Continuity of posets via Scott topology and sobrification

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Abstract

In this paper, posets which may not be dcpos are considered. The concept of embedded bases for posets is introduced. Characterizations of continuity of posets in terms of embedded bases and Scott topology are given. The main results are:

(1) A poset is continuous iff it is an embedded basis for a dcpo up to an isomorphism;
(2) A poset is continuous iff its Scott topology is completely distributive;
(3) A topological $T_0$ space is a continuous poset equipped with the Scott topology in the specialization order iff its topology is completely distributive and coarser than or equal to the Scott topology;
(4) A topological $T_1$ space is a discrete space iff its topology is completely distributive.

These results generalize the relevant results obtained by J.D. Lawson for dcpos.

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1. Introduction

Domain theory has traditionally had a standing hypothesis of directed completeness, i.e., dcpos are basically considered. However, there are important mathematical structures where arise posets such as the reals \( \mathbb{R} \) and the natures \( \mathbb{N} \) which fail to be dcpos. There are more and more demands to study posets which are not directed complete (see [5,6]). To pave these demands, this paper manages to embed continuous posets in larger domains and characterize continuity of posets by Scott topology.

In [3], Lawson proved that a dcpo is continuous iff its Scott topology is completely distributive. In [9], the author generalized this result to the realm of cdcpo’s (or, local dcpos in Mislove’s sense [7]). In [10], Zhang proved that a poset is continuous iff it is weakly locally compact in the Scott topology and has a basis of Scott open filters, stressing topological properties of the Scott topology of posets. We in this paper will establish some characterization theorems for continuity of general posets by the technique of embedded bases and sobrification via Scott topology, stressing order properties of the Scott topology of posets and rich interplay between topological and order-theoretical aspects of posets. We will see that continuous posets are all embedded bases for continuous dcpos (i.e., domains), and vice versa. Thus, one can then deduce properties of continuous posets directly from the known properties of continuous domains by treating them as embedded bases for continuous domains. We will see also that a poset is continuous iff its Scott topology is a complete completely distributive lattice. Interesting enough, in terms of specialization order, some related results for topological \( T_0 \) spaces and \( T_1 \) spaces can also be naturally obtained in this circumstance.

2. Embedded bases

Recall that in a poset \( P \), we say that \( x \) approximates \( y \), written \( x \preccurlyeq y \) if whenever \( D \) is a directed set that has a supremum \( \text{sup} D \geq y \), then \( x \leq d \) for some \( d \in D \). When confusion may arise, the relation \( \preccurlyeq \) in a poset \( P \) will be specifically written \( \preccurlyeq_P \). The poset \( P \) is said to be continuous if every element is the directed supremum of elements that approximate it.

**Proposition 2.1.** If \( P \) is a continuous poset, then the approximating relation \( \preccurlyeq \) has the interpolation property:

\[
x \preccurlyeq z \implies \exists y \in P \text{ such that } x \preccurlyeq y \preccurlyeq z.
\]

**(INT)**

**Proof.** Define \( D = \{u \in P : \exists y \in P \text{ such that } u \preccurlyeq y \preccurlyeq z\} \). It can be easily deduced from the directedness and the approximating property of \( d \) for every \( p \in P \) that \( D \) is directed and has \( z \) as its least upper bound. Thus it follows from \( x \preccurlyeq z \) that there is some \( u \in D \) such that \( x \preccurlyeq u \). By the construction of \( D \), there is some \( y \in P \) such that \( x \preccurlyeq u \preccurlyeq y \preccurlyeq z \), as desired. \( \Box \)

**Definition 2.2.** (For case of dcpos, see [2,12].) Let \( P \) be a poset \( P \) and \( B \subseteq P \), \( B \) is called a basis for \( P \) if \( \forall a \in P \), there is a directed set \( D_a \subseteq B \) such that \( \forall d \in D_a, d \preccurlyeq_P a \) and \( \text{sup}_P D_a = a \), where the subscript \( P \) means to take relevant operations in poset \( P \).
It is well known that a dcpo $P$ is continuous if and only if it has a basis. To go further, a new concept of embedded bases for posets will be needed.

