L-Fuzzy Sets

J. A. GOGUEN*

University of California, Berkeley, California

Submitted by L. Zadeh

1. INTRODUCTION

This paper explores the foundations of, generalizes, and continues the work of Zadeh in [1] and [2]. Perhaps the most important generalization is the consideration of order structures beyond the unit interval. Because of this we have been able to develop a new point of view toward optimization problems. The significance of this work may lie more in its point of view than in any particular results. The theory is still young, and no doubt many concepts have yet to be formulated, while others have yet to take their final form. However it should now be possible to visualize the outlines of the theory.

Throughout the development of the theory of fuzzy sets, pattern classification has been a seminal influence. One reason for this is the natural feeling that probability theory is not appropriate for treating the kind of uncertainty that appears in pattern classification; this uncertainty seems to be more of an *ambiguity* than a statistical variation.

Similar difficulties arise in a wide variety of problems. It is characteristic of attempts to apply probability theory to them that it is difficult or impossible to estimate the distributions assumed to be involved, that there is uncertainty about the nature of the statistical assumptions (independence, etc.), or that certain parameters are ignored, taken as given, or found difficult to estimate. Under these circumstances, the chief use of probability theory has been to partially justify intuitively appealing procedures, to suggest procedures already found useful in statistics, or to provide some sort of insight into the nature of things. We believe that fuzzy sets should be able to do at least this much.

Let us consider some specific examples. A housewife faces a fairly typical optimization problem in her grocery shopping: she must select among all possible grocery bundles one that meets as well as possible several conflicting criteria of optimality, such as cost, nutritional value, quality, and variety. The *partial ordering* of the bundles is an intrinsic quality of this problem.

145

Consider a machine for reading typewritten characters which computes correlations with various pattern prototypes, and extracts certain "features." Actual samples of the letter "*A*" may produce a variety of values of these criteria, and some of the criteria intended to detect "*A*"'s may conflict, producing an ambiguity. There may very well be *no* way of determining whether or not some character is an "*A*". Thus the set of characters intended to be apprehended as "*A*"'s is a *fuzzy* set, a set without a well-defined boundary. The fuzziness appears to be an essential aspect of this problem.

Partial orderings of optimality and fuzzy ambiguities are characteristics of many problems. Such problems are *ill-posed* in the sense that they do not admit unique solutions; in fact, they may not have solutions at all in the usual sense. The theory of fuzzy sets studies formal properties of ill-posed problems and ill-defined sets, much as ordinary set theory does for ordinary sets.

The so-called "hard sciences," such as physics, find *crisp* (as opposed to fuzzy) relations between their observables. The appropriate methodology for the so-called "soft sciences" (biology, psychology, etc.) may involve finding fuzzy relations between variables; even the variables may be fuzzy. We develop a theory of fuzzy relations, and discuss some applications. In particular, fuzzy relations enable one to study fuzzy systems.

Fuzziness is more the rule than the exception in engineering design problems: usually there is no well-defined best solution or design; increases in speed, compactness, or efficiency are paid for by increases in cost, difficulty of service, etc. The usual way out (other than ignoring the partial ordering or applying intuition) is to pick semiarbitrary "weighting factors" for the various design parameters, and designate as best the system with the greatest total "weight." The significance and justification of this scheme are unclear, and so is the way the "weighting factors" are to be chosen; but the ways of intuition are still more unclear. Fuzzy sets can operate with the problem as posed, and clarify various operations, including weighting.

We give a preliminary definition: A *fuzzy set* is a set with a function to a transitive partially ordered set (hereafter called a poset); a fuzzy set is therefore a sort of generalized characteristic function. We habitually denote the poset by L and call the fuzzy set an *L-fuzzy* set or an *L-set*. Because of the generality of the mathematical definition, some important applications of fuzzy sets do not involve the intuitive concept of fuzziness at all.

The use of posets imparts to the theory a special character which emphasizes order theoretic statements. In order for it to make sense to ask what the maximum and minimum values of a fuzzy set are, the poset ought to be in general at least a complete lattice. Distributivity is also useful.

It may be helpful to think of fuzzy sets as nonsimply ordered utility functions. It can then be seen that the theory is related to statistical decision

problems. For example, we might think of the poset as a "decision language," that is, a space of decisions or evaluations which can be combined by the logical operations "and" and "or" (or "min" and "max").

Some problems have natural multicomponent optimality criteria as formulated but require solutions in the poset {yes, no} = {1, 0}. To proceed toward a solution, we map from one poset to the other; it is particularly nice if the map preserves order, i.e., if it is a homomorphism of order structures. For example, the weighting process described above is a homomorphism from $\mathbf{R}^n$ (with the product ordering) to the simply ordered set $\mathbf{R}$.

Products of simply ordered sets are particularly common posets in the applications, and particularly natural mathematically. Nevertheless, in certain applications more exotic lattices may be found. In the pattern classification problem, k pattern decisions and their logical combinations might constitute our decision language.

Although the topics we consider might be important for certain applications, they do not involve any great mathematical depth. This paper develops a basic language and a few elementary properties, mainly formal and algebraic, and prepares for new points of view. However, there are related topics of greater mathematical depth. These include an information theory for fuzzy sets, convex fuzzy sets, the fuzzification of various mathematical structures, and a more detailed treatment of the pattern classification problem. We hope to consider some of these in future papers.

We have not tried to distinguish between philosophical, applied mathematical, and purely mathematical passages in this paper. It is our impression that most major assertions are susceptible to all three interpretations.

We have produced a "logic of inexact quantities" within the framework of modern pure mathematics; the results concerning fuzzy sets are proved as rigorous mathematical theorems. Yet we hold that these results are for "inexact quantities." This is an assertion that pure mathematics applies to certain philosophical and practical matters and is therefore an applied mathematical statement.

It is not necessary to know particular fuzzy sets as exact mathematical functions to be able to make about them certain assertions of theoretical character which may have philosophical and/or practical significance.

Our results, attitudes, and methods might raise some questions about foundations. It is somewhat unsatisfying philosophically to ground a logic of fuzziness in a logic of exactness; it would seem to ask for an independent postulational formulation. On the other hand, our method shows that if mathematics, as we use it, is consistent, so is fuzziness, as we formulate it.

We have used the axiomatic method, in the sense that our underlying assumptions, especially about L, are abstract; it can thus be ascertained to what extent our results apply to some new problem.

The author wishes to thank Professor L. A. Zadeh for several stimulating discussions which inspired the work of this paper, and Dr. Ralph McKenzie for a suggestion on Section 6.

2. L-FUZZY SETS

The fuzzy set concept deals with situations in which there are evaluations for elements of a set X. The elements may be typewritten letters (called characters), and the evaluation of how much they look like the letter "A"; or the elements may be grocery bundles, and the evaluation their utility to the housewife (or "appeal," if one wishes to consider subjective evaluation); or the elements may be acts and the evaluation corresponding payoffs or other results.

X generally has some structure beyond that of a set, although we shall not, in this paper, make use of any special assumptions about it. For example, if X consists of all possible grocery bundles, or outputs of a factory, etc., it is customary to view $x \in X$ as a vector (called a commodity bundle in economics) whose components are real numbers designating the amounts of the various commodities involved (a negative component would have the meaning of an input in the factory example). Thus X has the structure of a vector space.

In pattern classification we are most often concerned with the output of some machine. This output is usually a vector. For example, patterns might be reduced to $n \times n$ square arrays with the intensity of each square given (perhaps on a logarithmic scale); a pattern then appears as a vector in $\mathbf{R}^{n^2}$.

If we think of speech wave forms as square-integrable functions on, say, the unit interval, then the set of such elements can be given the structure of a Hilbert space, with correlation as the inner product (more properly, the elements of the space would be equivalence classes of functions). Thus the space X for speech recognition may be considered a Hilbert space.

In practical applications, there is usually some concept of "nearness" on X, so that it is a topological space, if not a metric space.

A fuzzy set on a set X is a sort of generalized "characteristic function" on X, whose "degrees of membership" may be more general than "yes" or "no." In fact, we assume a set, from here on denoted L, of degrees of membership. In an optimization problem, L may express the degree of optimality of the choice (in X); in a classification problem, it may express the degree of membership in a pattern class; in other contexts, other terminologies will appear.

DEFINITION.   *An L-fuzzy set A on a set X is a function $A : X \to L$.*

Thus fuzzy sets are to be considered *equal* iff they are equal as functions.

If the elements of X have a name (e.g., "characters"), A will be called an L-fuzzy set of such elements (e.g., "A is an L-fuzzy set of characters"). We may drop the prefix "L-" if convenient, euphoneous, or unconfusing; or

we may drop the prefix "fuzzy"; but we will *not* drop *both* "*L*-" and "fuzzy." This applies to other fuzzy concepts as well.

We use the notation XY for the set theoretic product of X and Y; that is, $XY = \{(x, y) \mid x \in X, y \in Y\}$. $X + Y$ will denote the *disjoint* union of X and Y, and X^Y the set of all functions from Y to X. It is straightforward to verify the following set theoretic isomorphisms.

PROPOSITION 1.

$$X + Y \cong Y + X; \quad X + (Y + Z) \cong (X + Y) + Z; \quad XY \cong YX;$$

$$X(YZ) \cong (XY)Z; \quad X(Y + Z) \cong XY + XZ; \quad (X + Y)Z \cong XZ + YZ;$$

$$X^{Y+Z} \cong X^Y X^Z; \quad (XY)^Z \cong X^Z Y^Z; \quad (X^Y)^Z \cong X^{YZ}.$$

There is no associative law for exponentiation, that is, $X^{(Y^Z)} \not\cong (X^Y)^Z$, and when we write X^{Y^Z}, we shall intend the stronger form $X^{(Y^Z)}$. Since the set of all L-sets on X is L^X, the "base set" in an exponent expression will usually be L. Elements of L are "values," so an element of L^L is a "fuzzy value."

PROPOSITION 2. *L^X can be given whatever operations L has, and these operations in L^X will obey any law valid in L which extends point by point.*

For example, if a binary operation $*$ is defined on L, it induces an operation $*$ on L^X by pointwise extension: $(A * B)(x) = A(x) * B(x)$, for $A, B \in L^X$ and $x \in X$. If some law, such as associative, commutative, or idempotent holds for $*$ on L, it also extends to L^X. The cancellation law will not extend pointwise; if L is an integral domain, and X has more than one point, L^X will be a ring, but the cancellation law will fail.

