Approximate and pseudo-amenability of various classes of Banach algebras

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Abstract

We continue the investigation of notions of approximate amenability that were introduced in work of the second and third authors together with R.J. Loy. It is shown that every boundedly approximately contractible Banach algebra has a bounded approximate identity, and that the Fourier algebra of the free group on two generators is not operator approximately amenable. Further examples are obtained of $\ell^1$-semigroup algebras which are approximately amenable but not amenable; using these, we show that bounded approximate contractibility need not imply sequential approximate amenability. Results are also given for Segal algebras on locally compact groups, and algebras of $p$-pseudo-functions on discrete groups.

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1. Introduction

In this article we continue the investigation of various notions of approximate amenability, initiated in [11] and continued in papers by various authors: see, for instance, [5, 14, 21, 25]. Most of this paper is taken up with consideration of certain classes of Banach algebras, such as Fourier algebras, Segal subalgebras of $L^1(G)$, certain $\ell^1$-semigroup algebras and the algebras $PF_p(\Gamma)$ — where $\Gamma$ is a discrete group — and the problem of determining when such algebras are approximately amenable or pseudo-amenable (the definitions will be given below).

The contents of the paper are as follows. After establishing some background definitions and notation in Section 2, we discuss some results for general Banach algebras. If $A$ is a Banach algebra with an approximate diagonal, the forced unitization $A^#$ need not possess an approximate diagonal (for example we may take $A$ to be $\ell^1$ with pointwise multiplication and apply [5, Theorem 4.1]). On the other hand, it follows from [14, Proposition 3.2] that if $A$ has a bounded approximate identity and an approximate diagonal, then $A^#$ also has an approximate diagonal. We present a partial extension of this result to the case of multiplier-bounded approximate diagonals: namely, we show that if $A$ has a central b.a.i. and a multiplier-bounded approximate diagonal, then so does $A^#$.

One outstanding basic question in this area is the following: does every approximately amenable algebra have a bounded approximate identity? Although we are not able to resolve this here, we obtain some general results (Section 3) showing that slightly stronger notions of approximate amenability guarantee the existence of a bounded approximate identity. As a consequence we are able to show that several classes of Banach algebras, which might plausibly be pseudo-amenable, cannot be boundedly approximately amenable: these include the Schatten classes $S_p(H)$ for $1 \leqslant p < \infty$ and $H$ a Hilbert space, and all proper Segal algebras on locally compact groups.

A related argument shows that for any infinite compact metric space $X$, the Lipschitz algebra $\text{lip}_\alpha(X)$, $0 < \alpha \leqslant 1$, is not boundedly approximately contractible.

The last four sections are largely independent of each other and can be read interchangeably. Section 4 resolves a question from [13], by showing that the Fourier algebra of $A(\mathbb{F}_2)$ is not (operator) approximately amenable. The proof uses direct manipulation of norm estimates, which rely on the “rapid decay” estimates known for $\mathbb{F}_2 \times \mathbb{F}_2$. It then follows from known restriction theorems that whenever a locally compact group $G$ contains $\mathbb{F}_2$ as a closed subgroup, $A(G)$ is not (operator) approximately amenable.

Section 5 collects some results on approximate notions of amenability for Segal algebras on locally compact groups. It is observed that Feichtinger’s Segal algebra on an infinite compact abelian group is not approximately amenable. We also show that if $S(G)$ is pseudo-contractible for some Segal subalgebra $S(G) \subseteq L^1(G)$, then $G$ must be compact. It is also shown that whenever $G$ is an SIN-group, every Segal subalgebra $S(G) \subseteq L^1(G)$ is approximately permanently weakly amenable: our proof uses the recent solution by Losert to the derivation problem for group algebras [23].

Section 6 is devoted to the $\ell^1$-convolution algebras of totally ordered sets: when the sets are infinite, these algebras are never amenable. We show that these algebras are always boundedly approximately contractible (and in particular are boundedly approximately amenable), but need not be sequentially approximately amenable. This strengthens the observation in [12] that the convolution algebra $\ell^1(\Omega, \lambda)$ is boundedly approximately contractible but not sequentially approximately contractible.

Finally, in Section 7 we consider the algebras of $p$-pseudo-functions on discrete groups. As a special case of the results in this section, we show that if $\Gamma$ is a discrete non-amenable group
then its reduced $C^*$-algebra is not approximately amenable. This gives some evidence for the tentative conjecture that all approximately amenable $C^*$-algebras are automatically amenable. Further evidence is provided by the fact that not only are the $C^*$-algebras $B(H)$ and $\prod_n M_n(\mathbb{C})$ not amenable, but they are not even approximately amenable — this observation appears to be due to Ozawa, see the remark after Definition 1.2 in [24].

2. Preliminaries

2.1. Definitions and notation

Throughout, if $A$ is a Banach algebra we shall write $A^#$ for the forced unitization of $A$. The adjoined identity element will usually be denoted by 1 unless stated otherwise.

We will frequently use $\pi$ to denote the bounded linear map $A \hat{\otimes} A \to A$ that is specified by $\pi(a \otimes b) = ab$ ($a, b \in A$).

Recall that a Banach algebra $A$ is said to be approximately amenable if for every $A$-bimodule $X$ and every bounded derivation $D : A \to X^*$ there exists a net $(D_\alpha)$ of inner derivations such that $\lim_\alpha D_\alpha(a) = D(a)$ for all $a \in A$.

$A$ is said to be:

- boundedly approximately amenable if the net $(D_\alpha)$ can always be taken to be bounded (in the usual norm of $L(A, X^*)$);
- sequentially approximately amenable if the net $(D_\alpha)$ can always be taken to be a sequence.

By the uniform boundedness principle (or a more direct Baire category argument) one sees that sequential approximate amenability implies bounded approximate amenability. The converse is not in general true, as will be shown in Section 6 by combining Theorems 6.1 and 6.4.

A Banach algebra $A$ is approximately contractible if for every continuous derivation $D : A \to X$, where $X$ is a Banach $A$-bimodule, there exists a net $(D_\alpha)$ of inner derivations such that $\lim_i D_i(a) = D(a)$ for all $a \in A$. The corresponding variants of bounded and sequential approximate contractibility are defined in analogous fashion to the corresponding notions of approximate amenability.

Remark. It is shown in [12, Theorem 2.1] that the concepts of approximate contractibility and approximate amenability are in fact equivalent. However, this is not true for the corresponding sequential variants, and remains unknown (at present) for the bounded variants.

It has proved very useful in the classical theory of amenability to have characterizations in terms of virtual diagonals or approximate diagonals. In much of this paper we shall work with approximate diagonals rather than nets of derivations. To fix terminology we recall the following definition.

Definition 2.1. Let $A$ be a Banach algebra. An approximate diagonal for $A$ is a net $(M_i)$ in $A \hat{\otimes} A$ such that, for each $a \in A$,

$$aM_i - M_ia \to 0 \quad \text{and} \quad a\pi(M_i) \to a.$$
We say that the approximate diagonal \((M_i)\) is \textit{multiplier-bounded} if there exists a constant \(K\) such that for all \(a \in A\) and all \(i\), each of
\[
\|aM_i - M_ia\|, \quad \|a\pi(M_i) - a\| \quad \text{and} \quad \|\pi(M_i)a - a\| \tag{2.1}
\]
is bounded by \(K\|a\|\).

The following equivalence is easily verified.

**Proposition 2.2.** A Banach algebra \(A\) is boundedly approximately contractible if and only if \(A^\#\) has a multiplier-bounded approximate diagonal.

We shall also make brief use of the notions of pseudo-amenability and pseudo-contractibility. For convenience we recall the relevant definitions from [14].

**Definition 2.3.** Let \(A\) be a Banach algebra. We say that \(A\) is \textit{pseudo-amenable} if it has an approximate diagonal, and \textit{pseudo-contractible} if it has an approximate diagonal \((M_i)\) which satisfies \(aM_i = M_ia\) for all \(a \in A\) and all \(i\).

2.2. Basics

**Proposition 2.4.** Let \(S\) be one of the following classes of Banach algebras: approximately amenable, approximately contractible, sequentially approximately amenable, sequentially approximately contractible, boundedly approximately amenable, boundedly approximately contractible.

Let \(A\) be a Banach algebra. Then \(A \in S\) if and only if \(A^\# \in S\).

**Proof.** The case of approximate amenability is given by [11, Proposition 2.4], and in fact the proofs for all the other cases follow the same argument. The key points are that (i) every derivation from \(A\) can be extended to a derivation from \(A^\#\), such that the extended derivation is inner if and only if the original one was; (ii) if \(D\) is a derivation from \(A^\#\) to an \(A\)-bimodule \(X\), and \(e\) denotes the identity of \(A^\#\), then there is an inner derivation \(D_1: A^\# \to X\) such that \((D - D_1)(e) = 0\).

**Remark.** Note that the proofs of “\(A\) approximately contractible \(\iff\) \(A^\#\) approximately contractible” and “\(A\) approximately amenable \(\iff\) \(A^\#\) approximately amenable” do not rely on the fact that approximate contractibility and approximate amenability are equivalent.

**Theorem 2.5.** Let \(A\) be a boundedly approximately contractible Banach algebra. Then there exist a constant \(C > 0\) and nets \((M_i)\) in \(A \otimes A\) and nets \((F_i), (G_i)\) in \(A\) such that

(i) \(\pi(M_i) = F_i + G_i\);
(ii) \(aF_i \to a\) for all \(a \in A\);
(iii) \(\|aF_i\| \leq C\|a\|\) for all \(a \in A\) and all \(i\);
(iv) \(G_ia \to a\) for all \(a \in A\);
(v) \(\|G_ia\| \leq C\|a\|\) for all \(a \in A\) and all \(i\);
(vi) \(aM_i - M_ia - a \otimes G_i + F_i \otimes a \to 0\) for all \(a \in A\);
(vii) \(\|aM_i - M_ia - a \otimes G_i + F_i \otimes a\| \leq C\|a\|\) for all \(a \in A\) and all \(i\).
For the sake of completeness we give the proof.

**Proof.** Regard $A^\# \hat{\otimes} A^\#$ as an $A$-bimodule in the usual way. Let $K$ be the kernel of the product map $A^\# \hat{\otimes} A^\# \to A^\#$ and let $D : A \to K$ be the derivation defined by $D(a) = a \otimes 1 - 1 \otimes a$.

Since $A$ is boundedly approximately contractible, there exists a net $(u_i)$ in $K$ such that

$$C := \sup_i \sup_{\|a\| \leq 1} \|au_i - u_ia\| < \infty$$

and $au_i - u_ia \to D(a)$ for all $a \in A$.

Identifying $A^\# \hat{\otimes} A^\#$ with the $\ell^1$-direct sum $A \hat{\otimes} A \oplus 1 \oplus 1 \otimes A \oplus \mathbb{C}1 \otimes 1$, we may write each $u_i$ in the form $u_i = (-M_i) \oplus (F_i \otimes 1) \oplus (1 \otimes G_i)$ for some $M_i \in A \hat{\otimes} A$ and some $F_i, G_i \in A$. We shall show that the nets $(M_i)$, $(F_i)$ and $(G_i)$ have the required properties.

First, note that since $u_i \in K$ for all $i$ we must have

$$0 = \pi(u_i) = -\pi(M_i) + F_i + G_i$$

for all $i$.

Next, since

$$au_i - u_ia = (-aM_i + M_ia + a \otimes G_i - F_i \otimes a) \oplus aF_i \otimes 1 \oplus (-1 \otimes G_i a),$$

where the left-hand side is bounded in norm by $C\|a\|$ for all $a$, we have $\|aF_i\| \leq C\|a\|$, $\|G_ia\| \leq C\|a\|$ and $\|aM_i - M_ia - a \otimes G_i + F_i \otimes a\| \leq C\|a\|$ for all $i$ and all $a$.

Finally, for each $a$ in $A$ we have

$$a \otimes 1 - 1 \otimes a = D(a)$$

and matching up terms we conclude that

$$a = \lim_i aF_i = \lim_i G_ia$$

as required. \[\square\]

**Remark.** It follows from this that every boundedly approximately contractible Banach algebra has a multiplier-bounded right approximate identity and a multiplier-bounded left approximate identity. We shall use this later, in Section 3.

Let $\kappa$ denote the canonical embedding of $A$ into $A^{**}$. We have the following analogue of Theorem 2.5.

**Theorem 2.6.** Let $A$ be a boundedly approximately amenable Banach algebra. Then there exist a constant $C > 0$ and nets $(M_i)$ in $(A \hat{\otimes} A)^{**}$ and nets $(F_i)$, $(G_i)$ in $A^{**}$ such that
\[ \pi(M_i) = F_i + G_i; \]
\[ a F_i \to \kappa(a) \text{ for all } a \in A; \]
\[ \|a F_i\| \leq C \|a\| \text{ for all } a \in A \text{ and all } i; \]
\[ G_i a \to \kappa(a) \text{ for all } a \in A; \]
\[ \|G_i a\| \leq C \|a\| \text{ for all } a \in A \text{ and all } i; \]
\[ a M_i - M_i a - a \otimes G_i + F_i \otimes a \to 0 \text{ for all } a \in A; \]
\[ \|a M_i - M_i a - a \otimes G_i + F_i \otimes a\| \leq C \|a\| \text{ for all } a \in A \text{ and all } i. \]

We omit the proof: the argument exactly follows the one for Theorem 2.5.

