Let $R$ be a regular noetherian local ring of dimension $n \geq 2$ and $(R_i) = R = R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_i \subset \cdots$ be a sequence of successive quadratic transforms along a regular prime ideal $p$ of $R$ (i.e. if $p_i$ is the strict transform of $p$ in $R_i$, then $p_i \neq R_i$, $i \geq 0$). We say that $p$ is maximal for $(R_i)$ if for every non-negative integer $j \geq 0$ and for every prime ideal $q_j$ of $R_j$ such that $(R_i)$ is a quadratic sequence along $q_j$ with $p_j \subset q_j$, we have $p_j = q_j$. We show that $p$ is maximal for $(R_i)$ if and only if $V = \bigcup_{i=0}^{\infty} R_i/p_i$ is a valuation ring of dimension one. In this case, the equimultiple locus at $p$ is the set of elements of the maximal ideal of $R$ for which the multiplicity is stable along the sequence $(R_i)$, provided that the series of real numbers given by the multiplicity sequence associated with $V$ diverges. Furthermore, if we consider an ideal $J$ of $R$, we also show that $\text{Spec}(R/J)$ is normally flat along $\text{Spec}(R/p)$ at the closed point if and only if the Hironaka's character $\nu^*(J, R)$ is stable along the sequence $(R_i)$. This generalizes well known results for the case where $p$ has height one (see [B.M. Bennett, On the characteristic functions of a local ring, Ann. of Math. Second Series 91 (1) (1970) 25–87]).
such that \( \pi_i \) is the blowing-up of \( X_{i-1} \) with center a closed point \( x_{i-1} \in X_{i-1}, i \geq 1 \). If we write \( R_i = O_{X_i,x_i}, i \geq 0 \), then we have commutative diagrams

\[
\begin{array}{ccc}
X_i & \xrightarrow{\pi_i} & X'_i \\
\downarrow & & \downarrow \\
X_{i-1} & \xrightarrow{\pi'_i} & \text{Spec}(R_{i-1})
\end{array}
\]

where \( \pi'_i \) is the blowing-up of \( \text{Spec}(R_{i-1}) \) with center the closed point \( x_{i-1} \) defined by the maximal ideal of \( R_{i-1}, i \geq 1 \).

Therefore, from sequence (\ast), we get to the sequence of noetherian local regular rings of the same dimension (notice that the centers are closed points

\[
(R_i) \equiv R = R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_i \subset \cdots
\]

such that \( R_i \) is a quadratic transform of \( R_{i-1} \), \( i \geq 1 \). (See [2], p. 141.) In particular, the residue field of \( R_i \) is an algebraic extension of the residue field of \( R_{i-1}, i \geq 1 \).

To study the sequence (\ast), we assume that there exists a non-negative integer \( k \) and a subvariety \( Y_k \) of \( X_k \) such that \( Y_k \) goes through all the \( x_i, i \geq k \), i.e. if \( Y_i \) is the strict transform of \( Y_k \) by \( \pi_i \circ \cdots \circ \pi_{k+2} \circ \pi_{k+1}, i \geq k + 1 \), then \( x_i \in Y_k, i \geq k \).

Thus, there exists \( p_k \in \text{Spec}(R_k) \) such that \( p_k \) defines locally \( Y_k \) at \( x_k \) and \( (R_i) \) is a quadratic sequence along \( p_k \). (See [3], p. 67.)

Note that \( \bigcup_{i \geq 0} R_i \subset (R_i) p_k \) and that \( p_k \) cannot be the maximal ideal of \( R_k \).

As we have said, the above situation is classical in resolution and classification of singularities. In fact, if we consider a closed subscheme \( Z_k \) of \( X_k \) with \( x_k \in Z_k \) and some invariant associated with \( (Z_k, x_k) \) that remains stable throughout the sequence (\ast) (i.e. the invariant is the same for all \( (Z_i, x_i) \), where \( Z_i \) is the strict transform of \( Z_k \) by \( \pi_i \circ \cdots \circ \pi_{k+2} \circ \pi_{k+1}, i \geq k + 1 \)), then there exists a variety \( Y_k \) as above. This is the case when the invariant is the Hilbert–Samuel function of \( (Z_k, x_k) \) and \( Y_k \) is a (smooth) variety that has maximal contact with \( Z_k \) at \( x_k \). If the Hilbert–Samuel function is stable throughout the sequence (\ast), then \( Y_k \) has also maximal contact with \( Z_k \) at \( x_k, i \geq k \). (See for example [4,5].)

The main results of this paper are obtained under the condition that \( R_k/p_k \) is a regular noetherian local ring (i.e. \( Y_k \) is not singular at \( x_k \)). In this case, we have the sequence

\[
O_{Y_k,x_k} = \frac{R_k}{p_k} \subset O_{Y_{k+1},x_{k+1}} = \frac{R_{k+1}}{p_{k+1}} \subset \cdots \subset O_{Y_{k+i},x_{k+i}} = \frac{R_{k+i}}{p_{k+i}} \subset \cdots
\]

of noetherian local regular rings of the same dimension such that \( R_{k+i}/p_{k+i} \) is a quadratic transform of \( R_{k+i-1}/p_{k+i-1}, i \geq 1 \), where \( p_{k+i} = R_{k+i} \cap p_k(R_k) p_k, i \geq 0 \).

Among the prime ideals \( p_k \) for which \( (R_i) \) is a quadratic sequence along \( p_k \), we consider those that we call maximal for the sequence, i.e. \( p_k \) is maximal for \( (R_i) \), if for every non-negative integer \( j \geq k \) and for every prime ideal \( q_j \) of \( R_j \) such that \( (R_i) \) is a quadratic sequence along \( q_j \) with \( p_k(R_k) p_k \cap R_j \subset q_j \), we have \( p_k(R_k) p_k \cap R_j = q_j \). (See Definition 6.)

Our first result (Theorem 8) gives the following characterization: \( p_k \) is maximal for \( (R_i) \) if and only if \( V_k = \bigcup_{i \geq 0} \frac{R_{k+i}}{p_{k+i}} \) is a valuation ring of dimension one.

The rest of the paper is devoted to study the relation between \( V_k \) and the stabilization of some invariants along the sequence \( (R_i) \). For this, we consider the multiplicity sequence \( \{n_i\}_{i \geq k} \) associated with the valuation \( v_k \) of the ring \( V_k \), where \( n_i = \min\{v_k(f) : f \text{ is the maximal ideal of } R_i/p_i\} \), \( i \geq k \). Note that \( \{n_i\}_{i \geq k} \) is a sequence of non-negative real numbers.

In Section 4, we study the stabilization of the usual multiplicity for elements \( f \) of the maximal ideal of \( R_k \). Thus, in Theorem 10 we show that the series \( \sum_{i \geq k} n_i \) diverges if and only if the equimultiple locus at \( p_k \) is the subset of elements \( f \) in the maximal ideal of \( R_k \) such that the multiplicity of the hypersurface \( Z_k(f) \) defined by \( f \) at \( x_k \) is stable along the sequence (\ast), i.e. if \( Z_{k+i}(f) \) is the strict transform of \( Z_k(f) \) by \( \pi_{k+i} \circ \cdots \circ \pi_{k+2} \circ \pi_{k+1}, i \geq 1 \), then \( x_{k+i} \in Z_{k+i}(f) \) and \( Z_{k+i}(f) \) has constant multiplicity \( d \) at \( x_{k+i}, i \geq 0 \).

A particular case is when \( p_k \) has height \( n - 1 \). Note that, in this case, \( V_k = \frac{R_{k+i}}{p_{k+i}}, i \geq 0 \) is a discrete valuation ring, \( n_k = n_{k+i}, i \geq 0 \) and the series \( \sum_{i \geq k} n_i \) diverges. In [3] some results about normal flatness and stabilization of the Hironaka’s character along the sequence \( (R_i) \) are given, provided that \( p_k \) has height \( n - 1 \). Theorem 10 extends these results when \( p_k \) is maximal for \( (R_i) \), \( R_i/p_k \) is a local regular ring and the series \( \sum_{i \geq k} n_i \) diverges, without any condition on the height of \( p_k \). In fact, we show that an ideal \( J_k \) of \( R_k \) is normally flat along \( p_k \) at the maximal ideal of \( R_k \) if and only if Hironaka’s character \( v^* (J_k, R_k) \) is stable along the sequence \( (R_i) \). We recall that \( v^* (J_k, R_k) \) is nothing but the sequence of multiplicities of elements of a standard base of \( J_k \) (see [3] 0(3.1) or [6], Remark 1).

Finally, we point out that similar results about stabilization along sequences (\ast), for example Theorem 1 of [7] can be obtained as a particular case of our results.