This could be made completely precise by considerations of "universal algebra" (see Birkhoff [3]), but it does not seem worth introducing this complication. L^X is the same as the product $\prod_{x \in X} L_x$ of L's indexed by X, and the above construction is essentially the usual "product process" found in algebra. If L had a topological structure, L^X could be given the product topology.

L might be a semigroup, a poset, a lattice, or a Boolean σ-ring. In the next section we shall argue that the most appropriate categories for most purposes are those of complete lattices or complete lattice ordered semigroups.

We shall use the notation 2 for the special and important lattice $\{0, I\}$, where $0 < I$. If L is a lattice with zero 0 and infinity I, 2 will denote the sublattice $\{0, I\}$. The case $L = 2$ is essentially set theory, for elements of 2^X are characteristic functions of subsets of X. The case $L = J$, the closed unit interval $[0, 1]$, is the special case of fuzzy sets in the original sense of Zadeh [1].

3. The Structure of L

In this section we discuss axioms which it might be desirable for L to satisfy, and we introduce important examples which we prove do satisfy various of these axioms. In general, the structure desired varies with the context.

In most applications, for example in pattern classification, L ought to have some sort of order structure; for we wish to say "x is better than y," "x is less expensive than y," "x looks more like an 'A' than y," etc. Although it has usually been assumed that the ordering is simple, we have argued that this is often inappropriate. On the other hand, it is hard to imagine being able to get along with an ordering that is not at least transitive. Thus L is at least a (reflexive) transitive partially ordered set, or *poset*.

In many examples, L was a product of simply ordered sets, with "vectors" $a = (a_1 ,..., a_n)$, ordered by a $a \geqslant b \Leftrightarrow a_i \geqslant b_i$, $i = 1,..., n$, as elements. This ordering is called the *product ordering*, and from here on we assume all products are ordered in this way. The most important properties of a product of simply ordered sets are included in the assertion that it is a distributive lattice, and is complete if each factor is. This follows from the following easily verified results.

PROPOSITION 1. *A simply ordered set is a distributive lattice. A product of distributive lattices is distributive, and a product of complete lattices is complete.*

For some applications, this type of L seems to be too special. For example, we might want L to be a language of decisions for a pattern classification problem, allowing combinations by "and" and "or." In any case the mathematical theory of fuzzy sets does not depend upon L having any special form, but upon certain general assumptions, which we are now developing.

The question of the existence of lub and glb (which we denote $\bigvee$ and $\bigwedge$ respectively) may be somewhat illuminated by the following: Let A be an L-fuzzy set on X, and let $Y \subseteq X$. For convenience of explanation, let us agree that "$a \geqslant b$" means "a is better than b." In the kinds of problems we are considering it is important to know how well or how badly things might turn out if a choice of some $x \in Y$ is made. It is clear that we will do better than glb $\{A(x) \mid x \in Y\}$ and worse than lub $\{A(x) \mid x \in Y\}$, if these are defined; and it is clear that glb and lub are the most reasonable ways of obtaining bounds in a poset.

If a poset has a glb and lub for each nonempty subset, it is a complete lattice, and from here on we assume L has at least this structure. In some applications we may not be interested in the glb, but only the lub; the algebraic structure relevant to this case is briefly discussed later.

The lub of an indexed family a_i of elements of L will be written $\bigvee_i a_i$ or $\bigvee a_i$, and dually for the glb. We will sometimes use the notation

$$\bigvee A = \bigvee \{A(x) \mid x \in X\} \quad \text{and} \quad \bigwedge A = \bigwedge \{A(x) \mid x \in X\}.$$

If L has zero 0 and infinity I, we will use the conventions that $\bigvee \emptyset = 0$ and $\bigwedge \emptyset = I$.

If L does not have 0 and I, they can be added without loss of generality, and we hereafter assume L does. It then becomes possible for a fuzzy set to be *crisp* in the sense that it takes only the values 0 and I.

Proposition 2 of the previous section enables us to conclude that L^X also has a lattice structure, namely that given by defining

$$(A \vee B)(x) = A(x) \vee B(x) \quad \text{and} \quad (A \wedge B)(x) = A(x) \wedge B(x),$$

for $A, B \in L^X$ and $x \in X$. The associative, commutative, etc. laws for $\vee$ and $\wedge$ on L extend to L^X. The ordering of L^X is given by defining $A \leqslant B$ iff $A(x) \leqslant B(x)$ for all $x \in X$. Many special laws which hold for L, such as completeness or distributivity, will extend to L^X.

The *principle of duality* for lattices says that if a result is true of L, so is its *dual*, obtained by interchanging $\wedge$ and $\vee$, and $\leqslant$ and $\geqslant$. It is important to notice that this extends to L^X, so that we have the *Principle of Duality for Fuzzy Sets*: If a theorem or identity is true of fuzzy sets, so is its dual.

This principle is a labor saving device in proving theorems and identities.

By analogy with the terminology of the special case of set theory, $L = 2$, we may call $A \vee B$ the *union* of A and B, $A \wedge B$ the *intersection* of A and B, and read $A \leqslant B$ as "A is *contained in* B."

Another assumption about L which seems intuitively natural, and which plays an important technical role, is distributivity. Actually, the law is neither natural nor very useful in a complete lattice unless it occurs in a strong form which we call the *complete distributive law*:

$$a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i).$$

By the principle of duality, this identity is equivalent to its dual. We will call a complete lattice which satisfies the complete distributive law a *completely distributive lattice*.

To summarize the discussion so far, we assume that L is a completely distributive lattice with zero and infinity elements.

Since a lattice is in particular a commutative semigroup (by a *semigroup* we mean a set with an associative binary operation) with respect to either lattice operation, it satisfies the usual generalized (finite) commutative and associative laws (see Chapter 1 of [4]). For a complete lattice, more general

forms of these laws are valid. A particularly useful form of the commutative law is

$$\bigvee_i \bigvee_j a_{ij} = \bigvee_j \bigvee_i a_{ij} \, .$$

A very general law of Birkhoff ([3], p. 53) is valid for all complete lattices and includes the general commutative, general associative, idempotent, and above commutative laws. For each index i let Φ_i be a set of indices, let $\Phi = \bigcup_i \Phi_i$, and let a_j be a family of elements of L indexed by $j \in \Phi$. Then Birkhoff's law is

$$\bigvee_i \bigvee_{j \in \Phi_i} a_j = \bigvee_{j \in \Phi} a_j \, .$$

PROPOSITION 2.   *The complete distributive law is equivalent to the following, also called the complete distributive law:*

$$\bigvee_{i \in \Phi} a_i \wedge \bigvee_{j \in \Psi} b_j = \bigvee_{(i,j) \in \Phi\Psi} a_i \wedge b_j \, .$$

PROOF.   Obviously the law of the proposition implies that of the definition (let $\Phi = \{1\}$). Conversely,

$$\bigvee_{i \in \Phi} a_i \wedge \bigvee_{j \in \Psi} b_j = \bigvee_i \left(a_i \wedge \bigvee_j b_j \right) = \bigvee_i \left(\bigvee_j (a_i \wedge b_i) \right)$$

$$= \bigvee_{(i,j) \in \Phi\Psi} (a_i \wedge b_j),$$

where the last step uses, say, Birkhoff's law.   □

The duals are equivalent to each other and to the original law.

We now continue the discussion which initiated this section by asking whether products of complete simply ordered sets obey the complete distributive law. The usual product argument shows it suffices to establish the law for the factors. Strangely enough, this involves topological considerations. The *interval topology* of a simply ordered set is that generated by the subbasis of half-open intervals, or by subsets of the form $\{x \mid x > a\}$ and $\{x \mid x < a\}$.

LEMMA 1.   *Let L be a complete simply ordered set with the interval topology. Then a montone increasing net in L with bounded range converges to the supremum of its range.*

LEMMA 2.   *The lattice operations $\bigvee$ and $\bigwedge$ are continuous in a simply ordered set with the interval topology.*

THEOREM 1.   *A complete simply ordered set L obeys the complete distributive law.*

PROOF.   It suffices to establish the first form of the law. Let $a, b_i \in L$ for $i \in \Phi$. Let D be the directed set of all *finite* subsets of Φ, directed by inclusion. Then $b_S = \bigvee_{i \in S} b_i$ and $c_S = \bigvee_{i \in S} a \wedge b_i$ for $S \in D$ define nets in L. Lemma 1 implies $b_S \to b = \bigvee_{i \in \Phi} b_i$ and $c_S \to c = \bigvee_{i \in \Phi} a \wedge b_i$.

By Proposition 1, L is distributive, and by induction the complete distributive law holds for all *finite* index sets. Thus

$$c_S = \bigvee_{i \in S} a \wedge b_i = a \wedge \bigvee_{i \in S} b_i = a \wedge b_S .$$

$b_S \to b$ and the continuity of $\wedge$ imply that $a \wedge b_S \to a \wedge b$. Since c_S has a unique limit, we conclude that $c = a \wedge b$. $\square$

The appropriate notion of homomorphism for complete lattices is more special than that for general lattices. A mapping $\ell : L_1 \to L_2$ between complete lattices is a *complete lattice homomorphism* iff it preserves arbitrary lub and glb, that is, iff $\ell \bigvee_i a_i = \bigvee_i \ell a_i$, and dually. Complete lattices and complete lattice homomorphisms form a category (see Section 5).

The *minimax inequality*

$$\bigwedge_{i=1}^{m} \left(\bigvee_{j=1}^{n} a_{i,j} \right) \geqslant \bigvee_{j=1}^{n} \left(\bigwedge_{i=1}^{m} a_{i,j} \right)$$

holds for any lattice, and generalizes to infinite sets of indices for complete lattices.

In many applications the simply ordered sets are subintervals of the reals. For most purposes things are unchanged by applying a preliminary order-preserving transformation and adding endpoints, so that we may assume we are dealing with $J = [0, 1]$. The intervals $[-\infty, +\infty]$, $[0, +\infty]$, $[-1, 1]$, and J are especially interesting because they are closed under multiplication (in the ordinary sense of real numbers). This multiplication is very important in applications for taking percentages, forming averages, etc. The lattice and multiplication structures are interrelated in a way which will also appear in other contexts, and to the abstract study of which we now turn.

