remark 1 of 2.3, for any $b \in B \cap \tilde{L}_{J,w}$, the G-orbit that contains $(P_J, {}^{\dot{w}_J^{-1}} P_{J^*}, \dot{w}b)$ corresponds to the $\tilde{L}_{J,w}$-orbit that contains $\dot{w}b$ via the bijection.

2.5. Since G is adjoint, the center of P/U_P is connected for any parabolic subgroup P. Let H_P be the inverse image of the (connected) center of P/U_P under $P \to P/U_P$. We can regard H_P/U_P as a single torus Δ_J independent of P. Now Δ_J acts (freely) on $\tilde{Z}_J$ by $\delta : (P, P', \gamma) \mapsto (P, P', \gamma z)$ where $z \in H_P$ represents $\delta \in \Delta_J$. The action of G on $\tilde{Z}_J$ commutes with the action of Δ_J and induces an action of G on $\Delta_J \setminus \tilde{Z}_J$. There exists a G-equivariant isomorphism from Z_J to $\Delta_J \setminus \tilde{Z}_J$ which sends $(g_1, g_2) \cdot h_J$ to $({}^{g_2}P_J, {}^{g_1}P_J^-, U_{{}^{g_1}P_J^-} g_1 g_2^{-1} H_{{}^{g_2}P_J})$. We will identify Z_J with $\Delta_J \setminus \tilde{Z}_J$.

It is easy to see that $\Delta_J(\tilde{Z}_{J,w}) = \tilde{Z}_{J,w}$. Set $Z_{J,w} = \Delta_J \setminus \tilde{Z}_{J,w}$. Then

$$Z_J = \bigsqcup_{w \in W^J} Z_{J,w}.$$

Moreover, we may identify Δ_J with a closed subgroup of the center of $\tilde{L}_{J,w}$. Set $L_{J,w} = \tilde{L}_{J,w}/\Delta_J$ and $C_{J,w} = \tilde{C}_{J,w}/\Delta_J$. Thus we obtain a bijection between the set of G-stable subvarieties of $Z_{J,w}$ and the set of $L_{J,w}$-stable subvarieties of $C_{J,w}$ (see [L4, 11.19]). Moreover, a G-stable subvariety of $Z_{J,w}$ is closed if and only if the corresponding $L_{J,w}$-stable subvariety of $C_{J,w}$ is closed and for any $b \in B \cap \tilde{L}_{J,w}$, the G-orbit that contains $(P_J, {}^{\dot{w}_J^{-1}} P_{J^*}, \dot{w}b)$ corresponds to the $L_{J,w}$-orbit that contains $\dot{w}b\Delta_J$ via the bijection.

2.6. Proposition. *For any* $w \in W^J$, $Z_{J,w} = G_{\text{diag}} \cdot [J, w, 1]$.

Proof. By 2.3, $(\dot{w}, b) \cdot h_J \in Z_{J,w}$ for all $b \in B$. Since $Z_{J,w}$ is G-stable, $G_{\text{diag}}[J, w.1] \subset Z_{J,w}$.

For any $z \in Z_{J,w}$, let C be the $L_{J,w}$-stable subvariety corresponding to $G_{\text{diag}} \cdot z$ and let c be an element in $\tilde{C}_{J,w}$ such that $c\Delta_J \in C$. By 2.2, $\dot{w}$ normalizes $B \cap \tilde{L}_{J,w}$. Thus c is $\tilde{L}_{J,w}$-conjugate to an element of $\dot{w}(B \cap \tilde{L}_{J,w})$. Therefore, z is G-conjugate to $(\dot{w}, b) \cdot h_J$ for some $b \in B \cap \tilde{L}_{J,w}$. The proposition is proved. □

2.7. Proposition. *For any* $w \in W^J$, $\overline{Z_{J,w}} = \overline{G_{\text{diag}}(\dot{w}T, 1) \cdot h_J}$.

Proof. Since $(\dot{w}T, 1) \cdot h_J \subset Z_{J,w}$ and $\overline{Z_{J,w}}$ is a G-stable closed variety, we have that $\overline{G_{\text{diag}}(\dot{w}T, 1) \cdot h_J} \subset \overline{Z_{J,w}}$.

Set $X = \{(\dot{w}t, u) \cdot h_J \mid t \in T, u \in U\}$. For any $u \in {}^{\dot{w}}U_J$ and $t \in T$, we have that $\text{Ad}(\dot{w}t)^{-1} u \in U_J$ and $u \in {}^{\dot{w}}U_J \subset U$. Consider the map $\phi : {}^{\dot{w}} U_J \times T \to X$ defined by $\phi(u, t) = (u, u)(\dot{w}t, 1) \cdot h_J = (\dot{w}t, (\dot{w}t)^{-1} u \dot{w} t u^{-1}) \cdot h_J$, for $u \in {}^{\dot{w}}U_J, t \in T$.

It is easy to see that there is an open subset T' of T, such that the restriction of ϕ to ${}^{\dot{w}}U_J \times T'$ is injective. Note that $\dim(X) = \dim(T) + \dim(U/U_{P_J}) = \dim(T) + \dim(U_J) = \dim({}^{\dot{w}}U_J \times T)$. Then the image of ϕ is dense in X. The proposition is proved. □

Remark. This argument was suggested by the referee.

2.8. For $w \in W$, denote by $\operatorname{supp}(w)$ the set of simple roots whose associated simple reflections occur in a reduced expression of w. An element $w \in W$ is called a Coxeter element if it is a product of the simple reflections, in some order, or in other words, $|\operatorname{supp}(w)| = l(w) = |I|$. We have the following properties.

2.9. Proposition. *Fix $i \in I$. Then all the Coxeter elements are conjugate under elements of $W_{I-\{i\}}$.*

Proof. Let c, c' be Coxeter elements. We say that c' can be obtained from c via a cyclic shift if $c = s_{i_1} s_{i_2} \cdots s_{i_n}$ is a reduced expression and $c' = s_{i_1} c s_{i_1}$. It is known that for any Coxeter elements c, c', there exists a finite sequences of Coxeter elements $c = c_0, c_1, \dots, c_m = c'$ such that c_{k+1} can be obtained from c_k via a cyclic shift (see [Bo, p. 116, Prop. 1]).

Now assume that $c = s_{i_1} s_{i_2} \cdots s_{i_n}$ is a reduced expression of a Coxeter element. If $i_1 \neq i$, then $s_{i_1} c s_{i_1}$ and c are conjugated by $s_{i_1} \in W_{I-\{i\}}$. If $i_1 = i$, then $s_{i_1} c s_{i_1} = s_{i_2} s_{i_3} \cdots s_{i_n} c (s_{i_2} s_{i_3} \cdots s_{i_n})^{-1}$. Therefore, if a Coxeter element can be obtained from another Coxeter element via a cyclic shift, then they are conjugated by elements of $W_{I-\{i\}}$. The proposition is proved. □

Remark. The proof of [Bo, p. 116, Prop. 1] also can be used to prove this proposition.

2.10. Proposition. *Let $J \subset I$ and $w \in W^J$ with* $\operatorname{supp}(w) = I$. *Then there exist a Coxeter element w', such that $w' \in W^J$ and $w' \leqslant w$.*

Proof. We prove the statement by induction on $l(w)$.

Let $i \in I$ with $s_i w < w$. Then $s_i w \in W^J$. If $\operatorname{supp}(s_i w) = I$, then the statement holds by induction hypothesis on $s_i w$. Now assume that $\operatorname{supp}(s_i w) = I - \{i\}$. By induction, there exists a Coxeter element w' of $W_{I-\{i\}}$, such that $w' \in W^{J-\{i\}}$ and $w' \leqslant s_i w$. Then $s_i w'$ is a Coxeter element of w and $s_i w' \leqslant w$.

Since $w' \in W_{I-\{i\}}$, $(w')^{-1}\alpha_i$ is either α_i or a non-simple positive root. We also have that w' is a Coxeter element of $W_{I-\{i\}}$. Thus if $(w')^{-1}\alpha_i = \alpha_i$, then $< \alpha_i, \alpha_j^\vee >= 0$ for all $j \neq i$. It contradicts the assumption that G is simple. Hence $(w')^{-1}\alpha_i$ is a non-simple positive root. Note that if $s_i w' \notin W^J$, then $s_i w' = w' s_j$ for some $j \in J$, that is, $(w')^{-1}\alpha_i = \alpha_j$. Therefore, $s_i w' \in W^J$. The proposition is proved. □

2.11. Corollary. *Let $i \in I$, $J = I - \{i\}$ and w be a Coxeter element of W with $w \in W^J$. Then $\bigsqcup_{K \subset J} \bigsqcup_{w' \in W^K, \text{supp}(w')=I} Z_{K,w'} \subset \overline{Z_{J,w}}$.*

Proof. By 1.4, $[K, wv, v] \subset \overline{[J, w, 1]}$ for $K \subset J$ and $v \in W_J$. Since $\overline{Z_{J,w}}$ is G-stable, $(\dot{v}^{-1}\dot{w}\dot{v}T, 1) \cdot h_K \subset \overline{Z_{J,w}}$. By 2.9, $(\dot{w}'T, 1) \cdot h_K \subset \overline{Z_{J,w}}$ for all Coxeter element w'. By 2.7, $Z_{K,w'} \subset \overline{Z_{J,w}}$ for all Coxeter element w' with $w' \in W^K$. For any $u \in W^K$ with $\text{supp}(u) = I$, there exists a Coxeter element w', such that $w' \in W^K$ and $w' \leqslant u$. Thus by 1.4, we have that $[K, u, 1] \subset \overline{Z_{J,w}}$. By 2.6, $Z_{K,u} \subset \overline{Z_{J,w}}$. The corollary is proved. □

Remark. In 4.4, we will show that the equality holds.