2. Notations and preliminaries

All the rings considered are commutative and with unit element. For a noetherian local ring \( R \), we denote by \( M(R) \) the maximal ideal of \( R \) and by \( \dim(R) \) the Krull dimension of \( R \). Also, for each non-zero ideal \( J \) of \( R \) we denote by \( \text{Ord}_d(J) \) the non-negative integer \( d \) such that \( J \subset (M(R))^d \) and \( J \nsubseteq (M(R))^{d+1} \).
From now on we assume that $R$ is a regular noetherian local ring of dimension $n \geq 2$ and write $X_0 = \text{Spec}(R)$. We denote by $\pi_1 : X_1 \longrightarrow X_0$ the blowing-up of $X_0$ with center the closed point $x_0$ defined by $M(R)$. For any $x_1 \in X_1$ with $\pi_1(x_1) = x_0$, the ring $R_1 = O_{X_1,x_1}$ is a quadratic transform of $R$. (See [2], p. 141.) Note that we can take a base $(y_1, \ldots, y_n)$ of $M(R)$ such that $R_1 = \left( R \left[ \frac{y_1}{\gamma_1}, \ldots, \frac{y_n}{\gamma_n} \right] \right)_Q$, where $Q$ is a prime ideal of $R \left[ \frac{y_1}{\gamma_1}, \ldots, \frac{y_n}{\gamma_n} \right]$ with $M(R) \subset Q$.

In general, we have $\dim(R) \geq \dim(R_1)$, see (1.4.2) (3) of [8]. Throughout this paper we will only consider quadratic transforms such that $\dim(R) = \dim(R_1)$, in particular, $R_1/M(R_1)$ is an algebraic extension of $R/M(R)$.

Let $J$ be a (coherent) non-zero ideal of $R$ and write $Z_0 = \text{Spec}(R/J)$. Then $Z_0$ is a closed subscheme of $X_0$ and $x_0 \in Z_0$. We denote by $Z_1$ the strict transform of $Z_0$, which is a closed subscheme of $X_1$. Following, for example, Proposition 1.6, p. II-7 of [9], if we take $f_1, \ldots, f_s \in J$ such that the initial forms $l_{M(R)}(f_1), \ldots, l_{M(R)}(f_s)$ are a set of generators of $\sqrt{M(R)} \cdot R$ as a $\sqrt{M(R)}(R)$-module, then $Z_1$ is locally defined by $f_1', \ldots, f_s'$, where $f_j' = f_j/\gamma_j$). Hence, $\dim(Z_1) = n_j, 1 \leq j \leq s$.

$$\sqrt{M(R)}(R) = \bigoplus_{i \geq 0}(M(R))^{i}/(M(R))^{i+1},$$

$$\sqrt{M(R)}(J, R) = \bigoplus_{i \geq 0}(J \cap (M(R))^{i}/(J \cap (M(R))^{i+1}) and \gamma_i defines locally the exceptional divisor $\pi^{-1}(x_0)$. Sometimes it is useful to consider the strict transform of $Z_0$ when $x_0 \not\in Z_0$. In this case, we have $Z_1 = Z_0$, notice that $\pi_1$ is an isomorphism outside its center.

Now, let us consider a sequence of regular noetherian local rings of the same dimension $R = R_0 \subset R_1 \subset \cdots \subset R_N$ such that $R_i$ is a quadratic transform of $R_{i-1}, 1 \leq i \leq N$. We have a sequence

$$X_N \xrightarrow{\pi_N} X_{N-1} \xrightarrow{\pi_{N-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0$$

such that $\pi_i$ is the blowing-up of $X_{i-1}$ with center a closed point $x_{i-1}, 1 \leq i \leq N$ and $R_i = O_{X_i,x_i}, 0 \leq i \leq N$. Since blowing-up commutes with flat morphisms, we have commutative diagrams

$$\begin{align*}
X_i & \xleftarrow{\pi_i'} \quad X_i' \\
\pi_{i-1}' & \xrightarrow{\pi_i'} \\
X_{i-1} & \xleftarrow{\pi_i} \quad \text{Spec}(R_{i-1})
\end{align*}$$

where $\pi_i'$ is the blowing-up of $\text{Spec}(R_{i-1})$ with center the closed point $x_{i-1}$ defined by $M(R_{i-1}), 1 \leq i \leq N$. In particular, $R_i = O_{X_i,x_i} = O_{X_i,x_i}, 0 \leq i \leq N$ with $\pi_i(x_{N}) = x_{N-1}$. Furthermore, we can consider $Z_2$ the strict transform of $Z_1$ and so on. Note that $Z_2 = Z_1$ if $x_{N-1} \not\in Z_2, 1 \leq i \leq N$ and if $x_i \not\in Z_i$, then $x_k \not\in Z_k$ for $k \geq i$.

We point out that $Z_i$ is locally defined at $x_i$ by an ideal $J_i$ of $R_i = O_{X_i,x_i}$ as follows:

1. If $i = 0$, then $J_0 = J$.
2. If $i > 0$ and $J_{i-1}$ is given, then $J_i$ is the ideal of $R_i$ generated by all $\alpha/y^m$ with $\alpha \in J_{i-1}$ and $m \leq \text{Ord}_{R_{i-1}}(\alpha R_{i-1}),$ where $y R_i = M(R_{i-1})/R_i$.

We will say that $(R_i, J_i)$ is the strict transform of $(R, J)$ in $R_i, 0 \leq i \leq N$. (See [3], (3.2.4).) Notice that $x_i \not\in Z_i$ if and only if $J_i = R_i$.

In particular, if $Z$ is given by a non-zero principal ideal $J$ of $R$, then $J_i$ is also a non-zero principal ideal of $R_i = O_{X_i,x_i}$ such that

$$J_i(M(R_{i-1}))^{m_{i-1}} R_i = J_{i-1} R_i,$$

with $m_{i-1} = \text{Ord}_{R_{i-1}}(J_{i-1}), i \geq 1$.

At this point, let us recall some facts about valuations.

Let $v$ be a 0-dimensional valuation of the quotient field $K(R)$ of $R$ dominating $R$. Thus, if $V$ is the valuation ring associated with $v$, we have $R \subset V$ and $M(R) = M(V) \cap R$. Moreover, since $v$ is 0-dimensional, then $V/M(V)$ is an algebraic (possibly infinite) extension of $R/M(R)$.

Associated with the pair $(R, V)$ we have the sequence of noetherian local regular rings of the same dimension

$$(R_i) \equiv R = R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_i \subset \cdots \subset V,$$

where $R_i$ is the quadratic transform of $R_{i-1}$ along $V, i \geq 1$. (See [2], p. 141.) As $\dim(R_i) = \dim(R_{i+1}),$ for $i \geq 0,$ then $R_{i+1}/M(R_{i+1})$ is an algebraic extension of $R_i/M(R_i), i \geq 0.$ (See [8], (1.4.2) p. 17.)

Therefore, we have an infinite sequence

$$\cdots \xrightarrow{\pi_{i+1}} X_i \xrightarrow{\pi_i} X_{i-1} \xrightarrow{\pi_{i-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0$$

such that $\pi_i$ is the blowing-up of $X_{i-1}$ with center a closed point $x_{i-1}, i \geq 1$ and $R_i = O_{X_i,x_i}, i \geq 0.$ Notice that $v$ has center $x_i$ on $X_i, i \geq 0$.

We write $\tilde{R} = \bigcup_{i \geq 0} R_i$. We point out that the valuation rings dominating $\tilde{R}$ are not univocally determined by the sequence $(R_i), i.e.,$ in general, $\tilde{R}$ is not a valuation ring. A characterization of when $\tilde{R}$ is a valuation ring in terms of the sequence $(R_i)$ is given in [10], Theorem 13. This characterization completes Shannon’s one for the case of real rank one valuations. (See [11].)
On the other hand, let us assume that rank \((v) = r \geq 1\), where rank \((v)\) denotes the real rank of the valuation \(v\). Thus, we can write \(v : K(R) - \{0\} \rightarrow \Gamma'\), where \(\Gamma'\) is a totally ordered abelian subgroup of \(\mathbb{R}^r\) lexicographically ordered. (See [12].)

Note that \(n \geq r\).

The multiplicity sequence \(\{n_i\}_{i \geq 0} \subset \Gamma'\) associated with \((R, V)\) is defined by \(n_i = \min\{v(z) : z \in M(R_i) - \{0\}\}, n_i \geq 0\). Furthermore, since \(M(R_i) \subset M(R_{i+1})\), we have \(n_{i+1} \leq n_i, i \geq 0\).

The properties of the multiplicity sequence as well as other invariants are stated in [13–15]. In particular, Proposition 23 of [13] gives a sufficient condition on the multiplicity sequences to get that \(R\) is a valuation ring associated with a valuation of real rank one.

To finish this section we give a technical result.

**Lemma 1.** Let \(R_1\) be a quadratic transform of \(R\) and let \(p\) be a prime ideal of \(R\) such that \(R_1 \subset R_p\). Then \(y + f \notin p\) for each \(y \in R\) such that \(yR_1 = M(R)R_1\) and for each \(f \in M(R)^2\). In particular, \(y \notin p\).