DEFINITION.   *A multiplicative lattice is a lattice M closed under an operation $*$ which satisfies the "distributive" laws, for all $a, b, c \in M$*

$$a * (b \vee c) = (a * b) \vee (a * c) \qquad and \qquad (a \vee b) * c = (a * c) \vee (b * c).$$

Identities involving $*$'s in multiplicative lattices will *not* remain identities

when $\wedge$ and $\vee$, and $\geqslant$ and $\leqslant$ are interchanged; thus the Principle of Duality is not valid. But there is a principle of left-right symmetry, as exhibited in the two "distributive" laws above.

PROPOSITION 3. *Let a, b, c be elements of a multiplicative lattice M. Then:*

(1) $a \leqslant b \Rightarrow c * a \leqslant c * b$ *and* $a * c \leqslant b * c$;

(2) $(a \wedge b) * (a \vee b) \leqslant (b * a) \wedge (a * b)$.

PROOF.

$$a \leqslant b \Rightarrow b * c = (a \vee b) * c = (a * c) \vee (b * c) \Rightarrow a * c \leqslant b * c;$$

$$(a \wedge b) * (a \vee b) = [(a \wedge b) * a] \vee [(a \wedge b) * b] \leqslant (b * a) \vee (a * b). \qquad \square$$

The structure we had in mind for the above examples is still more restrictive, and is given by the following

DEFINITION. *A complete lattice L which is also a semigroup with identity under $*$ and which satisfies the "distributive" laws,*

$$a * \bigvee_i b_i = \bigvee_i (a * b_i) \qquad and \qquad \left(\bigvee_i a_i \right) * b = \bigvee_i (a_i * b)$$

is a complete lattice ordered semigroup, abbreviated closg. The semigroup identity is called the identity of L. If there is an element 0 of L such that, for all $a \in L$

$$0 \wedge a = 0 * a = a * 0 = 0$$

it is called a zero for L. If there is an element I in L such that, for all $a \in L$

$$I \vee a = I * a = a * I = I$$

it is called an infinity for L. If $*$ is commutative, we call L a commutative closg, if L is distributive as a lattice we call it a distributive closg, and if L satisfies the complete distributive law as a lattice, we call it a completely distributive closg.

It is easy to verify that all the above-mentioned intervals are commutative completely distributive closg's with identity. All have 0 and I in the lattice sense, and J, $[-\infty, +\infty]$, and $[0, +\infty]$ have zeroes in the closg sense if we make the appropriate conventions regarding multiplications of infinities. By the usual product argument, any product of these intervals will also be a commutative completely distributive closg with identity and lattice 0 and I.

PROPOSITION 4. **A complete lattice is a closg with $\wedge$ as $*$ if it is completely distributive.**

This easily verified remark shows that there will be no loss of generality, and indeed a slight gain, if we assume that L is a completely distributive closg with lattice (but not necessarily closg) zero and infinity, and this assumption is made throughout most of the rest of this paper. It is also necessary to assume that the (lattice) infinity I is a semigroup identity.

As with multiplicative lattices, there is no principle of duality, but there is a principle of left-right symmetry for closg's. If $* = \wedge$ the principle of duality holds again, and the dual of this $*$ is $\vee$.

In all of these intervals $*$ has no *zero divisors* (that is, $a * b = 0 \Rightarrow a = 0$ or $b = 0$); however products of intervals will have zero divisors.

We point out that since L^X has the algebraic structure of L, the operation $*$ is defined for $A, B \in L^X$ by $A * B(x) = A(x) * B(x)$.

We will see later that the algebraic structure of the set of fuzzy relations (or systems) on a set is that of a closg.

DEFINITION. *A mapping $\ell : L_1 \to L_2$ between closg's is a closg homomorphism iff it preserves $\vee$, $\wedge$, and $*$.*

closg's with closg homomorphisms form a category.

To summarize, we will assume that L is a complete lattice ordered semigroup with zero 0 and semigroup identity I, which is also a lattice infinity.

Most of the results of this paper are valid under the more general assumption that L has an operation $\vee$ defined on subsets and a binary operation $*$ such that:

(1)   Birkhoff's generalized law holds for $\vee$;

(2) $*$ is associative and has a zero 0 and identity I;

(3)   the two complete distributive laws relate $\vee$ and $*$;

(4) $0 \vee a = a$ and $I \vee a = I$.

We then define a transitive reflexive order $\leqslant$ on L by $a \leqslant b \Leftrightarrow a \vee b = b$, and it is easy to see that $\vee$ is a least upper bound operator, 0 is a zero, and I is an infinity. Also $a \leqslant b \Rightarrow a * c \leqslant b * c$. For the proofs of these assertions, recall that Birkhoff's generalized law implies the general commutative, associative, and idempotent laws for $\vee$. A principle of left-right symmetry holds. Probably the L's which arise in practice will satisfy more than these minimal requirements; for example, they may be closg's.

Notice that our L's, viewed as abstract logics, are (generally) negationless, and that the order relation can be interpreted as implication. However, if L is a complete distributive lattice (with 0 and I), it is relatively pseudo-complemented (see [3], p. 147). If complemented it is, of course, a Boolean algebra. Cohen's independence results can be obtained, I understand, from Boolean algebra valued models for set theory.

We conclude this section with some additional examples. The set of all left ideals of an (associative) ring with unit is a suitable complete lattice ordered semigroup. The set of regular events (words) in a free semigroup is a (noncomplete) lattice ordered semigroup. The set of open sets of a topological space is an (upper semi-complete) semi-lattice ordered semigroup with zero and suitable identity. If one attempts to consider it to be a complete lattice by defining $\bigwedge_i U_i = $ Interior $(\bigcap_i U_i)$, the distributive law fails.

4. FUZZIFICATION

Fuzzification is the process of imparting a fuzzy structure to a definition (concept), theorem, or even a whole theory. This process is not necessarily unique, as will be seen. In a sense, most of this paper is devoted to fuzzifying set theory. A fundamental method is given by the *Principle of Fuzzification*: A fuzzy (or L-fuzzy or L-) something is an L-set of somethings (i.e., an L-fuzzy set on the set of somethings).

This is not a mathematical result; rather it is an heuristic aid in formulating and subsequently interpreting definitions. We will use the terminology it suggests whenever possible, but it must be noted that it does not apply to all cases, and even when it applies, it does not necessarily give the most appropriate fuzzification.

As an example, we formulate the terminology for fuzzy sets and subsets. Recall that an element of L^X is an L-fuzzy set on X, or more simply an *L-set on X*, or a fuzzy set on X, but *not* a set on X. By the Principle, an *L-fuzzy subset of X* is an L-set of subsets of X, that is, an element of L^{2^X}. An *L-fuzzy L-set on X* (or a fuzzy L-set, or even a fuzzy fuzzy set on X) is an element of L^{L^X}.

A somewhat analogous situation obtains for relations. An *L-relation between sets X and Y* is an element of L^{XY}. A *fuzzy L-relation* between X and Y is an element of $L^{L^{XY}}$. "Relation" with no prefix means "2-relation," so an L-fuzzy relation might mean an element of $L^{2^{XY}}$. This terminology is confusing and fortunately is not used later in this paper; but notice that the principle of fuzzification suggests the L-fuzzy 2-relation, although it is not the most natural fuzzification for many purposes. Because $2 \subseteq L$, L-fuzzy 2-relations are in particular L-fuzzy L-relations. The L-relation was first suggested by Zadeh [3] for $L = J$ and $X = Y$.

An *L-fuzzy mapping* from X to Y is an L-set of mappings from X to Y, that is, a mapping from Y^X to L, or an element L^{Y^X}. If X and Y have more struture than sets, it is appropriate to correspondingly restrict the mappings.

For example, if X and Y are vector spaces, an *L-fuzzy linear map from X to Y* is a fuzzy set of linear maps from X to Y, i.e., an element of $L^{\text{Lin}(X,Y)}$.

It is very important to notice that all fuzzy things are in particular fuzzy sets, i.e., functions from some set to L; fuzzy sets are the basic building blocks for all fuzzy objects. It therefore follows that all fuzzy things enjoy most of the same algebraic properties L does.

For example, let $\mathscr{A}$ and $\mathscr{B}$ be L-fuzzy L-sets on X. Then $\mathscr{A} \vee \mathscr{B}$ is defined by $(\mathscr{A} \vee \mathscr{B})(A) = \mathscr{A}(A) \vee \mathscr{B}(A)$, where A is an L-set on X, $\mathscr{A} \wedge \mathscr{B}$ is defined dually, and the set of all L-fuzzy L-sets on X is a lattice; it has 0, I and is completely distributive since L is, the Principle of Duality holds, etc.

Things unfuzzified or only trivially fuzzified are *crisp*; crispness is the qualitative opposite of fuzziness, although technically it is a special case.

DEFINITION. *Let L have 0, I. An L-fuzzy set on X is crisp if it takes only the values 0, I on X.*

It is always possible to specialize some fuzzy object or structure back to a (crisp) set theoretic one by setting $L = 2$. Sometimes, in order to obtain the original set theoretical model, it is also necessary to impose a restriction to singletons. For example, a fuzzy L-subset may be made conceptually equivalent to an ordinary subset by requiring it to be crisp, and non-0 only on singleton subsets.

DEFINITION. *An L-fuzzy set on X is a singleton if it takes the value 0 for all $x \in X$ except one. Let $x \in X$; then x^+ is the crisp singleton on X defined by $x^+(y) = 0$ if $x \neq y$ and $x^+(y) = I$ if $x = y$, for $y \in X$.*

For example, let $A \subseteq X$. Then A^+ is the crisp L-fuzzy subset of X which is non-0 only on A. x^+ is parallel to $\{x\}$ in ordinary set theory.

We shall not always be explicit about specializations to and analogies with ordinary set theory in the following.

$\vee$ and $\wedge$ are fuzzy fuzzy sets on X; for $\vee : A \rightsquigarrow \vee A$, where $A \in L^X$, and $\vee A \in L$.

There is no reason not to have fuzzy fuzzy fuzzy sets, L_1-fuzzy L_2-fuzzy sets, etc. Indeed, there is no end to the possible levels of fuzzification.