2.3. Two lemmas using approximate diagonals

We record some lemmas here which will be used later. Both are natural adaptations of routine arguments from the setting of amenable Banach algebras.

**Lemma 2.7.** Let \( B \) be a unital Banach algebra with identity element \( 1 \), \( A \subseteq B \) a closed subalgebra that contains \( 1 \), and suppose that there exists a tracial continuous functional \( \tau \) on \( A \) such that \( \tau(1) = 1 \). If \( A \) is pseudo-amenable, then there exists a net \((\psi_\alpha)\) in \( B^* \) such that \( \psi_\alpha(1) \to 1 \) and

\[ \sup_{b \in B, \|b\| \leq 1} |\psi_\alpha(ab - ba)| \to 0 \text{ for any } a \in A. \]

Note that by a trivial rescaling, the net \((\psi_\alpha)\) in the conclusion of our lemma can be chosen such that \( \psi_\alpha(1) = 1 \) for all \( \alpha \). However, the formally weaker property \( \psi_\alpha(1) \to 1 \) will suffice for our intended application.

**Proof.** Let \((u_\alpha)\) be an approximate diagonal for \( A \): note that since \( A \) has an identity element \( 1 \), \( \pi(u_\alpha) \to 1 \). For each \( \alpha \) we may write \( u_\alpha = \sum_i c_\alpha^i \otimes d_\alpha^i \), where \( c_\alpha^i, d_\alpha^i \in A \) for all \( i \) and \( \sum_i \|c_\alpha^i\| \|d_\alpha^i\| < \infty \). Let \( \tilde{\tau} \in B^* \) be any bounded extension of \( \tau \) to a functional on \( B \), and define

\[ \psi_\alpha(S) = \tilde{\tau} \left( \sum_i d_\alpha^i S c_\alpha^i \right) \quad (S \in B). \]

Then since \( \tau \) is a trace on \( A \),

\[ \psi_\alpha(1) = \tau \left( \sum_i d_\alpha^i c_\alpha^i \right) = \tau \left( \sum_i c_\alpha^i d_\alpha^i \right) = \tau \pi(u_\alpha) \to \tau(1) \]

and by hypothesis \( \tau(1) = 1 \).

For fixed \( b \in B \), define a functional \( \phi_b \in (A \otimes A)^* \) by

\[ \phi_b(x \otimes y) = \tilde{\tau}(ybx) \quad (x, y \in A). \]

By definition of the projective tensor norm, we have \( \|\phi_b\| \leq \|\tilde{\tau}\| \|b\| \).
Now for each \( a \in A \), the tracial property of \( \tau \) gives
\[
\phi_b(u_a a) = \phi_b(\sum_i c_i^\alpha \otimes d_i^\alpha a) = \tilde{\tau}(\sum_i d_i^\alpha abc_i^\alpha) = \psi_\alpha(ab)
\]
and
\[
\phi_b(a u_a) = \phi_b(\sum_i ac_i^\alpha \otimes d_i^\alpha) = \tilde{\tau}(\sum_i d_i^\alpha bac_i^\alpha) = \psi_\alpha(ba).
\]
Therefore
\[
\sup_{b \in B, \|b\| \leq 1} |\psi_\alpha(ab - ba)| = \sup_{b \in B, \|b\| \leq 1} |\phi_b(a u_a - u_a a)| \leq \|\tilde{\tau}\| \|au_a - u_a a\| \to 0
\]
for each \( a \in A \). Thus \((\psi_\alpha)\) has the required properties.

Our second lemma will be needed for the proof of Theorem 6.4. It says, loosely, that the Gelfand transforms of an approximate diagonal must converge pointwise to the indicator function of the diagonal of the character space.

Lemma 2.8. Let \( A \) be a Banach algebra with non-empty character space \( \Phi_A \), and suppose \( A \) has a (two-sided) bounded approximate identity. If \( A \) is approximately amenable, then there exists a net \((\Delta_\alpha)\) in \((A \hat{\otimes} A)^{**}\) with the following properties:

(i) \( \lim_{\alpha} \langle \Delta_\alpha, \varphi \otimes \chi \rangle = 0 \) for any \( \varphi, \chi \in \Phi_A \) with \( \varphi \neq \chi \);

(ii) \( \langle \Delta_\alpha, \varphi \otimes \varphi \rangle = 1 \) for all \( \alpha \) and any \( \varphi \in \Phi_A \).

Moreover, if \( A \) is sequentially approximately amenable, we can take \((\Delta_\alpha)\) to be a sequence.

Proof of Lemma 2.8. We shall only prove the statement in the case where \( A \) is sequentially approximately amenable (the case where we merely assume \( A \) to be approximately amenable is completely analogous).

Thus, suppose \( A \) has a bounded approximate identity and is sequentially approximately amenable. Let \( E \) be any weak* -limit point in \( A^{**} \) of the bounded approximate identity of \( A \), so that \( a E = E a = \kappa(a) \in A^{**} \), \( \kappa \) denoting the canonical embedding of \( A \) in its bidual.

Let \( \pi : A \hat{\otimes} A \to A \) be the product map and let \( K = \ker \pi \); this is a sub-\( A \)-bimodule of \( A \hat{\otimes} A \). Define a bounded derivation \( D : A \to K^{**} \) by \( D(a) = a \otimes E - E \otimes a \) (\( a \in A \)). Since \( A \) is sequentially approximately amenable there exists a sequence \((u_n)\) in \( K^{**} \) such that \( a \cdot u_n - u_n \cdot a \to D(a) \) for all \( a \in A \).
Define $\Delta_n = E \otimes E - u_n \in (A \hat{\otimes} A)^{**}$. We have

$$\langle \Delta_n, \varphi \otimes \varphi \rangle = \langle \varphi, E \rangle^2 - \langle u_n, \varphi \otimes \varphi \rangle = 1 - \langle u_n, \pi^*(\varphi) \rangle = 1 - \langle \pi^{**}(u_n), \varphi \rangle = 1$$

for all $n$. Moreover, if $\varphi$ and $\chi$ are distinct characters on $A$, then there exists $a \in A$ with $\varphi(a) \neq \chi(a)$. Then

$$\langle a \cdot u_n - u_n \cdot a, \varphi \otimes \chi \rangle = \langle u_n, (\varphi \otimes \chi) \cdot a \rangle - \langle u_n, a \cdot (\varphi \otimes \chi) \rangle$$

$$= \langle u_n, \varphi(a) \varphi \otimes \chi \rangle - \langle u_n, \chi(a) \varphi \otimes \chi \rangle$$

$$= (\varphi(a) - \chi(a)) \langle u_n, \varphi \otimes \chi \rangle,$$

while

$$\langle D(a), \varphi \otimes \chi \rangle = \langle a \otimes E - E \otimes a, \varphi \otimes \chi \rangle = \varphi(a) - \chi(a)$$

so that, since $a \cdot u_n - u_n \cdot a \to D(a)$,

$$\varphi(a) - \chi(a) = (\varphi(a) - \chi(a)) \lim_n \langle u_n, \varphi \otimes \chi \rangle.$$

Since $\varphi(a) - \chi(a) \neq 0$, this implies that $1 = \lim_n \langle u_n, \varphi \otimes \chi \rangle$, and so

$$\lim_n \langle \Delta_n, \varphi \otimes \chi \rangle = \langle E \otimes E, \varphi \otimes \chi \rangle - \lim_n \langle u_n, \varphi \otimes \chi \rangle = 0,$$

as required. □

3. General results

Recall that $A$ is approximately contractible if and only if $A^\#$ has an approximate diagonal (this is [11, Proposition 2.6(a)]).

We would like to have a better understanding of just when the presence of an approximate diagonal in $A$ guarantees an approximate diagonal in $A^\#$, and to obtain corresponding results for multiplier-bounded approximate diagonals. Note that by combining the proof of (ii) $\Rightarrow$ (iii) in [14, Proposition 3.2] with (3) $\Rightarrow$ (1) of [12, Theorem 2.1], we obtain the following result.

**Proposition 3.1.** Let $A$ be a Banach algebra which has a bounded approximate identity and an approximate diagonal. Then $A$ is approximately contractible, and so $A^\#$ has an approximate diagonal.

A natural hope is that the result just stated remains true if we replace ‘approximate diagonal’ with ‘multiplier-bounded approximate diagonal’. We have been unable to verify this: the problem seems to be that while approximate amenability implies approximate contractibility, it is not known if bounded approximate amenability implies bounded approximate contractibility. The following result gives some partial answers.
Proposition 3.2. Let $A$ be a Banach algebra with a central b.a.i. $(e_{\lambda})_{\lambda \in \Lambda}$. Suppose that $A$ has a multiplier-bounded approximate diagonal $(M_i)$. Then $A^\#$ has a multiplier-bounded approximate diagonal, and so $A$ is boundedly approximately contractible.

Proof. Throughout we denote the adjoined unit of $A^\#$ by $1$, and the linearized product map $A^\# \otimes A^\# \to A^\#$ by $\pi$. We shall abuse notation and also use $\pi$ to denote the restricted map $A^\# \otimes A^\# \to A$.

We shall construct a net $(n_j)$ in $A^\# \otimes A^\#$ and a constant $K > 0$ such that:

$$\|\pi(n_j) - 1\| \leq K \quad \text{and} \quad \|b \cdot n_j - n_j \cdot b\| \leq K \|b\| \quad \text{for all } b \in A \text{ and all } j; \quad (3.1)$$

and

$$\lim_{j} \pi(n_j) = 1 \quad \text{and} \quad \lim_{j} (b \cdot n_j - n_j \cdot b) = 0 \quad \text{for all } b \in A \text{ and all } j. \quad (3.2)$$

If these properties are satisfied, it is then straightforward to show that the net $(n_j)$ has the required properties in Definition 2.1.

By hypothesis there exist constants $C$ and $K_1$ such that $\|e_{\lambda}\| \leq C$ for all $\lambda$ and such that, for all $a \in A$ and all $i$,

$$\|a \pi(M_i) - a\| \leq K_1 \|a\| \quad \text{and} \quad \|a \cdot M_i - M_i \cdot a\| \leq K_1 \|a\|. \quad (3.3)$$

Moreover, for any $a \in A$, we have

$$\lim_{i} a \pi(M_i) = a \quad \text{and} \quad \lim_{i} a \cdot M_i - M_i \cdot a = 0. \quad (3.4)$$

To simplify the ensuing formulas slightly, we let $u_{\lambda} := 2e_{\lambda} - e^2_{\lambda}$ for each $\lambda$: note that $u_{\lambda} + (1 - e_{\lambda})^2 = 1$ and $\|u_{\lambda}\| \leq 2C + C^2$, for all $\lambda$. We now set

$$m_{\lambda,i} := u_{\lambda} \cdot M_i + (1 - e_{\lambda}) \otimes (1 - e_{\lambda}), \quad (3.5)$$

so that

$$\pi(m_{\lambda,i}) = u_{\lambda} \pi(M_i) + 1 - u_{\lambda}. \quad (3.6)$$

Let $\mathcal{I}$ and $\Lambda$ be the index sets for the nets $(M_i)$ and $(e_{\lambda})$, respectively. We construct the required net $(n_j)$ using an iterated limit construction (see [19, p. 26]). Our indexing directed set is defined to be $J = \Lambda \times \prod_{\lambda \in \Lambda} \mathcal{I}$, equipped with the product ordering, and for each $j = (\lambda, f) \in J$ we define $n_j = m_{\lambda,f(\lambda)}$.

Fix $\lambda$ and $i$. Using (3.3) and (3.6) gives

$$\|\pi(m_{\lambda,i}) - 1\| = \|u_{\lambda} \pi(M_i) - u_{\lambda}\| \leq K_1 \|u_{\lambda}\| \leq K_1(2C + C^2).$$

Also, since each $e_{\lambda}$ lies in the centre of $A$, we have for any $b \in A$ the identity

$$b \cdot m_{\lambda,i} - m_{\lambda,i} \cdot b = u_{\lambda} b \cdot M_i - u_{\lambda} \cdot M_i \cdot b + (b - be_{\lambda}) \otimes (1 - e_{\lambda})$$

$$- (1 - e_{\lambda}) \otimes (b - be_{\lambda}). \quad (3.7)$$
Using (3.3) again gives
\[ \|b \cdot m_{\lambda,i} - m_{\lambda,i} \cdot b\| \leq \|u_{\lambda}\| \|b \cdot M_i - M_i \cdot b\| + 2 \|b\| \|1 - e_{\lambda}\|^2 \]
\[ \leq (CK_1 + 2(1 + C)^2) \|b\| \]
for any \( b \in A \). Since \( \lambda \) and \( i \) were arbitrary, we have shown that (3.1) holds with, say, \( K = 2(1 + K_1)(1 + C)^2 \).