**Proof.** We can write \(R_1 = \left( R \left[ \frac{y_1}{y}, \ldots, \frac{y_n}{y} \right] \right)_{\mathfrak{q}}, \) where \((y, y_2, \ldots, y_n)\) is a base of \(M(R)\).

Let us assume that \(x = y + f \in p\) for some \(f \in M(R)^2\). We have \(\frac{x}{y} = 1 + \frac{f}{y} \in R_1 - M(R_1)\). Since \(f \in M(R)^2\), we have \(f/y \in M(R_1)\).

On the other hand, \(y \notin p\). Otherwise, \(y \in p, y_2/y \in R_1 \subset R_p\) and \(\frac{y_2}{y} = \frac{h}{g}, h, g \in R\) without common factors and \(g \notin p\).

Hence, \(y_2g = yh\) and \(g \in yR \subset p\), which is a contradiction.

Therefore, \(\frac{x}{y} \notin pR_p\) and \(\frac{x}{y}\) is invertible in \(R_1\) and, hence, in \(R_p\), which is a contradiction. \(\Box\)

**Remark 2.** We point out that if \(p\) is a prime ideal such that \(y \in R(M(R))\) \(-\mathbb{M}(R)\) with \(y + f \notin p\) for all \(f \in M(R)^2\), then there exists a quadratic transform \(R_1\) of \(R\) such that \(yR_1 = M(R)R_1\) and \(R_1 \subset R_p\).

On the other hand, if, as above, \(\pi_1 : X_1 \rightarrow X_0 = \text{Spec}(R)\) is the blowing-up of \(X_0\) with center the closed point \(x_0\) defined by \(M(R)\) and if \(Y_0 = \text{Spec}(R/p)\) is the closed subscheme of \(X_0 = \text{Spec}(R)\) defined by \(p\), then the condition \(R_1 \subset R_p\) is equivalent to \(x_1 \in Y_1\), where \(Y_1\) is the strict transform of \(Y_0\) by \(\pi_1\) and \(R_1 = O_{X_1, x_1}\). Note that since \(p \subset M(R)\), then \(x_0 \notin Y_0\).

Moreover, \(O_{Y_1, x_1} = \frac{O_{X_1, x_1}}{p_1}\), where \(p_1 = R_1 \cap pR_p\), i.e. \((R_1, p_1)\) is the strict transform of \((R, p)\) in \(R_1\) and \(p_1\) defines \(Y_1\) locally at \(x_1\).

### 3. Quadratic sequences along prime ideals

In this section we study sequences of quadratic transforms along prime ideals. (See [3], pp. 67.)

Let \(R\) be a noetherian local regular ring of dimension \(n \geq 2\). Let \((R_i) \equiv R = R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_i \subset \cdots \subset \) be a sequence of noetherian local regular rings of the same dimension such that \(R_i\) is a quadratic transform of \(R_{i-1}\) for \(i \geq 1\).

As in the above section, we have an infinite sequence

\[
\cdots \rightarrow X_j \xrightarrow{\pi_{i-1}} X_{i-1} \xrightarrow{\pi_{i-2}} \cdots \rightarrow X_1 \xrightarrow{\pi_1} X_0
\]

such that \(\pi_i\) is the blowing-up of \(X_{i-1}\) with center a closed point \(x_{i-1}, i \geq 1\) and \(R_i = O_{X_i, x_{i}}, i \geq 0\).

**Definition 3.** Let \(p_k\) be a prime ideal of \(R_k, k \geq 0\). We say that \((R_i)\) is a **quadratic sequence along** \(p_k\), if \(p_k^{(j)} \neq R_j\) for \(j \geq k\), where \((R_j, p_k^{(j)})\) is the strict transform of \((R_k, p_k)\) in \(R_j, j \geq k\).

**Remark 4.** We note that \((R_i)\) is a quadratic sequence along \(p_k\) if and only if \(\bigcup_{j \geq k} R_j \subset (R_k)_{p_k}\). (See Lemma 1 and Remark 2.) Furthermore, always \((R_i)\) is a quadratic sequence along the zero ideal of \(R_k, k \geq 0\) and \((R_i)\) is never a quadratic sequence along \(M(R_k), k \geq 0\).

On the other hand, assuming that \((R_i)\) is a quadratic sequence along the prime ideal \(p_k\) of \(R_k\), we have:

1. \((R_i)\) is a quadratic sequence along each prime ideal \(q_k\) of \(R_k\) such that \(q_k \subset p_k\).
2. \((R_i)\) is a quadratic sequence along \(p_j\), where \(p_j = p_kR_k \cap R_j, j \geq k\).
3. If \(p_k \cap R_{k-1} = p_{k-1} \neq M(R_{k-1})\), then \((R_i)\) is a quadratic sequence along \(p_{k-1}\).

Finally, if we write \(W_k = \{p_k \in \text{Spec}(R_k) : (R_i)\) is a quadratic sequence along \(p_k\}\), \(k \geq 0\), then \((1)\) means nothing else that \(W_k\) is generically stable for \(k \geq 0\), i.e., if \(y \in W_k\) and \(x \in X_k\) with \(y \in \overline{\{x\}}\), the closure of \(\{x\}\) in \(X_k\), imply \(x \in W_k\). Moreover, by \((2)\) \(W_k \subset \pi_{k+1}(W_{k+1})\) and by \((3)\) \(W_k = \pi_{k+1}(W_{k+1})\) if and only if \(M(R_k)R_{k+1} \notin W_{k+1}, k \geq 0\).
In terms of blowing-ups we have the following result for sequences of quadratic transforms along prime ideals.

**Lemma 5.** Let $p$ be a prime ideal of $R = R_0$ and let $Y_0 = \text{Spec}(R/p)$ be the closed irreducible subscheme of $X_0$ defined by $p$. Then $(R_i)$ is a quadratic sequence along $p$ if and only if $X_i \subseteq Y_i$ for $i \geq 0$, where $Y_i$ is the strict transform of $Y_{i-1}$ by $\pi_i$, $i \geq 1$. Moreover, for $j \geq 0$ we have $O_{Y_j,x_j} = \frac{O_{Y_j,x_j}}{p_j}$, where $p_j = p_{R_j} \cap R_j$. In particular, if $x_i$ is a non-singular point of $Y_i$, then $x_i$ is a non-singular point of $Y_k$ for $k \geq i$. In this case, $O_{Y_{k+1},x_{k+1}}$ is a quadratic transform of $O_{Y_k,x_k}$, $k \geq j$.

**Proof.** The proof is straightforward. (See also Remark 2.) □

**Definition 6.** We say that a prime ideal $p_k$ of $R_k$, $k \geq 0$ is maximal for the sequence $(R_i)$ if

1. $(R_i)$ is a quadratic sequence along $p_k$.
2. For every non-negative integer $j \geq k$ and for every prime ideal $q_j$ of $R_j$ such that $(R_i)$ is a quadratic sequence along $q_j$ with $p_k(R_k)q_k \cap R_j \subseteq q_j$, we have $p_k(R_k)p_k \cap R_j = q_j$.

**Remark 7.** With the notation as in Remark 4, $p_k \in \text{Spec}(R_k)$ is maximal for the sequence $(R_i)$ if and only if $p_k \in W_k$ and $W_k \cap (p_k) = (p_k)$. Furthermore, in the situation of Lemma 5, $p$ is maximal for the sequence $(R_i)$ if and only if $x_i \in Y_i$ for $i \geq 0$ and for any proper closed irreducible subscheme $Y'$ of $Y_0$ there exists a non-negative integer $i_0$ such that $x_j \notin Y'_j$ for $j \geq i_0$, where $Y'_j$ is the strict transform of $Y'_{j-1}$ by $\pi_j$, $j \geq 1$.

Moreover, we note that if $(R_i)$ is a quadratic sequence along $p_k$ and $R_k$ has height $n - 1$ (i.e. $\text{dim}(R_k/p_k) = n - 1$), then $p_k$ is necessarily maximal for the sequence $(R_i)$.

The next result characterizes maximal prime ideals $p_k$ for sequences $(R_i)$, provided that $R_k/p_k$ is a regular ring.

**Theorem 8.** Let us assume that $(R_i)$ is a quadratic sequence along the prime ideal $p_k$ of $R_k$, $k \geq 0$ and that $R_k/p_k$ is a regular ring. Then the following statements are equivalent:

1. $p_k$ is maximal for the sequence $(R_i)$.
2. $\bigcup_{j \geq k} \frac{R_j}{p_j}$ is a valuation ring of dimension one (i.e. its associated valuation has real rank one).

Here $p_j = p_k(R_k)p_k \cap R_j$ for $j \geq k$.