5. CATEGORIES AND THE COMPOSITION OF FUZZY MAPS

A *category* $\mathscr{C}$ is a collection of *objects*, such as groups, together with, for each pair of objects X, Y, a (possibly empty) set Mor (X, Y) of "maps" or *morphisms* from X to Y of the "appropriate" kind, such as group homo-

morphisms. If X is an object of $\mathscr{C}$, write $X \in \mathscr{C}$. The appropriate maps for $\mathscr{C}$ are called *$\mathscr{C}$-morphisms*. Mor (X, Y) and Mor (X', Y') are supposed to be disjoint if $(X, Y) \neq (X', Y')$. It is assumed that a map from X to Y can be *composed* with one from Y to Z to yield one from X to Z; if $f \in$ Mor (X, Y) and $g \in$ Mor (Y, Z), then their composition is denoted $g \circ f \in$ Mor (X, Z). Composition is assumed to be associative (when it makes sense for it to be), and finally, there is supposed to be, for each X in $\mathscr{C}$, an *identity* map $1_X \in$ Mor (X, X), such that for each $f \in$ Mor (X, Y), each $g \in$ Mor (Y, X) and each $Y \in \mathscr{C}$, $f \circ 1_X = f$ and $1_X \circ g = g$. If $f \in$ Mor (X, Y), write $f : X \to Y$ or $X \xrightarrow{f} Y$. To emphasize $\mathscr{C}$ write Mor$_\mathscr{C}$ (X, Y).

This concept, though very abstract and general, is not very deep. It will enable us to describe a fuzzification of whole theories in a unified manner, and it motivates some of our results.

Categories are very common in mathematics. Groups with group homomorphisms, sets with maps, lattices with lattice homomorphisms, complete lattices with complete lattice homomorphisms, closg's with closg homomorphisms, topological spaces with continuous maps, and measure spaces with measurable transformations are, for example, all categories. Lang [4] has further discussion of category theory with notation similar to ours.

DEFINITION. *Let $\mathscr{C}$ be a category, and let $X, Y \in \mathscr{C}$. Then an L-fuzzy $\mathscr{C}$-morphism from X to Y is an L-fuzzy set of $\mathscr{C}$-morphisms from X to Y, that is, an element of $L^{\mathrm{Mor}(X,Y)}$. If $F \in L^{\mathrm{Mor}(X,Y)}$, write $F : X \to Y$ or $X \xrightarrow{F} Y$. We may also call F a fuzzy $\mathscr{C}$-map.*

The above definition is in accord with the Principle of Fuzzification. If $\mathscr{C}$ is some specific category, say linear spaces and linear maps, we use a more specific name, *L-fuzzy linear maps*.

DEFINITION. *If $F : Y \to Z$ and $G : X \to Y$ are L-fuzzy $\mathscr{C}$-morphisms, their composition $F \circ G : X \to Z$ is defined, for $h \in$ Mor (X, Z) by*

$$F \circ G(h) = \bigvee_{f,g} \left\{ F(f) * G(g) \mid g : X \to Y, f : Y \to Z, \text{ and } f \circ g = h \right\}.$$

If $L = 2$, and if F and G are (crisp) singletons, $\bigvee$ becomes existential quantification, $*$ becomes and, and the above reduces to the ordinary composition. Notice particularly that for $L = J$ the two main $*$'s we allow, $\wedge$ and multiplication, agree on the sub-closg 2, and agree with logical and; thus $*$ in either case generalizes "and."

PROPOSITION 1. *Composition of fuzzy $\mathscr{C}$-maps is associative.*

PROOF. Let $X \xrightarrow{H} Y \xrightarrow{G} Z \xrightarrow{F} W$ be fuzzy $\mathscr{C}$-morphisms, and let $i \in \mathrm{Mor}\,(X, W)$. Then

$$((F \circ G) \circ H)(i) = \bigvee \left\{ \bigvee \{F(f) * G(g) \mid f \circ g = j\} * H(h) \,\middle|\, j \circ h = i \right\}$$

$$= \bigvee \left\{ \bigvee \{F(f) * G(g) * H(h) \mid f \circ g = j\} \,\middle|\, j \circ h = i \right\}$$

$$= \bigvee \{F(f) * G(g) * H(h) \mid f \circ g \circ h = i\}.$$

Similarly, $(F \circ (G \circ H))$ (i) can be shown to be equal to the same expression. $\square$

DEFINITION. *The identity L-fuzzy map on $X \in \mathscr{C}$ is the crisp singleton $(1_X)^+$ in $L^{\mathrm{Mor}(X, Y)}$. We will also denote it 1_X if this is not confusing.*

PROPOSITION. 1_X *is an identity for fuzzy $\mathscr{C}$-morphisms; that is, for every $F \in \mathrm{Mor}\,(X, Y)$, $1_Y \circ F = F \circ 1_X = F$.*

PROOF.

$$(1_Y \circ F)(h) = \bigvee \{1_Y(f) * F(g) \mid f \circ g = h\} = \bigvee \{F(g) \mid 1_Y \circ g = h\} = F(h),$$

and the other identity is proved similarly. $\square$

These two basic properties of composition are summarized by the following, for which they constitute a proof.

THEOREM 1. *Let $\mathscr{C}$ be a category. Then the objects of $\mathscr{C}$ with the L-fuzzy $\mathscr{C}$-morphisms and the above definition of composition constitute a category, which we denote by $L^{\mathscr{C}}$.*

This construction fuzzifies, in particular, set theory, but the fuzzification involves fuzzy maps and crisp sets. Ways of combining fuzzy sets and maps are discussed in later sections, and set theory is fuzzified more satisfactorily. The complete fuzzification of other categories is a difficult and important problem which we hope to discuss elsewhere.

A fuzzy $\mathscr{C}$-map is to be thought of as a fuzzy processor of objects of $\mathscr{C}$; the composition of two fuzzy maps then corresponds to processing first by one and then by the other.

In the case $*$ is $\wedge$ there is, as usual, a dual composition, defined by the dual of the above expression, which we denote by $\bar{\circ}$. It satisfies the duals of the above properties; since they are self-dual, it satisfies the conclusion of Theorem 1. Since $*$ does not distribute over $\wedge$ in general, these considerations do not apply to arbitrary closg's.

This section ends with some purely categoric theoretic results and concepts which will be of use, or at least of interest, in the sequel. Let $\mathscr{C}$ be a fixed category.

It is possible to characterize $1 - 1$ and onto maps, say for sets, in a way which makes sense in any category. However in other categories the maps which satisfy these conditions will not necessarily be the maps which are $1 - 1$ and onto in the set theoretic sense. Thus it becomes an interesting problem to determine just what they are in each given category. These maps are given new names, as follows: $f \in \mathrm{Mor}\,(X, Y)$ is a *coretraction* in $\mathscr{C}$ iff $\exists g \in \mathrm{Mor}\,(Y, X)$ such that $g \circ f = 1_X$. $f \in \mathrm{Mor}\,(X, Y)$ is a *retraction* in $\mathscr{C}$ iff $\exists g \in \mathrm{Mor}\,(Y, X)$ such that $f \circ g = 1_Y$. $f \in \mathrm{Mor}\,(X, Y)$ is an *isomorphism* in $\mathscr{C}$ iff $\exists g \in \mathrm{Mor}\,(Y, X)$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.

f is an isomorphism iff it is a coretraction and a retraction. One half of this assertion is obvious, and the other can be seen as follows: Suppose $\exists g, \bar{g}$ such that $g \circ f = 1_X$ and $f \circ \bar{g} = 1_Y$. Then

$$g = g \circ 1_Y = g \circ (f \circ \bar{g}) = (g \circ f) \circ \bar{g} = 1_X \circ \bar{g} = \bar{g}.$$

Elements of $\mathrm{Mor}\,(X, X)$ are called *endomorphisms* of X. An endomorphism which is an isomorphism is called an *automorphism*. The set of endomorphisms of X is denoted $\mathrm{End}\,(X)$ and of automorphisms $\mathrm{Aut}\,(X)$.

PROPOSITION 3. $\mathrm{End}\,(X)$ *is a semigroup with identity and* $\mathrm{Aut}\,(X)$ *is a group.*

PROOF.   The composition of any two endomorphisms is defined and is another endomorphism. By definition of category, composition is associative and there is an identity in $\mathrm{Mor}\,(X, X)$. Thus $\mathrm{End}\,(X)$ is a semigroup with identity. To check that $\mathrm{Aut}\,(X) \subseteq \mathrm{End}\,(X)$ is a group, it suffices to note that each automorphism of X has an inverse, by definition of isomorphism, and that said inverse is also an automorphism.   □

If for $f \in \mathrm{Mor}\,(X, Y)$ there exists some $g \in \mathrm{Mor}\,(Y, X)$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$, then f is *invertible* and g is an *inverse* of f; we may write $g = f^{-1}$. We then have the following

PROPOSITION 4. f *is invertible iff* f *is an isomorphism. The inverse is unique if it exists.* f *invertible* $\Rightarrow f^{-1}$ *invertible and* $(f^{-1})^{-1} = f$. *Furthermore, if* f *and* g *are invertible and* $f \circ g$ *is defined, then* $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

PROOF.   The first assertion is a translation of definitions. The second follows from the same argument used to prove that f is an isomorphism iff f is a coretraction and a retraction, for of two inverses $g, \bar{g}$ to f, each will be both a left and a right inverse. The third assertion follows immediately from

the definition. The last is proved by noticing that $(f \circ g)(g^{-1} \circ f^{-1}) = 1$ and $(g^{-1} \circ f^{-1}) \circ (f \circ g) = 1$, by using associativity. $\square$

A *subcategory* of a category $\mathscr{C}$ is a category which has as objects and morphisms subcollections of those of $\mathscr{C}$.

The notions of direct sum and direct product also can be formulated so as to make sense in any category. Not all categories will have direct sums and products.

Since we deal with both order and category concepts, it is interesting to notice that a poset L can be regarded as a category, as follows: the objects are the elements of L; and for $a, b \in L$, Mor (a, b) is some one point set, say $\{\emptyset\}$, if $a = b$, and Mor $(a, b) = \emptyset$ otherwise. Associativity corresponds to transitivity and the existence of identities to reflexivity. L has direct sums and products as a category iff it is a complete lattice as a poset.