3. Some combinatorial results

3.1. Fix $i \in I$. Define subsets I_k of I for all $k \in \mathbf{N}$ in the following way. Set $I_1 = \{i\}$. Assume that I_k is already defined. Set

$$I_{k+1} = \{\alpha_j \mid j \in I - \cup_{l=1}^{k} I_l, < \alpha_j^\vee, \alpha_m > \neq 0 \text{ for some } m \in I_k\}.$$

It is easy to see that if $j_1, j_2 \in I_k$ with $j_1 \neq j_2$, then $< \alpha_{j_1}, \alpha_{j_2}^\vee >= 0$. Thus $s_{I_k} = \prod_{j \in I_k} s_j$ is well-defined. For sufficiently large n, we have $I_n = I_{n+1} = \cdots = \varnothing$ and $s_{I_n} = s_{I_{n+1}} = \cdots = 1$. Now set $w_k = s_{I_n} s_{I_{n-1}} \cdots s_{I_k}$ for $k \in \mathbf{N}$. We will write w^J for w_1. Set $J_{-1} = I$ and $J_0 = J = I - \{i\}$. Then w^J is a Coxeter element and $w^J \in W^J$. Let (J_n, J'_n, u_n, y_n) be the sequence determined by w^J and $w_0 w_0^J$. Then we can show by induction that for $k \geqslant 0$, $J_k = J_{k-1} - I_{k+1}$, $u_k = w_0^{J_{k-1}} w_0^{J_k} s_{I_{k+1}} w_0^{J_{k+1}} w_0^{J_k}$, $y_k = w_0^{J_{k-1}} w_0^{J_k} s_{I_k} s_{I_{k-1}} \cdots s_{I_1}$ and $J'_k = \text{Ad}(y_k) J_k$. In particular, $J_n = \varnothing$. Thus $\tilde{L}_{J,w^J} = T$ and $\tilde{C}_{J,w^J} = \dot{w}^J T$. Since w is a Coxeter element, the homomorphism $T \to T$ sending $t \in T$ to $(\dot{w}^J)^{-1} t \dot{w}^J t^{-1}$ is surjective. Thus $\tilde{L}_{J,w^J}$ acts transitively on $\tilde{C}_{J,w^J}$. By 2.5, G acts transitively on Z_{J,w^J}.

For $k \in \mathbf{N}$, we set $v_k = w_0^{J_{k-1}} w_0^{J_k} w_{k+1}^{-1}$. Then it is easy to see that

$$^{\dot{v}_k^{-1}}(U_{P_{J'_k}} \cap U_{J_{k-1}}) = {}^{w_{k+1}}(U_{P^-_{J_k}} \cap U^-_{J_{k-1}}).$$

Therefore by 2.3, for $b \in {}^{w^{n-1} w_{n+1}}(U_{P^-_{J_n}} \cap U^-_{J_{n-1}}) \cdots {}^{w_2}(U_{P^-_{J_1}} \cap U^-_{J_0})T$, we have that $(\dot{w}^J b, 1) \cdot h_J \in Z_{J,w^J}$.

In the rest of this section, we will keep the notations of J, J_k, w^J and w_k as above. We will prove the following statement.

Proposition. *Let X be a closed subvariety of $\bar{G}$ satisfying the following condition: for any admissible pair (w, λ) and $x \in W$ with $x^{-1}\lambda$ is dominant, there exist some $t \in T$, such that $G_{\text{diag}}(U \times U)(\dot{w}\dot{x}t, \dot{x}) \cdot h_{I(x^{-1}\lambda)} \subset X$. Then $Z_{J,w^J} \subset X$.*

An example of such X is $\bar{\mathcal{U}}$. There are some other interesting examples, which we will discuss in 4.5. The proof is based on case-by-case checking.

Remark. The outline of the case-by-case checking is as follows.

For $\lambda \in Y$, we write $\lambda \geqslant 0$ if $\lambda \in \sum_{l \in I} \mathbf{R}_{\geqslant 0}\alpha_l^\vee$.

We start with the fundamental coweight $\omega_i^\vee$. Find $x \in W$ that satisfies the conditions (1) $x\omega_i^\vee \geqslant 0$ and (2) for $l \in I$, either $(s_l - 1)x\omega_i^\vee \geqslant 0$ or $s_l x\omega_i^\vee \ngeqslant 0$. Such x always exists, as we will see by case-by-case checking. The elements $x\omega_i^\vee$ that we obtain in this way are not unique, in general. Fortunately, there always exists some $x \in W$ that satisfies the conditions (1) and (2) and allows us to do the procedures that we will discuss below.

In the rest of the remark, we fix such x. Since $x\omega_i^\vee \in Y$, there exists $n \in \mathbf{N}$, such that $nx\omega_i^\vee$ is contained in the coroot lattice. Set $\lambda = nx\omega_i^\vee$. Now we can find $v \in W$ such that (v, λ) is admissible. (In practice, we find $v \in W$ with $l(v) = |\text{supp}(v)|$ and $-v\lambda \geqslant 0$. Then we can use Lemma 3.2 to check that if (v, λ) is admissible.) By the assumption on X, $G_{\text{diag}}(U \times U)(\dot{v}\dot{x}t, \dot{x}) \cdot h_J \subset X$ for some $t \in T$.

In some cases, $x^{-1}vx = w_J$. Since w_J is a Coxeter element, $(\dot{w}_J T, 1) \cdot h_J = T_{\text{diag}}(\dot{w}_J t, 1) \cdot h_J \subset X$. By 2.7, $Z_{J,w_J} \subset X$.

In other cases, the situation is more complicated. We need to choose some $u \in U$, such that $(u\dot{v}\dot{x}t, \dot{x}) \cdot h_J \in Z_{J,w_J}$. This is the most difficult part of the case-by-case checking. The Lemmas 3.3 and 2.3 will be used to overcome the difficulties.

Throughout this section, we will use the same labelling of Dynkin diagram as in [Bo]. For $a, b \in I$, we denote by $s_{[a,b]}$ the element $s_b s_{b-1} \cdots s_a$ of the Weyl group W and $\dot{s}_{[a,b]} = \dot{s}_b \dot{s}_{b-1} \cdots \dot{s}_a$. (If $b < a$, then $s_{[a,b]} = 1$ and $\dot{s}_{[a,b]} = 1$.)

3.2. Lemma. *Let $x = s_{i_1} s_{i_2} \cdots s_{i_n}$ with $|\text{supp}(x)| = n$. Then $(1 - x^{-1})\omega_k^\vee = 0$ if $k \notin \{i_1, i_2, \ldots, i_n\}$ and $(1 - x^{-1})\omega_{i_j}^\vee = s_{i_n} s_{i_{n-1}} \cdots s_{i_{j+1}} \alpha_{i_j}^\vee$. Thus (x, λ) is admissible for all $\lambda \in \sum_{j=1}^n \mathbf{N} s_{i_n} s_{i_{n-1}} \cdots s_{i_{j+1}} \alpha_{i_j}^\vee$.*

The lemma is a direct consequence of [Bo, p. 226, Ex. 22a], which was pointed out to me by the referee.

3.3. Lemma. *Let $w, x, y_1, y_2 \in W$ and $t \in T$. Assume that $y_1 = s_{i_1} s_{i_2} \cdots s_{i_l}$, $y_2 = s_{i_{l+1}} s_{i_{l+2}} \cdots s_{i_{l+k}}$ with $k + l = |\text{supp}(y_1 y_2)|$. If moreover, $< \alpha_{i_{l_1}}^\vee, \alpha_{i_{l_2}} >= 0$ for all $1 \leqslant l_1 < l_2 \leqslant l$ and $(1 - y_1 y_2)x\omega_i^\vee, (1 - y_1)w\omega_i^\vee \in \sum_{j=1}^k \mathbf{R}_{>0}\alpha_{i_j}^\vee$, then there exists $u \in U_{-w^{-1}\alpha_{i_{l+1}}} U_{-w^{-1}\alpha_{i_{l+2}}} \cdots U_{-w^{-1}\alpha_{i_{l+k}}}$ such that $(\dot{x}^{-1}\dot{w}ut, 1) \cdot h_J \in G_{\text{diag}}(U \times U)(\dot{w}t, \dot{y}_1\dot{y}_2\dot{x}) \cdot h_J$.*

Proof. We have that $(1-y_1y_2)x\omega_i^\vee = \sum_{j=1}^{k+l}(1-s_{i_j})s_{i_{j+1}}\cdots s_{i_{l+k}}x\omega_i^\vee$. Note that $i_1, i_2, \ldots, i_{k+l}$ are distinct and $(1-s_{i_j})s_{i_{j+1}}\cdots s_{i_{l+k}}x\omega_i^\vee \in \mathbf{R}\alpha_{i_j}^\vee$ for all j. Hence $(1-s_{i_j})s_{i_{j+1}}\cdots s_{i_{l+k}}x\omega_i^\vee \in \mathbf{R}_{>0}\alpha_{i_j}^\vee$ for all j, i.e., $< s_{i_{j+1}}\cdots s_{i_k}x\omega_i^\vee, \alpha_{i_j} > \in \mathbf{R}_{>0}$. Therefore $\dot{x}^{-1}\dot{s}_{i_{l+k}}^{-1}\cdots\dot{s}_{i_{j+1}}^{-1}U_{\alpha_{i_j}}\dot{s}_{i_{j+1}}\cdots\dot{s}_{i_{l+k}}\dot{x} \subset U_{P_J}$. Similarly, we have that $\dot{w}^{-1}U_{-\alpha_{i_j}}\dot{w} \in U_{P_J^-}$ for $j \leqslant l$.