It remains to show that (3.2) holds. Using (3.4) and (3.6) we have, for every \( \lambda \),
\[ \lim_i \pi(m_{\lambda,i}) = \lim_i u_{\lambda} \pi(M_i) + 1 - u_{\lambda} = 1; \]
hence, by [19, Theorem 2.4],
\[ \lim_j \pi(n_j) = \lim_\lambda \lim_i \pi(m_{\lambda,i}) = 1. \]
Using (3.4) and (3.7) we have, for every \( \lambda \),
\[ \lim_i b \cdot m_{\lambda,i} - m_{\lambda,i} \cdot b = -(b - be_{\lambda}) \otimes (1 - e_{\lambda}) + (1 - e_{\lambda}) \otimes (b - be_{\lambda}); \]
therefore, since \( (e_{\lambda}) \) is a bounded approximate identity for \( A \), applying [19, Theorem 2.4] we obtain
\[ \lim_j (b \cdot n_j - n_j \cdot b) = \lim_\lambda \lim_i (b \cdot m_{\lambda,i} - m_{\lambda,i} \cdot b) = 0. \]
Thus (3.2) holds and our proof is complete. \( \square \)

**Remark.** The result is false if we do not require the central approximate identity in \( A \) to be bounded: an example is given by \( \ell^1(\mathbb{N}) \) equipped with pointwise multiplication [5, Theorem 4.1].

It is still open whether an approximately amenable Banach algebra must have a bounded approximate identity. If this were the case then one could extend many of the known hereditary properties of amenability to hold for approximate amenability. All presently known examples of approximately amenable Banach algebras have a bounded approximate identity. In addition, all known examples of approximately amenable Banach algebras are in fact boundedly approximately contractible.

These last two observations are connected by the following result: every boundedly approximately contractible algebra has a bounded approximate identity (Corollary 3.4 below). We are able to prove a slightly stronger technical result, that allows us to rule out bounded approximate amenability for several classes of Banach algebras.

**Theorem 3.3.** Suppose that the Banach algebra \( A \) is boundedly approximately amenable, and has both a multiplier-bounded left approximate identity and a multiplier-bounded right approximate identity. Then \( A \) has a bounded approximate identity.
**Proof.** Let \((e_\alpha)\) and \((f_\beta)\) be, respectively, right and left multiplier-bounded approximate identities for \(A\), so that there exists a constant \(K > 0\) such that

\[
\|a e_\alpha\| \leq K \|a\| \quad \text{and} \quad \|f_\beta a\| \leq K \|a\| \quad \text{for all} \ a \in A \ \text{and all} \ \alpha, \beta.
\]

From this we obtain the following estimates:

(i) \(\|f_\beta \cdot m\| \leq K \|m\|\) and \(\|m \cdot e_\alpha\| \leq K \|m\|\), for every \(m \in A \hat{\otimes} A\) and every \(\alpha, \beta\);
(ii) \(\|f_\beta \cdot T\| \leq K \|T\|\) and \(\|T \cdot e_\alpha\| \leq K \|T\|\), for every \(T \in (A \hat{\otimes} A)^{**}\) and every \(\alpha, \beta\).

(The first pair of estimates follows easily from the definition of the projective tensor norm. The second pair follows from the first pair using Goldstine’s theorem and the weak*-continuity of the actions of \(A\) on \((A \hat{\otimes} A)^{**}\).)

Let \((F_i)\), \((G_i)\), \((M_i)\) and \(C\) be the nets (respectively, the constant) satisfying (ii)–(vi) of Theorem 2.6.

Suppose that the net \((f_\beta)\) is (norm) unbounded. We derive a contradiction as follows. For every \(i\) and every \(\beta\) we have

\[
\|f_\beta \cdot M_i - M_i \cdot f_\beta - f_\beta \otimes G_i + F_i \otimes f_\beta\| \leq C \|f_\beta\|,
\]

and so by (ii) above, we have

\[
\|(f_\beta \cdot M_i - M_i \cdot f_\beta - f_\beta \otimes G_i + F_i \otimes f_\beta)e_\alpha\| \leq KC \|f_\beta\| \tag{3.8}
\]

for every \(\alpha, \beta\) and \(i\).

Using the triangle inequality and the left multiplier-boundedness of the set \(\{f_\beta\}\), from (3.8) we have

\[
\|f_\beta\| \|G_i \cdot e_\alpha\| \leq KC \|f_\beta\| + \|f_\beta \cdot (M_i \cdot e_\alpha)\| + \|M_i \cdot (f_\beta e_\alpha)\| + \|F_i\| \|f_\beta e_\alpha\|
\]

\[
\leq KC \|f_\beta\| + K \|M_i \cdot e_\alpha\| + K \|M_i\| \|e_\alpha\| + K \|F_i\| \|e_\alpha\| \tag{3.9}
\]

for every \(\alpha, \beta\) and \(i\). Hence

\[
\|G_i \cdot e_\alpha\| \leq KC + \frac{K}{\|f_\beta\|} \left(\|M_i \cdot e_\alpha\| + \|M_i\| \|e_\alpha\| + \|F_i\| \|e_\alpha\|\right) \tag{3.10}
\]

for every \(\alpha, \beta\) and \(i\).

For fixed \(\alpha\) and \(i\), combining (3.10) with our assumption that \(\{f_\beta\}\) is an unbounded set yields \(\|G_i \cdot e_\alpha\| \leq KC\). Taking limits with respect to \(i\), we then obtain \(\|e_\alpha\| \leq KC\) for each \(\alpha\). But since \((e_\alpha)\) is a right approximate identity and \((f_\gamma)\) is a left multiplier-bounded set, we obtain

\[
\|f_\gamma\| = \lim_{\alpha} \|f_\gamma e_\alpha\| \leq \limsup_{\alpha} K \|e_\alpha\| \leq K^2 C
\]

for all \(\gamma\). This contradicts our assumption that the net \((f_\beta)\) is unbounded.

A similar argument, with left and right interchanged, shows that the net \((e_\alpha)\) is also bounded; the existence of a bounded approximate identity is now standard. \(\square\)
Corollary 3.4. Let $A$ be boundedly approximately contractible. Then $A$ has a bounded approximate identity.

Proof. This is an immediate consequence of Theorem 3.3, since by Theorem 2.5 every boundedly approximately contractible Banach algebra has a right and a left multiplier-bounded approximate identity. □

Corollary 3.5. Suppose that $A$ and $B$ are boundedly approximately contractible Banach algebras. Then the direct sum $A \oplus B$ is boundedly approximately contractible.

Proof. This follows from the proof of [11, Proposition 2.7] and Corollary 3.4. □

The following result is similar to Theorem 3.3, but seems not to imply it nor be implied by it.

Proposition 3.6. Suppose that the Banach algebra $A$ is boundedly approximately amenable. Let $S$ be a subset of $A$ which is (left and right) multiplier-bounded, i.e. for some $K > 0$, we have $\|sa\| \leq K \|a\|$ and $\|as\| \leq K \|a\|$ for all $a \in A$, $s \in S$. Then $S$ is norm bounded.

Proof sketch. Arguing as at the start of the proof of Theorem 3.3, we note that for every $s \in S$:

(i) $\|s \cdot m\| \leq K \|m\|$ and $\|m \cdot s\| \leq K \|m\|$, for every $m \in A \hat{\otimes} A$;
(ii) $\|s \cdot T\| \leq K \|m\|$ and $\|T \cdot s\| \leq K \|s\|$, for every $T \in (A \hat{\otimes} A)^{**}$.

Suppose that $S$ is (norm) unbounded, so that there exists a sequence $(s_n)$ in $S$ with $\|s_n\| \to \infty$. Then we may argue as in the proof of Theorem 3.3, replacing $e_\alpha$ with $s_m$ and $f_\beta$ with $s_n$ in Eqs. (3.8)–(3.10), to show that the sequence $(s_m)$ is bounded, giving us a contradiction as before. Hence $S$ is (norm) bounded as claimed. □

Examples 3.7. The following algebras have multiplier-bounded approximate identities but have no bounded approximate identities.

(a) $c_0(\omega)$, the space of all sequences such that $|a_n|\omega_n \to 0$, equipped with pointwise multiplication, where $\lim_n \omega_n = +\infty$.
(b) $\ell^1(\mathbb{N}_{\text{min}}, \omega)$, the weighted convolution algebra of the semilattice $\mathbb{N}_{\text{min}}$, where $\lim_m \omega_m = \infty$.
(c) The Schatten ideals $S_p(H)$ ($H$ a Hilbert space) where $1 \leq p < \infty$.
(d) The Fourier algebras of weakly amenable, non-amenable groups (see [6] for the definition and examples).
(e) Proper symmetric Segal subalgebras (in the sense of Reiter [27]) of $L^1(G)$.

It therefore follows from Theorem 3.3 that none of the above algebras can be boundedly approximately amenable.

Remark. It has recently been shown (H.G. Dales and R.J. Loy, private communication) that the algebras of Example 3.7(b) are not even approximately amenable.

We can exploit Corollary 3.4 further to show that certain unital Banach algebras are not boundedly approximately contractible.
Corollary 3.8. Let $X$ be an infinite, compact metric space and let $0 < \alpha \leq \frac{1}{2}$. Then the Lipschitz algebra $\text{lip}_\alpha(X)$ is not boundedly approximately contractible.

Proof. Since $X$ is infinite and compact it contains a non-isolated point, $x_0$ say. Let $M = \{ f \in \text{lip}_\alpha(X): f(x_0) = 0 \}$: then $M^\# \cong \text{lip}_\alpha(X)$. If $\text{lip}_\alpha(X)$ were boundedly approximately contractible, then by Proposition 2.4 $M$ would also be boundedly approximately contractible, and hence by Corollary 3.4 would have a bounded approximate identity. By Cohen’s factorization theorem, this would imply that $M^2 = M$, which is easily seen to be false by considering the function $f : x \mapsto d(x, x_0)^\beta$ where $\alpha < \beta < 2\alpha$ (see also the remarks at the end of [1]). \qed

Remark. The above proof works also for $\frac{1}{2} < \alpha < 1$. However, it is already known that in the latter case $\text{lip}_\alpha(X)$ is not even approximately amenable since it is abelian and not weakly amenable [1].

4. $A(\mathbb{F}_2)$ is not approximately amenable

Let $\mathbb{F}_2$ denote the free group on two generators. It was observed in [13, Remark 3.4(b)] that $A(\mathbb{F}_2)$ is pseudo-amenable, and the authors asked if it is approximately amenable. In this section we shall answer this question in the negative; indeed, we prove the formally stronger result that $A(\mathbb{F}_2)$ is not even operator approximately amenable. Our techniques are based on direct estimates, exploiting the fact that the norm in $A(\mathbb{F}_2 \times \mathbb{F}_2)$ majorizes a certain weighted $\ell^2$-norm. Some consequences for Fourier algebras of more general groups will be given at the end of the section.

Background material

We state the required definitions and basic properties in the setting of discrete groups, since we will eventually specialize to $\mathbb{F}_2$: some hold in greater generality, but we shall not discuss this here.

Let $\Gamma$ be a discrete group and $C_{\text{c}}(\Gamma)$ the space of compactly supported functions on $\Gamma$. The Fourier algebra $A(\Gamma)$ can be defined as the completion of $C_{\text{c}}(\Gamma)$ with respect to the norm

$$
\| f \|_{A(\Gamma)} = \inf \{ \| \xi \|_2 \| \eta \|_2 : \xi, \eta \in \ell^2(\Gamma); \ f = \xi \ast \eta \} \quad (f \in C_{\text{c}}(\Gamma)).
$$

Let $\lambda : \ell^1(\Gamma) \to B(\ell^2(\Gamma))$ denote the (faithful) left regular representation of $\ell^1(\Gamma)$ on $\ell^2(\Gamma)$. The WOT-closure of the image of $\lambda$ is the von Neumann algebra of $\Gamma$, and will here be denoted by $VN(\Gamma)$. We can identify $A(\Gamma)$ with the predual of the group von Neumann algebra $VN(\Gamma)$: the pairing between the two satisfies

$$
[\lambda(T), f] = \sum_{g \in \Gamma} T(g)f(g) \quad (f \in A(\Gamma), \ T \in C_{\text{c}}(\Gamma)),
$$

from which the following is immediate.

Lemma 4.1. For every $f \in A(\Gamma)$ and every $T \in C_{\text{c}}(\Gamma)$ we have

$$
\left| \sum_{x \in \Gamma} f(x)T(x) \right| \leq \| f \|_{A(\Gamma)}\| \lambda(T) \|.
$$
The norm on $A(\Gamma)$ is in general hard to describe, but there are easy upper and lower bounds. Specifically we have $\ell^2(\Gamma) \subseteq A(\Gamma)$ and

\[ \|f\|_\infty \leq \|f\|_{A(\Gamma)} \leq \|f\|_2 \quad \text{for all } f \in \ell^2(\Gamma). \]  

(4.1)

The following definition first appeared in [13].