6. Relations

The importance of relations is almost self-evident. Science is, in a sense, the discovery of relations between observables. Zadeh [5] has shown the study of relations to be equivalent to the general study of systems (a system is a relation between an "input" space and an "output" space). In addition, relations play a central role in the purely mathematical development of our theory.

Difficulties arise in the so-called "soft" sciences because the relations involved do not appear to be "hard," as they are, say, in classical physics. A thoroughgoing application of probability theory has relieved many difficulties, but it is clear that others remain. We suggest that further difficulties might be cleared up through a systematic exploitation of fuzziness.

Introspection will show that one's concept of the relation "x is better that y" for x and y cars, novels, or politicians, is "fuzzy" in some sense. We suggest that the sense discussed here may be appropriate for problems of biology, psychology, engineering design, and economics.

Recall that an L-relation R is an element of L^{XY}, and an L-fuzzy L-relation is a function $\mathscr{R} : L^{XY} \to L$. Let $R(X, Y; L)$ denote the set of L-relations from X to Y, and $\mathscr{R}(X, Y; L)$ the set of fuzzy L-relations from X to Y.

In system theory a fuzzy L-relation corresponds to a fuzzy class of fuzzy systems (see Zadeh [2]). This section develops an algebraic theory or calculus of fuzzy relations, which is also an algebraic theory or calculus of fuzzy systems.

If R is an L-relation from Y to Z and S is an L-relation from X to Y, the composition $R \circ S$ of R and S is defined by

$$(R \circ S)(x, z) = \bigvee_y \{S(x, y) * R(y, z)\}.$$

PROPOSITION 1. *Composition of L-relations is associative.*

PROOF.

$$[(R \circ S) \circ T](x, w) = \bigvee_{z} \left\{ \bigvee_{y} \{R(x, y) * S(y, z)\} * T(z, w) \right\}$$

$$= \bigvee_{y, z} \{R(x, y) * S(y, z) * T(z, w)\}$$

$$= \bigvee_{y} \left\{ R(x, y) * \bigvee_{z} \{S(y, z) * T(z, w)\} \right\}$$

$$= [R \circ (S \circ T)](x, w). \qquad \square$$

Let X be a set. Then E_X denotes the (crisp) identity relation of X defined by $E_X(x, x') = 0$ if $x \neq x'$, and $E_X(x, x) = I$. We may write E for E_X.

PROPOSITION 2. *The E's are identities; i.e., if R is an L-relation from X to Y, $E_Y \circ R = R \circ E_X = R$.*

PROOF. $E_Y \circ R(x, y) = \bigvee_{y'} \{R(x, y') * E_Y(y', y)\} = R(x, y)$, and similarly for the other. $\square$

These two propositions show that there is a category $R(L)$ whose objects are sets, and whose morphisms are L-relations; that is, for sets X, Y, Mor $(X, Y) = R(X, Y; L)$.

An L-relation R from X to Y is *functional* iff for every $x \in X$ there is a unique $y \in Y$ such that $R(x, y) > 0$. Then a crisp functional relation is an ordinary function. The above construction can be repeated using functional relations as morphisms, obtaining a category which specializes to the category of sets and maps for $L = 2$.

If $*$ is $\wedge$, the construction can be carried out for the dual composition $\bar{\circ}$ of relations.

By the general fuzzification process of the previous section, $L^{R(L)}$ is also a category, whose objects are sets and whose morphisms are fuzzy L-relations; that is, for X, Y sets

$$\text{Mor}(X, Y) = L^{R(X, Y; L)} = \mathscr{R}(X, Y; L).$$

The identity in $L^{R(L)}$ is $(E_X)^+$ which we write as $\mathscr{E}_X$. *Composition*, to be explicit, is defined, for $\mathscr{R} \in \mathscr{R}(Y, Z; L)$, $\mathscr{S} \in \mathscr{R}(X, Y; L)$, and $T \in R(X, Z; L)$, by

$$(\mathscr{R} \circ \mathscr{S})(T) = \bigvee_{R, S} \{\mathscr{R}(R) * \mathscr{S}(S) \mid R \circ S = T\},$$

and is associative. We will write $\mathscr{R}(L)$ for $L^{R(L)}$.

Actually, the algebraic structure of these collections of relations is still richer. Both $R(X, Y; L)$ and $\mathscr{R}(X, Y; L)$ are of the form L^S, for some set S (S is XY and $R(X, Y; L)$ respectively), and therefore are, in particular, L-sets. Thus they take on the algebraic structure of L, as usual.

In particular, they are posets, and it makes sense to consider the following *alternate* definition of composition for fuzzy L-relations

$$(\mathscr{R} \circ \mathscr{S})(T) = \bigvee \{\mathscr{R}(R) * \mathscr{S}(S) \mid R \circ S \leqslant T\}.$$

It is easy to verify that there is a category whose objects are sets, whose morphisms are fuzzy L-relations, and whose composition is defined in this way.

An L-fuzzy L-relation from X to Y is *functional* if it is non-0 only on functional L-relations. Then there is a sub-category of $\mathscr{R}(L)$ whose morphisms are the functional fuzzy L-relations.

Composition and the lattice operations in $R(X, Y; L)$ and $\mathscr{R}(X, Y; L)$ are related as follows:

PROPOSITION 3A. *Let* $R, R_i, T \in R(Y, Z; L)$, $S, S_i, U \in R(X, Y; L)$. *Then*

(1) *If 0 is the relation identically* 0, $0 \circ R = R \circ 0 = 0$;

(2) $R \leqslant T \Rightarrow R \circ S \leqslant T \circ S$, *and* $S \leqslant U \Rightarrow R \circ S \leqslant R \circ U$;

(3) $R \circ \bigvee_i S_i = \bigvee_i (R \circ S_i)$, *and* $(\bigvee_i R_i) \circ S = \bigvee_i (R_i \circ S)$;

(4) $R \circ \bigwedge_i S_i \leqslant \bigwedge_i (R \circ S_i)$, *and* $(\bigwedge_i R_i) \circ S \leqslant \bigwedge_i (R_i \circ S)$, for $* = \wedge$.

PROOF. The first and second assertions are essentially obvious. The two assertions in each of (3) and (4) are symmetric, so it suffices to prove one of each. This is done as follows:

$$\left(R \circ \bigvee_i S_i\right)(x, z) = \bigvee_y \left\{\left[\bigvee_i S_i(x, y)\right] * R(y, z)\right\}$$

$$= \bigvee_i \bigvee_y \{S_i(x, y) * R(y, z)\} = \bigvee_i (R \circ S_i)(x, z);$$

$$\left(R \circ \bigwedge_i S_i\right)(x, z) = \bigvee_y \left\{\left[\bigwedge_i S_i(x, y)\right] * R(y, z)\right\}$$

$$= \bigvee_y \bigwedge_i \{S_i(x, y) * R(y, z)\}$$

$$\leqslant \bigwedge_i \bigvee_y \{S_i(x, y) * R(y, z)\} = \bigwedge_i (R \circ S_i)(y, z). \quad \square$$

PROPOSITION 3B. *Exactly the same results as in Proposition 3A hold for* $\mathscr{R}, \mathscr{R}_i, \mathscr{T} \in \mathscr{R}(Y, Z; L)$ *and* $\mathscr{S}, \mathscr{S}_i, \mathscr{U} \in \mathscr{R}(X, Y; L)$, *with either definition of composition.*

The *converse* of $R \in R(X, Y; L)$ is the L-relation $\check{R} \in R(Y, X; L)$ defined by $\check{R}(y, x) = R(x, y)$. The *converse* of $\mathscr{R} \in \mathscr{R}(X, Y; L)$ is the fuzzy L-relation $\check{\mathscr{R}} \in \mathscr{R}(Y, X; L)$ defined by $\check{\mathscr{R}}(R) = \mathscr{R}(\check{R})$.

PROPOSITION 4.   *Let* $R, R_i \in R(X, Y; L)$ *and* $S \in R(Y, Z; L)$. *Then*:

(1) $(S \circ R)^{\vee} = \check{R} \circ \check{S}$;

(2) $(\bigvee_i R_i)^{\vee} = \bigvee_i \check{R}_i$, *and dually*;

(3) $(\check{R})^{\vee} = R$.

Furthermore the same results hold for $\mathscr{R}$, $\mathscr{R}_i$, $\mathscr{S}$ *(with either composition)*.

PROOF.

$$\left(\bigvee_i R_i\right)^{\cup}(y, x) = \left(\bigvee_i R_i\right)(x, y) = \bigvee_i (R_i(x, y))$$

$$= \bigvee_i (\check{R}_i(y, x)) = \left(\bigvee_i \check{R}_i\right)(y, x).$$

$$(\mathscr{R} \circ \mathscr{S})^{\cup}(T) = \bigvee \{\mathscr{R}(R) * \mathscr{S}(S) \mid R \circ S \leqslant \check{T}\}$$

$$= \bigvee \{\mathscr{R}(R) * \mathscr{S}(S) \mid \check{S} \circ \check{R} \leqslant T\} = (\mathscr{S} \circ \check{\mathscr{R}})(T).$$

The other assertions are either very easy or similar to the above.   □
    Since $R(L)$ and $\mathscr{R}(L)$ are categories, the abstract notion of inverse, as defined in the previous section, becomes available as a notion of inverse for relations. To be more concrete, an L-relation R is *invertible* iff $\exists$ an L-relation S such that $R \circ S = E$ and $S \circ R = E$; then S is an inverse to R, and we write $R^{-1} = S$. Similarly for an L-fuzzy L-relation $\mathscr{R}$. The general category results on inverses (Proposition 4 of Section 5) give us the following

PROPOSITION 5.   *The inverse L-relation is unique if it exists. If R is invertible, so is R^{-1}, and $(R^{-1})^{-1} = R$. If R and S are invertible and $R \circ S$ is defined, then $R \circ S$ is invertible and $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$. And similarly for fuzzy L-relations.*
    It would seem that invertible L-relations and fuzzy L-relations must be very special and are probably not very fuzzy. Furthermore, the similarity of the formal properties of converse and inverse, and the facts for $L = 2$ suggest that there should be a strong relationship between converse and inverse. These conjectures are proved below provided $*$ has no zero divisors in L. Theorem 1A fails if there are: if $L = J^2$, $* = \wedge$, $(1, 0) = u$, $(0, 1) = v$, $X = Y = \{1, 2\}$, and R is defined by $R(i, j) = u$ if $i \neq j$ and $R(i, i) = v$ for $i, j \in \{1, 2\}$, then $R^{-1} = R$, but R is not crisp.