There exists $u_j \in U_{\alpha_{i_j}}$ and $u_j' \in U_{-\alpha_{i_j}}$ such that $u_j\dot{s}_{i_j}u_j = u_j'$. Note that $u_1'u_2'\cdots u_{l+k-1}' \in L_{I-\{i_{l+k}\}}$, $u_{l+k} \in U_{P_{I-\{i_{l+k}\}}}$ and $\dot{x}^{-1}u_{l+k}\dot{x} \subset U_{P_J}$. Thus

$$\begin{aligned} u_1'u_2'\cdots u_{l+k}'\dot{x} &= u_1'u_2'\cdots u_{l+k-1}'u_{l+k}\dot{s}_{i_k}u_{l+k}\dot{x} \in U_{P_{I-\{i_k\}}}u_1'u_2'\cdots u_{l+k-1}'\dot{s}_{i_k}\dot{x}U_{P_J} \\ &\subset Uu_1'u_2'\cdots u_{l+k-1}'\dot{s}_{i_k}\dot{x}U_{P_J}. \end{aligned}$$

We can show in the same way that $u_1'u_2'\cdots u_{l+k}'\dot{x} \in U\dot{y}_1\dot{y}_2\dot{x}U_{P_J}$. Therefore, $(\dot{w}t, u_1'u_2'\cdots u_{l+k}'\dot{x})\cdot h_J \in (U\times U)(\dot{w}t, \dot{y}_1\dot{y}_2\dot{x})\cdot h_J$. Set $u = \dot{w}^{-1}u_{l+1}'u_{l+2}'\cdots u_{l+k}'\dot{w}$ and $u' = t^{-1}\dot{w}^{-1}$
$(u_1'u_2'\cdots u_l')^{-1}\dot{w}t \in U_{P_J^-}$. Then

$$\begin{aligned} (\dot{x}^{-1}\dot{w}ut, 1)\cdot h_J &= (\dot{x}^{-1}\dot{w}utu', 1)\cdot h_J = \left(\dot{x}^{-1}(u_1'u_2'\cdots u_{l+k}')^{-1}\dot{w}t, 1\right)\cdot h_J \\ &\in G_{\mathrm{diag}}(U\times U)(\dot{w}t, \dot{y}_1\dot{y}_2\dot{x})\cdot h_J. \qquad \square \end{aligned}$$

3.4. In 3.4–3.7, we assume that G is $PGL_n(k)$. Without loss of generality, we assume that $i \leqslant n/2$. In this case, $w^J = s_{[i+1,n-1]}s_{[1,i]}^{-1}$. For any $a \in \mathbf{R}$, we denote by $[a]$ the maximal integer that is less than or equal to a.

For $1 \leqslant j \leqslant i$, set $a_j = [(j-1)n/i]$. For convenience, we will set $a_{i+1} = n-1$. Note that for $j \leqslant i-1$, $a_{j+1} - a_j = [jn/i] - [(j-1)n/i] \geqslant [n/i] \geqslant 2$. Therefore, we have that $0 = a_1 < a_1 + 1 < a_2 < a_2 + 1 < \cdots < a_i < a_i + 1 \leqslant a_{i+1} = n-1$. Now set $b_0 = 0$. For $k \in \{1, 2, \ldots, n-1\} - \{a_2, a_3, \ldots, a_i\} - \{a_2+1, a_3+1, \ldots, a_i+1\}$, set $b_k = i$. For $j \in \{2, 3, \ldots, i\}$, set $b_{a_j} = (j-1)n - ia_k$ and $b_{a_j+1} = i - b_{a_j}$. In particular, $b_{n-1} = i$.

Now set $v = s_{[a_1+1, a_2-\delta_{b_{a_2},0}]}s_{[a_2+1, a_3-\delta_{b_{a_3},0}]}\cdots s_{[a_i+1, a_{i+1}-\delta_{b_{a_{i+1}},0}]}$, where $\delta_{a,b}$ is the Kronecker delta. Set $v_j = s_{[a_j+1, a_{j+1}]}s_{[a_{j+1}+1, a_{j+2}]}\cdots s_{[a_i+1, a_{i+1}]}$ for $1 \leqslant j \leqslant i$. Set $\lambda = \sum_{j=1}^{i}\sum_{k=1}^{a_{j+1}-a_j} b_{a_j+k}(s_{[a_j+1, a_j+k-1]}v_{j+1})^{-1}\alpha_{a_j+k}^\vee$. It is easy to see that for $1 \leqslant a \leqslant b \leqslant n-1$ and $1 \leqslant k \leqslant n-1$,

$$s_{[a,b]}\alpha_k^\vee = \begin{cases} \sum_{l=a-1}^{b}\alpha_l^\vee & \text{if } k = a-1, \\ -\sum_{l=a}^{b}\alpha_l^\vee & \text{if } k = a, \\ \alpha_{k-1}^\vee & \text{if } a < k \leqslant b, \\ \alpha_b^\vee + \alpha_{b+1}^\vee & \text{if } k = b+1, \\ \alpha_k^\vee & \text{otherwise .} \end{cases}$$

If $b_{a_j+k} \neq 0$, then $(s_{[a_j+1,a_j+k-1]} s_{[a_{j+1}+1,a_{j+2}-\delta_{b_{a_{j+2}},0}]} \cdots s_{[a_i+1,a_{i+1}]})^{-1} \alpha^\vee_{a_j+k} = (s_{[a_j+1,a_j+k-1]} v_{j+1})^{-1} \alpha^\vee_{a_j+k}$. By 3.2, (v, λ) is admissible.

We have that

$$\begin{aligned}
\lambda &= \sum_{j=1}^{i} \sum_{k=1}^{a_{j+1}-a_j-1} b_{a_j+k} v_{j+1}^{-1} s^{-1}_{[a_j+1,a_j+k-1]} \alpha^\vee_{a_j+k} + \sum_{j=1}^{i} b_{a_{j+1}} v_{j+1}^{-1} s^{-1}_{[a_j+1,a_{j+1}-1]} \alpha^\vee_{a_{j+1}} \\
&= \sum_{j=1}^{i} \sum_{k=1}^{a_{j+1}-a_j-1} \sum_{l=1}^{k} b_{a_j+k} \alpha^\vee_{a_j+l} + \sum_{j=1}^{i-1} b_{a_{j+1}} \sum_{l=1}^{a_{j+1}-a_j+1} \alpha^\vee_{a_j+l} + b_{a_{i+1}} \sum_{l=1}^{a_{i+1}-a_i} \alpha^\vee_{a_i+l} \\
&= \sum_{j=1}^{i} \sum_{k=1}^{a_{j+1}-a_j} \sum_{l=1}^{k} b_{a_j+k} \alpha^\vee_{a_j+l} + \sum_{j=1}^{i-1} b_{a_{j+1}} \alpha^\vee_{a_{j+1}+1} \\
&= \sum_{j=1}^{i} \sum_{l=1}^{a_{j+1}-a_j} \sum_{k=l}^{a_{j+1}-a_j} b_{a_j+k} \alpha^\vee_{a_j+l} + \sum_{j=1}^{i-1} b_{a_{j+1}} \alpha^\vee_{a_{j+1}+1} \\
&= \sum_{j=1}^{i} \sum_{l=2}^{a_{j+1}-a_j} \big((a_{j+1}-a_j-l)i + b_{a_{j+1}}\big) \alpha^\vee_{a_j+l} + \big((a_2-1)i + b_{a_2}\big) \alpha^\vee_1 \\
&\quad + \sum_{j=2}^{i} \big(b_{a_j} + (a_{j+1}-a_j-2)i + b_{a_{j+1}} + b_{a_j+1}\big) \alpha^\vee_{a_j+1} \\
&= \sum_{j=1}^{i} \sum_{l=1}^{a_{j+1}-a_j} \big((a_{j+1}-a_j-l)i + b_{a_{j+1}}\big) \alpha^\vee_{a_j+l} = nx\omega^\vee_i.
\end{aligned}$$

Note that $a_j \geqslant j$ for $j \geqslant 2$. Set $x_i = 1$ and $x_j = s_{[j+1,a_{j+1}]} s_{[j+2,a_{j+2}]} \cdots s_{[i,a_i]}$ for $1 \leqslant j \leqslant i-1$. If $j = 1$, we will simply write x for x_1.