**Definition 2.3.** Let $B$ and $P$ be posets. If there is a map $j : B \to P$ satisfying

1. $j$ preserves existing directed sups;
2. $j : B \to j(B)$ is an order isomorphism;
3. $j(B)$ is a basis for $P$,

then $(B, j)$ is called an embedded basis for $P$. If $B \subseteq P$ and $(B, i)$ is an embedded basis for $P$, where $i$ is the inclusion map, then we say also that $B$ is an embedded basis for $P$.

It is easy to see that if $B \subseteq P$, then $B$ is an embedded basis for $P$ iff $B$ is a basis for $P$ and for every directed set $D \subseteq B$ with existing $\sup_B D$, one has $\sup_B D = \sup_P D$. We can also observe that if $(B, j)$ is an embedded basis for $P$, then $j(B) \subseteq P$ is an embedded basis for $P$.

**Proposition 2.4.** If $B$ is an embedded basis for $T$, then for all $x, y \in B$, $x \ll_B y$ if and only if $x \ll_T y$.

**Proof.** ($\Rightarrow$) Let $x, y \in B$ with $x \ll_B y$. Let $D_y \subseteq B$ be the existing directed set in Definition 2.2. For a directed $D \subseteq T$ with $\sup_T D \geq y$, since $\sup_B D_y = \sup_T D_y = y \in B$ and $x \ll_B y$, there is some $b \in D_y \subseteq B$ such that $x \leq b$. Thus by $b \in D_y$ and $b \ll_T y$, there is some $d \in D$ such that $x \leq b \leq d$, showing that $x \ll_T y$.

($\Leftarrow$) Let $x, y \in B$ with $x \ll_T y$. Let $D \subseteq B$ with $\sup_B D \geq y$. Then by Definition 2.3(1), $\sup_T D = \sup_B D \geq y$. Thus, by $x \ll_T y$, there is $d \in D$ such that $x \leq d$. This shows that $x \ll_B y$, as desired. 

**Proposition 2.5.** If $B$ is a basis for a poset $P$, then $P$ is continuous; if $B$ is an embedded basis for $P$ (up to an isomorphism), then $B$ itself is also continuous.

**Proof.** It is easy to see that if $P$ has a basis $B$, then $P$ is continuous. We next show that if $B$ is an embedded basis for $P$, then $B$ is also continuous. Let $a \in B \subseteq P$. Then $\downarrow_P a$ is directed and $\sup_P \downarrow_P a = a$. Let $D_a \subseteq B$ be the directed set in Definition 2.2. Then $D_a$ is directed in $B$ with existing $\sup_B D_a = \sup_P D_a = a \in B$. By Proposition 2.4, we have for every $d_a \in D_a$, $d_a \ll_B a$. Thus $B$ is a basis for itself, and by the first part of this proposition $B$ is continuous.

**Example 2.6.** It is easy to see that the rationales $\mathbb{Q}$ is an embedded basis for the reals $\mathbb{R}$. So, by Proposition 2.5, $\mathbb{R}$ and $\mathbb{Q}$ are all continuous posets. Actually, a direct verification can show that every linear ordered set is a continuous poset.

**Example 2.7.** Let $\mathbb{N}$ be the natures and $\mathbb{N}^T = \mathbb{N} \cup \{t, \top\}$ with the linear order such that $n < t < \top$ for all $n \in \mathbb{N}$. Then $\mathbb{N}^T$ is an algebraic lattice with a basis $B = \mathbb{N} \cup \{\top\}$. Clearly
the directed join of $\mathbb{N}$ in $B$ is the element $\top$, while the join of $\mathbb{N}$ in $\mathbb{N}^\tau$ is the element $t \neq \top$. Thus $B$ is not an embedded basis for $\mathbb{N}^\tau$. However, $B$ is an algebraic lattice itself.

**Example 2.8.** Let $S = \{(0, y) : y \in [0, 1]\} \cup \{(x, 0) : x \in [0, 1]\} \cup \{(x, 1) : x \in [0, 1]\}$ with the partial order $\leq$ defined for all $(x, y), (u, v) \in S$, $(x, y) \leq (u, v) \iff x \leq u$ and $y \leq v$. Then $(S, \leq)$ is a continuous lattice. It is clear that $S\setminus\{(1, 0)\}$ is a basis for $S$. This basis with the restricted order of $S$ is not continuous itself.

**Proposition 2.9.** If $B$ is an embedded basis for $P$ and $P$ is an embedded basis for $Q$, then $B$ is also an embedded basis for $Q$.