**Definition 4.2.** Let $A$ be a quantized Banach algebra. $A$ is *operator approximately amenable* if, for each quantized Banach $A$-bimodule $X$, every completely bounded derivation $A \to X^*$ is approximately inner.

Clearly, if $A$ is a quantized Banach algebra which happens to be approximately amenable, then it is operator approximately amenable.

The following is a ‘quantized’ version of one direction of [5, Proposition 3.3], specialized to the cases of interest.

**Proposition 4.3.** Let $\Gamma$ be a discrete group and suppose that $A(\Gamma)$ is operator approximately amenable. Then for every finite set $S \subseteq A(\Gamma)$ and every $\varepsilon > 0$, there exists $F \in c_{00}(\Gamma \times \Gamma)$ such that

(i) $\|a \cdot F - F \cdot a - a \otimes \pi(F) + \pi(F) \otimes a\|_{A(\Gamma \times \Gamma)} \leq \varepsilon$;

(ii) $\|a - a\pi(F)\|_{A(\Gamma)} \leq \varepsilon$

for every $a \in S$.

For convenience we give a brief outline of how Proposition 4.3 follows from existing results.

**Proof sketch.** We use $\hat{\otimes}_{\text{op}}$ to denote the operator projective tensor product of two operator spaces. Since $A(\Gamma)$ is a quantized Banach algebra, $A(\Gamma)^\# \hat{\otimes}_{\text{op}} A(\Gamma)^\#$ is a quantized Banach $A(\Gamma)$-bimodule. Let $K$ be the kernel of the (surjective, completely bounded) product map $A(\Gamma)^\# \hat{\otimes}_{\text{op}} A(\Gamma)^\# \to A(\Gamma)^\#$; then $K$ and hence $K^\ast$ are also quantized Banach $A(\Gamma)$-bimodules.

Let $D : A(\Gamma) \to K^{\ast\ast}$ be the completely bounded derivation defined by $D(a) = a \otimes 1 - 1 \otimes a$ ($a \in A(\Gamma)$). Since $K^{\ast\ast}$ is the dual of a quantized Banach $A(\Gamma)$-bimodule, by hypothesis $D$ is approximately inner. Therefore, by combining the proofs of [11, Corollary 2.2] and [5, Proposition 2.1], we obtain the following: for any finite subset $S \subseteq A(\Gamma)$ and any $\varepsilon > 0$, there exist $F \in A(\Gamma) \otimes A(\Gamma)$ and $u, v \in A(\Gamma)$ such that

1. $\|a \cdot F - F \cdot a + u \otimes a - a \otimes v\|_{A(\Gamma) \hat{\otimes}_{\text{op}} A(\Gamma)} < \varepsilon$;
2. $\|a - au\|_{A(\Gamma)} < \varepsilon$ and $\|a - va\|_{A(\Gamma)} < \varepsilon$.

By results of Effros and Ruan [8], the operator projective tensor norm on $A(\Gamma) \otimes A(\Gamma)$ coincides with its norm as a linear subspace of $A(\Gamma \times \Gamma)$. The rest of the proof now follows [5, Propositions 2.3 and 3.3] and we omit the details. □
Specializing to $F_2$

**Notation.** For $t \in F_2$ we denote by $|t|$ the *word length* of $t$, with the convention that the identity element has length 0. For each $n \in \mathbb{Z}_+$ let $S(n)$ denote the set $\{t \in F_2 : |t| = n\}$. Elementary calculations show that $|S(n)| = 4 \cdot 3^{n-1}$ for each $n \geq 1$.

The following Sobolev-type estimate, which we state without proof, is crucial for the argument to follow. It is a special case of [26, Theorem 1.1], and as such really belongs to the province of geometric group theory.

**Proposition 4.4.** Fix $m,n \in \mathbb{N}$. Let $T \in C_{00}(F_2 \times F_2)$ be supported on $S(m) \times S(n)$. Then

$$\|\lambda(T)\| \leq (m+1)(n+1) \|T\|_2.$$  

**Corollary 4.5.** Let $F \in A(F_2 \times F_2)$. Then

$$\|F\|_{A(F_2 \times F_2)} \geq \sup_{m,n \in \mathbb{Z}_+} \frac{1}{(m+1)(n+1)} \left( \sum_{x \in S(m)} \sum_{y \in S(n)} |F(x,y)|^2 \right)^{1/2}.$$  

**Proof.** This is a routine deduction from Proposition 4.4, using duality. Let $(m, n) \in \mathbb{Z}_+^2$ and let $T_{m,n} \in C_{00}(F_2 \times F_2)$ be defined by

$$T_{m,n}(x, y) = \begin{cases} (m+1)^{-1}(n+1)^{-1} \overline{F(x,y)} & \text{if } x \in S(m) \text{ and } y \in S(n), \\ 0 & \text{otherwise}. \end{cases}$$

Then by Proposition 4.4,

$$\|\lambda(T_{m,n})\| \leq \left( \sum_{(x,y) \in S(m) \times S(n)} |F(x,y)|^2 \right)^{1/2},$$

and since

$$\sum_{(x,y) \in F_2 \times F_2} F(x, y)T_{m,n}(x, y) = \frac{1}{(m+1)(n+1)} \sum_{(x,y) \in S(m) \times S(n)} |F(x,y)|^2,$$

applying Lemma 4.1 completes the proof. $\square$

**Proof that $A(F_2)$ is not operator approximately amenable**

We start with some notation. In view of the lower bound (4.2), we introduce the following norm on $c_{00}(F_2 \times F_2)$: given $H \in c_{00}(F_2 \times F_2)$, let

$$\|H\|_{\omega \times \omega} = \sup_{m,n \geq 0} \frac{1}{(m+1)(n+1)} \left( \sum_{x \in S(m)} \sum_{y \in S(n)} |H(x,y)|^2 \right)^{1/2}.$$  

For each $n \in \mathbb{N}$, we fix a partition of $S(n)$ into two disjoint subsets $A(n)$ and $B(n)$ of equal cardinality, so that $|A(n)| = |B(n)| = \frac{1}{2}|S(n)|$. We also fix a sequence $(\gamma_n)_{n \geq 1}$ of strictly positive
reals, such that
\[
\sum_{n \geq 1} \gamma_n^2 |\mathcal{S}(n)| < \infty \quad (4.3)
\]
(the \( \gamma_n \) will be chosen later with appropriate hindsight). Now define elements \( a \) and \( b \) of \( \ell^2(\mathbb{F}_2) \) by
\[
a := \sum_{m \geq 1} \gamma_m 1_{A(m)}, \quad b := \sum_{n \geq 1} \gamma_n 1_{B(n)}. \quad (4.4)
\]
Finally, let \( \varepsilon > 0 \).

**Suppose that \( A(\mathbb{F}_2) \) is operator approximately amenable.** Applying Proposition 4.3 with \( S = \{a, b\} \) and using the lower bounds from (4.1) and (4.2), we obtain \( F \in c_{00}(\mathbb{F}_2 \times \mathbb{F}_2) \) such that, if we set \( u = \pi(F) \in c_{00}(\mathbb{F}_2) \):
\[
\|a - ua\|_\infty \leq \varepsilon \quad \text{and} \quad \|b - ub\|_\infty \leq \varepsilon; \quad (4.5)
\]
\[
\|a \cdot F - F \cdot a - a \otimes u + u \otimes a\|_{\omega \times \omega} \leq \varepsilon; \quad (4.6a)
\]
\[
\|b \cdot F - F \cdot b - b \otimes u + u \otimes b\|_{\omega \times \omega} \leq \varepsilon. \quad (4.6b)
\]
For the moment we shall ignore the relations (4.5), and work exclusively with the information given by (4.6a) and (4.6b). Our task will be simplified by the fact that we have chosen the functions \( a \) and \( b \) to have ‘large’ yet disjoint supports (this theme, if not the actual calculations, is inspired by the proof of [5, Theorem 4.1]).

**Remark.** Since \( a \) and \( b \) are fixed in advance of \( F \), both (4.6a) and (4.6b) can always be satisfied by taking \( F \) to be of the form \( c1_{W \times W} \) for some \( c \in \mathbb{C} \) and some suitably large, finite subset \( W \subset \mathbb{F}_2 \); hence we will need to use (4.5) at some point if we want to obtain the required contradiction.

Our task would be simplified if we furthermore assume that \( F \) is constant on sets of the form \( \mathcal{S}(m) \times \mathcal{S}(n) \), and indeed the calculations that follow are motivated by this special case. The key step is contained in the following proposition.

**Proposition 4.6.** For each \( k \in \mathbb{N} \) let
\[
g(k) := \frac{1}{|A(k)|} \sum_{p \in A(k)} u(p) \quad \text{and} \quad h(k) := \frac{1}{|B(k)|} \sum_{q \in B(k)} u(q).
\]
Then for every \( m, n \in \mathbb{N} \) we have
\[
|A(m)|^{1/2}|B(n)|^{1/2}|g(m) - h(n)| \leq (m + 1)(n + 1)(\gamma_m^{-1} + \gamma_n^{-1})\varepsilon. \quad (4.7)
\]
For our proof we need a technical lemma, whose essential content is well known, but is stated here for convenience.
Lemma 4.7. Let \( \mathbb{I}, \mathbb{J} \) be finite index sets and let \( c_i, d_j \in \mathbb{C} \) for all \( i \in \mathbb{I} \) and all \( j \in \mathbb{J} \). Let
\[
\mu_c := \frac{1}{|\mathbb{I}|} \sum_{i} c_i \quad \text{and} \quad \mu_d := \frac{1}{|\mathbb{J}|} \sum_{j} d_j.
\]
Then
\[
\frac{1}{|\mathbb{I}|} \frac{1}{|\mathbb{J}|} \sum_{i \in \mathbb{I}} \sum_{j \in \mathbb{J}} |c_i - d_j|^2 \geq |\mu_c - \mu_d|^2.
\]

**Proof sketch.** One can prove this by direct calculation. Alternatively, we can use the language of probability theory, as follows. If \( X \) and \( Y \) are independent complex-valued random variables defined on a common finite probability space (in this case, \( \mathbb{I} \times \mathbb{J} \)) then
\[
\mathbb{E}|X - Y|^2 = \mathbb{E}(X - Y)(\overline{X - Y})
\]
\[
= \mathbb{E}X\overline{X} - (\mathbb{E}X)(\mathbb{E}Y) - (\mathbb{E}X)(\mathbb{E}Y) + \mathbb{E}Y\overline{Y}
\]
\[
= (\mathbb{E}X - \mathbb{E}Y)^2 + \mathbb{E}|X|^2 - |\mathbb{E}X|^2 + \mathbb{E}|Y|^2 - |\mathbb{E}Y|^2 \geq (\mathbb{E}X - \mathbb{E}Y)^2.
\]
(In the last step we used the fact that \( |\mathbb{E}X|^2 \leq \mathbb{E}|X|^2 \) and \( |\mathbb{E}Y|^2 \leq \mathbb{E}|Y|^2 \).) Taking \( X \) to be the random variable \( (i, j) \mapsto c_i \) and \( Y \) to be the random variable \( (i, j) \mapsto d_j \), the proof is complete. \( \square \)

**Proof of Proposition 4.6.** Eq. (4.6a) implies that
\[
(m + 1)^2(n + 1)^2 \varepsilon^2 \geq \sum_{x \in \mathbb{S}(m)} \sum_{y \in \mathbb{S}(n)} \left| (a(x) - a(y))F(x, y) - a(x)u(y) + a(y)u(x) \right|^2
\]
\[
\geq \sum_{x \in \mathbb{A}(m)} \sum_{y \in \mathbb{B}(n)} \left| (a(x) - a(y))F(x, y) - a(x)u(y) + a(y)u(x) \right|^2
\]
\[
= \sum_{x \in \mathbb{A}(m)} \sum_{y \in \mathbb{B}(n)} \left| a(x)F(x, y) - a(x)u(y) \right|^2
\]
\[
= \gamma_n^2 \sum_{x \in \mathbb{A}(m)} \sum_{y \in \mathbb{B}(n)} \left| F(x, y) - u(y) \right|^2.
\]
Therefore
\[
(m + 1)(n + 1) \varepsilon \geq \gamma_n \left( \sum_{x \in \mathbb{A}(m)} \sum_{y \in \mathbb{B}(n)} \left| F(x, y) - u(y) \right|^2 \right)^{1/2} \tag{4.8a}
\]

Similarly, using Eq. (4.6b) instead of (4.6a), we have
\[
(m + 1)(n + 1) \varepsilon \geq \left( \sum_{x \in \mathbb{A}(m)} \sum_{y \in \mathbb{B}(n)} \left| -b(y)F(x, y) + b(y)u(x) \right|^2 \right)^{1/2}
\]
\[
= \gamma_n \left( \sum_{x \in \mathbb{A}(m)} \sum_{y \in \mathbb{B}(n)} \left| F(x, y) - u(x) \right|^2 \right)^{1/2} \tag{4.8b}
\]
Hence, by using the triangle inequality for the 2-norm on $\ell^2(A(m) \times B(n))$, we see that (4.8a) and (4.8b) together imply
\[
(m + 1)(n + 1)(\gamma_m^{-1} + \gamma_n^{-1})\varepsilon \geq \left( \sum_{x \in A(m)} \sum_{y \in B(n)} |u(x) - u(y)|^2 \right)^{1/2}.
\] (4.9)

The desired estimate (4.7) now follows by applying Lemma 4.7 to (4.9).