**Proof.** After taking the sequence $R_k \subseteq R_{k+1} \subseteq R_{k+2} \subseteq \cdots \subseteq R_{k+i} \subseteq \cdots$ instead of $(R_i)$, that $R = R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots \subseteq R_i \subseteq \cdots$, we can assume $k = 0$ without loss of generality.

Since $(R_i)$ is a quadratic sequence along $p_0$ and $R_0/p_0$ is a regular ring, then $\frac{R_j}{p_j}$ is a regular ring and $\frac{R_j}{p_j} \cap R_j$ is a quadratic transform of $\frac{R_j}{p_j}$, $j \geq 0$. (See Lemma 5.)

First, let us assume that $\bigcup_{j \geq 0} \frac{R_j}{p_j}$ is not a valuation ring of dimension one. By Theorem 13 of [10] and Proposition 6 of [13], there exists a non-negative integer $l$ and a height one prime ideal $\eta$ of $\frac{R_j}{p_j}$ such that $\bigcup_{j \geq 0} \frac{R_j}{p_j} \subseteq \left(\frac{R_j}{\eta}\right)$. Let us write

$$\left(\frac{R_j}{\eta}\right) = \left(\frac{R_j}{p_j}\right) \cap \left(\frac{R_j}{p_j}\right),$$

$j \geq l$. We point out that

$$\left(\frac{R_j}{p_j}\right) = \left(\frac{R_j}{p_j}\right) \text{ for } j \geq l.$$

Let $q_j$ be the prime ideal of $R_j$ such that $\eta_j(q_j) = \frac{R_j}{p_j}$ for $j \geq l$, where $\eta_j : R_j \rightarrow R_j/p_j$ denotes the canonical epimorphism, $j \geq 0$. Note that $\eta_j$ is the restriction of $\eta_j$ to $R_j$, $s \geq j$ and $\left(\frac{R_j}{p_j}, \eta_j\right)$ is the strict transform of $\left(\frac{R_j}{p_j}, \eta\right)$ in $\frac{R_j}{p_j}$, $j \geq j$.

Next we claim that $\eta_j R_j = (R_j + 1)q_j + 1$, for $j \geq l$.

Since $\frac{R_j}{p_j} \cap q_j + 1 = \frac{R_j}{p_j}$, we have $q_j + 1 \cap R_j = q_j$ and $(R_j)_q \subseteq (R_j + 1)q_j + 1$, for $j \geq l$.

On the other hand, let $y_j \in R_j$ be such that $y_jR_j + 1 = M(R_jR_j + 1)$. We have

$$\eta_j(y_j) \frac{R_j + 1}{p_j + 1} = M \left(\frac{R_j}{p_j}\right) \frac{R_j + 1}{p_j + 1}.$$

By Lemma 1, $\eta_j(y_j) \notin \eta_j$. Thus, $y_j \notin q_j = q_j + 1 \cap R_j$ and $y_j \notin q_j + 1$. Hence, $y_j \notin p_j$, also by Lemma 1.

Let us consider $h \in (R_j + 1)q_j + 1$. We can write $h = f/(y_j)^\alpha g$ with $f, g \in R_j$, $\alpha, \beta$ non-negative integers and $(g/y_j)^\beta \notin q_j + 1$. Therefore, either $h = f/(y_j)^{\alpha - \beta}g$ with $(y_j)^{\alpha - \beta}g \notin q_j$ or $h = (y_j)^{\beta - \alpha}f/g$ with $g \notin q_j$. So in both cases $h \in (R_j)_q$, $(R_j + 1)q_j + 1 \subseteq (R_j)_q$ and this proves the claim.
Now, we have $R_j \subset (R_j)_{q_j} = (R_i)_{q_i}, j \geq l$ and $\bigcup_{j \geq l} R_j \subset (R_i)_{q_i}$. Hence, $p_0$ is not maximal for the sequence $(R_i)$, which is a contradiction. This proves $(1) \implies (2)$.

To show $(2) \implies (1)$, let us assume that $p_0$ is not maximal for the sequence $(R_i)$. Then there exists a non-negative integer $l$ and a prime ideal $q_i$ of $R_i$ with $p_l \subset q_i, p_l \neq q_i$ and $\bigcup_{j \geq 0} R_j \subset (R_i)_{q_i}$. Therefore, $\bigcup_{j \geq 0} \frac{R_j}{p_j} \subset \left( \frac{R}{p} \right)_{\eta_1(q_i)}$ and $\bigcup_{j \geq 0} \frac{R_j}{p_j}$ is not a valuation ring of dimension one. (See Theorem 13 of [10].) □

**Remark 9.** We note that statement (2) in Theorem 8 is equivalent to

(3) The sequence $\frac{R_k}{p_k} \subset \frac{R_{k+1}}{p_{k+1}} \subset \cdots \subset \frac{R_{k+i}}{p_{k+i}} \subset \cdots$ switches strongly infinitely often (i.e. there does not exist an integer $j$ and a height one prime ideal $\mathfrak{q}_{i+j}$ of $\frac{R_{k+i}}{p_{k+i}}$ with the property that $\bigcup_{i=0}^{\infty} \frac{R_{k+i}}{p_{k+i}} \subset \left( \frac{R}{p} \right)_{\mathfrak{q}_{i+j}}$.

See [10].

To finish this section, let us go back to the case where $(R_l)$ is a quadratic sequence along $p \in \text{Spec}(R)$ and $p$ has height $n - 1$, so $p$ is maximal for the sequence $(R_l)$. In addition, let us assume that $R$ is a pseudogeometric or Nagata ring, hence $R/p$ is also a pseudogeometric or Nagata ring and, in particular, its integral closure is a finite $R/p$-module. Let $Y_0 = \text{Spec}(R/p)$ be the irreducible closed subscheme of $X_0$ defined by $p$, then $Y_0$ has dimension one, i.e. $Y_0$ is a curve. By the Resolution Theorem for curves (see, for example, either Proposition 1.10 of [9], p. II-2 or (4.2) of [3]), there exists a non-negative integer $i_0 \geq 0$ such that $x_i$ is a non-singular point of $Y_i, i \geq i_0$, where $Y_i$ is the strict transform of $Y_{i-1}$ by $\pi_j, j \geq 1$. Therefore, $R_k/p_k$ is a regular noetherian local ring of dimension one for $k \geq i_0$, i.e. $R_k/p_k$ is a discrete valuation ring $k \geq i_0$ and, in fact, $R_k/p_k = R_{i_0}/p_{i_0}$ for $k \geq i_0$.

In the next section, we extend this situation to the case for which $p$ is maximal for the sequence $(R_l)$ and $R/p$ is a regular local ring.

### 4. Equimultiplicity and multiplicity sequence

For the rest of the paper, let $R$ be a noetherian local regular ring of dimension $n \geq 2$ and let $(R_i) \equiv R \subset R_1 \subset R_2 \subset \cdots \subset R_l \subset \cdots$ be a sequence of noetherian local regular rings of the same dimension such that $R_l$ is a quadratic transform of $R_{l-1}$ for $l \geq 1$.

In this section we study the stabilization of the usual multiplicity along the sequence $(R_i)$.

Let $p$ be a prime ideal of $R$ such that $(R_l)$ is a quadratic sequence along $p$ and $R/p$ is a regular ring. Let us also assume that $p$ is maximal for the sequence $(R_l)$.

By Theorem 8, $V = \bigcup_{j \geq 0} \frac{R_j}{p_j}$ is a valuation ring of dimension one, where $p_j = p(R) \cap R_j$ for $j \geq 0$. Let us denote by $v$ the real rank one valuation associated with $V$ and by $\{n_i\}_{i \geq 0}$ the multiplicity sequence associated with $\left( \frac{R}{p}, V \right)$. Note that $n_i \in \mathbb{R}$ and $n_i \geq 0$ for $i \geq 0$.

On the other hand, for $i \geq 0$ we write

$$A_i = \{ f_i \in M(R_l) ; \text{Ord}_{R_l}(f_i^{(j)}) = \text{Ord}_{R_l}(f_i) , j \geq i \},$$

where $(R_l, f_i^{(j)})$ is the strict transform of $(R_l, f_i)R_l$ in $R_{l+j}$. Note that if $f_i \in A_i$, then $f_i^{(j)} \in A_j, j \geq i$.

We recall that an ideal $f$ is **equimultiple** at the prime ideal $q$ of $R$ if $\text{Ord}_q(f) = \text{Ord}_{R_q}(fR_q)$. In particular, a subset $\Delta \subset R$ is said to be **equimultiple** at the prime ideal $q$ of $R$ if $\text{Ord}_q(R\Delta) = \text{Ord}_{R_q}(R\Delta)$ for all $f \in \Delta$.