THEOREM 1A. *If* $*$ *has no zero divisors, the invertible L-relations are the set isomorphisms (i.e., the crisp* $1 - 1$ *onto set mappings).*

PROOF. Let $R \in R(X, Y; L)$, $S \in R(Y, X; L)$ such that $S \circ R = E_X$ and $R \circ S = E_Y$, i.e., such that $\bigvee_y \{R(x', y) * S(y, x)\} = E_X(x', x)$ and $V_x \{S(y, x) * R(x, y')\} = E_Y(y, y')$. These imply:

(1) $\forall x' \neq x \forall y\ R(x', y) * S(y, x) = 0$ and

(2) $\forall x \bigvee_y \{R(x, y) * S(y, x)\} = I$; and

(3) $\forall y \neq y' \forall x\ R(x, y') * S(y, x) = 0$ and

(4) $\forall y \bigvee_x \{S(y, x) * R(x, y)\} = I$.

Now suppose R is not crisp, i.e., suppose $\exists x_0, y_0$ such that

$$R(x_0, y_0) = a \neq 0, I.$$

By (1)

$$\forall x \neq x_0 R(x_0, y_0) * S(y_0, x) = 0,$$

i.e., $a * S(y_0, x) = 0$. Since $a < I$, by (4) there must $\exists$ an $\tilde{x} \neq x_0$ such that $S(y_0, \tilde{x}) * R(\tilde{x}, y_0) \neq 0$. Then $a * [S(y_0, \tilde{x}) * R(\tilde{x}, y_0)] \neq 0$ since there are no zero divisors. But also $[a * S(y_0, \tilde{x})] * R(\tilde{x}, y_0) = 0$, contradicting associativity. Thus R is crisp, and by symmetry, so is S.

Then (2) translates to

$$\forall x \exists y : R(x, y) = S(y, x) = I,$$

and (4) to

$$\forall y \exists x : R(x, y) = S(y, x) = I.$$

Now suppose $R(x, y) = S(y, x) = I$, and also $R(x, y') = I$ for some $y \neq y'$. Then (3) $\Rightarrow R(x, y') * S(y, x) = 0$, and one of $R(x, y')$, $S(y, x)$ is 0, again a contradiction. Thus ($\exists |$ means $\exists$ a unique)

$$\forall x \exists | y : R(x, y) = S(y, x) = I,$$

and by symmetry

$$\forall y \exists | x : R(x, y) = S(y, x) = I,$$

and of course $R(x, y) = S(y, x) = 0$ otherwise, so that R and S are functions and isomorphisms, as well as crisp, and $R^{-1} = S$. □

THEOREM 1B. *The invertible fuzzy L-relations in* $\mathscr{R}(L)$ *are of the form* R^+, *where R is an invertible L-relation; inverse agrees with converse when it exists.*

PROOF. Suppose $\mathscr{R} \in \mathscr{R}(X, Y; L)$, $\mathscr{S} \in \mathscr{R}(Y, X; L)$ such that $\mathscr{R} \circ \mathscr{S} = \mathscr{E}_Y$ and $\mathscr{S} \circ \mathscr{R} = \mathscr{E}_X$. Then: (1) if $R \circ S \neq E_X$, $\mathscr{R}(R) * \mathscr{S}(S) = 0$; (2) $\exists R, S$ such that $R \circ S = E_X$ and $\mathscr{R}(R) * \mathscr{S}(S) > 0$; (3) if $S \circ R \neq E_Y$, $\mathscr{R}(R) * \mathscr{S}(S) = 0$; (4) $\exists R, S$ such that $S \circ R = E_Y$ and $\mathscr{R}(R) * \mathscr{S}(S) > 0$.

Let R, S be as in (2). Then $\mathscr{R}(R) * \mathscr{S}(S) > 0$, so that $S \circ R = E_Y$ by (3). Thus R and S are inverses and R is invertible.

Let S be as above and suppose $\exists R'$ such that $\mathscr{R}(R') > 0$. Then $\mathscr{R} \circ \mathscr{S}(R' \circ S) > 0$, so $R' \circ S = E_X$. Also $\mathscr{S} \circ \mathscr{R}(S \circ R') > 0$, so $S \circ R' = E_Y$. By uniqueness of the inverse of S, $R = R'$.

Thus $\mathscr{R}$ is I on R and 0 elsewhere, that is, $\mathscr{R} = R^+$, and symmetrically for $\mathscr{S}$, S. $\square$

If we use the *alternate* definition of composition (with $R \circ S \leqslant T$ instead of $R \circ S = T$) there are no invertible fuzzy L-relations, for it is impossible that $\mathscr{R} \circ \mathscr{S} = \mathscr{E}$, since $F \geqslant E \Rightarrow \mathscr{R} \circ \mathscr{S}(F) \geqslant \mathscr{R} \circ \mathscr{S}(E) = \mathscr{E}(E) = I$, but we are supposed to have $E \neq F \Rightarrow \mathscr{R} \circ \mathscr{S}(F) = 0$.

In particular, we have determined Aut (X) if L has no zero divisors; it is essentially the same as in the category of sets. Even if L has no zero divisors, coretractions and retractions are not necessarily crisp.

The conjecture that inverse and converse agree is true in full generality.

THEOREM 2. *The inverse of an L-relation (when it exists) equals the converse.*

PROOF. Say $\mathscr{R}$ is invertible and $\mathscr{R}^{-1} = \mathscr{S}$. Then

$$S \circ R(x, \bar{x}) = \bigvee_y R(x, y) * S(y, \bar{x}) = 0 \qquad \text{if} \qquad x \neq \bar{x},$$

and

$$R \circ S(y, \bar{y}) = \bigvee_x S(y, x) * R(x, \bar{y}) = 0 \qquad \text{if} \qquad y \neq \bar{y}.$$

Now $R = R \circ S \circ R$, so we obtain

$$R(\bar{x}, \bar{y}) = \bigvee_{x,y} R(\bar{x}, y) * S(y, x) * R(x, \bar{y}) = R(\bar{x}, \bar{y}) * S(\bar{y}, \bar{x}) * R(\bar{x}, \bar{y}),$$

the last step because $R(\bar{x}, y) * S(y, x) * R(x, \bar{y}) = 0$ unless $y = \bar{y}$ and $x = \bar{x}$, by the observations above. Therefore by (1) of Proposition 3 of Section 3,

$$R(\bar{x}, \bar{y}) \leqslant S(\bar{y}, \bar{x}),$$

and by symmetry $S(\bar{y}, \bar{x}) \leqslant R(\bar{x}, \bar{y})$. Therefore $R(\bar{x}, \bar{y}) = S(\bar{y}, \bar{x})$, for all $\bar{x}, \bar{y}$; i.e. $S = \check{R}$. $\square$

In the previous section we showed that the set of endomorphisms of an object in a category is a semigroup with identity (Proposition 3). Thus $R(X, X; L)$ and $\mathscr{R}(X, X; L)$, which we hereafter denote by $R(X; L)$ and

$\mathscr{R}(X;L)$ respectively, are semigroups with identity. Combining this with earlier results in this section (Proposition 3), we see that we have already proved the following:

THEOREM 3. $R(X;L)$ and $\mathscr{R}(X;L)$ are completely distributive closg's with identity, zero, and lattice theoretic infinity, when we use composition.

This is in addition, of course, to the closg structures they have already inherited from L, using $*$ as multiplication. It now follows ([3], p. 202) that they are residuated lattices (with respect to composition).

An L-relation R on X is symmetric iff $R = \breve{R}$, and is transitive iff $R \circ R \leqslant R$. Similarly for fuzzy L-relations.

PROPOSITION 6. The intersection (i.e., infimum) of symmetric L-relations is symmetric, and of transitive L-relations is transitive; the union (i.e., supremum) of symmetric L-relations is symmetric. And similarly for fuzzy L-relations.

The assumption $* = \bigwedge$ is required for the results on transitivity.

PROOF. Let $R_i \in R(X; L)$ such that $R_i = \breve{R}_i$. Then

$$\left(\bigvee_i R_i \right)^{\vee} = \bigvee_i \breve{R}_i = \bigvee_i R_i \,,$$

and dually. Now let $R_i \in R(X;L)$ such that $R_i \circ R_i \leqslant R_i$. Then

$$\bigwedge_i R_i \circ \bigwedge_i R_i \leqslant \bigwedge_i \left(R_i \circ \bigwedge_j R_j \right) \leqslant \bigwedge_i (R_i \circ R_i) \leqslant \bigwedge_i R_i \,.$$

And similarly for $\mathscr{R}(X;L)$. □

COROLLARY. Every $R \in R(X;L)$ has a least symmetric L-relation containing it, a least transitive L-relation containing it, and a greatest symmetric L-relation contained in it. Similarly for $\mathbf{R}(X;L)$.

PROOF. All assertions are proved similarly to the first, which goes as follows: Let Q be the set of all symmetric L-relations S such that $R \leqslant S$. $Q \neq \emptyset$ since $I \in Q$. Then $S_0 = \bigwedge \{S \mid S \in Q\}$ is the desired least symmetric relation $\geqslant R$. □

The same argument will establish that every relation is contained in (i.e., $\leqslant$) a unique least relation which is both transitive and symmetric.

7. IMAGES

In this section we define and discuss properties of various images. The direct image is a generalization of Zadeh's concept of shadow [1]; using it,

we fuzzify set theory in a way which involves fuzzy maps and fuzzy sets. The various images also have important phenomenological interpretations.

Let p denote a fixed single point, or (ambiguously) the singelton $\{p\}$ containing p (context will indicate which is intended).

If A is an L-relation from p to X, we will let A also denote the L-set on X which assigns $A(p, x)$ to $x \in X$; conversely, if A is an L-set on X, we let A also denote the L-relation from p to X defined by $A(p, x) = A(x)$. This correspondence is equivalent to the isomorphism $L^{pX} \cong L^X$. It is also equivalent to a correspondence between fuzzy sets and those fuzzy systems which produce a fuzzy output whenever their one input p is "pushed."