We now show that by fixing our sequence $(\gamma_n)$ appropriately, we can force $g$ to be “slowly varying at infinity”, and play this off against the fact that $g$ has finite support (since $u$ does). For each $n$, take $\gamma_n := n^{-1}|S(n)|^{-1/2}$ (this certainly satisfies the condition in (4.3)). If we substitute this into the estimate (4.7) and take $m = k, n = k + 1$ we get
\[
|A(k)|^{1/2}|B(k + 1)|^{1/2}|g(k) - h(k + 1)| \leq \varepsilon(k + 1)(k + 2)(|S(k)|^{1/2} + (k + 1)|S(k + 1)|^{1/2}) \leq \varepsilon(k + 2)^3(|S(k)|^{1/2} + |S(k + 1)|^{1/2});
\]
and since $|A(n)| = |B(n)| = \frac{1}{2}|S(n)| = 2 \cdot 3^n - 1$, we find that
\[
|g(k) - h(k + 1)| \leq \varepsilon(k + 2)^3 \frac{2 \cdot 3^{(k-1)/2} + 2 \cdot 3^{k/2}}{2 \cdot 3^{(k-1)/2} \cdot 3^{k/2}} = (1 + \sqrt{3})\varepsilon(k + 2)^3 3^{-k/2}.
\]

On the other hand, taking $m = n = k + 1$ in (4.7), an exactly similar argument gives
\[
|g(k + 1) - h(k + 1)| \leq 2\varepsilon(k + 2)^3 3^{-k/2},
\]
and we thus obtain the estimate
\[
|g(k + 1) - g(k)| \leq 5\varepsilon(k + 2)^3 3^{-k/2}. (4.10)
\]

By the comparison test, the infinite sum $\sum_{k \geq 1}(k + 2)^3 3^{-k/2}$ converges, with value $M$ say. Moreover, since $u$ has finite support, there exists $N \geq 2$ such that $g(j) = h(j) = 0$ for all $j \geq N$. Hence, using (4.10), we get
\[
|g(1)| = |g(N) - g(1)| \leq \sum_{k=0}^{N-1} |g(k + 1) - g(k)| \leq 5\varepsilon \sum_{k=0}^{N-1} (k + 2)^3 3^{-k/2} < 5M\varepsilon. (4.11)
\]

Now observe that, by (4.5),
\[
\varepsilon \geq \|a - au\|_{\infty} \geq \max_{x \in A(1)} |a(x) - a(x)u(x)| = \frac{1}{2} \max_{x \in A(1)} |1 - u(x)|.
\]
Let $x, y$ be the two elements of $A(1)$. Then the estimate just given implies that

$$2\varepsilon \geq \frac{1}{2}(|1 - u(x)| + |1 - u(y)|) \geq \frac{1}{2}|2 - u(x) - u(y)| = |1 - g(1)|,$$

and combining this with (4.11), we finally arrive at

$$1 \leq |1 - g(1)| + |g(1)| \leq 2\varepsilon + 5M\varepsilon.$$

As $M$ is, by definition, independent of $\varepsilon$, we obtain a contradiction by taking $\varepsilon$ to be sufficiently small. Hence our assumption that $A(\mathbb{F}_2)$ is operator approximately amenable must be false, and the proof is complete. □

Corollary 4.8. Let $G$ be a locally compact group, into which $\mathbb{F}_2$ embeds as a closed subgroup. Then $A(G)$ is not operator approximately amenable.

Proof. The hypothesis on $G$ ensures that the restriction homomorphism $A(G) \to A(\mathbb{F}_2)$ is completely bounded and a quotient map of Banach spaces [16, Theorem 1]. If $A(G)$ were operator approximately amenable, then $A(\mathbb{F}_2)$ would be also, since operator approximate amenability is inherited by completely bounded quotient algebras. This gives a contradiction. □

Remark. For a discrete, amenable group $G$, it was shown in [13] that $A(G)$ is approximately amenable. However, there are discrete groups which are non-amenable yet contain no copy of $\mathbb{F}_2$: Ol’shanskii’s groups, or Burnside groups of sufficiently large rank and exponent. So any attempt to prove that approximate amenability of $A(G)$ implies amenability of $G$ must use different, or additional, methods.

5. Results for Segal algebras

Following on from Example 3.7(e) above, we give some results on other notions of approximate amenability in the setting of Segal algebras.

Let $G$ be a locally compact group with a left-invariant Haar measure $\lambda$. Throughout this section, $S(G)$ denotes a Segal subalgebra of $L^1(G)$ (in the sense of Reiter [27]). We have already seen that a symmetric $S(G)$ is boundedly approximately amenable if and only if it is equal to the whole of $L^1(G)$ and $G$ is amenable. For Feichtinger’s Segal algebra (see [28] for the definition) on a compact abelian group we easily obtain the following:

Proposition 5.1. The Feichtinger algebra on an infinite compact abelian group is not approximately amenable.

Proof. When $G$ is compact and abelian, the Feichtinger algebra on $G$ is

$$S_0(G) = \left\{ f = \sum_{\gamma \in \hat{G}} c_{\gamma} \chi_{\gamma}: \|f\| = \sum |c_{\gamma}| < \infty \right\},$$

where $\chi_{\gamma}$ is the character of $G$ associated with $\gamma \in \hat{G}$. Hence,

$$S_0(G) \cong \ell^1(\hat{G}),$$

where $\ell^1(\hat{G})$ is the space of absolutely summable functions on the Pontryagin dual of $G$. □
where the right-hand side is equipped with the pointwise product. But $\ell^1(S)$ is not approximately amenable if $S$ is an infinite set, due to [5, Theorem 4.1]. So $S_0(G)$ is not approximately amenable.

It has been shown in [14] that a Segal algebra on a compact group is pseudo-contractible. The converse is also true and is a consequence of the next proposition.

**Proposition 5.2.** If there is $N \in S(G) \hat{\otimes} S(G)$ such that $\pi(N) \neq 0$ and $f \cdot N = N \cdot f$ for $f \in S(G)$, then $G$ is compact.

**Proof.** Let $\theta: S(G) \to L^1(G)$ be the inclusion injection. Then the following diagram commutes

$$
\begin{array}{ccc}
S(G) \hat{\otimes} S(G) & \xrightarrow{\theta \otimes \theta} & L^1(G) \hat{\otimes} L^1(G) \\
\downarrow \pi & & \downarrow \pi \\
S(G) & \xrightarrow{\theta} & L^1(G).
\end{array}
$$

Let $N \in S(G) \hat{\otimes} S(G)$ be such that $\pi(N) \neq 0$ and $f \cdot N = N \cdot f$ for $f \in S(G)$. Let $M = \theta \otimes \theta(N) \in L^1(G) \hat{\otimes} L^1(G)$. We have $f \cdot M = M \cdot f$ for all $f \in L^1(G)$, and therefore $\mu \cdot M = M \cdot \mu$ ($\mu \in M(G)$). In particular, $M = \delta_{x^{-1}} \cdot M \cdot \delta_x$ ($x \in G$). Let $K$ be a compact subset of $G \times G$. If we regard $M$ as a function in $L^1(G \times G)$, then

$$
\int_K |M(s,t)| ds \, dt = \int_K |\delta_{x^{-1}} \cdot M \cdot \delta_x(s,t)| ds \, dt \\
= \int_{(x,e)K(e,x^{-1})} \Delta(x) |M(s,t)| ds \, dt,
$$

where $(x,e)K(e,x^{-1})$ denotes the set $\{(xs,tx^{-1}) : (s,t) \in K\}$, and $\Delta$ is the modular function of the group $G$. Given $\varepsilon > 0$, let $R \subset G \times G$ be a compact set such that

$$
\int_{G \times G \setminus R} |M(s,t)| ds \, dt < \varepsilon.
$$

If $G$ is not compact, then there is $x \in G$ such that $(x,e)K(e,x^{-1}) \subset G \times G \setminus R$ and $\Delta(x) \leq 1$, so that

$$
\int_{(x,e)K(e,x^{-1})} \Delta(x) |M(s,t)| ds \, dt < \varepsilon.
$$

We then have $\int_K |M(s,t)| ds \, dt < \varepsilon$. This implies that $\int_K |M(s,t)| ds \, dt = 0$, for all compact $K \subset G \times G$, and so $M = 0$ in $L^1(G) \hat{\otimes} L^1(G)$. On the other hand, $\pi(N) \neq 0$ in $S(G)$ and hence $\pi(M) = \theta \pi(N) \neq 0$ in $L^1(G)$, a contradiction. Thus, $G$ must be compact. \qed
Remark. Proposition 5.2 holds with $S(G)$ replaced by any Banach algebra $B$ which admits a continuous injective homomorphism $B \to L^1(G)$ whose range is dense. Therefore, if such a $B$ exists and is pseudo-contractible, $G$ must be compact.

Combining Proposition 5.2 and [14, Theorem 4.5], we then have a characterization of a compact group.

Theorem 5.3. The following are equivalent for a locally compact group $G$.

(i) The group $G$ is compact.
(ii) There is a Segal algebra on $G$ which is pseudo-contractible.
(iii) All Segal algebras on $G$ are pseudo-contractible.

(A different proof of the part “(ii) \Rightarrow (i)” can be seen in [29].)

It is natural to ask for an analogous characterization of amenability of $G$ in terms of approximate amenability or pseudo-amenability of Segal algebras on $G$. First we recall some material from the theory of abstract Segal algebras.

A dense left ideal $B$ of a Banach algebra $(A, \| \cdot \|_A)$ is called an abstract Segal algebra in $A$, or simply a Segal algebra in $A$, with respect to some norm $\| \cdot \|_B$ if it is a Banach algebra with respect to the norm $\| \cdot \|_B$ and if $\|b\|_A \leq \|b\|_B$ ($b \in B$) [3,22]. It was shown in [22] that if $B$ is a Segal algebra in $A$, then the mapping $J \mapsto \overline{J}^A$ is a bijection from the set of all closed right (two-sided) ideals in $B$ onto the set of all closed right (two-sided) ideals in $A$ and the inverse mapping is $I \mapsto I \cap B$, where for a set $J \subset B$ the notation $\overline{J}^A$ stands for the closure of $J$ in $A$. The same machinery as in [22, Proposition 2.7] yields the following:

Proposition 5.4. Let $B$ be an abstract Segal algebra in a Banach algebra $A$, let $J$ be a closed two-sided ideal of $B$ and let $I = \overline{J}^A$.

(i) Suppose that $A$ and $J$ both have right approximate identities. Then $I$ has a right approximate identity.
(ii) Suppose that $B$ and $I$ both have right approximate identities. Then $J$ has a right approximate identity.

Since we only need part (i) of Proposition 5.4, we shall give an independent proof of (i), which is more direct in the sense that it avoids the duality machinery of [22].

Proof of Proposition 5.4. Let $F \subset I$ be a finite subset, and let $\varepsilon > 0$. It suffices to find $s \in I$ such that $\max_{y \in F} \|ys - y\|_A \leq \varepsilon$.

Since $A$ has a right approximate identity, there exists $u \in A$ with $\|yu - y\|_A < \varepsilon/2$ for all $y \in F$. Therefore, since $B$ is dense in $A$, there exists $u' \in B$ such that $\|yu' - y\|_A < \varepsilon/2$ for all $y \in F$.

Since $J$ is a right ideal and $B$ is a left ideal in $A$, $yu' \in I \cap B = J$ for every $y \in F$. Therefore, as $J$ has a right approximate identity, there exists $w \in J$ such that $\|yu'w - yu'\|_B \leq \varepsilon/2$ for all $y \in F$. 

Let \( s = u'w \): since \( w \in J \subseteq I \) and \( I \) is also a left ideal, we have \( s \in I \); and for every \( y \in F \),
\[
\|ys - y\|_A \leq \|yu'w - yu'\|_A + \|yu' - y\|_A \\
\leq \|yu'w - yu'\|_B + \|yu' - y\|_A < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\]
as required. \( \square \)

**Remark.** Part (ii) of Proposition 5.4 can also be proved by direct \( \varepsilon-\delta \) arguments, similar to the ones just given; again, the duality machinery from [22] can be bypassed.