3.5. Lemma. *For $1 \leqslant j \leqslant i$, we have that*

$$\begin{aligned}
nx_j \omega^\vee_i &= \sum_{l=1}^{j-1} l(n-i) \alpha^\vee_l + \sum_{l=j}^{a_{j+1}} \big(jn - il\big) \alpha^\vee_l \\
&\quad + \sum_{k=j+1}^{i} \sum_{l=1}^{a_{k+1}-a_k} \big((a_{k+1}-a_k-l)i + b_{a_{k+1}}\big) \alpha^\vee_{a_k+l}.
\end{aligned}$$

In particular, $nx\omega^\vee_i = \sum_{j=1}^{i} \sum_{l=1}^{a_{j+1}-a_j} \big((a_{j+1}-a_j-l)i + b_{a_{j+1}}\big) \alpha^\vee_{a_j+l}$.

Proof. We argue by induction on j. Note that $n\omega_i^\vee = \sum_{l=1}^{i-1} l(n-i)\alpha_l^\vee + \sum_{l=i}^{n-1} i(n-l)\alpha_l^\vee$. Thus the lemma holds for $j = i$.

Note that $jn - i(a_j + l) = jn - ia_{j+1} + i(a_{j+1} - a_j - l) = b_{a_{j+1}} + i(a_{j+1} - a_j - l)$.

Assume that the lemma holds for j. Then

$$\begin{aligned}
nx_{j-1}\omega_i^\vee &= s_{[j,a_j]}\sum_{l=1}^{j-1} l(n-i)\alpha_l^\vee + s_{[j,a_j]}\sum_{l=j}^{a_{j+1}}(jn-il)\alpha_1^\vee \\
&\quad + s_{[j,a_j]}\sum_{k=j+1}^{i}\sum_{l=1}^{a_{k+1}-a_k}\big((a_{k+1}-a_k-l)i + b_{a_{k+1}}\big)\alpha_{a_k+l}^\vee \\
&= \sum_{l=1}^{j-2} l(n-i)\alpha_l^\vee + (j-1)(n-i)\sum_{l=j-1}^{a_j}\alpha_l^\vee - j(n-i)\sum_{l=j}^{a_j}\alpha_l^\vee \\
&\quad + \sum_{l=j+1}^{a_j}(jn-il)\alpha_{l-1}^\vee \\
&\quad + \big(jn - i(a_j+1)\big)(\alpha_{a_j}^\vee + \alpha_{a_{j+1}}^\vee) + \sum_{l=a_j+2}^{a_{j+1}}(jn-il)\alpha_l^\vee \\
&\quad + \sum_{k=j+1}^{i}\sum_{l=1}^{a_{k+1}-a_k}\big((a_{k+1}-a_k-l)i + b_{a_{k+1}}\big)\alpha_{a_k+l}^\vee \\
&= \sum_{l=1}^{j-2} l(n-i)\alpha_l^\vee + (j-1)(n-i)\sum_{l=j-1}^{a_j}\alpha_l^\vee - j(n-i)\sum_{l=j}^{a_j}\alpha_l^\vee \\
&\quad + \sum_{l=j+1}^{a_j}(jn-il)\alpha_{l-1}^\vee + \big(jn - i(a_j+1)\big)\alpha_{a_j}^\vee \\
&\quad + \sum_{k=j}^{i}\sum_{l=1}^{a_{k+1}-a_k}\big((a_{k+1}-a_k-l)i + b_{a_{k+1}}\big)\alpha_{a_k+l}^\vee \\
&= \sum_{l=1}^{j-2} l(n-i)\alpha_l^\vee + (j-1)(n-i)\alpha_{j-1}^\vee + \sum_{l=j}^{a_j}\big((j-1)(n-i) \\
&\quad - j(n-i) + jn - i(l+1)\big)\alpha_l^\vee \\
&\quad + \sum_{k=j}^{i}\sum_{l=1}^{a_{k+1}-a_k}\big((a_{k+1}-a_k-l)i + b_{a_{k+1}}\big)\alpha_{a_k+l}^\vee
\end{aligned}$$

$$= \sum_{l=1}^{j-2} l(n-i)\alpha_l^\vee + \sum_{l=j-1}^{a_j} \big((j-1)n - il\big)\alpha_l^\vee$$

$$+ \sum_{k=j}^{i} \sum_{l=1}^{a_{k+1}-a_k} \big((a_{k+1} - a_k - l)i + b_{a_{k+1}}\big)\alpha_{a_k+l}^\vee.$$

Thus the lemma holds for j. $\square$

3.6. Lemma. *We have that* $x^{-1}v_1 x = w^J$.

Proof. If $a_j \geqslant j+1$, then $s_{[j+1,a_{j+1}]}^{-1} s_{[a_j+1,a_{j+1}]} = s_{[j+1,a_j]}^{-1}$. If $j \geqslant 2$ and $a_j < j+1$, then $j = 2$, $a_j = 2$ and $s_{[3,a_3]}^{-1} s_{[a_2+1,a_3]} = 1 = s_{[3,a_2]}^{-1}$. In conclusion, $s_{[j+1,a_{j+1}]}^{-1} s_{[a_j+1,a_{j+1}]} = s_{[j+1,a_j]}^{-1}$ for $j \geqslant 2$. Moreover, $s_{[2,a_2]}^{-1} s_{[a_1+1,a_2]} = s_1$. Thus

$$s_{[2,a_2]}^{-1} v_1 s_{[2,a_2]} = s_{[2,a_2]}^{-1} s_{[a_1+1,a_2]} v_2 s_{[2,a_2]} = s_1 v_2 s_{[2,a_2]} = v_2 s_1 s_{[2,a_2]} = v_2 s_{[3,a_2]} s_1 s_2.$$

$$\begin{aligned} s_{[j+1,a_{j+1}]}^{-1} v_j s_{[j+1,a_j]} s_{[1,j]}^{-1} s_{[j+1,a_{j+1}]} &= s_{[j+1,a_{j+1}]}^{-1} s_{[a_j+1,a_{j+1}]} v_{j+1} s_{[j+1,a_j]} s_{[1,j]}^{-1} s_{[j+1,a_{j+1}]} \\ &= s_{[j+1,a_j]}^{-1} v_{j+1} s_{[j+1,a_j]} s_{[1,j]}^{-1} s_{[j+1,a_{j+1}]} \\ &= v_{j+1} s_{[1,j]}^{-1} s_{[j+2,a_{j+1}]} s_{j+1} = v_{j+1} s_{[j+2,a_{j+1}]} s_{[1,j+1]}^{-1}. \end{aligned}$$

Thus, we can prove by induction on j that $x^{-1} v_1 x = x_j^{-1} v_j s_{[j+1,a_j]} s_{[1,j]}^{-1} x_j$ for $1 \leqslant j \leqslant i$. In particular, $x^{-1} v_1 x = s_{[i+1,n-1]} s_{[1,i]}^{-1}$. The lemma is proved. $\square$

3.7. By 3.4 and 3.5, there exists $t \in T$, such that $(U \times U)(\dot{v}\dot{x}t, \dot{x}) \cdot h_J \subset X$. Consider $K = \{a_j \mid b_{a_j} = 0\}$. Then for any $j, j' \in K$ with $j \neq j'$, we have that $|j - j'| \geqslant 2$ and $< \alpha_j^\vee, \alpha_{j'} >= 0$. Set $y = \prod_{j \in K} s_j$. Then y is well-defined. Note that $(1-y)yx\omega_i^\vee, (1-y)vx\omega_i^\vee \in \sum_{j \in K} \mathbf{R}_{>0}\alpha_j^\vee$. By 3.3, $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$. Therefore, $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$. By 3.6, $x^{-1}yvx = x^{-1}v_1 x = w^J$. Therefore, $Z_{J,w^J} \cap X \neq \emptyset$. By 3.1, G acts transitively on Z_{J,w^J}. Therefore $Z_{J,w^J} \subset X$.

3.8. In this subsection, we assume that G is of type C_n and set

$$\varepsilon = \begin{cases} 1 & \text{if } 2 \mid i, \\ 0 & \text{otherwise.} \end{cases}$$

Set $v = s_{n-i+1}s_{n-i+3}\cdots s_{n-\varepsilon}$, $x_1 = s_{[n-i,n-1]}^{-1}s_{[n-i-1,n-2]}^{-1}\cdots s_{[1,i]}^{-1}$ and $x_2 = s_{[n+\varepsilon-1,n]}^{-1}s_{[n+\varepsilon-3,n]}^{-1}\cdots s_{[n-i+2,n]}^{-1}$. Set $\lambda = \alpha_{n-i+1}^\vee + \alpha_{n-i+3}^\vee + \cdots + \alpha_{n-\varepsilon}^\vee$. Then we have that (v, λ) is admissible.