**Proof.** Straightforward. \(\Box\)

**Proposition 2.10.** If $B_i$ is a(n) (embedded) basis for $P_i$ ($i = 1, 2$), then $B_1 \times B_2$ is a(n) (embedded) basis for $P_1 \times P_2$, where the product poset is in pointwise order.

**Proof.** Straightforward. \(\Box\)

### 3. Abstract bases and round ideal completions

Embedded bases have closely relations with abstract bases and round ideal completions. We recall the concept of abstract bases and related results appeared in [1,4] first.

**Definition 3.1.** (See [1,4].) Let $(P, \prec)$ be a set equipped with a binary relation. The binary relation $\prec$ is called fully transitive if it is transitive $(x \prec y, y \prec z \implies x \prec z)$ and satisfies the strong interpolation property:

$$\forall |F| < \infty, F \prec z \implies \exists y \prec z \text{ such that } F \prec y,$$

where $F \prec y$ means $\forall t \in F, t \prec y$. If $(B, \prec)$ is a set equipped with a binary relation which is fully transitive, then $(B, \prec)$ is called an abstract basis.

**Definition 3.2.** [1,4] Let $(B, \prec)$ be an abstract basis. A nonempty subset $I$ of $B$ is a round ideal if

1. $\forall y \in I, x \prec y \implies x \in I$;
2. $\forall x, y \in I, \exists z \in I$ such that $x \prec z$ and $y \prec z$.

The set of all round ideals of $B$ ordered by set inclusion is called the round ideal completion of $B$, denoted by $RI(B)$.

We observe that if $B$ is a basis for a continuous dcpo $P$, then $(B, \ll)$, the restriction of the approximation relation to $B$, is an abstract basis. And it is known (see Proposition 2.2.25(1) in [1]) that $P$ in this case is isomorphic to $RI(B)$. 

**Proposition 3.3.** [1,4] Let $(B,\prec)$ be an abstract basis. Define map $j : B \to RI(B)$ such that $\forall y \in B$, $j(y) = \downarrow y := \{x : x \prec y\}$. Then $j(B)$ is a basis for $RI(B)$, and thus $RI(B)$ is a continuous domain.

The following proposition gives a general example of embedded bases and relations with abstract bases.

**Proposition 3.4.** If $P$ is a continuous poset, then $(P, j)$ is an embedded basis for $RI(P)$, where $j$ is of Proposition 3.3 for the abstract basis $(P, \ll)$.

**Proof.** By the proposition above, $j(P)$ is a basis for $RI(P)$. That $j$ is continuous and $j : P \to j(P) \subseteq RI(P)$ is an order isomorphism can be deduced from the continuity of $P$. So, by Definition 2.3, the corollary holds. ∎

**Theorem 3.5.** Let $P$ be a continuous poset and $x, y \in P$. Then $x \ll_P y$ in $P$ iff $j(x) \ll_{RI(P)} j(y)$ in $RI(P)$, where $j : P \to RI(P)$ is the map defined for all $p \in P$, $j(p) = \downarrow p \in RI(P)$.

**Proof.** Straightforward or can be quickly given by applying [1, Proposition 2.2.22(2)] to the abstract basis $(P, \ll)$. ∎

**Theorem 3.6.** Let $P$ be a poset. Then $P$ is continuous iff $(P, j)$ is an embedded basis for the round ideal completion $RI(P)$.

**Proof.** If $(P, j)$ is an embedded basis for the round ideal completion $RI(P)$, then $j(P)$ is an embedded basis for $RI(P)$. By Proposition 2.5, $P \cong j(P)$ and $RI(P)$ are all continuous. Conversely, if $P$ is continuous, then by Proposition 3.4, we have that $(P, j)$ is an embedded basis for $RI(P)$. ∎

We now arrive at our first characterization theorem for the continuity of posets.

**Theorem 3.7.** A poset $P$ is continuous iff $P$ is order isomorphic to an embedded basis for a dcpo.

**Proof.** If $P$ is continuous, then by Theorem 3.6, $P \cong j(P)$ is an embedded basis for the round ideal completion $RI(P)$ which is a dcpo. Conversely, if $P$ is order isomorphic to an embedded basis for a dcpo $Q$, then by Proposition 2.5, $P$ is continuous. ∎

In view of these, one can then deduce properties of continuous posets directly from the known properties of continuous domains by treating them as embedded bases for continuous domains.