Similarly L-relations from Y to p are in $1 - 1$ correspondence with L-sets on Y; $B(y, p) \leftrightarrow B(y)$. And relations R from p to p are in $1 - 1$ correspondence with elements of L, since $L^{pp} \cong L$.

Let R be an L-relation from X to Y, and let A be an L-set on X regarded as an L-relation from p to X. Then the composition $R \circ A$ is an L-relation from p to Y, which may be regarded as an L-set on Y. We call this set the *direct R-image of A* or the *R-image of A*, or *the image of A under R*, and we denote it by $R(A)$ or $R \circ A$.

If we think of R as a system and A as an input, then the image $R(A)$ is a fuzzy output, assigning evaluations or degrees of membership, etc., to the point outputs $y \in Y$. The study of relations of the previous section enables us to immediately deduce a number of properties of the direct image.

PROPOSITION 1. *Let A, B and A_i be L-sets on X and let R, S and R_i be L-relations from X to Y. Then:*

(1) $A \leqslant B \Rightarrow R(A) \leqslant R(B)$;

(2) $R(\bigvee_i A_i) = \bigvee_i R(A_i)$;

(3) $R(\bigvee_i A_i) \leqslant \bigvee_i R(A_i)$;

(4) $R(0) = 0$;

(5) $R \leqslant S \Rightarrow R(A) \leqslant S(A)$;

(6) $[\bigvee_i R_i](A) = \bigvee_i [R_i(A)]$;

(7) $[\bigwedge_i R_i](A) \leqslant \bigwedge_i [R_i(A)]$, for $* = \wedge$;

(8) $0(A) = 0$.

There is also the following useful consequence of the associativity of the composition of relations:

PROPOSITION 2. *Let R be an L-relation from Y to Z, S an L-relation from X to Y, and A an L-set of X. Then*

$$(R \circ S)(A) = R(S(A)).$$

If R is an L-relation from X to Y and B is an L-set on Y, the composition $B \circ R$ defines an L-set of X, which we call the *converse R-image of B*, or the *converse image of B under R*, and denote by $\check{R}(B)$. This terminology is justified by noticing that the converse image is actually $(\check{R} \circ \check{B})^{\smile}$, and $^{\smile}$ makes little difference when one set is p. The results on converses in the previous section allow the properties of Propositions 1 and 2 of this section to be asserted for the converse image.

If $L = 2$, these results become the familiar assertions usually made of set theoretic mappings (converse image is then usually called inverse image).

Let R be an L-relation from X to Y, A an L-set on X, and B an L-set on Y. Then the *R-image of A and B* is the element of L given by the composition $B \circ R \circ A$ (if you prefer, it is $B \circ R \circ A(p,p)$). It measures the extent to which A and B are related under R. Properties parallel to the above hold for this *double image*.

In fact, direct and converse images are special cases of the double image: $\check{R}(A)(y) = y^{+} \circ R \circ A$; and $R(B)(x) = B \circ R \circ x^{+}$.

We now give the promised fuzzification of set theory. Let the objects of the category $S(L)$ be L-sets $A : X \to L$, and let the morphisms from $A : X \to L$ to $B : Y \to L$ be the triples (A, B, R), where R is an L-relation from X to Y such that $R(A) \leqslant B$; that is,

$$\text{Mor}\,(A, B) = \{(A, B, R) \mid R(A) \leqslant B\}.$$

Let $(A, B, R) \in \text{Mor}\,(A, B)$ and let $(B, C, S) \in \text{Mor}\,(B, C)$. Then $(B, C, S) \circ (A, B, R)$ is defined to be $(A, C, S \circ R)$, and is seen to be an element of $\text{Mor}\,(A, C)$ by the associativity property of Proposition 2. The required category associativity is verified by reference to associativity of composition of relations, and it is clear that $(A, A, 1_X) \in \text{Mor}\,(A, A)$ is an identity for $A : X \to L$. We call $S(L)$ the *category of L-sets and L-relations*. The subcategory $F(L)$ whose morphisms involve only functional relations is the category of fuzzy sets and fuzzy functions.

These considerations generalize to fuzzy L-relations and fuzzy L-sets.

Let $\mathscr{R}$ be a fuzzy L-relation from X to Y, and let $\mathscr{A}$ be a fuzzy L-set on X, regarded as a fuzzy L-relation from p to X. Then the composition $\mathscr{R} \circ \mathscr{A}$ is a fuzzy L-relation from p to Y which may be regarded as a fuzzy L-set on Y. As such, it is denoted $\mathscr{R}(\mathscr{A})$, and called the *(direct) $\mathscr{R}$-image of $\mathscr{A}$*, or *(direct) image of $\mathscr{A}$ under $\mathscr{R}$*. To be quite specific, we have

$$\mathscr{R}(\mathscr{A})(B) = \bigvee\{\mathscr{R}(R) * \mathscr{A}(A) \mid R(A) = B\},$$

where B is an L-set on Y. Properties (1)–(8) and associativity hold (we could also use the alternate composition).

If $\mathscr{B}$ is a fuzzy L-set on Y, the composition $\mathscr{B} \circ \mathscr{R}$ defines a fuzzy L-set on X, called the *converse image of $\mathscr{B}$ under $\mathscr{R}$* and denoted $\check{\mathscr{R}}(\mathscr{B})$. Properties (1)-(8) and associativity hold again.

Given $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{R}$, the double image $\mathscr{B} \circ \mathscr{R} \circ \mathscr{A}$ is an L-fuzzy L-relation from p to p, that is, an element of L^L, or a "fuzzy value."

Given $\mathscr{R}$ and $\mathscr{A}$, $B \rightsquigarrow B^+ \circ \mathscr{R} \circ \mathscr{A}$ gives a map from L^Y to L^L. Similarly consider $y \rightsquigarrow y^{++} \circ \mathscr{R} \circ \mathscr{A}$. If $* = \wedge$, one could use dual composition and get dual properties. If L were complemented, one could season definitions with occasional complements, obtaining appropriately complemented properties.

There is a category $\mathscr{S}(L)$ with objects $\mathscr{A}$ and morphisms defined by $(\mathscr{A}, \mathscr{B}, \mathscr{R}) \in \mathrm{Mor}\,(\mathscr{A}, \mathscr{B}) \Leftrightarrow \mathscr{R}(\mathscr{A}) \leqslant \mathscr{B}$, and there is a subcategory $\mathscr{F}(L)$ involving only functional relations.

PROPOSITION 3.   *Let A be an L-set on X, S a functional L-relation from Y to X, and R its converse. Then*

$$S \circ R(A) \leqslant A.$$

PROOF.

$$S \circ R \circ A(x) = \bigvee_{y,\bar{x}} \{A(\bar{x}) * R(\bar{x}, y) * S(y, x)\}$$

and

$$A(\bar{x}) * R(\bar{x}, y) * S(y, x) \leqslant A(x) \qquad \forall x, \bar{x}, y. \qquad \square$$

That is $R \circ \check{R} \circ A \leqslant A$ if R is functional. If $* = \wedge$ the dual is $R \circ\!\!\!\circ \check{R} \circ\!\!\!\circ A \geqslant A$. Thus we have proved the following estimate (originally due to Zadeh [4]):

PROPOSITION 4.   *Let A be an L-set on X, let R_i be functional L-relations from Y to X, and let $* = \wedge$. Then*

$$\bigvee_i R_i \circ \check{R}_i \circ A \leqslant A \leqslant \bigwedge_i R_i \,\hat{\circ}\, \check{R}_i \,\hat{\circ}\, A.$$

Exactly the same results hold for fuzzy L-set $\mathscr{A}$ and fuzzy L-relations $\mathscr{R}$, $\mathscr{S}$, $\mathscr{R}_i$.

Let R and S be functional L-relations from X to Y. If $R(A) = S(A)$ for all L-sets A on X, or even all crisp singletons A on X, then $R = S$. Similarly if $B \circ R \circ A = B \circ S \circ A$ for all crisp singletons B, A, $R = S$.

8.   DECISIONS AND MAPPINGS BETWEEN LATTICES

This section sketches an algebraic theory of decisions under uncertainty which is fuzzy rather than statistical. We give a few elementary results and

some general ideas. Most of it also applies to crisp decision processes. We do not treat the problem of optimizing decision processes.

While a fuzzy set $A : X \to L$ may constitute a very adequate description of a problem, the interesting thing in practice is to reach a decision or solution. A *decision function* on a set X is a mapping $d : X \to L_0$, and we call L_0 a *decision language*; that is, d is an L_0-fuzzy set on X. This kind of fuzzy set has little to do with the intuitive conception of fuzziness, but it fits into our framework very nicely. Often the appropriate X is L^Y for some other set Y.

It appears natural to assume L_0 is at least a complete lattice with 0 and I. If d decides "yes" or "no," $L_0 = 2$. If d ranks alternatives in X, L_0 is simply ordered. L_0 could be a free distributive lattice or Boolean algebra generated by a set D of basic decisions, or, if nothing else seems appropriate, a *discrete decision language*, with Hasse diagram of the form
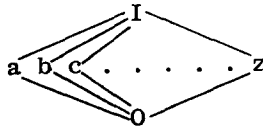
even though it is not distributive. The inverse image of $a \in L_0$ under d is the subset of X for which the decision a is rendered. In many cases the decisions 0 and I will be "reject" decisions.

The most natural way to go from a function $A : X \to L$ to a function $d : X \to L_0$ is via a function $\ell : L \to L_0$. Thus we are interested in mappings between lattices in this section.

A map ℓ from one closg L_1 to another L_2 could be of many kinds, for example: a set map; a poset homomorphism; a map preserving $\vee$ or $\wedge$; a lattice homomorphism; a complete lattice homomorphism; a closg homomorphism; and for any of these it could preserve 0 and/or I. This (partially ordered) hierarchy of homomorphisms gives an algebraic classification scheme for decision processes. It can be slightly extended by considering preservation of $\vee$, $\wedge$, and $*$ separately.