Now set $\lambda' = \sum_{j\in I}\min(i,j)\alpha_j^\vee \in \mathbf{N}\omega_i^\vee$. Set $x_{1,j} = s_{[j-i+1,j]}^{-1}s_{[j-i,j-1]}^{-1}\cdots s_{[1,i]}^{-1}$ for $i-1 \leqslant j \leqslant n-1, s$. Then we can show by induction that $x_{1,j}\lambda' = \sum_{k=1}^{i} k\alpha_{j-i+1+k}^\vee + i\sum_{l=j+2}^{n}\alpha_l^\vee$. In particular, $x_1\omega_i^\vee = \sum_{k=1}^{i} k\alpha_{n-i+k}^\vee$.

For $0 \leqslant j \leqslant (i+\varepsilon-1)/2$, set $x_{2,j} = s_{[n-i+2j,n]}^{-1}s_{[n-i+2j-2,n]}^{-1}\cdots s_{[n-i+2,n]}^{-1}$. Then we can show by induction that $x_{2,j}x_1\lambda' = \sum_{k=0}^{j-1}\alpha_{n-i+1+2k}^\vee + \sum_{l=1}^{i-2j} l\alpha_{n-i+2j+l}^\vee$. In particular, we have that $x_2x_1\lambda' = \lambda$. Therefore, there exists $t \in T$, such that $(U,U)(\dot{v}\dot{x}_2\dot{x}_1t, \dot{x}_2\dot{x}_1)\cdot h_J \subset X$.

Now set $y_1 = s_{n+\varepsilon-1}s_{n+\varepsilon-3}\cdots s_{n-i}$ and $y_2 = s_{[1,n-i-1]}$. For $1 \leqslant j \leqslant n-i-1$, set $\beta_k = -(vx_2x_1)^{-1}\alpha_k = -\alpha_{k+i}$. Thus by 3.3, there exists $u \in U_{\beta_1}U_{\beta_2}\cdots U_{\beta_{n-i}}$, such that $(\dot{x}_1^{-1}\dot{x}_2^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}_2\dot{x}_1ut, 1)\cdot h_J \in X$.

For $0 \leqslant j \leqslant (i+\varepsilon-1)/2$, set

$$v_{2,j} = s_{[1,n-i]}(s_{n-i+2}s_{n-i+4}\cdots s_{n-i+2j})(s_{n-i+1}s_{n-i+3}\cdots s_{n-i+2j-1})s_{[n-i+2j+1,n]}^{-1}.$$

It is easy to see that $s_{[n-i+2j,n]}v_{2,j}s_{[n-i+2j,n]}^{-1} = v_{2,j-1}$. Therefore, we can show by induction that $x_2^{-1}y_1y_2vx_2 = x_{2,j}^{-1}v_{2,j}x_{2,j}$ for $0 \leqslant j \leqslant (i+\varepsilon-1)/2$. In particular, $x_2^{-1}y_1y_2vx_2 = s_{[1,n-i]}s_{[n-i+1,n]}^{-1}$.

For $i-1 \leqslant j \leqslant n-1$, set $v_{1,j} = s_{[1,j-i+1]}s_{[j+2,n]}s_{[j-i+2,j+1]}^{-1}$. Then we have that $s_{[j-i+1,j]}v_{1,j}s_{[j-i+1,j]}^{-1} = v_{1,j-1}$. Therefore, we can show by induction that $x_1^{-1}s_{[1,n-i]}s_{[n-i+1,n]}^{-1}x_1 = x_{1,j}^{-1}v_{1,j}x_{1,j}$ for $i-1 \leqslant j \leqslant n-1$. In particular, $x_2^{-1}y_1y_2vx_2 = s_{[i+1,n]}s_{[1,i]}^{-1} = w^J$.

Moreover, $w_{n-i-k+1}^{-1}w^{-n+i+k+1}\beta_k = w_{n-i-k+1}^{-1}(-\alpha_{n-1}) = -\sum_{l=n-k}^{n}\alpha_l$. Since $n-k \in J_{n-i-k-1} - J_{n-i-k}$, $U_{\beta_k} \subset {}^{\dot{w}^{n-i-k-1}\dot{w}_{n-i-k+1}}(U_{P_{J_{n-i-k}}}^- \cap U_{J_{n-i-k-1}}^-)$. By 3.1, $(\dot{x}_1^{-1}\dot{x}_2^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}_2\dot{x}_1ut, 1)\cdot h_J \in Z_{J,w^J}$. Therefore, $Z_{J,w^J} \subset X$.

For type B_n, we have the similar results.

3.9. In 3.9 and 3.10, we assume that G is of type D_n. In this subsection, assume that $i \leqslant n-2$.

If $2 \mid i$, set $v = s_{n-i}s_{n-i+2}\cdots s_{n-2}$, $\lambda = \alpha_{n-i}^\vee + \alpha_{n-i+2}^\vee + \cdots + \alpha_{n-2}^\vee$ and $x = (s_{[n-1,n]}^{-1}s_{[n-3,n]}^{-1}\cdots s_{[n-i+1,n]}^{-1})(s_{[n-i-1,n-2]}^{-1}s_{[n-i-2,n-3]}^{-1}\cdots s_{[1,i]}^{-1})$.

If $2 \nmid i$, set $v = (s_{n-i}s_{n-i+2}\cdots s_{n-1})s_n$, $\lambda = \sum_{l=0}^{(i-3)/2}\alpha_{n-i+2l}^\vee + 1/2(\alpha_{n-1}^\vee + \alpha_n^\vee)$ and $x = (s_{[n-2,n]}^{-1}s_{[n-4,n]}^{-1}\cdots s_{[n-i+1,n]}^{-1})(s_{[n-i-1,n-2]}^{-1}s_{[n-i-2,n-3]}^{-1}\cdots s_{[1,i]}^{-1})$.

By the similar calculation to what we did for type C_{n-1}, we have that in both cases (v, λ) is admissible and $x^{-1}\lambda = \omega_i^\vee$. Moreover, by the similar argument to what we did for type C_{n-1}, we can show that $Z_{J,w^J} \subset X$.

3.10. Assume that $i = n$. Set

$$\varepsilon = \begin{cases} 1 & \text{if } 2 \mid [n/2], \\ 0 & \text{otherwise.} \end{cases}$$

If $2 \nmid n$, set $v = s_{n+\varepsilon-1}(s_1 s_3 \cdots s_{n-2})s_{n-\varepsilon}$, $x = s_{n+\varepsilon-1}(s^{-1}_{[n-3,n]} s^{-1}_{[n-5,n]} \cdots s^{-1}_{[2,n]})s_{n-1}$ and $\lambda = \frac{3}{2}\alpha^\vee_{n-\varepsilon} + \frac{1}{2}\alpha^\vee_{a+\varepsilon-1} + \sum_{j=0}^{(n-3)/2} \alpha^\vee_{2j+1}$. Then $\lambda = 2x\omega^\vee_n$ and (v, λ) is admissible. Set $y = s_2 s_4 \cdots s_{n-3}$. Then $(\dot{v}\dot{x}t, \dot{y}^{-1}\dot{x}) \cdot h_J \in X$ for some $t \in T$. By 3.3, $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$. Since $x^{-1}yvx = s_{n-1}s^{-1}_{[1,n-2]}s_n = w^J$, $Z_{J,w^J} \subset X$.

If $2 \mid n$, set $v = (s_1 s_3 \cdots s_{n-3})s_{n-\varepsilon}$, $\lambda = \alpha^\vee_{n-\varepsilon} + \sum_{j=0}^{n/2-2} \alpha^\vee_{1+2j}$ and

$$x = \begin{cases} s_2 s_4 & \text{if } n = 4, \\ s_{n-2}s_{n+\varepsilon-1}(s^{-1}_{[n-4,n]} s^{-1}_{[n-6,n]} \cdots s^{-1}_{[2,n]})s_{n-1} & \text{otherwise.} \end{cases}$$

Then $\lambda = 2x\omega^\vee_n$ and (v, λ) is admissible. Therefore, there exists $t \in T$, such that $(U, U)(\dot{v}\dot{x}t, \dot{x}) \cdot h_J \subset X$. Set $y_1 = s_2 s_4 \cdots s_{n-2}$, $y_2 = s_{n+\varepsilon-1}$ and $\beta = -(vx)^{-1}\alpha_{n+\varepsilon-1} = -\alpha_{n/2}$. By 3.3, there exists $u \in U_\beta$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. It is easy to see that $x^{-1}y_1 y_2 vx = s_{n-1}s^{-1}_{[1,n-2]}s_n = w^J$ and

$$w_2^{-1}\beta = \begin{cases} -\sum_{l=1}^{3} \alpha_l & \text{if } n = 4, \\ -\sum_{l=n/2-1}^{n-2} \alpha_l & \text{otherwise.} \end{cases}$$

Note that $J_0 = I - \{n\}$ and $J_1 = I - \{n-2, n\}$. Thus $U_\beta \subset {}^{w_2}(U_{P^-_{J_1}} \cap U^-_{J_0})$. By 3.1, $Z_{J,w^J} \subset X$.

Similarly, $Z_{I-\{i-1\}, s_n s^{-1}_{[1,n-2]} s_{n-1}} \subset X$.