**Theorem 5.5.** If \( S(G) \) is approximately amenable or pseudo-amenable then \( G \) is an amenable group.

**Proof.** Let \( I_0 = \{ f \in L^1(G): \int_G f(x) \, dx = 0 \} \) be the augmentation ideal in \( L^1(G) \), and let \( J = I_0 \cap S(G) \). Then \( J \) is a codimension-1 two-sided closed ideal in \( S(G) \). If \( S(G) \) is approximately amenable or pseudo-amenable, then \( J \) has a right approximate identity by [11, Corollary 2.4] or [14, Proposition 2.5] respectively. By Proposition 5.4(i), \( I_0 \) must also have a right approximate identity. This implies that \( G \) is amenable due to [30, Theorem 5.2]. \( \square \)

**Remark.** We do not know whether there is a Segal algebra \( S(G) \) that is approximately amenable and that is not identical with \( L^1(G) \). It is also an open question whether a Segal algebra on an amenable group is always pseudo-amenable. Partial results can be found in [14].

We now turn to results that do not depend on amenability or compactness of \( G \). While \( L^1(G) \) is weakly amenable for every locally compact group \( G \), the same need not be true for Segal algebras; see [9, Remark 3.2] for examples. Following on from results in [9,10] on approximate weak amenability of Segal algebras, we now look at approximate permanent weak amenability.

Recall from [4] that a Banach algebra \( A \) is said to be \( n \)-weakly amenable if every continuous derivation from \( A \) into the \( n \)th dual space \( A^{(n)} \) is inner. \( A \) is permanently weakly amenable if it is \( n \)-weakly amenable for all \( n \in \mathbb{N} \).

It was shown in [4] that every \( C^* \)-algebra is permanently weakly amenable, and that every \( L^1(G) \) is \( n \)-weakly amenable for all odd, positive integers \( n \). In [18] B.E. Johnson proved that for every free group \( G \), the group algebra \( \ell^1(G) \) is \( n \)-weakly amenable for all even, positive \( n \). Combined with [4, Theorem 4.1], this shows that for such groups, \( \ell^1(G) \) is permanently weakly amenable. In an unpublished paper Johnson also showed that for any discrete word-hyperbolic group, the group algebra is permanently weakly amenable.

In fact, for any locally compact group \( G \), \( L^1(G) \) is permanently weakly amenable. Our proof relies heavily on the following result, proved recently by V. Losert [23, Theorem 1.1].

**Theorem 5.6.** Let \( G \) be a (discrete) group and \( X \) a locally compact space on which \( G \) has a 2-sided action by homeomorphisms. Then any bounded derivation \( D : G \to M(X) \) is inner.

(The statement in [23] refers only to those \( X \) with a left action; however, by standard arguments of Johnson one can reduce the 2-sided case to the 1-sided case, see e.g. [17, §2].)

**Proof that \( L^1(G) \) is permanently weakly amenable.** In light of [4] it suffices to show that \( L^1(G) \) is \( 2n \)-weakly amenable for all \( n \in \mathbb{N} \).
Let \( D : L^1(G) \to L^1(G)^{(2n)} \) be a continuous derivation. By the techniques of [17, §1.d] \( D \) extends to a derivation \( \overline{D} : M(G) \to L^1(G)^{(2n)} \), where the measure algebra \( M(G) \) acts on \( L^1(G)^{(2n)} \) through dualizations of its actions on \( L^1(G) \).

Now \( L^1(G)^{(2n)} \) is isomorphic, as an \( M(G) \)-bimodule, to \( M(X) \) for some compact space \( X \). The action of point masses on \( M(X) \) is equivalent to an action of \( G \) on \( M(X) \); and \( g \mapsto \overline{D}(\delta_g) \) is a bounded derivation from \( G \) into \( M(X) \). Hence by Theorem 5.6 this derivation is inner, and this suffices for us to conclude that \( \overline{D} : M(G) \to L^1(G)^{(2n)} \) is inner, by \( w^* \)-continuity of \( \overline{D} \).

**Theorem 5.7.** Let \( G \) be a locally compact SIN-group and let \( S(G) \) be a Segal algebra on \( G \). Then \( S(G) \) is approximately permanently weakly amenable (i.e. for each \( n \in \mathbb{N} \), every continuous derivation \( S(G) \to S(G)^{(n)} \) is approximately inner).

Note that the case “\( n = 0 \)” was proved in [9, Theorem 2.1(i)] under the extra hypothesis that our Segal algebra is symmetric.

**Proof of Theorem 5.7.** Since \( G \) is SIN, it follows from the results of [20] that \( S(G) \) has a central approximate identity \( (e_i) \) which is bounded in the \( L^1 \)-norm.

Let \( n \in \mathbb{N} \) and let \( D : S(G) \to S(G)^{(n)} \) be a continuous derivation. Our approach is to construct from \( D \) a net of continuous derivations \( L^1(G) \to L^1(G)^{(n)} \), so that we can appeal to Theorem 5.6.

The centrality of \( (e_i) \), together with the derivation property of \( D \), imply that

\[
D\left(S(G)\right) \subseteq X_n := \operatorname{lin}\{a \cdot S(G)^{(n)} \cdot b : a, b \in S(G)\}.
\]

Moreover, \( (e_i) \) is a two-sided, multiplier-bounded, central approximate identity for \( X_n \). In particular

\[
\lim_i e_i^2 \cdot D(f) = D(f) \quad (f \in S(G)), \tag{5.1}
\]

where the limit is taken in the norm topology of \( S(G)^{(n)} \).

For each \( i \), define a continuous linear mapping \( \tau_i : L^1(G) \to S(G) \) by

\[
\tau_i(f) = f \ast e_i \quad (f \in L^1(G)),
\]

and let \( \theta : S(G) \to L^1(G) \) denote the (continuous) inclusion map. Both \( \tau_i \) and \( \theta \) are left \( L^1(G) \)-module morphisms and are also \( S(G) \)-bimodule morphisms. Clearly, \( \tau_i \theta(f) = f \ast e_i = e_i \ast f \) for \( f \in S(G) \); so by induction, for each \( n \in \mathbb{N} \) the map \( (\tau_i \theta)^{(n)} : S(G)^{(n)} \to S(G)^{(n)} \) satisfies

\[
(\tau_i \theta)^{(n)}(F) = F \cdot e_i = e_i \cdot F \quad (F \in S(G)^{(n)}).
\]

Define \( \Delta_i : L^1(G) \to L^1(G)^{(n)} \) by

\[
\Delta_i(f) = \begin{cases} 
\theta^{(n)}[D(f \ast e_i) - f \cdot D(e_i)] & \text{if } n \text{ is even}, \\
\tau_i^{(n)}[D(e_i \ast f) - D(e_i) \cdot f] & \text{if } n \text{ is odd}
\end{cases} \quad (f \in L^1(G)).
\]

Then \( \Delta_i \) is a continuous linear map, and for \( f \in S(G) \) we have, using the derivation property of \( D \),
\[ \Delta_1(f) = \begin{cases} \theta^{(n)}(D(f) \cdot e_i) & \text{if } n \text{ is even}, \\ \tau_i^{(n)}(e_i \cdot D(f)) & \text{if } n \text{ is odd}. \end{cases} \] (5.2)

Since \( e_i \) is central, it is straightforward to verify using (5.2) that

\[ \Delta_1(f * g) = f \cdot \Delta_1(g) + \Delta_1(f) \cdot g \]

for all \( f, g \in S(G) \). Therefore, since \( S(G) \) is dense in \( L^1(G) \), it follows that \( \Delta_1 \) is a derivation from \( L^1(G) \) to \( L^1(G)^{(n)} \).

By Theorem 5.6 there exists \( \varphi_i \in L^1(G)^{(n)} \) such that

\[ \Delta_1(f) = f \cdot \varphi_i - \varphi_i \cdot f \quad (f \in L^1(G)). \]

In particular, for \( f \in S(G) \), Eq. (5.2) implies that for even \( n \) we have

\[ D(f) \cdot e_i^2 = (\tau_i \theta)^{(n)}(D(f) \cdot e_i) = \tau_i^{(n)} \Delta_1(f) \]

\[ = f \cdot \tau_i^{(n)}(\varphi_i) - \tau_i^{(n)}(\varphi_i) \cdot f, \]

while for odd \( n \) we have

\[ e_i^2 \cdot D(f) = (\tau_i \theta)^{(n)}(e_i \cdot D(f)) = \theta^{(n)} \Delta_1(f) \]

\[ = f \cdot \theta^{(n)}(\varphi_i) - \theta^{(n)}(\varphi_i) \cdot f. \]

Take \( \psi_i = \tau_i^{(n)}(\varphi_i) \) if \( n \) is even, and take \( \psi_i = \theta^{(n)}(\varphi_i) \) if \( n \) is odd. Then \( \psi_i \in S(G)^{(n)} \) for all \( i \), and for every \( f \in S(G) \) we have, by (5.1),

\[ D(f) = \lim_i e_i^2 \cdot D(f) = \lim_i f \cdot \psi_i - \psi_i \cdot f. \]

Thus \( D \) is approximately inner, as required. \( \square \)

**Remark.** Our construction actually provides a bounded net of inner derivations which approximate \( D \), although the net of implementing elements need not be bounded.

### 6. \( \ell^1 \)-Convolution algebras of totally ordered sets

Recall that a *semilattice* is a commutative semigroup in which every element is idempotent. The \( \ell^1 \)-convolution algebras of semilattices provide interesting examples of commutative Banach algebras. However, amenability is too strong a notion for such algebras: if \( S \) is a semilattice then the convolution algebra \( \ell^1(S) \) is amenable if and only if \( S \) is finite [7, Theorem 10].

It is not clear to the authors exactly which semilattices have approximately amenable \( \ell^1 \)-convolution algebras. In the case where the semilattice is *totally ordered* we can do better.

Let \( \Lambda \) be a non-empty, totally ordered set, and regard it as a semigroup by defining the product of two elements to be their maximum. The resulting semigroup, which we denote by \( \Lambda_\vee \), is a semilattice. We may then form the \( \ell^1 \)-convolution algebra \( \ell^1(\Lambda_\vee) \). For every \( t \in \Lambda_\vee \) we denote the point mass concentrated at \( t \) by \( e_t \). The definition of multiplication in \( \ell^1(\Lambda_\vee) \) ensures that \( e_s e_t = e_{\max(s,t)} \) for all \( s \) and \( t \).
Remark. One could also turn $\Lambda$ into a semilattice $\Lambda\land$ by defining the product of two elements to be their minimum. This is in some sense more natural, for reasons we shall not discuss here; we have chosen to work with $\Lambda\lor$ as this fits our main example (in Theorem 6.4) better.

**Theorem 6.1.** Let $I$ be any totally ordered set. Then $\ell^1(I\lor)$ is boundedly approximately contractible.

**Remark.** The special case of $I = \mathbb{N}$ or $\mathbb{N}^{op}$ was done in [12]. Our arguments are a more abstract version of the ones there.

We prove the theorem in several steps. First, by following the proof of [12, Theorem 5.10], it suffices to prove that $\ell^1(I\lor)$ has a multiplier-bounded approximate diagonal, in the sense of Definition 2.1. Moreover, we can identify $\ell^1(I\lor)$ with $\ell^1(\tilde{I})$, where $\tilde{I}$ denotes the disjoint union of $I$ with an adjoined least element. Clearly $\tilde{I}$ is also a totally ordered set, and so to prove Theorem 6.1 it suffices to prove the following claim: for any totally ordered set $I$, $\ell^1(I\lor)$ has a multiplier-bounded approximate diagonal.

It is useful to first consider the case of a finite totally ordered set. More precisely, let $F$ be a finite subset of $I$, and enumerate its elements in increasing order as $\min(F) = c(0) < c(1) < \cdots < c(n) = \max(F)$ say. We then define $\Delta_F \in \ell^1(I\lor) \otimes \ell^1(I\lor)$ by

$$\Delta_F = \left( \sum_{j=1}^{n} (e_{c(j-1)} - e_{c(j)}) \otimes (e_{c(j-1)} - e_{c(j)}) \right) + e_{c(n)} \otimes e_{c(n)}. \quad (6.1)$$

A small calculation shows that $\pi(\Delta_F) = e_{c(0)}$, so that

$$e_\lambda \pi(\Delta_F) = e_\lambda \quad \text{for all } \lambda \in F. \quad (6.2)$$

It is also easily checked that

$$e_\lambda \cdot \Delta_F = \Delta_F \cdot e_\lambda \quad \text{for all } \lambda \in F, \quad (6.3)$$

and thus $\Delta_F$ is a diagonal for the subalgebra $\ell^1(F\lor) \subseteq \ell^1(I\lor)$.

Having seen how to construct a diagonal for the finite case, we now proceed to the general case. Let $\text{FIN}$ be the set of all non-empty finite subsets of $I$, and order $\text{FIN}$ with respect to inclusion, so that for any $E$ and $F$ in $\text{FIN}$, $E \preceq F$ if and only if $E \subseteq F$.