Furthermore, we note that if $f_i \in M(R_l)$ with $\text{Ord}_{R_l}(f_i) = \text{Ord}_{R_{l+i}}(f_i)$, then $f_i \in A_i, i \geq 0$. In particular, $\varepsilon(p_i) \subset A_i$, where

$$\varepsilon(p_i) = \{ f \in M(R_l) ; \text{Ord}_R(f) = \text{Ord}_{R_{l+i}}(f) \}$$

is the **equimultiple locus** at $p_i$ in $R_l, i \geq 0$.

**Theorem 10.** With the above assumptions and notation, the following statements are equivalent:

1. $A_i$ is equimultiple at $p_i$ (i.e. $\varepsilon(p_i) = A_i$), $i \geq 0$.
2. $A_0$ is equimultiple at $p_0 = p$ (i.e. $\varepsilon(p) = A_0$).
3. The series $\sum_{j \geq 0} n_j$ diverges.
4. The series $\sum_{j \geq i} n_j$ diverges, $i \geq k$.

**Proof.** Since $(1) \implies (2)$ and $(3) \iff (4)$ are obvious, it is sufficient to show that $A_i$ is equimultiple at $p_i$ if and only if $\sum_{j \geq i} n_j$ diverges, for $i \geq 0$. This would prove $(2) \iff (3)$ and $(1) \iff (4)$ simultaneously.

First let us assume that $\sum_{j \geq i} n_j$ diverges for some $i \geq 0$ and that $A_i$ is not equimultiple at $p_i$. Thus, there exists $f \in A_i$ such that $e = \text{Ord}_{R_l}(R_l) \neq \text{Ord}_{R_{l+i}}(f)$, $f(R_{l+i}) = d$. Note that $e \geq d$ by [2] (38.3), hence $e > d$. 
Since $R_j/p_j$ is a regular ring for $j \geq 0$, we have $(p_j(R_j)/p_j)^1 \cap R_j = (p_j)^1$ for $l \geq 1$ and $j \geq 0$. (See [8], (1.4).) Moreover, we can assume $M(R_j) = (x_1^{(j)} \ldots , x_n^{(j)})$ such that $p_j = (x_1^{(j)}, \ldots , x_l^{(j)})$ and $x_{i+1}^{(j)} R_{i+1} = M(R_j)R_{i+1}, j \geq 0$, see Lemma 1. (Note that $s < n$.) Therefore, by Lemma 5, we have $x_i^{(j)} = x_i^{(j)}/x_{i+1}^{(j)}, 1 \leq i \leq s$ and $j \geq 0$.

Thus, we can write
\[
f = \sum_{d \leq \alpha_1 + \cdots + \alpha_s < e} f_{x_1^{(j)}, \ldots , x_s^{(j)}} (x_1^{(j)})^{\alpha_1} \cdots (x_s^{(j)})^{\alpha_s} + g
\]
such that $g \in (p_j)^e,f_{x_1^{(j)}, \ldots , x_s^{(j)}} \notin p_j$ or $f_{x_1^{(j)}, \ldots , x_s^{(j)}} = 0$ for $d \leq \alpha_1 + \cdots + \alpha_s < e$ and $f_{x_1^{(j)}, \ldots , x_s^{(j)}} \neq 0$ for some $\alpha_1, \ldots , \alpha_s$ with $\alpha_1 + \cdots + \alpha_s = d$.

Since $f \in A_i$, then $\text{Ord}_{R_j}(f_j R_j) = d, j \geq i$, where $(R_j, f_j R_j)$ is the strict transform of $(R_l, f_l R_l)$ in $R_j$.

On the other hand, for $j > i$ we have
\[
f_j = \frac{f}{(x_{s+1}^{(i)})^e} \cdots (x_{s+1}^{(i)})^e = \sum_{d \leq \alpha_1 + \cdots + \alpha_s < e} \frac{f_{x_1^{(j)}, \ldots , x_s^{(j)}}}{(x_{s+1}^{(i)})^{e-\alpha}} (x_1^{(j)})^{\alpha_1} \cdots (x_s^{(j)})^{\alpha_s} + g_j,
\]
where $g_j = \frac{g}{(x_{s+1}^{(i)})^{e-\alpha}}$ and $\alpha = \alpha_1 + \cdots + \alpha_s$.

We point out that
\[
\text{Ord}_{R_l} \left( \frac{f_{x_1^{(j)}, \ldots , x_s^{(j)}}}{(x_{s+1}^{(i)})^{e-\alpha}} \cdots (x_{s+1}^{(i)})^{e-\alpha} \right) \geq e - \alpha,
\]
for $j \geq i$. In particular,
\[
v(\eta_{R_l}(f_{x_1^{(j)}, \ldots , x_s^{(j)}})) \geq (e - \alpha) \sum_{l=i}^{j-1} v(\eta_{R_l}(x_{s+1}^{(l)})),
\]
where $\eta_{R_l} : R_l \to R_l/p_l$ denotes the canonical epimorphism for $l \geq 0$. Therefore,
\[
v(\eta_{R_l}(f_{x_1^{(j)}, \ldots , x_s^{(j)}})) \geq (e - \alpha) \sum_{l=i}^{j-1} v(\eta_{R_l}(x_{s+1}^{(l)})) \geq \sum_{l=i}^{j-1} n_l,
\]
which is a contradiction (notice that $f_{x_1^{(j)}, \ldots , x_s^{(j)}} \notin p_l$ for some $\alpha_1, \ldots , \alpha_s$ with $\alpha_1 + \cdots + \alpha_s = d$). Hence, if the series $\sum_{j \geq i} n_j$ diverges, then $A_i$ is equimultiple at $p_i$.

Now, let us assume that the series $\sum_{j \geq i} n_j$ converges. Let $\beta$ be a non-negative integer such that $n_i \beta > \sum_{j \geq i} n_j$ and let us consider $f = x_1^{(i)} + (x_{s+1}^{(i)})^{\beta}$.

We claim that $f \in A_i$.

Let $(R_j, f_j R_j)$ be the strict transform of $(R_l, f_l R_l)$ in $R_j, j \geq i$. Since
\[
v(\eta_{R_l}(x_{s+1}^{(i)})^\beta) > \sum_{j \geq i} n_j,
\]
then $\eta_{R_l}(x_{s+1}^{(i)})^\beta \notin M(R_l) \setminus \frac{R_l^l}{R_l}$, $j \geq i$, where as always $\eta_{R_l} : R_l \to R_l/p_l$ denotes the canonical epimorphism for $l \geq 0$.

Since $\text{Ord}_{R_l}(f) = 1$ and
\[
f_j = x_1^{(i)} + \frac{(x_{s+1}^{(i)})^\beta}{x_{s+1}^{(i)} \cdots x_{s+1}^{(i)}}
\]
then $f_j \in M(R_j)$ and $\text{Ord}_{R_j}(f_j) = 1$ for $j \geq i$. Hence, $f \in A_i$.

Finally, as $R_i$ is not equimultiple at $p_i$, then $A_i$ is not equimultiple at $p_i$ and this proves the result. Note that $x_{s+1}^{(i)} \notin p_i$ by Lemma 1. □

Remark 11. With the above assumptions and notation, let us also assume that the series $\sum_{j \geq 0} n_j$ diverges, then $A_j \subset p_j$, $j \geq 0$; and, in fact, $A_j$ generates $p_j$, $j \geq 0$. Note that $x_i^{(i)} \in A_j, 1 \leq i \leq s$ and $p_j = (x_1^{(i)}, \ldots , x_s^{(i)})$.

On the other hand, for $j \geq 0$ let us consider $f \in M(R_j)$ with $f \notin A_j$, then $e = \text{Ord}_{R_j}(f_j R_j) > \text{Ord}_{R_l}(f_j R_l) = d$.

Furthermore, if $d \geq 1$, then there exists a non-negative integer $h_0 \geq j$ such that $f_{h_0} \in A_h$ for $h \geq h_0$, where $(R_h, f_{h_0} R_h)$ is the strict transform of $(R_j, f_j R_j)$ in $R_h, h \geq j$. In particular, we have $\text{Ord}_{(R_h)^{h_0}}(f_{h_0}(R_{h_0})) = d$ for $h \geq 0$ and $\text{Ord}_{R_h}(f_{h_0} R_h) = d$.\[\]
Ord\((R_0, p_0)\) \(= d\) for \(h \geq h_0\), this last statement by **Theorem 10**. Note that in the case where \(d = 0\), there exists also a non-negative integer \(h_0 \geq j\) such that \(f_h \notin M(R_0)\) for \(h \geq h_0\).