**Theorem 3.8.** If $P$ is an embedded basis for a dcpo $Q$, then $RI(P) \cong Q$.

**Proof.** By Proposition 2.5, $Q$ is continuous. Thus, by [1, Proposition 2.2.25(1)] that $RI(P) \cong Q$, as desired. ∎
Corollary 3.9. If $P$ is a embedded basis for $T$ and $T$ is a embedded basis for a dcpo $Q$, then $RI(P) \cong RI(T) \cong Q$.

Proof. It follows from Proposition 2.9 and Theorem 3.8. □

Example 3.10. Let $X$ be an infinite set. Let $\text{Fin}(X)$ be the set of all finite subset of $X$ ordered by set inclusion. Then $\text{Fin}(X)$ is an algebraic poset. It is easy to verify that $\text{Fin}(X)$ is an embedded basis for $\mathcal{P}(X)$, the power set of $X$. So, by Theorem 3.8, the round ideal completion $RI(\text{Fin}(X))$ is just $\mathcal{P}(X)$ up to an isomorphism.

4. Characterization theorems by Scott topology and sobrification

We recall that (see [2,4]) the specialization order for a $T_0$ space $X$ is defined by that $\forall x, y \in X, x \leq y$ iff $x \in \text{cl}(\{y\})$. A subset $A$ of a poset $P$ is said to be Scott closed if $\downarrow A = A$ and for any directed set $D \subseteq A$, sup $D \in A$ whenever sup $D$ exists. The complement of a Scott closed set is a Scott open set. All the Scott open sets of $P$ forms a topology called the Scott topology, denoted by $\sigma(P)$. A set $F$ of a space $X$ is said to be irreducible, if $F \neq \emptyset$ and for any pair of closed sets $F_1$ and $F_2$ in $X$, $F \subseteq F_1 \cup F_2$ implies that $F \subseteq F_1$ or $F \subseteq F_2$. A topological $T_0$ space is said to sober if every its irreducible closed set is a closure of a unique point. For a topological space $(X, \mathcal{O}(X))$, a pair $(X^s, j)$ is called a sobrification of $X$ if $X^s$ is a sober space and $j : X \to X^s$ is a continuous map such that $j^{-1} : \mathcal{O}(X^s) \to \mathcal{O}(X)$ is a lattice isomorphism. In this sobrification case, the map $j$ is called the sobrification embedding.

Proposition 4.1. Let $P$ be a continuous poset. For each $x \in P$, the set $\uparrow x = \{y \in P : x \ll y\}$ is open in the Scott topology, and these form a topological basis for the Scott topology.

Proof. The standard proof for continuous dcpos (see, for example [2, Proposition II-1.6]) carries over to continuous posets; see also [10, Proposition 4]. □

The following two results about sobrifications appear as exercises in [2] where hints to them are given. Here we quote them with the proofs omitted.

Lemma 4.2. [2, Ex. V-5.26] Let $X$ be a sober topological space and $\Omega(X)$ the set $X$ considered as a poset with the specialization order. Then $\mathcal{O}(X)$ is completely distributive iff $X$ is locally compact, $\sigma(\Omega(X)) \subseteq \mathcal{O}(X)$ and $\Omega(X)$ is a domain. Moreover, if one of the conditions is satisfied, then $\sigma(\Omega(X)) = \mathcal{O}(X)$.

Lemma 4.3. [2, Ex. V-5.34] A map $j : X \to \hat{X}$ between $T_0$ spaces is the sobrification (embedding) of $X$ (up to an $X$-homeomorphism) iff $\hat{X}$ is sober and $j$ is a strict embedding in the sense that given any $y$ in $\hat{X}$ and any open set $U$ containing $y$, there exists $x \in X$ such that $j(x) \in U$ and $j(x) \leq y$ in the specialization order.
Proposition 4.4. Let $B$ be an embedded basis for $T$. Then the inclusion map $i : B \to T$, $\forall a \in B, i(a) = a$ induces an isomorphism $i^{-1}: \sigma(T) \cong \sigma(B)$. As a consequence, $\sigma(B) = i^{-1}(\sigma(T)) = \{U \cap B : U \in \sigma(T)\} = \sigma(T)|B$ is the subspace topology.

Proof. Define $j^*: \sigma(B) \to \sigma(T)$ such that $\forall G \in \sigma(B), j^*(G) = \uparrow i(G)$.