3.11. Type G_2.

Set $v = s_i$, $x = w^J$ and $\lambda = \alpha^\vee_i = x\omega^\vee_i$. Then (v, λ) is admissible. Set $y = s_{3-i}$, then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Therefore, $Z_{J,w^J} \subset X$.

3.12. Type F_4.

If $i = 1$, then set $v = s_2$, $x = s_1 s_4 w^2$ and $\lambda = \alpha^\vee_2 = x\omega^\vee_1$. Thus (v, λ) is admissible. Set $y_1 = s_1 s_3$, $y_2 = s_4$ and $\beta = -(vx)^{-1}\alpha_4 = -(\alpha_2 + \alpha_3)$. Then there exists $u \in U_\beta$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1 y_2 vx = w^J$ and $w_2^{-1}\beta = -(\alpha_2 + 2\alpha_3 + \alpha_4)$. By 3.1, $Z_{J,w^J} \subset X$.

If $i = 2$, then set $v = s_1 s_3$, $x = s_2 w^2$ and $\lambda = \alpha^\vee_1 + \alpha^\vee_3 = x\omega^\vee_2$. Thus (v, λ) is admissible. Set $y = s_2 s_4$, then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

If $i = 3$, then set $v = s_2s_4$, $x = s_3w^2$ and $\lambda = 2\alpha_2^\vee + \alpha_4^\vee = x\omega_3^\vee$. Thus (v, λ) is admissible. Set $y = s_1s_3$, then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

If $i = 4$, then set $v = s_3$, $x = s_1s_4w^2$ and $\lambda = \alpha_3^\vee = x\omega_1^\vee$. Thus (v, λ) is admissible. Set $y_1 = s_2s_4$, $y_2 = s_1$ and $\beta = -(vx)^{-1}\alpha_1 = -(\alpha_2 + 2\alpha_3)$. Then there exists $u \in U_\beta$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$ and $w_2^{-1}\beta = -(\alpha_1 + 2\alpha_2 + 2\alpha_3)$. By 3.1, $Z_{J,w^J} \subset X$.

3.13. Type E_6.

If $i = 1$, then set $v = s_1s_5s_3s_6$, $x = s_1s_4s_3s_1s_6w^J$ and $\lambda = \alpha_1^\vee + 2\alpha_3^\vee + \alpha_5^\vee + 2\alpha_6^\vee = 3x\omega_1^\vee$. Thus (v, λ) is admissible. Set $y_1 = s_4$, $y_2 = s_2$ and $\beta = -(vx)^{-1}\alpha_2 = -(\alpha_3 + \alpha_4 + \alpha_5)$. Then there exists $u \in U_\beta$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$ and $w_2^{-1}\beta = -(\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)$. By 3.1, $Z_{J,w^J} \subset X$.

Similarly, $Z_{I-\{6\},s_2s_1s_3s_4s_5s_6} \subset X$.

If $i = 2$, then set $v = s_4$, $x = s_2s_3s_5s_4s_2w^J$ and $\lambda = \alpha_4^\vee = x\omega_1^\vee$. Thus (v, λ) is admissible. Set $y_1 = s_2s_3s_5$, $y_2 = s_1s_6$, $\beta_1 = -(vx)^{-1}\alpha_1 = -(\alpha_4 + \alpha_5)$ and $\beta_2 = -(vx)^{-1}\alpha_6 = -(\alpha_3 + \alpha_4)$. Then there exists $u \in U_{\beta_1}U_{\beta_2}$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$, $w_2^{-1}\beta_1 = -\sum_{l=3}^{6}\alpha_l$ and $w_2^{-1}\beta_2 = -(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)$. By 3.1, $Z_{J,w^J} \subset X$.

If $i = 3$, then set $v = s_3s_6s_1s_4s_5$, $x = s_2s_3s_4s_1s_3w^J$ and $\lambda = 2\alpha_1^\vee + \alpha_3^\vee + 3\alpha_4^\vee + 5\alpha_5^\vee + \alpha_6^\vee = 3x\omega_3^\vee$. Thus (v, λ) is admissible. Set $y = s_2$, then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

Similarly, $Z_{I-\{5\},s_2s_1s_3s_4s_6s_5} \subset X$.

If $i = 4$, then set $v = s_2s_3s_5$, $x = s_4(w^J)^2$ and $\lambda = \alpha_2^\vee + \alpha_3^\vee + 5\alpha_5^\vee = x\omega_3^\vee$. Thus (v, λ) is admissible. Set $y = s_1s_4s_6$, then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

3.14. Type E_7.

If $i = 1$, then set $v = s_4$, $x = s_3s_1s_2s_5s_4s_3s_1s_7(w^J)^2$ and $\lambda = \alpha_4^\vee = x\omega_1^\vee$. Thus (v, λ) is admissible. Set $y_1 = s_3s_2s_5$, $y_2 = s_1s_6s_7$, $\beta_1 = -(vx)^{-1}\alpha_1 = -\sum_{l=3}^{6}\alpha_l$, $\beta_2 = -(vx)^{-1}\alpha_6 = -(\alpha_4 + \alpha_5)$ and $\beta_3 = -(vx)^{-1}\alpha_7 = -(\alpha_2 + \alpha_3 + \alpha_4)$. Then there exists $u \in U_{\beta_3}U_{\beta_2}U_{\beta_1}$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$, $w_2^{-1}\beta_1 = -\alpha_4 - \sum_{l=2}^{7}\alpha_l$, $w_2^{-1}\beta_2 = -\sum_{l=2}^{6}\alpha_l$ and $w_3^{-1}(w^J)^{-1}\beta_3 = -(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6)$. By 3.1, $Z_{J,w^J} \subset X$.

If $i = 2$, then set $v = s_2s_3s_5s_7$, $x = s_4s_2s_7(w^J)^3$ and $\lambda = \alpha_2^\vee + 2\alpha_3^\vee + \alpha_5^\vee + \alpha_7^\vee = 2x\omega_2^\vee$. Thus (v, λ) is admissible. Set $y = s_1s_4s_6$. Then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

If $i = 3$, then set $v = s_2s_3s_5$, $x = s_1s_4s_3s_7(w^J)^3$ and $\lambda = \alpha_2^\vee + \alpha_3^\vee + \alpha_5^\vee = x\omega_3^\vee$. Thus (v, λ) is admissible. Set $y_1 = s_1s_4s_6$, $y_2 = s_7$ and $\beta = -(vx)^{-1}\alpha_7 = -(\alpha_4 + \alpha_5)$. Then there exists $u \in U_{\beta_3}U_{\beta_2}U_{\beta_1}$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$ and $w_2^{-1}\beta = -(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6)$. By 3.1, $Z_{J,w^J} \subset X$.

If $i = 4$, then set $v = s_1s_4s_6$, $x = s_2s_3s_5s_4(w^J)^3$ and $\lambda = \alpha_1^\vee + 2\alpha_4^\vee + \alpha_6^\vee = x\omega_4^\vee$. Thus (v, λ) is admissible. Set $y = s_2s_3s_5s_7$. Then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

If $i = 5$, then set $v = s_2s_3s_5s_7$, $x = s_4s_6s_5(w^J)^3$ and $\lambda = \alpha_2^\vee + 2\alpha_3^\vee + 3\alpha_5^\vee + \alpha_7^\vee = 2x\omega_5^\vee$. Thus (v, λ) is admissible. Set $y = s_1s_4s_6$. Then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

If $i = 6$, then set $v = s_4s_6$, $x = s_1s_5s_7s_6(w^J)^3$ and $\lambda = \alpha_4^\vee + \alpha_6^\vee = x\omega_6^\vee$. Thus (v, λ) is admissible. Set $y_1 = s_2s_3s_5s_7$, $y_2 = s_1$ and $\beta = -(vx)^{-1}\alpha_1 = -(\alpha_3 + \alpha_4 + \alpha_5)$. Then there exists $u \in U_\beta$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$ and $w_2^{-1}\beta = -\alpha_4 - \sum_{l=1}^{5} \alpha_l$. By 3.1, $Z_{J,w^J} \subset X$.

If $i = 7$, then set $v = s_2s_5s_7$, $x = s_6s_7s_4s_5s_6s_7s_1(w^J)^2$ and $\lambda = \alpha_2^\vee + \alpha_5^\vee + \alpha_7^\vee = 2x\omega_7^\vee$. Thus (v, λ) is admissible. Set $y_1 = s_4s_6$, $y_2 = s_3s_1$, $\beta_1 = -(vx)^{-1}\alpha_3 = -(\alpha_3 + \alpha_4 + \alpha_5)$ and $\beta_2 = -(vx)^{-1}\alpha_1 = -(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6)$. Then there exists $u \in U_{\beta_2}U_{\beta_1}$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$, $w_2^{-1}\beta_1 = -\alpha_4 - \sum_{l=1}^{6} \alpha_l$, $w_3^{-1}(w^J)^{-1}\beta_2 = -\alpha_4 - \sum_{l=1}^{5} \alpha_l$. By 3.1, $Z_{J,w^J} \subset X$.