Finally, if \(q_j\) is any prime ideal of \(R_j, j \geq 0\) such that \((R_j, \mathfrak{q}_j)\) is a quadratic sequence along \(q_j\), then \(q_j \subset p_j\). To see this, let us consider \(f \in q_j\) such that \(f\) is an irreducible element of \(R_j\). Since \((R_j, \mathfrak{q}_j)\) is a quadratic sequence along \(q_j\), \((R_j, \mathfrak{q}_j)\) is a quadratic sequence along \(R_0\). If \(f \notin p_j\), then \(Ord\((R_0, p_0)\) \(= 0\) and there exists a non-negative integer \(h_0 \geq j\) such that \(f_{h_0} \notin M(R_0)\) for \(h \geq h_0\), where \((R_0, f_{h_0}R_0)\) is the strict transform of \((R_j, \mathfrak{q}_j)\) in \(R_0\), \(h \geq j\). Hence, \((R_j, \mathfrak{q}_j)\) is not a quadratic sequence along the prime ideal \(\mathfrak{p}_j\), which is a contradiction.

Therefore, we have shown that if the series \(\sum_{j \geq 0} n_j\) diverges, then \(p = p_0\) is the only prime ideal of \(R\) such that is maximal for the sequence \((R_j)\).

### 5. Normal flatness and quadratic sequences

The assumptions and notation are the same as in the preceding section.

In **Theorem 10** we have shown the relation between equimultiple locus at \(p\) and the multiplicity sequences, provided that \(p\) is maximal for \((R_j)\). Now, we can ask about a similar relation when we consider ideals \(J\) of \(R\) and normal flatness along \(p\) at \(M(R)\). This is the aim of this section.

Let \(J\) be an ideal of \(R\) such that \(J \subset p\) and let \(f_1, \ldots, f_r\) be a standard base of \(J\), i.e., the initial forms \(In_M(J)(f_1), \ldots, In_M(J)(f_r)\) of \(f_1, \ldots, f_r\) form a minimal generating set of \(gr_M(J, R)\) of \(gr_M(J, R)\) (see, for example, [3], 0.12.1) or [6], Remark 1). Let us write \(v_i = Ord\((R_0)(f_i)\), 1 \leq i \leq r\). Notice that \(v_i\) is the degree of \(In_M(J)(f_i)\) in \(gr_M(J, R)\), 1 \(\leq i \leq r\) and the set \(\{v_1, v_2, \ldots, v_r\}\) does not depend on the standard base. We denote by \(\nu^*(J, R) = (v_1, v_2, \ldots, v_r, \infty, \infty, \ldots)\), provided that \(v_1 \leq v_2 \leq \cdots \leq v_r < \infty\). With the terminology of [6], Section 3, we have \(\nu^*(J, R) = R^*_V(X, Z)\), where \(X = Spec(R/J)\) and \(Z = Spec(R)\).

Finally, we refer either to [3], chap. 0, Section 2 or to [1], chap. 2 for the concept and properties of normal flatness. In particular, we will use the following numerical criterion for normal flatness (see [3], 0.2.1, (iii)): \(J\) is normally flat along \(p\) at \(M(R)\) if there exists a standard base \(f_1, \ldots, f_r\) of \(J\) such that \(Ord\((R_0)(f_i)\) = Ord\((R_0)(f_j)\), 1 \(\leq i \leq r\).

Now, let us assume that \(J\) is normally flat along \(p\) at \(M(R)\), then, by the numerical criterion, there exists a standard base \(f_1, \ldots, f_r\) of \(J\) such that \(Ord\((R_0)(f_i)\) = Ord\((R_0)(f_j)\), 1 \(\leq i \leq r\). Let \((R_j, J_1)\) be the strict transform of \((R_j, J)\) in \(R_0\), then \(J_1 = (f_1^{(1)}, \ldots, f_r^{(1)})\). Moreover, since \(R/p\) is a regular ring and \(Ord\((R_0)(f_i)\) = Ord\((R_0)(f_j)\), 1 \(\leq i \leq r\), we have that, in fact, \(f_1^{(1)}, \ldots, f_r^{(1)}\) is a standard base of \(J_1\). Thus, we have shown the following:

**Proposition 12.** With the above assumptions and notation, let us assume that \(J\) is normally flat along \(p\) at \(M(R)\), then \(\nu^*(J, R) = \nu^*(J, R_0)\), where \((R_j, J)\) is the strict transform of \((R, J)\) in \(R_0\), \(i \geq 0\).

**Remark 13.** The converse of **Proposition 12** is, in general, false. Namely, let us assume that \(n = 3 = \text{dim}(R)\) and let \((x, y, z)\) be a base of \(M(R)\). We consider the quadratic sequence \((R_j)\) along \(z\) defined inductively as follows. Assume that \(R_i\) is defined together with an ordered base \((x_i, y_i, z_i)\) of \(M(R_j)\), then we define \(x_{i+1} = y_i/x_i, y_{i+1} = x_i, z_{i+1} = z_i/x_i\) and

\[
R_{i+1} = \left( R_i \begin{bmatrix} y_i & z_i \\ x_i & x_i^2 \end{bmatrix} \right)_{x_i^3} = \left( R_i[x_i+1, z_i+1] \right)_{x_i+1, y_i+1, z_i+1}.
\]

Here, \(x_0 = x, y_0 = y\) and \(z_0 = z\). Note that the exceptional divisor of the quadratic transform of \(R_i\) is given by \(y_i+1R_{i+1} = x_iR_{i+1}, i \geq 0\).

Therefore, \((R_0)\) is a quadratic sequence along \(z\) and \(R_0\) is maximal for \((R_0)\). Hence, \(V = \bigcup_{j \geq 0} \frac{R_j}{z_jR_j}\) is valuation ring of dimension one by **Theorem 8**, where \(z_jR_j = R_j \cap (R, z)R_j\) (or equivalently \((R_j, z_jR_j)\) is the strict transform of \((R, zR_j)\) in \((R_j, J)\), \(j \geq 0\).

Let \(\{n_i\}_{i \geq 0}\) be the multiplicity sequence associated with \((\frac{R_j}{z_jR_j}, V)\). Since \(n_0 > n_1 > n_2 > \cdots\) and \(n_j = n_{j+1} + n_{j+2}\), then \(n_{2j+1} < n_{2j} < (1/2)n_0, i \geq 1\). Thus, \(\sum_{j \geq 0} n_j \leq n_0 + n_1 + 2n_0\) and the series \(\sum_{j \geq 0} n_j\) converges.

Now, let \(\beta\) be a non-negative integer such that \(\beta > \sum_{j \geq 0} n_j\) and \(J = (z + y^\beta)R\). We have \(\nu^*(J, R) = \nu^*(J, R_0) = (1, \infty, \infty, \ldots)\), where \((R_j, J)\) is the strict transform of \((R, J)\) in \(R_0, i \geq 0\). Finally, \(J\) is not normally flat along \(z\) at \(M(R)\), note that \(Ord\((z + y^\beta)R\) = 1 \neq Ord\((z + y^\beta)R_0\).

We point out that \(n_0 + n_1\) are rationally independent, in fact, the continued fraction of \(n_0/n_1\) is

\[
\frac{n_0}{n_1} = 1 + \frac{1}{1 + \frac{1}{1 + \ldots}}.
\]

Furthermore, the sequence \((R_0)\) is a particular case of Example 4.17 of [11].

Next we show the converse of **Proposition 12**, when \(p\) is maximal for \((R_j)\) and the series \(\sum_{j \geq 0} n_j\) diverges.

**Theorem 14.** With the above assumptions and notation, let us assume that \(p\) is maximal for the sequence \((R_j)\) and that the series \(\sum_{j \geq 0} n_j\) diverges, then the following statements are equivalent:

1. \(J\) is normally flat along \(p\) at \(M(R)\).
2. \(\nu^*(J, R) = \nu^*(J, R_0)\), where \((R_j, J)\) is the strict transform of \((R, J)\) in \(R_0, i \geq 0\).
3. \(\sum_{j \geq 0} n_j\) diverges.
(1) \( v^*(f, R) = v^*(f_i, R_i), i \geq 0. \)
(2) \( f \) is normally flat along \( p \) at \( M(R) \).

Here \((R_i, f_i)\) is the strict transform of \((R, f)\) in \( R_i, i \geq 0. \)

**Proof.** From Proposition 12 we have \((2) \implies (1)\), even when \( p \) is not maximal for \((R_i)\).

Thus, let us assume \( v^*(f, R) = v^*(f_i, R_i), i \geq 0 \) and write

\[
 v^*(f, R) = v^*(f_i, R_i) = (v_1, v_2, \ldots, v_r, \infty, \infty, \ldots), \]

\( i \geq 0 \). Note that \( v_1 \leq v_2 \leq \cdots \leq v_r \).

Let \( f \in J \) such that \( \text{Ord}_R(f_R) = v_1 \). We have \( \text{Ord}_{R_i}(f^{(i)}(R_i)) = v_1 \) for \( i \geq 0 \), where \((R_i, f^{(i)}(R_i))\) is the strict transform of \((R, f)\) in \( R_i, i \geq 0 \). Note that \( f^{(i)}(R_i) \in J_i \) for \( i \geq 0 \).