The following result will, by the remarks above, imply Theorem 6.1.

**Proposition 6.2.** The net $(\Delta_F)_{F \in \text{FIN}}$ is a multiplier-bounded approximate diagonal for $\ell^1(I\lor)$.

We isolate the key technical estimate as a lemma.

**Lemma 6.3.** Let $b \in \ell^1(I\lor)$, $F \in \text{FIN}$. Then $\|b \cdot \Delta_F - \Delta_F \cdot b\| \leq 6\|b\|$. 

Proof. By the triangle inequality and the definition of the \( \ell^1 \)-norm, we can without loss of generality assume that \( b = e_\lambda \) for some \( \lambda \in I \). Thus it suffices to prove that

\[
\|e_\lambda \cdot \Delta_F - \Delta_F \cdot e_\lambda\| \leq 6 \quad \text{for all } F \in \text{FIN}.
\] (6.4)

This estimate holds trivially if \( F \) consists of only one point, so we shall henceforth assume that \( |F| \geq 2 \).

As before we enumerate the elements of \( F \) in increasing order as \( c(0) < c(1) < \cdots < c(n) \). We consider three possibilities. If \( \lambda \geq c(n) \), then \( e_\lambda \cdot \Delta_F = e_\lambda \otimes e_{c(n)} \) and \( \Delta_F \cdot e_\lambda = e_{c(n)} \otimes e_\lambda \), so that (6.4) certainly holds. At the other extreme, if \( \lambda \leq c(0) \) then \( e_\lambda \cdot \Delta_F = \Delta_F = \Delta_F \cdot e_\lambda \), so that (6.4) once again holds.

The third possibility is that \( c(0) < \lambda < c(n) \). Let \( m = \min\{k: c(k) > \lambda\} \) so that \( 1 \leq m \leq n \) and \( c(m-1) < \lambda < c(m) \). When we calculate \( e_\lambda \cdot \Delta_F - \Delta_F \cdot e_\lambda \) using the formula (6.1), most of the terms cancel and we obtain

\[
e_\lambda \cdot \Delta_F - \Delta_F \cdot e_\lambda = \begin{cases} e_\lambda(e_{c(m-1)} - e_{c(m)}) \otimes (e_{c(m-1)} - e_{c(m)}), \\ -(e_{c-1(m)} - e_{c(m)}) \otimes (e_{c(m-1)} - e_{c(m)}) e_\lambda, \\ (e_\lambda - e_{c(m)}) \otimes (e_{c(m-1)} - e_{c(m)}), \\ -(e_{c(m-1)} - e_{c(m)}) \otimes (e_\lambda - e_{c(m)}). \end{cases}
\]

Expanding out and using the triangle inequality gives \( \|e_\lambda \cdot \Delta_F - \Delta_F \cdot e_\lambda\| \leq 6 \), as required. \( \square \)

Proof of Proposition 6.2. Fix \( a \in \ell^1(I) \). We have already seen in Lemma 6.3 that

\[
\|a \cdot \Delta_F - \Delta_F \cdot a\| \leq 6\|a\| \quad \text{for every } F \in \text{FIN}.
\]

Also, since \( \pi(\Delta_F) = e_{\min(F)} \), we have

\[
\|a \pi(\Delta_F) - a\| \leq 2\|a\| \quad \text{for every } F \in \text{FIN}.
\] (6.5)

Thus the ‘multiplier-bounded’ part of the defining condition (2.1) is satisfied.

It remains to show that, given \( \varepsilon > 0 \), there exists \( F_0 \in \text{FIN} \) such that

\[
\|a \pi(\Delta_F) - a\| < \varepsilon \quad \text{and} \quad \|a \cdot \Delta_F - \Delta_F \cdot a\| < \varepsilon
\]

for any \( F \in \text{FIN} \) with \( F \supseteq F_0 \).

Fix \( \varepsilon > 0 \) and choose \( F_0 \in \text{FIN} \) such that \( \sum_{\lambda \in I \setminus F_0} |a_{\lambda}| \leq \varepsilon /6 \), and let \( F \in \text{FIN} \) with \( F \supseteq F_0 \). Let \( \tilde{a} \) denote the obvious truncation of \( a \) to the subset \( F \) (i.e. \( \tilde{a}_\lambda = a_\lambda \) if \( \lambda \in F \) and \( \tilde{a}_\lambda = 0 \) otherwise). Note that \( \|a - \tilde{a}\| \leq \varepsilon /6 \).

Since \( \tilde{a} \in \ell^1(F) \), we deduce from Eq. (6.3) and the estimate (6.5) that

\[
\|a \cdot \Delta_F - \Delta_F \cdot a\| = \|(a - \tilde{a}) \cdot \Delta_F - \Delta_F \cdot (a - \tilde{a})\| \leq 6\|a - \tilde{a}\| \leq \varepsilon.
\]

Finally, using Eq. (6.2) and Lemma 6.3 we obtain

\[
\|a \pi(\Delta_F) - a\| = \|(a - \tilde{a}) \pi(\Delta_F) - (a - \tilde{a})\| \leq 2\|a - \tilde{a}\| = \varepsilon /3,
\]

and the proof is complete. \( \square \)
Remark. If the set $I$ is countable, then the net $(\Delta_F)_{F \in \text{FIN}}$ has a subnet which is a sequence (take any enumeration of $I$ as $\{t_1, t_2, \ldots\}$ and let $\widetilde{\Delta}_n := \Delta_{\{t_1, \ldots, t_n\}}$). So if $I$ is countable, then $\ell^1(I_\vee)$ is sequentially approximately contractible.

A counter-example

While sequential approximate amenability implies bounded approximate amenability, the converse is false. This is proved by combining Theorem 6.1 with the following result.

Theorem 6.4. Let $\Lambda$ be an uncountable well-ordered set. Then $\ell^1(\Lambda_\vee)$ is not sequentially approximately amenable.

(Recall that a totally ordered set is well-ordered if every non-empty subset has a least element: well-ordered sets are precisely those ordered sets which are order-isomorphic to ordinals.)

In proving Theorem 6.4 we shall use some basic facts on the character theory of $\ell^1(\Lambda_\vee)$. It is clear that the characters on $\ell^1(\Lambda_\vee)$ correspond to the non-zero semigroup homomorphisms from $\Lambda_\vee$ to the two-element semigroup $\{0, 1\}$; and a little thought gives the following characterization.

Proposition 6.5. When regarded as elements of $\ell^\infty(\Lambda)$, the characters on $\ell^1(\Lambda_\vee)$ are all of the form $1_{\Lambda \setminus U}$, where $U$ is a proper (and possibly empty) subset of $\Lambda$ that is upwards-directed with respect to the given order on $\Lambda$.

Example 6.6. Take $\Lambda$ to be the real line with its usual ordering. Then the characters on $\ell^1(\Lambda_\vee)$ are either of the form $1_{(-\infty, t]}$ or $1_{(-\infty, t)}$.

If $U$ is a non-empty, upwards-directed subset of a well-ordered set $\Lambda$, then $U$ has a least element, $u$ say: hence $U = \{x \in \Lambda: x \geq u\}$. Thus the complements of upwards-directed sets are all of the form $\{y: y < u\}$.

If $\lambda$ is an element of a well-ordered set and it is not maximal, then there is a unique minimal element greater than $\lambda$, which we shall denote by $\lambda + 1$.

Notation. Let $\Lambda$ be a well-ordered set and consider the algebra $\ell^1(\Lambda_\vee)$. If $\lambda \in \Lambda$ we denote by $\widetilde{\lambda}$ the character $1_{\langle \lambda \rangle}$. If $\lambda$ is maximal in $\Lambda$ then we adopt the convention that $\widetilde{\lambda + 1}$ is the augmentation character $1_{\Lambda}$.

The following is then obvious: we isolate it as a lemma for later reference.

Lemma 6.7. Let $\Lambda$ be a well-ordered set and let $\lambda \in \Lambda$. Then

$$\delta_\lambda = \widetilde{\lambda + 1} - \widetilde{\lambda},$$

where $\delta_\lambda$ denotes the point mass at $\lambda$, regarded as an element of $\ell^1(\Lambda_\vee)^*$.

Our proof of Theorem 6.4 uses our earlier observations on the characters of $\ell^1(\Lambda_\vee)$, together with Lemma 2.8. Intuitively, the idea is that the Gelfand transforms of elements in $\ell^1(\Lambda_\vee) \otimes \ell^1(\Lambda_\vee)$ are bad approximations to the indicator function of the set $\{(\lambda, \lambda): \lambda \in \Lambda\}$, so that if $\Lambda$ is uncountable then no countable net $(\Delta_n)$ can have the properties described in Lemma 2.8. We make this idea precise as follows.
Lemma 6.8. Let $\mathcal{I}$ be an uncountable index set and let $(F_n)$ be a countable family in $\ell^1(\mathcal{I} \times \mathcal{I})^{**}$. Then there exist uncountably many $t \in \mathcal{I}$ such that

$$\langle F_n, \delta_t \otimes \delta_t \rangle = 0 \quad \text{for all } n.$$  

Proof. In view of the direct sum decomposition

$$\ell^1(\mathcal{I} \times \mathcal{I})^{**} = \ell^1(\mathcal{I} \times \mathcal{I}) \oplus c_0(\mathcal{I} \times \mathcal{I})^\perp$$

we may write each $F_n$ as $\kappa(f_n) + G_n$ where $G_n \in c_0(\mathcal{I} \times \mathcal{I})^\perp$, $f_n \in \ell^1(\mathcal{I} \times \mathcal{I})$ and $\kappa$ is the natural embedding of $\ell^1(\mathcal{I} \times \mathcal{I})$ in its bidual.

Let $S = \bigcup_n \{ t \in \mathcal{I}: f_n(t,t) \neq 0 \}$. Since each $f_n$ has countable support, $S$ is countable. In particular $\mathcal{I} \setminus S$ is uncountable, and for any $t \in \mathcal{I} \setminus S$ we have

$$\langle F_n, \delta_t \otimes \delta_t \rangle = (f_n)_{t,t} + \langle G_n, \delta_t \otimes \delta_t \rangle = 0,$$

as claimed. □

Proof of Theorem 6.4. Suppose $\ell^1(\Lambda_\vee)$ is sequentially approximately amenable. Since $\Lambda$ is well-ordered it has a least element, and consequently $\ell^1(\Lambda_\vee)$ has an identity element. Hence Lemma 2.8 applies and there is a sequence $\Delta_n \in (\ell^1(\Lambda_\vee) \hat{\otimes} \ell^1(\Lambda_\vee))^{**}$ such that

$$\langle \Delta_n, \varphi \otimes \varphi \rangle = 1 \quad \text{for all } n \quad \text{and} \quad \lim_n \langle \Delta_n, \varphi \otimes \chi \rangle = 0 \quad (6.6)$$

for every pair of distinct characters $\varphi, \chi$.

By Lemma 6.8 there exists $\lambda \in \Lambda$ such that $\langle \Delta_n, \delta_\lambda \otimes \delta_\lambda \rangle = 0$ for all $n$, and hence by Lemma 6.7,

$$0 = \begin{cases} 
\langle \Delta_n, \tilde{\lambda} \otimes \tilde{\lambda} \rangle - \langle \Delta_n, \tilde{\lambda} \otimes \tilde{\lambda} + 1 \rangle, \\
-\langle \Delta_n, \tilde{\lambda} + 1 \otimes \tilde{\lambda} \rangle + \langle \Delta_n, \tilde{\lambda} + 1 \otimes \tilde{\lambda} + 1 \rangle.
\end{cases}$$

But by Eq. (6.6) the right-hand side converges to 2 as $n \to \infty$, which is a flagrant contradiction. □

Remark. The proof just given yields something formally stronger, namely that $\ell^1(\Lambda_\vee)$ cannot have an approximate diagonal with countable indexing set. We do not pursue this further in this paper, chiefly because we know of no Banach algebra which has a countably-indexed approximate diagonal and yet has no sequential approximate diagonal.

7. Algebras of pseudo-functions on discrete groups

Let $\Gamma$ be a discrete group, with convolution algebra $\ell^1(\Gamma)$. Given $p \in (1, \infty)$ we may consider the left regular representation of $\Gamma$ on $\ell^p(\Gamma)$, and this gives an injective continuous algebra homomorphism $\theta_p : \ell^1(\Gamma) \to B(\ell^p(\Gamma))$. We denote by $PF_p(\Gamma)$ the norm-closure in $B(\ell^p(\Gamma))$ of the range of $\theta_p$. Note that $PF_2(\Gamma)$ is nothing but the reduced $C^*$-algebra of $\Gamma$.
If $\Gamma$ is amenable, then by Johnson’s theorem the convolution algebra $\ell^1(\Gamma)$ is amenable, and since amenability is inherited by closures under Banach algebra norms we deduce that $\text{PF}_p(\Gamma)$ is amenable; in particular the reduced $C^*$-algebra $C^*_r(\Gamma)$ is amenable. The converse result — that amenability of $C^*_r(\Gamma)$ implies amenability of $\Gamma$ — was proved by Bunce in [2]. With some modifications one can adapt his proof to show that amenability of any one of the algebras $\text{PF}_p(\Gamma)$ is enough to force amenability of $\Gamma$.