3.15. Type E_8.

If $i = 1$, then set $v = s_4s_6$, $x = s_3s_1s_2s_5s_4s_3s_1s_8(w^J)^5$ and $\lambda = \alpha_4^\vee + \alpha_6^\vee = x\omega_1^\vee$. Thus (v, λ) is admissible. Set $y_1 = s_2s_3s_5s_7$, $y_2 = s_1s_8$, $\beta_1 = -(vx)^{-1}\alpha_1 = -\alpha_4 - \sum_{l=2}^{6} \alpha_l$ and $\beta_2 = -(vx)^{-1}\alpha_8 = -\sum_{l=3}^{7} \alpha_l$. Then there exists $u \in U_{\beta_2}U_{\beta_1}$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$, $w_2^{-1}\beta_1 = -\alpha_4 - \alpha_5 - \sum_{l=2}^{7} \alpha_l$ and $w_2^{-1}\beta_2 = -\alpha_4 - \sum_{l=2}^{8} \alpha_l$. By 3.1, $Z_{J,w^J} \subset X$.

If $i = 2$, then set $v = s_2s_3s_5s_7$, $x = s_4s_2s_7s_8(w^J)^6$ and $\lambda = \alpha_2^\vee + \alpha_3^\vee + \alpha_5^\vee + \alpha_7^\vee = x\omega_2^\vee$. Thus (v, λ) is admissible. Set $y = s_1s_4s_6s_8$. Then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

If $i = 3$, then set $v = s_2s_3s_5s_7$, $x = s_1s_4s_3s_7s_8(w^J)^6$ and $\lambda = \alpha_2^\vee + \alpha_3^\vee + 2\alpha_5^\vee + \alpha_7^\vee = x\omega_3^\vee$. Thus (v, λ) is admissible. Set $y = s_1s_4s_6s_8$. Then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

If $i = 4$, then set $v = s_1s_4s_6s_8$, $x = s_2s_5s_3s_4s_8(w^J)^6$ and $\lambda = \alpha_1^\vee + 3\alpha_4^\vee + 2\alpha_6^\vee + \alpha_8^\vee = x\omega_4^\vee$. Thus (v, λ) is admissible. Set $y = s_2s_3s_5s_7$. Then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

If $i = 5$, then set $v = s_2s_3s_5s_7$, $x = s_4s_6s_5(w^J)^6$ and $\lambda = \alpha_2^\vee + 2\alpha_3^\vee + 2\alpha_5^\vee + \alpha_7^\vee = x\omega_5^\vee$. Thus (v, λ) is admissible. Set $y = s_1s_4s_6s_8$. Then $(\dot{x}^{-1}\dot{y}\dot{v}\dot{x}t, 1) \cdot h_J \in X$ for some $t \in T$. Note that $x^{-1}yvx = w^J$. Thus $Z_{J,w^J} \subset X$.

If $i = 6$, then set $v = s_1s_4s_6$, $x = s_1s_5s_7s_6(w^J)^6$ and $\lambda = \alpha_1^\vee + 2\alpha_4^\vee + \alpha_6^\vee = x\omega_6^\vee$. Thus (v, λ) is admissible. Set $y_1 = s_2s_3s_5s_7$, $y_2 = s_8$ and $\beta = -(vx)^{-1}\alpha_8$. Then there exists $u \in U_\beta$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$ and $w_2^{-1}\beta = -\alpha_4 - \sum_{l=1}^{5} \alpha_l$. By 3.1, $Z_{J,w^J} \subset X$.

If $i = 7$, then set $v = s_2s_3s_5$, $x = s_6s_7s_8s_4s_5s_6s_7(w^J)^5$ and $\lambda = \alpha_2^\vee + \alpha_3^\vee + \alpha_5^\vee = x\omega_7^\vee$. Thus (v, λ) is admissible. Set $y_1 = s_1s_4s_6$, $y_2 = s_7s_8$, $\beta_1 = -(vx)^{-1}\alpha_7 = -(\alpha_3 + \alpha_4 + \alpha_5)$ and $\beta_2 = -(vx)^{-1}\alpha_8 = -(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6)$. Then there exists $u \in U_{\beta_2}U_{\beta_1}$ and

$t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$, $w_2^{-1}\beta_1 = -\alpha_4 - \sum_{l=1}^{6} \alpha_l$ and $w_3^{-1}(w^J)^{-1}\beta_2 = -\alpha_4 - \sum_{l=1}^{5} \alpha_l$. By 3.1, $Z_{J,w^J} \subset X$.

If $i = 8$, then set $v = s_4$, $x = s_1s_5s_6s_7s_8(w^J)^5$ and $\lambda = \alpha_4^\vee = x\omega_8^\vee$. Thus (v, λ) is admissible. Set $y_1 = s_5s_2s_3$, $y_2 = s_1s_6s_7s_8$, $\beta_1 = -(vx)^{-1}\alpha_1 = -\alpha_4 - \sum_{l=2}^{7} \alpha_l$, $\beta_2 = -(vx)^{-1}\alpha_6 = -(\alpha_3 + \alpha_4 + \alpha_5)$, $\beta_3 = -(vx)^{-1}\alpha_7 = w^J\beta_2$ and $\beta_4 = -(vx)^{-1}\alpha_8 = (w^J)^2\beta_2$. Then there exists $u \in U_{\beta_4}U_{\beta_3}U_{\beta_2}U_{\beta_1}$ and $t \in T$, such that $(\dot{x}^{-1}\dot{y}_1\dot{y}_2\dot{v}\dot{x}ut, 1) \cdot h_J \in X$. Note that $x^{-1}y_1y_2vx = w^J$, $w_2^{-1}\beta_1 = -\sum_{l=3}^{6} \alpha_l - \sum_{l=1}^{7} \alpha_l$, $w_2^{-1}\beta_2 = -\alpha_4 - \sum_{l=1}^{7} \alpha_l$, $w_3^{-1}(w^J)^{-1}\beta_3 = -\alpha_4 - \sum_{l=1}^{6} \alpha_l$ and $w_4^{-1}(w^J)^{-2}\beta_4 = -\alpha_4 - \sum_{l=1}^{5} \alpha_l$. By 3.1, $Z_{J,w^J} \subset X$.

4. The explicit description of $\bar{\mathcal{U}}$

4.1. We assume that G^1 is a disconnected algebraic group such that its identity component G^0 is reductive. Following [St, 9], an element $g \in G^1$ is called quasi-semisimple if $gBg^{-1} = B, gTg^{-1} = T$ for some Borel subgroup B of G^0 and some maximal torus T of B. We have the following properties.

(a) *if g is semisimple, then it is quasi-semisimple*. See [St, 7.5 & 7.6].
(b) *Let $g \in G^1$ is a quasi-semisimple element and T_1 be a maximal torus of $Z_{G^0}(g)^0$, where $Z_{G^0}(g)^0$ is the identity component of $\{x \in G^0 \mid xg = gx\}$. Then any quasi-semisimple element in gG^0* is *G^0-conjugate to some element of gT_1*. See [L4, 1.14].
(c) *g is quasi-semisimple if and only if the G^0-conjugacy class of g is closed in G^1*.

See [Spa, 1.15(f)] for the if-part, the only-if-part is due to Lusztig in an unpublished note. His proof is as follows.

Proposition(Lusztig). *Let $g \in G^1$. Let $cl_{G^0}g$ be the G^0-conjugacy class of g. Assume that $cl_{G^0}g$ is closed. Then g is quasi-semisimple.*

Proof. The proof is due to Lusztig.

By [St, 7.2], we can find a Borel subgroup B such that $gBg^{-1} = B$. Let cl_Bg be the B-conjugacy class of g. Since $cl_Bg \subset cl_{G^0}g$ and $cl_{G^0}g$ is closed, we see that the closure of cl_Bg is contained in $cl_{G^0}g$. By [Spa, 1.15(e)], the closure of cl_Bg contains a quasi-semisimple element. Hence $cl_{G^0}g$ contains a quasi-semisimple element. Hence g is quasi-semisimple. □

4.2. Let $\rho_i : G \to GL(V_i)$ be the irreducible representation of G with lowest weight $-\omega_i$ and $\bar{\rho}_i : \bar{G} \to P(\mathrm{End}(V_i))$ be the morphism induced from ρ_i (see [DS, 3.15]). Let $\mathcal{N}$ be the subvariety of $\bar{G}$ consisting of elements such that for all $i \in I$, the images under $\bar{\rho}_i$ are represented by nilpotent endomorphisms of V_i. We have the following result.

4.3. Theorem. *We have that*

$$\bar{\mathcal{U}} - \mathcal{U} = \mathcal{N} = \bigsqcup_{J \subsetneqq I} \bigsqcup_{w \in W^J, \mathrm{supp}(w)=I} Z_{J,w}.$$

Proof. By 2.11 and the results in Section 3, we have that

$$\bigsqcup_{J \subsetneqq I} \bigsqcup_{w \in W^J, \mathrm{supp}(w)=I} Z_{J,w} \subset \bar{\mathcal{U}} - \mathcal{U}.$$

For $i \in I$, let X_i be the subvariety of $P\big(\mathrm{End}(V_i)\big)$ consisting of the elements that can be represented by unipotent or nilpotent endomorphisms of V_i. Then X_i is closed in $P(\mathrm{End}(V_i))$. Thus, $\bar{\rho}_i(z) \in X_i$ for $z \in \bar{\mathcal{U}}$. Moreover, since G is simple, for any $g \in \bar{G}$, $\bar{\rho}_i(g)$ is represented by an automorphism of V_i if and only if $g \in G$. Thus if $z \in \bar{\mathcal{U}} - \mathcal{U}$, then $\bar{\rho}_i(z)$ is represented by an nilpotent endomorphism of V_i. Therefore $\bar{\mathcal{U}} - \mathcal{U} \subset \mathcal{N}$.