A poset homomorphism preserves $\leqslant$. Not every poset homomorphism preserves $\vee$ or $\wedge$, and some preserve one but not the other. But if a map preserves $\vee$ or $\wedge$, it is easily seen to be a poset homomorphism. If it preserves both $\vee$ and $\wedge$, it is (by definition) a lattice homomorphism. If both lattices are simply ordered, ℓ is a poset homomorphism iff it is a lattice homomorphism.

The mappings which arise in practice tend to be rather low in the hierarchy, although the higher ones tend to have the nicest algebraic properties. We consider some examples.

Let L be simply ordered, and define $t : L \to 2$ by $t(b) = 0$ if $b < a$ and $t(b) = I$ if $b \geqslant a$. t is called the *a-threshold mapping*, and it is a complete

lattice homomorphism, preserving I and preserving 0 except in the trivial case where $a = 0$. If $L = J$, t is a closg homomorphism. This is about as nice a map (algebraically speaking) as one is likely to meet (excepting isomorphisms).

t as above induces a homomorphism of the same kind $t^n : L^n \to 2^n$, or more generally $t^X : L^X \to 2^X$, for any set X, by the usual componentwise process; one could even use different t's for each component.

An interesting example occurs in pattern classification. Suppose a pattern is represented by an N-vector (say $N = n^2$ as in Section 2) of real numbers. Then the threshold transformation t^N reduces patterns to binary N-vectors in a particularly nice way.

We may view an element of J^n as a J-fuzzy set on $X = \{0, 1,..., n-1\}$; denote $\{0, 1,..., n-1\}$ by n. Let $B \in J^n$, and consider the mapping from J^n to $\mathbf{R}$ defined by $A \rightsquigarrow \Sigma_i B(i) * A(i)$ for $* =$ multiplication. We could, without loss of order-theoretic generality, normalize B so that $\Sigma_i(B_i) \leqslant 1$; then $w : J^n \to J$. w is a *weighting* of the components of A, as in Section 1. It is a poset homomorphism which preserves neither $\vee$ nor $\wedge$. w can also be viewed as computing the *correlation* of $A \in J^n$ with B. For infinite index sets X summation could be replaced by integration. We could also use the closg's $\mathbf{R}$ or 2 instead of J. Component-wise extension yields poset homomorphisms of the form $W = (w_0 ,..., w_{k-1}) : J^n \to J^k$.

A mapping $W : J^N \to J^k$ of the form described above might occur as an intermediate step of a pattern classification machine, in which w_i computes the correlation of $A \in J^N$ with some pattern prototype $B_i \in J^N$.

Let $A \in L^X$, and consider the mapping $m : L^X \to L$ defined by

$$A \overset{m}{\rightsquigarrow} \bigvee \{A(x) \mid x \in X\}.$$

Then m preserves $\vee$, and $m(A * B) \leqslant m(A) * m(B)$; this holds in particular for $* = \wedge$, even if L has a given $*$ different from $\wedge$.

In the pattern classification machine which we have been constructing, m would tell what the maximum correlation with a stored prototype B_i was, and to some extent would measure the effectiveness of the decision scheme which assigned A to the class with a representative of which it was maximally correlated. Let $d(A)$ denote this class, one of $1,..., k$, or say one of $(k = 26)$ $a,..., z \in L_0$, where L_0 has the Hasse diagram given earlier in this section.

Then the mappings $\ell = m \circ w$ (or $\ell = m \circ w \circ t$) : $J^N \to J$ and $d : J^N \to L_0$ are crisp functional systems which describe the operation of this machine. The first is order preserving. The practically important questions of course relate to such quantities as error rate, which we have agreed not to discuss here.

We now inquire how the various fuzzy structures we have defined in previous sections behave under closg homomorphisms. The typical result

is that a fuzzy structure behaves "functorially", in the sense of category theory.

A *covariant functor* T from a category $\mathscr{C}$ to a category $\mathscr{C}'$ is an assignment of an object $T(A)$ in $\mathscr{C}'$ to each object A in C, and an assignment of a morphism $T(f) : T(A) \to T(B)$ in $\mathscr{C}'$ to each morphism $f : A \to B$ in $\mathscr{C}$ in such a way that $T(1_A) = 1_{T(A)}$ and $T(f \circ g) = T(f) \circ T(g)$. We will write $T : \mathscr{C} \to \mathscr{C}'$.

A covariant functor T induces a map from $\mathrm{Mor}_{\mathscr{C}} (A, B)$ to $\mathrm{Mor}_{\mathscr{C}'}(T(A), T(B))$, and it follows immediately from the definition that T induces a semigroup homomorphism from $\mathrm{End}\,(A)$ to $\mathrm{End}\,(T(A))$, and a group homomorphism from $\mathrm{Aut}\,(A)$ to $\mathrm{Aut}\,(T(A))$. Also, if $f : A \to B$ is an isomorphism in $\mathscr{C}$ then $T(f) : T(A) \to T(B)$ is an isomorphism in $\mathscr{C}'$.

Let $\mathscr{C}$ be a category, and let $\ell : L_1 \to L_2$ be a closg homomorphism. Then ℓ induces a functor $\ell^{\mathscr{C}} : L_1^{\mathscr{C}} \to L_2^{\mathscr{C}}$ defined by $\ell^{\mathscr{C}}(A) = A$ for A an object of $\mathscr{C}$, and $\ell^{\mathscr{C}}(F)(f) = \ell \circ F(f)$, for $F \in L_1^{\mathrm{Mor}(A,B)}$ and $f \in \mathrm{Mor}\,(A, B)$, so that $\ell^{\mathscr{C}}(F) \in L_2^{\mathrm{Mor}(A,B)}$. Generally speaking we will write ℓF for $\ell^{\mathscr{C}}(F)$. It is easy to verify that $\ell(F \circ G) = \ell F \circ \ell G$ and $\ell(1_A) = 1_A$ so that we indeed have a functor.

In the category $R(L)$ of sets and L-relations, $\ell : L_1 \to L_2$ induces a functor $R(\ell)$ taking each object to itself and the L_1-relation $R : XY \to L_1$ to the L_2-relation $\ell R : XY \to L_2$.

It is easy to see that the map $\ell : R(X, Y; L_1) \to R(X, Y; L_2)$ defined above is a complete lattice homomorphism.

We already know that $\ell : R(X; L_1) \to R(X; L_2)$ is a semigroup homomorphism, since it comes from a functor from $R(L_1)$ to $R(L_2)$, and since it is a complete lattice homomorphism, it is a closg homomorphism.

Given, as usual, $\ell : L_1 \to L_2$, we define $\ell : \mathscr{R}(X, Y; L_1) \to \mathscr{R}(X, Y; L_2)$ by $\ell(\mathscr{R})(\bar{R}) = \ell \bigvee \{\mathscr{R}(R) \mid \ell R = \bar{R}\}$, for $\mathscr{R} \in \mathscr{R}(X, Y; L_1)$, $R \in R(X, Y; L_1)$, and $\bar{R} \in R(X, Y; L_2)$. This defines a lattice homomorphism. Then ℓ gives a functor $\mathscr{R}(\ell) : \mathscr{R}(L_1) \to \mathscr{R}(L_2)$ mapping each set to itself and mapping morphisms as above.

The situation for categories with fuzzy sets and morphisms is basically the same. Let $\ell : L_1 \to L_2$ be a homomorphism. Then we define a covariant functor from $S(L_1)$ to $S(L_2)$ by mapping the object $A : X \to L_1$ to the object $\ell A : X \to L_2$, and mapping the morphism $R : A \to B$, for $R \in R(X, Y; L_1)$, to $\ell R : \ell A \to \ell B$, $\ell R \in R(X, Y; L_2)$. One has only to check that

$$R(A) \leqslant B \Rightarrow \ell R(\ell A) \leqslant \ell B,$$

and this is easy.

The situation for $\mathscr{S}(L)$ is analogous. We define $\ell \mathscr{A}$ by

$$\ell \mathscr{A}(\bar{A}) = \bigvee \{\ell \mathscr{A}(A) \mid \ell A = \bar{A}\}$$

and proceed as above.

The same results are obtained in the same way if we restrict to functional relations, or if, for $* = \wedge$, we use dual composition.

One can think of $L^{\mathscr{C}}$ and $\ell^{\mathscr{C}}$ as defining a covariant functor from the category of lattices to the "category of categories." Strictly speaking, there is no such category (Would it be a member of itself?), but all the axioms of a category are satisfied when we take $\text{Mor}\,(\mathscr{C}, \mathscr{C}')$ to be all covariant functors $T : \mathscr{C} \to \mathscr{C}'$. Then mapping L to $L^{\mathscr{C}}$ and $\ell : L_1 \to L_2$ to $\ell^{\mathscr{C}} : L_1^{\mathscr{C}} \to L_2^{\mathscr{C}}$ satisfies the axioms for a covariant functor between these categories. The best way to view such an assertion is probably as a mathematical metaphor.

It is generally possible to view these fuzzy structures as bifunctors, contravariant in one variable, and covariant in the lattice variable.

It actually suffices for ℓ to preserve $\vee$ and $*$ for all the above to be true. Furthermore, if $\ell \bigvee_i a_i \leqslant \bigvee_i \ell a_i$ and $\ell(a * b) \leqslant \ell a * \ell b$, we get, for example, $\ell(R \circ S) \leqslant \ell(R) \circ \ell(S)$, rather than equality, and ℓ is no longer quite a functor.

REFERENCES

1. L. A. ZADEH. Fuzzy sets. *Information Control* 8 (1965), 338-353.
2. L. A. ZADEH. Fuzzy Sets and Systems. "Proceedings of the Symposium on System Theory," Polytechnic Press of the Polytechnic Institute of Brooklyn, New York, 1965.
3. G. D. BIRKHOFF. "Lattice Theory," Vol. XXV, 2nd ed. AMS Colloquium Publications, Providence, 1948.
4. S. LANG. "Algebra." Addison-Wesley, Reading, Massachusetts, 1965.
5. L. A. ZADEH AND C. A. DROESER. "Linear System Theory." McGraw-Hill, New York, 1963.
6. R. BELLMAN, R. KALABA, AND L. A. ZADEH. Abstraction and pattern classification. *J. Math. Anal. Appl.* 13 (1966), 1-7.
7. L. A. ZADEH. Shadows of fuzzy sets. *Probl. Transmission of Information* 2 (1966), 37-44 (in Russian).