Therefore, \( f \in A_0 = A \) and \( \text{Ord}_R(f_R) = \text{Ord}_{R_i}(f^{(i)}(R_i)) \), by Theorem 10. Hence, if \( f_1, f_2, \ldots, f_r \) is a standard base of \( J \) and \( v_1 = v_2 = \cdots = v_1 < v_{1+1} \), then \( \text{Ord}_R(f_R) = \text{Ord}_{R_i}(f^{(i)}(R_i)) \) for \( 1 \leq j \leq r_1 \).

At this point, let \( k \) be the greatest non-negative integer such that there exists a standard base \( f_1, f_2, \ldots, f_r \) of \( J \) with \( \text{Ord}_R(f_R) = \text{Ord}_{R_i}(f^{(i)}(R_i)) \) for \( 1 \leq j \leq k \). We have \( 1 \leq r_1 \leq k \leq r \).

If \( k = r \), then \( f \) is normally flat along \( p \) at \( M(R) \), by applying the numerical criterion for normal flatness. Thus, assume \( k < r \) and let \( f_1, f_2, \ldots, f_{k+1} \) be a standard base of \( J \) such that \( \text{Ord}_R(f^{(i)}(R_i)) = \text{Ord}_{R_i}(f^{(i)}(R_i)) \) for \( 1 \leq k \leq k \) and \( v_{k+1} = \text{Ord}_{R_i}(f^{(k+1)}(R_i)) \geq \text{Ord}_R(f_R) \).

On the other hand, since \( R_p \) is a regular ring for \( j \geq 0 \), we can write \( M(R_i) = (x^{(i)}_1, \ldots, x^{(i)}_s) \) such that \( p_j = (x^{(i)}_1, \ldots, x^{(i)}_s) \) and \( x^{(i+1)}_j = M(R_i)R_{j+1}, j \geq 0 \), see Lemma 1. (Note that \( s < n \).) Therefore, by Lemma 5, we have \( x^{(i+1)}_j = x^{(i)}_j / x^{(i+1)}_{j+1}, 1 \leq l \leq s \) and \( j \geq 0 \).

We can write

\[
 f_{k+1} = \sum_{\alpha_1 + \cdots + \alpha_s = d} f_{\alpha_1, \ldots, \alpha_s}^{k+1}(x^{(i)}_1)^{\alpha_1} \cdots (x^{(i)}_s)^{\alpha_s} + g
\]

such that \( g \in p^{d+1}f_{\alpha_1, \ldots, \alpha_s}^{k+1} \notin p \) or \( f_{\alpha_1, \ldots, \alpha_s}^{k+1} = 0 \) for some \( \alpha_1, \ldots, \alpha_s \) with \( \alpha_1 + \cdots + \alpha_s = d \).

We claim that there exists \( m \geq 0 \) such that

\[
 \text{Ord}_{R_i}(f^{(i)}(R_i)) = d = \text{Ord}_{R_i}(f^{(i)}(R_i)) \quad \text{or} \quad f^{(i)}(R_i) = 0
\]

for \( i \geq m \) and \( \text{Ord}_{R_i}(f^{(i)}(R_i)) > d \) for \( 0 \leq i < m \), where \((R_i, f^{(i)}(R_i))\) is the strict transform of \((R, f_{k+1}(R_i))\) in \( R_i, i \geq 0 \).

Since \( \text{Ord}_{R_i}(f^{(i)}(R_i)) \geq \text{Ord}_{R_i}(f^{(i)}(R_i)) \), \( i \geq 0 \), there exists \( m \geq 0 \) such that \( \text{Ord}_{R_i}(f^{(i)}(R_i)) \geq \text{Ord}_{R_i}(f^{(i)}(R_i)) \), for \( i \geq m \).

Thus, \( f^{(i)}(R_i) \in A_i \) for \( i \geq m \) and the claim follows from Theorem 10. Note that \( d = \text{Ord}_{R_i}(f^{(i)}(R_i)) \) for all \( i \geq 0 \). (See Remark 11.)

Furthermore, we can write

\[
 gr_{M(R)}(R) = \frac{R}{M(R)}[T_1, \ldots, T_n],
\]

as a polynomial ring over the residue field \( R/M(R) \). Thus, if, as always, \( \text{ln}_{M(R)}(f) \) denotes the initial form of \( f \in R \) in \( gr_{M(R)}(R) \), then we can assume \( T_i = \text{ln}_{M(R)}(x^{(i)}_1), 1 \leq i \leq n \) and we have

\[
 \text{ln}_{M(R)}(f) \in \frac{R}{M(R)}[T_1, \ldots, T_n] = A = A_0 \subset gr_{M(R)}(R),
\]

\( 1 \leq j \leq k \).

Now, let us fix a graded lexicographic ordering on \( N^s \), for example, we can take \((\alpha_1, \ldots, \alpha_s) < (\beta_1, \ldots, \beta_s) \) if and only if \( \sum_{j=1}^s \alpha_j < \sum_{j=1}^s \beta_j \) or \( \sum_{j=1}^s \alpha_j = \sum_{j=1}^s \beta_j \) and there exists some \( j \in \{1, \ldots, s\} \) such that \( \alpha_j < \beta_j \) and \( \alpha_i = \beta_i \) for \( 1 \leq i < j \).

This graded lexicographic ordering extends to an ordering on the monomials of \( A = \frac{R}{M(R)}[T_1, \ldots, T_n] \).

Let \( H_k \) be the homogeneous ideal of \( A \) generated by \( \{\text{ln}_{M(R)}(f)\} \) and denote by \( dH_k \) the homogeneous elements of \( H_k \) of degree \( d \). Thus, \( H_k \) is a \( R/M(R) \)-vector subspace of \( (T_1, \ldots, T_s)^d/(T_1, \ldots, T_s)^{d+1} \), so of finite dimension.

Let \( \overline{T}_1, \ldots, \overline{T}_l \) be a base of \( dH_k \) as \( R/M(R) \)-vectorial space such that

\[
 (\alpha^{(1)}_1, \ldots, \alpha^{(1)}_s) < (\alpha^{(2)}_1, \ldots, \alpha^{(2)}_s) < \cdots < (\alpha^{(l)}_1, \ldots, \alpha^{(l)}_s),
\]

where \( T^{(1)}_1 \cdots T^{(l)}_s \) is the smallest monomial for the graded lexicographic ordering that appears in the expression of \( \overline{T}_i, 1 \leq i \leq l \).

Notice that \( \sum_{j=1}^s \alpha^{(i)}_j = d, 1 \leq i \leq l \) and \( \{(\alpha^{(1)}_1, \ldots, \alpha^{(1)}_s)\}_{i=1}^l \) are univocally determined by \( dH_k \). We also assume that the coefficient of \( T^{(1)}_1 \cdots T^{(l)}_s \) in the expression of \( \overline{T}_i \) is \( 1, 1 \leq i \leq l \).
In general, we have \( \ln_{M(R)}(h_j^{(0)}) \in \mathcal{M}(R) = \frac{R}{M(R)}[T_1, \ldots, T_n] \) and, since \( \text{Ord}_R(f_jR) = \text{Ord}_R(f_jR) \) for \( 1 \leq j \leq k \), then
\[
\ln_{M(R)}(f_j^{(0)}) = \ln_{M(R)}(f_i) = \frac{R}{M(R)}[T_1, \ldots, T_1] = A_i, \]
i \geq 0, where \( (R, f_j^{(0)}R) \) is the strict transform of \( (R, f_jR) \) in \( R_i \), \( i \geq 0 \), \( 1 \leq j \leq k \).

Let \( H_k^{(i)} \) be the homogeneous ideal of \( A_i \), generated by \( \{\ln_{M(R)}(f_j^{(0)})\}_{j=1}^k \) and, as above, let us denote by \( dH_k^{(i)} \) the homogeneous elements of \( H_k^{(i)} \) of degree \( d \), \( i \geq 0 \). Since \( dH_k^{(i)} = dR \otimes_{R(M(R))} (R_i/M(R)) \), then \( \tilde{T}_1, \ldots, \tilde{T}_i \) is also a base of \( dH_k^{(i)} \) as \( R_i/M(R_i) \)-vector space, \( i \geq 0 \).

Let us consider \( h_1, \ldots, h_l \in f_1R + \cdots + f_kR \) such that \( \ln_{M(R)}(h_i) = \tilde{T}_i, 1 \leq i \leq l \). We can write
\[
h_i = \sum_{\beta_1 + \cdots + \beta_d = d, (\beta_1, \ldots, \beta_d) < (\alpha'_1, \ldots, \alpha'_r)} h_{\beta_1, \ldots, \beta_d}^i (x_1)^{\beta_1} \cdots (x_d)^{\beta_d} + (x_1)^{\alpha'_1} \cdots (x_r)^{\alpha'_r}
+ \sum_{\beta_1 + \cdots + \beta_d = d, (\beta_1, \ldots, \beta_d) > (\alpha'_1, \ldots, \alpha'_r)} h_{\beta_1, \ldots, \beta_d}^i (x_1)^{\beta_1} \cdots (x_d)^{\beta_d},
\]
where \( h_{\beta_1, \ldots, \beta_d}^i \in R \) for all \( (\beta_1, \ldots, \beta_d) \) and \( h_{\beta_1, \ldots, \beta_d}^i \in M(R) \) for \( (\beta_1, \ldots, \beta_d) < (\alpha'_1, \ldots, \alpha'_r), 1 \leq i \leq l \).