Assume that $w \in W^J$ with $\mathrm{supp}(w) \neq I$ and $\mathcal{N} \cap Z_{J,w} \neq \varnothing$. Let C be the closed $L_{J,w}$-stable subvariety that corresponds to $\mathcal{N} \cap Z_{J,w}$. We have seen that $\dot{w}$ is a quasi-semisimple element of $N_G(L_{J,w})$. Moreover, there exists a maximal torus T_1 in $Z_{L_{J,w}}(w)^0$ such that $T_1 \subset T$. Since C is an $L_{J,w}$-stable non-empty closed subvariety of $C_{J,w}$, $\dot{w}t \in C$ for some $t \in T_1$. Set $z = (\dot{w}t, 1) \cdot h_J$. Then $z \in \mathcal{N}$.

Since $\mathrm{supp}(w) \neq I$, there exists $i \in I$ with $i \notin \mathrm{supp}(w)$. Then $-w\omega_i = -\omega_i$. Let v be a lowest weight vector in V_i. Assume that $\bar{\rho}_i(z)$ is represented by an endomorphism A of V. Then $Av \in k^* v$. Thus $z \notin \mathcal{N}$. That is a contradiction. Therefore $\mathcal{N} \subset \bigsqcup_{J \subsetneqq I} \bigsqcup_{w \in W^J, \mathrm{supp}(w)=I} Z_{J,w}$. The theorem is proved. $\square$

Remark. Let $G = PGL_4(k)$ and $I = \{1, 2, 3\}$. Then the theorem implies that $Z_{\{1,3\}, s_2 s_1 s_3 s_2} \subset \bar{\mathcal{U}}$. By 2.5, we can see that $Z_{\{1,3\}, s_2 s_1 s_3 s_2}$ contains infinitely many G-orbits. Therefore $\bar{\mathcal{U}}$ contains infinitely many G-orbits.

4.4. Corollary. *Let $i \in I$ and $J = I - \{i\}$ and w be a Coxeter element of W with $w \in W^J$. Then $\overline{Z_{J,w}} = \bigsqcup_{K \subset J} \bigsqcup_{w' \in W^K, \mathrm{supp}(w')=I} Z_{K,w'}$.*

Proof. Note that $Z_{J,w} \subset \bar{\mathcal{U}} \cap (\bigsqcup_{K \subset J} Z_K)$. Since $\bar{\mathcal{U}}$ and $\bigsqcup_{K \subset J} Z_K$ are closed, $\overline{Z_{J,w}} \subset \bar{\mathcal{U}} \cap (\bigsqcup_{K \subset J} Z_K) = \bigsqcup_{K \subset J} \bigsqcup_{w' \in W^K, \mathrm{supp}(w')=I} Z_{K,w'}$. Therefore by 2.11, $\overline{Z_{J,w}} = \bigsqcup_{K \subset J} \bigsqcup_{w' \in W^K, \mathrm{supp}(w')=I} Z_{K,w'}$. $\square$

4.5. Let $\sigma : G \to T/W$ be the morphism which sends $g \in G$ to the W-orbit in T that contains an element in the G-conjugacy class of the semisimple part g_s. The map

σ is called Steinberg map. The fibers of σ are called Steinberg fibers. The unipotent variety is an example of Steinberg fiber. Some other interesting examples are the regular semisimple conjugacy classes of G.

Let F be a fiber of σ. It is known that $\bar{F}$ is a union of finitely many G-conjugacy classes. Let t be a representative of $\sigma(F)$ in T, then $F = G_{\text{diag}} \cdot tU$ and $\bar{F} = G_{\text{diag}} \cdot t\bar{U}$ (see [Spr2, 1.4]). It is easy to see that $t(\bar{U} - U) \subset \mathcal{N}$. Thus $\bar{F} - F = G_{\text{diag}} \cdot t(\bar{U} - U) \subset \mathcal{N}$. Therefore, if (w, λ) is admissible and $x^{-1} \cdot \lambda$ dominant, then there exists some $t' \in T$ such that $(U \times U)(\dot{w}\dot{x}t', \dot{x}) \cdot h_{I(x^{-1}\lambda)} \subset t\bar{U}$. Thus by 2.11 and the results in Section 3, $\bigsqcup_{J \subsetneq I} \bigsqcup_{w \in W^J, \text{supp}(w)=I} Z_{J,w} \subset \bar{F} - F$. Therefore, we have

$$\bar{F} - F = \mathcal{N} = \bigsqcup_{J \subsetneq I} \bigsqcup_{w \in W^J, \text{supp}(w)=I} Z_{J,w}.$$

Thus $\bar{F} - F$ is independent of the choice of the Steinberg fiber F. As a consequence, in general, $\bar{F}$ contains infinitely many G-orbits (answering a question that Springer asked in [Spr2]).

4.6. For any variety X that is defined over the finite field $\mathbf{F}_q$, we write $|X|_q$ for the number of $\mathbf{F}_q$-rational points in X.

If G is defined and split over the finite field $\mathbf{F}_q$, then for any $w \in W^J$, $|\tilde{Z}_{J,w}|_q = |G|_q q^{-l(w)}$ (see [L4, 8.20]). Thus

$$|Z_{J,w}|_q = |G|_q q^{-l(w)} (q-1)^{-|I-J|} = \left(\sum_{u \in W} q^{l(u)}\right) (q-1)^{|J|} q^{l(w_0 w)}.$$

Set $L(w) = \{i \in I \mid ws_i < w\}$. Then $w \in W^J$ if and only if $J \subset L(w_0 w)$. Moreover, if $w \neq 1$, then $L(w_0 w) \neq I$. Therefore

$$\begin{aligned}
|\bar{\mathcal{U}} - \mathcal{U}|_q &= \sum_{J \neq I} \sum_{w \in W^J, \text{supp}(w)=I} |Z_{J,w}|_q \\
&= \left(\sum_{w \in W} q^{l(w)}\right) \sum_{J \neq I} \sum_{w \in W^J, \text{supp}(w)=I} (q-1)^{|J|} q^{l(w_0 w)} \\
&= \sum_{w \in W} q^{l(w)} \sum_{\text{supp}(w)=I} \sum_{J \subset L(w_0 w)} q^{l(w_0 w)} (q-1)^{|J|} \\
&= \sum_{w \in W} q^{l(w)} \sum_{\text{supp}(w)=I} q^{l(w_0 w) + |L(w_0 w)|}.
\end{aligned}$$

Remark. Note that $|\bar{G}|_q = \sum_{w \in W} q^{l(w)} \sum_{w \in W} q^{l(w_0 w)+|L(w_0 w)|}$ (see [DP, 7.7]). Our formula for $|\bar{\mathcal{U}} - \mathcal{U}|_q$ bears some resemblance to the formula for $|\bar{G}|_q$.

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References

[Bo] N. Bourbaki, Groupes et Algèbres de Lie, Hermann, Paris, 1968 (Chapter 4–6).
[DP] C. De Concini, C. Procesi, Complete symmetric varieties, Invariant Theory (Montecatini 1982), Lecture Notes in Mathematics, vol. 996, Springer, Berlin, 1983, pp. 1–44.
[DS] C. De Concini, T.A. Springer, Compactification of symmetric varieties, Transform. Groups 4 (2–3) (1999) 273–300.
[EGA] A. Grothendieck, J. Dieudonné, Éléments de Géométrie Algébrique, Publ. Math. I.H.E.S. (1960–1967).
[L1] G. Lusztig, Total positivity in reductive groups, Lie Theory and Geometry: in Honor of Bertram Kostant, Prog. Math. 123 (1994) 531–568.
[L2] G. Lusztig, Character sheaves on disconnected groups I, Represent. Theory 7 (2003) 374–403.
[L3] G. Lusztig, Parabolic character sheaves I, Moscow Math. J 4 (2004) 153–179.
[L4] G. Lusztig, Parabolic character sheaves II, Moscow Math. J 4 (2004) 869–896.
[Spa] N. Spaltenstein, Classes unipotents et sous-groupes de Borel, Lecture Notes in Mathematics, vol. 946, Springer, New York, 1982.
[Spr1] T.A. Springer, Intersection cohomology of $B \times B$-orbits closures in group compactifications, J. Algebra 258 (2002) 71–111.
[Spr2] T.A. Springer, Some subvarieties of a group compactification, to appear.
[St] R. Steinberg, Endomorphisms of linear algebraic groups, Mem. Amer. Math. Soc. 80 (1968).
[Str] E. Strickland, A vanishing theorem for group compactifications, Math. Ann. 277 (1) (1987) 165–171.