In [11] it was shown that approximate amenability of the group algebra $L^1(G)$ implies amenability of $G$, by generalizing the well-known argument for amenability of $L^1(G)$. We shall now show that by combining arguments from [2] and [11] we have the following theorem.

**Theorem 7.1.** Let $\Gamma$ be a discrete group. Then the following are equivalent:

1. $\Gamma$ is amenable;
2. $\text{PF}_p(\Gamma)$ is amenable for all $p \in (1, \infty)$;
3. $\text{PFT}_p(\Gamma)$ is approximately amenable for some $p \in (1, \infty)$;
4. $\text{PF}_p(\Gamma)$ is pseudo-amenable for some $p \in (1, \infty)$.

As mentioned above, the implications (i) $\iff$ (ii) are already known, while the implication (ii) $\Rightarrow$ (iii) is trivial; the implication (iii) $\Rightarrow$ (iv) follows from [14, Proposition 3.2] and only uses the fact that $\text{PF}_p(\Gamma)$ has an identity element. Therefore our contribution is to prove the implication (iv) $\Rightarrow$ (i). Taking $p = 2$, our proof will give a slightly streamlined version of Bunce’s arguments, in that we are able to forgo technical arguments with states on $C^*$-algebras in favour of more direct positivity arguments with measures on compact spaces.

Our idea is to follow Bunce’s construction up to the point where he produces, from the assumption that $C^*_r(\Gamma)$ is amenable, a non-zero element $\rho$ in $\ell^\infty(\Gamma)^*$ which satisfies

$$\rho(g \cdot f) = \rho(f) \quad \text{for all } f \in \ell^\infty(\Gamma).$$

(In [2] $\rho$ is described as being ‘left-invariant’: we adopt the opposite and more usual convention, and say $\rho$ is right-invariant.) In our setting we merely obtain a net $(\phi_\alpha)$ of functionals on $\ell^\infty(\Gamma)$ which satisfies

$$\phi_\alpha(1) \to 1 \quad \text{and} \quad \|\phi_\alpha \cdot g - \phi_\alpha\| \to 0 \quad \text{for each } g \in \Gamma.$$

We then use this net to obtain a “genuine” invariant mean on $\ell^\infty(\Gamma)$, by following the last part of the proof of [11, Theorem 3.2]. For convenience we isolate the relevant argument and state it as the following lemma.

**Lemma 7.2.** Let $G$ be a locally compact group, and let $T$ be a compact $G$-space on which $G$ acts from the right by homeomorphisms. Equip $M(T)$ with its usual norm, and regard it as a right Banach $G$-module.

Suppose we have a net $(\varphi_i)$ of Radon measures on $T$ which satisfies the following conditions:

1. $\inf_i \|\varphi_i\| > 0$;
2. $\|\varphi_i \cdot g - \varphi_i\| \to 0$ for all $g \in G$.

Then there exists a probability measure $n$ on $T$ such that $n \cdot g = n$. 
We give the proof for sake of completeness (cf. the proof of [11, Theorem 3.2]).

Proof. Set \(n_i = \|\varphi_i\|^{-1}\varphi_i\). The hypothesis that \(\|\varphi_i\|\) is bounded below then ensures that

\[
\|n_i \cdot g - n_i\| \to 0 \quad \text{for all } g \in G.
\]

For any two Radon measures \(\mu, \nu\) on \(T\) we have

\[
\|\mu - \nu\| \geq \|\mu| - |\nu|\|,
\]

an inequality which can easily be deduced from the definition of the total variation of a measure. Therefore, since \(|\mu \cdot g| = |\mu| \cdot g\) for any \(\mu \in M(T)\), we have

\[
\|n_i \cdot g - |n_i|\| \leq \|n_i \cdot g - n_i\| \to 0
\]

for every \(g \in G\).

Take \(n\) to be any \(w^*\)-cluster point of the net \((|n_i|)\). Since \(|n_i|(1) = 1\) for all \(i\), we have \(n(1) = 1\); and for any \(g \in G\) and \(f \in C(T)\), we have

\[
\left| (n \cdot g - n)(f) \right| \leq \limsup_i \left| (|n_i| \cdot g - |n_i|)(f) \right| = 0,
\]

so that \(n \cdot g = n\) for all \(g \in G\). \(\Box\)

Proof of Theorem 7.1, (iv) \(\Rightarrow\) (i). Our aim is to construct a right-invariant mean on \(\ell^\infty(\Gamma)\). To fix notation, we recall that the usual left action of \(\Gamma\) on \(\ell^\infty(\Gamma) = \ell^1(\Gamma)^*\) is defined by

\[
(g \cdot f)(x) = f(g^{-1}x) \quad \text{for } f \in \ell^\infty(\Gamma) \text{ and } g, x \in \Gamma.
\]

For each \(g \in \Gamma\) let \(L_g\) be the isometric, invertible operator on \(\ell^p(\Gamma)\) given by left translation, i.e.

\[
(L_gk)(x) = k(g^{-1}x) \quad \text{for all } k \in \ell^p(\Gamma).
\]

We regard \(PF_p(\Gamma)\) as a subalgebra of \(B(\ell^p(\Gamma))\). Take \(\tau\) to be the functional given by

\[
\tau(T) = \langle T\delta_e, \delta_e \rangle \quad (T \in B(\ell^p(\Gamma))),
\]

where \(\delta_e\) is the basis vector of \(\ell^p(\Gamma)\) that takes the value 1 at \(e\) and the value 0 everywhere else. Clearly \(\tau(I) = 1\). A simple calculation shows that for any \(a, b\) in the group algebra \(\mathbb{C}\Gamma\), we have

\[
\tau(\theta_p(a)\theta_p(b)) = \tau(\theta_p(b)\theta_p(a)),
\]

and so by continuity the restriction of \(\tau\) to \(PF_p(\Gamma)\) defines a non-zero trace.

Suppose that \(PF_p(\Gamma)\) is pseudo-amenable. By Lemma 2.7, there exists a net \((\psi_\alpha)\) in \(B(\ell^p(\Gamma))^*\) such that

\[
\lim_\alpha \psi_\alpha(I) = 1 \quad (7.1)
\]

and

\[
\lim_\alpha \left( \sup_{T \in B(\ell^p(\Gamma)), \|T\| \leq 1} |\psi_\alpha(aT - Ta)| \right) = 0 \quad \text{for all } a \in PF_p(\Gamma).
\]
In particular, for any \( g \in \Gamma \) we have

\[
\sup_{M \in B(\ell^p(\Gamma)), \|M\| \leq 1} \left| \psi_\alpha(L_g M L_{g^{-1}}) - \psi_\alpha(M) \right| \\
\leq \sup_{T \in B(\ell^p(\Gamma)), \|T\| \leq 1} \left| \psi_\alpha(L_g T - T L_g) \right| \to 0. \tag{7.2}
\]

Regard \( \ell^\infty(\Gamma) \) as an algebra with pointwise multiplication and supremum norm. There is an embedding of \( \ell^\infty(\Gamma) \) as a closed unital subalgebra of \( B(\ell^p(\Gamma)) \), defined by sending a bounded function \( f \in \ell^\infty(\Gamma) \) to the “diagonal multiplication” operator \( M_f \) where \( (M_f k)(x) = f(x) k(x) \) for all \( k \in \ell^p(\Gamma) \) and \( x \in \Gamma \). Then a direct calculation shows that

\[
M(g \cdot f) = L_g M_f (L_g)^{-1} \quad \text{for all } f \in \ell^\infty(\Gamma) \text{ and } g \in \Gamma. \tag{7.3}
\]

For each \( \alpha \) define \( \phi_\alpha \in \ell^\infty(\Gamma)^* \) by \( \phi_\alpha(f) = \psi_\alpha(M_f), f \in \ell^\infty(\Gamma) \). It follows from Eqs. (7.1), (7.2) and (7.3) that \( \lim_\alpha \phi_\alpha(1) = 1 \), and that

\[
\lim_\alpha \|\phi_\alpha \cdot g - \phi_\alpha\| = \sup_{f \in \ell^\infty(\Gamma), \|f\| \leq 1} \left| \phi_\alpha(g \cdot f) - \phi_\alpha(f) \right| = 0 \quad \text{for all } g \in \Gamma.
\]

To finish we observe that \( \ell^\infty(\Gamma) \) may be identified with the space of continuous functions on a compact \( \Gamma \)-space \( T \) (namely, take \( T \) to be the Stone–Čech compactification of \( \Gamma \)), and hence we may identify each \( \phi_\alpha \) with a Radon measure on \( T \). By Lemma 7.2, there exists a positive functional \( n \in \ell^\infty(\Gamma)^* \) satisfying \( n(1) = 1 \) and \( n \cdot g = n \) for all \( g \in \Gamma \), and hence \( \Gamma \) is amenable as claimed. \( \square \)

Specializing to the case \( p = 2 \) (i.e. the reduced \( C^* \)-algebra \( C^*_r(\Gamma) \)), we have the following corollary.

**Corollary 7.3.** The full group \( C^* \)-algebra \( C^*(\Gamma) \) is approximately amenable if and only if \( \Gamma \) is amenable.

**Proof.** We first recall without proof some basic facts about \( C^*(\Gamma) \): firstly, it is by definition the completion of \( \ell^1(\Gamma) \) in a certain \( C^* \)-norm; and secondly, there is a canonical quotient homomorphism from \( C^*(\Gamma) \) onto \( C^*_r(\Gamma) \).

Now, suppose that \( \Gamma \) is amenable: then \( \ell^1(\Gamma) \) is amenable. As just mentioned, the inclusion homomorphism \( \ell^1(\Gamma) \to C^*(\Gamma) \) is continuous with dense range, and therefore \( C^*(\Gamma) \) must also be amenable.

Conversely, suppose that \( C^*(\Gamma) \) is approximately amenable. By [11, Proposition 2.2], approximate amenability passes to quotient algebras, and so \( C^*_r(\Gamma) \) is approximately amenable. Now apply Theorem 7.1 in the case \( p = 2 \). \( \square \)

**Remark.** Using the fact that the canonical tracial state \( \tau \) on \( C^*_r(\Gamma) \) actually extends to a tracial state on the von Neumann algebra \( VN(\Gamma) \), we can adapt the proof of Theorem 7.1 to show the following result: if \( A \) is a closed unital subalgebra of \( VN(\Gamma) \), with \( C^*_r(\Gamma) \subseteq A \), and furthermore \( A \) is approximately amenable, then \( \Gamma \) is amenable.
Appendix A. Implications

In this appendix we give some schematic diagrams. The first illustrates the known implications between various notions of approximate amenability; the second illustrates what is known for commutative Banach algebras; and the third illustrates some partial results concerning approximate identities. We hope that these pictorial representations clarify some of the relationships between the notions considered in this paper.

In these diagrams, approximate amenability has been abbreviated to ‘AA’, and approximate contractibility to ‘AC’; similarly for their bounded variants. ‘PsA’ and ‘PsC’ denote pseudo-amenability and pseudo-contractibility, respectively. ‘BAI’, ‘CAI’ and ‘MBAI’ respectively denote the presence of a bounded, central and multiplier-bounded approximate identity. In the second diagram, ‘comm.’ is an abbreviation for commutativity.

Implications are denoted by solid arrows: dashed arrows with a × in the middle denote the failure of an implication. The label def on an arrow means that the corresponding implication holds “by definition” or a fortiori. Labels in square brackets refer to items in the bibliography. In the third diagram, the labels 3.3 and 3.4 refer, respectively, to Theorem 3.3 and Corollary 3.4 of the present article.

A.1. General implications

Here the counter-example (⋆) to ‘pseudo-amenability implies pseudo-contractibility’ follows because unital pseudo-contractible algebras must be contractible [14, Theorem 2.4], while there are unital pseudo-amenable algebras which are not even amenable: perhaps the simplest example is $\ell^1(\mathbb{N}, \text{max})$.

A.2. Commutative settings

Here the counterexample to ‘pseudo-contractible implies approximately amenable’ is given by $\ell^1(\mathbb{N})$ with pointwise multiplication.
A.3. Approximate identities

\[
\begin{align*}
\text{PsA} + \text{BAI} & \overset{[14, \text{Proposition 3.2}]}{\leftrightarrow} \text{AA} + \text{BAI} \overset{\text{def}}{=} \text{BAA} + \text{MBAI} \\
\text{BAC} & \overset{3.4}{\leftrightarrow} \text{BAC} + \text{BAI} \overset{\text{def}}{=} \text{BAA} + \text{BAI}
\end{align*}
\]

References