Without loss of generality, we can also assume \( h_{\alpha'_1, \ldots, \alpha'_r}^i = 0, 1 \leq j < i \leq l \) and \( f_{\alpha'_1, \ldots, \alpha'_r}^{j+1} = 0, 1 \leq i \leq l \). Indeed, if \( h_{\alpha'_1, \ldots, \alpha'_r}^i \neq 0 \) for some \( 1 < i < l \), then \( h_{\alpha'_1, \ldots, \alpha'_r}^i \in M(R) \) and we can take
\[
h_i = (1 - h_{\alpha'_1, \ldots, \alpha'_r}^i h_{\alpha'_1, \ldots, \alpha'_r}^{j+1})^{-1}(h_i - h_{\alpha'_1, \ldots, \alpha'_r}^i h_{\alpha'_1, \ldots, \alpha'_r}^j).
\]
We have \( \ln_{M(R)}(h_i') = \ln_{M(R)}(h_i) \).
\[
h_i' = \sum_{\beta_1 + \cdots + \beta_d = d, (\beta_1, \ldots, \beta_d) < (\alpha'_1, \ldots, \alpha'_r)} h_{\beta_1, \ldots, \beta_d}^{i'} (x_1)^{\beta_1} \cdots (x_d)^{\beta_d} + (x_1)^{\alpha'_1} \cdots (x_r)^{\alpha'_r}
+ \sum_{\beta_1 + \cdots + \beta_d = d, (\beta_1, \ldots, \beta_d) > (\alpha'_1, \ldots, \alpha'_r)} h_{\beta_1, \ldots, \beta_d}^{i'} (x_1)^{\beta_1} \cdots (x_d)^{\beta_d},
\]
and \( h_{\alpha'_1, \ldots, \alpha'_r}^{i'} = 0, \) where \( h_{\alpha'_1, \ldots, \alpha'_r}^{i'} \) is \( R \) for all \( (\beta_1, \ldots, \beta_d) \) and \( h_{\alpha'_1, \ldots, \alpha'_r}^{i'} \in M(R) \) for \( (\beta_1, \ldots, \beta_d) < (\alpha'_1, \ldots, \alpha'_r), 1 \leq i \leq l \).

We now repeat the above procedure with \( (\alpha'_2, \ldots, \alpha'_r) \) to get \( h_{\alpha'_2, \ldots, \alpha'_r}^{i''} = 0, 2 < i \leq l \) and so on. Thus, after \( l \) steps we can assume \( h_{\alpha'_1, \ldots, \alpha'_r}^{i'''} = 0, 1 \leq j < i \leq l \).

On the other hand, let us consider
\[
f_{k+1} = f_{k+1} + \sum_{i=1}^l f_{i+1} h_i,
\]
then \( f_1, \ldots, f_{k+1}, f_{k+2}, \ldots, f_l \) is a standard base of \( f \) (note that \( h_1, \ldots, h_l \in f_1R + \cdots + f_kR \)). Furthermore, we can write
\[
f_{k+1} = \sum_{\alpha_1 + \cdots + \alpha_s = d} f_{\alpha_1, \ldots, \alpha_s}^{k+1} (x_1)^{\alpha_1} \cdots (x_s)^{\alpha_s} + g',
\]
where \( g' \in p^{d+1} \) and \( f_{\alpha_1, \ldots, \alpha_s}^{k+1} \not\in p \) or \( f_{\alpha_1, \ldots, \alpha_s}^{k+1} = 0 \) for \( \alpha_1 + \cdots + \alpha_s = d \). Note that \( f_{\alpha_1, \ldots, \alpha_s}^{k+1} = 0, 1 \leq i \leq l \). Therefore, we can also assume \( f_{\alpha_1, \ldots, \alpha_s}^{k+1} = 0, 1 \leq i \leq l \).

Finally, we have \( \text{Ord}_R f_{k+1}d \) > 0. Otherwise, \( \text{Ord}_R f_{k+1}d = 0 \) and we can write
\[
f_{k+1} = \sum_{\alpha_1 + \cdots + \alpha_s = d} f_{\alpha_1, \ldots, \alpha_s}^{k+1} (x_1)^{\alpha_1} \cdots (x_s)^{\alpha_s} + \frac{g}{(x_1^{(0)})^{s_1} \cdots (x_s^{(m)})^{s_m}} \in p_\infty^d,
\]
where
\[
\frac{g}{(x_1^{(0)})^{s_1} \cdots (x_s^{(m)})^{s_m}} \in p_\infty^d
\]
and \( \gamma_i = \text{Ord}_R f_{i+1}^{(0)}R_i, 0 \leq i \leq m - 1 \).
Since \( f_{k+1}^{(m)} \in J_m \), \( \text{Ord}_{R_m}(f_{k+1}^{(m)}R_m) = d = \text{Ord}_{R_m}(f_{k+1}^{(m)}R_m) < \nu_{k+1} \), \( v^*(J, R_m) = v^*(J, R) \) and since we can complete \( f_1^{(m)}, \ldots, f_k^{(m)} \) to a standard base of \( J_m \), we have \( \nu_{k+1} \) is the smallest monomial in the expression of \( \text{In}_{M(R_m)}(f_{k+1}^{(m)}) \), then \( (\beta_1, \ldots, \beta_i) = (a_1', \ldots, a_j') \) for some \( i \) with \( 1 \leq i \leq l \). Thus, we get to a contradiction since \( f_{k+1}^{(m)}a_1' \ldots a_j' = 0, 1 \leq i \leq l \).

Therefore, we get a standard base \( f_1, \ldots, f_k, f_{k+1}, f_{k+2}, \ldots, f_i \) of \( J \) such that \( \text{Ord}_{R_p}(f_{k+1}(R_p)) > d \). After repeating the above process (say \( \nu_{k+1} - d \) steps) we get to a standard basis \( f_1, \ldots, f_i \) of \( J \) such that \( \text{Ord}_{R_p}(f_i(R_p)) = \nu_i, 1 \leq i \leq k + 1 \), which contradicts the assumption on \( k \). Hence, necessarily \( k = r \) and the proof is complete. □

**Remark 15.** With the assumptions and notation as above, the condition that \( v^*(J, R_i) = v^*(J, R) \) for \( i \geq i_0 \) is ensured by Theorem 4, p. 234 of [1], provided that \( R_i \) is residually separable algebraic over \( R_{i-1} \), \( i \geq 0 \) (i.e, \( R_i/M(R_i) \) is a separable algebraic extension of \( R_{i-1}/M(R_{i-1}) \), \( i \geq 1 \)).

On the other hand, we note that by Theorem III of [6], \( v^*(J, R_i) = v^*(J, R) \) for \( i \geq 0 \) is equivalent to \( H_{R_i/J}^{(1)} = H_{R_{i-1}/J}^{(1)} \) for \( i \geq 0 \), where \( H_{R_i/J}^{(1)} \) denotes the Hilbert–Samuel function of a local noetherian ring \( \Theta \). (See [3], 0(1.3).) Thus, we can rewrite Theorem 14 in terms of the stabilization by blowing-ups of the Hilbert–Samuel function.

Furthermore, if \( p \) has height \( n - 1 \), then Theorem 14 is nothing but Proposition 3.1 of [3]. Also Theorem 1 of [7] can be obtained as a particular case of Theorem 14. In fact, in [7], the blowing-ups considered have as centers permissible variables of dimension \( n - r \) in such a way that there exists some transversal parameters to the ideal defining the center that are stable along the sequence. Thus, we can consider the generic section along the sequence of blowing-ups to obtain a sequence of blowing-ups centered at closed points (i.e locally we get our sequence \( (R_i) \)), then the stationary paths of [7] correspond to primes \( p \) that are maximal for the sequence \( (R_i) \), provided that \( p \) has height \( n - 1 \).

Finally, we point out that in characteristic zero the condition of maximal contact for varieties is stable under the blowing-ups for which the Hilbert–Samuel function is stable (or equivalently for which \( v^*(J, R) \) is stable) (see [4,5]). Therefore, if \( v^*(J, R_i) = v^*(J, R) \) for \( i \geq 0 \), then \( (R_i) \) is a quadratic sequence along the prime ideal that defines the variety of maximal contact. This has been used in [14] to give a structure Theorem for valuations and also can be used to determine prime ideals \( p \) that are maximal for \( (R_i) \) such that \( R/p \) is a regular ring.

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**References**