(1) Claim that $\uparrow i(G)$ is Scott open and $j^*$ is well defined. In fact, for all directed set $D \subseteq T$ with $\sup_T D \in \uparrow i(G)$, there is some $g \in G = \bigcup_{b \in G} \uparrow b h$ such that $g \leq \sup_T D$. For this $g$, there is some $h \in G$ such that $h \ll_B g$. By Proposition 2.4, we have $h \ll_T g$. Thus there is some $d \in D$ such that $i(h) = h \leq d$ and $d \in \uparrow i(G)$, showing the Scott openness of $\uparrow i(G)$.

(2) $j^*$ and $i^{-1}$ are both order preserving: Clear.

(3) $j^*i^{-1} = \text{id}_{\sigma(T)}$: For all $U \in \sigma(T)$, clearly $j^*i^{-1}(U) \subseteq U$. On the other hand, let $t \in U$, then there is some $b \in D_t$ such that $i(b) = b \in U$, where $D_t \subseteq B$ is the directed set in Definition 2.2. Thus, $t \geq b \in i^{-1}(U)$ and $t \in j^*i^{-1}(U)$, showing that $U \subseteq j^*i^{-1}(U)$.

(4) $i^{-1}j^* = \text{id}_{\sigma(B)}$: For all $G \in \sigma(B)$, clearly $i^{-1}j^*(G) \supseteq G$. Conversely, let $b \in i^{-1}j^*(G)$, then $b = i(b) \in j^*(G) = \uparrow i(G)$. Thus there is some $g \in G$ such that $b = i(b) \geq i(g) = g$. Since $G$ is an upper set in $B$, $b \in G$, showing $i^{-1}j^*(G) \subseteq G$.

To sum up, $i^{-1}$ and $j^*$ are mutually inverse lattice isomorphisms, and $\sigma(B)$ are the images of $\sigma(T)$ under $i^{-1}$, as desired. $\square$

Corollary 4.5. If $P$ is a continuous poset and $\text{RI}(P)$ its round ideal completion, then $j^{-1}: \sigma(\text{RI}(P)) \to \sigma(P)$ is a lattice isomorphism, where $j: P \to \text{RI}(P)$ defined by $\forall p \in P, j(p) = \downarrow p \in \text{RI}(P)$. As a consequence, $(\text{RI}(P), j)$ is a sobrification of $P$ in the Scott topology whenever $P$ is continuous.

Proof. By Theorem 3.6, $j(P)$ is an embedded basis for $\text{RI}(P)$. Then it follows from Proposition 4.4 that $j^{-1}: \sigma(\text{RI}(P)) \to \sigma(P)$ is a lattice isomorphism. $\square$

Proposition 4.6. If $P$ is a continuous poset, then $\text{OFilt}(P) \cong \text{OFilt}(\text{RI}(P))$ is a continuous domain, where $\text{OFilt}(P)$ denotes the set of all Scott open filters of $P$ in the order of set inclusion.

Proof. By Proposition 4.4, we have $\text{OFilt}(P) \cong \text{OFilt}(\text{RI}(P))$. Then it follows from Theorem II-1.17 of [2] that $\text{OFilt}(\text{RI}(P))$ is a continuous domain. $\square$

Theorem 4.7. If $P$ is a continuous poset, then its Scott topology $\sigma(P)$ is a completely distributive lattice.

Proof. By Corollary 4.5 or by [4, Theorem 3.3], we have $\sigma(P) \cong \sigma(\text{RI}(P))$. Since $RI(P)$ is a continuous dcpo, by the well-known result of Lawson in [3], $\sigma(\text{RI}(P))$ is a completely distributive lattice, as desired. $\square$

Theorem 4.8. If the Scott topology $\sigma(P)$ of a poset $P$ is a completely distributive lattice, then $(P, j)$ is an embedded basis for a sobrification $\text{sob}(P)$ in the specialization order and $P$ is continuous, where $j: (P, \sigma(P)) \to \text{sob}(P)$ is the related sobrification embedding.
Proof. Since \((P, \sigma(P))\) is a \(T_0\) space with a topology of complete completely distributive lattice, by Lemma 4.2, the sobrification topology in \(\text{sob}(P)\) coincides with the Scott topology \(\sigma(\Omega(\text{sob}(P)))\) and is completely distributive, where \(\Omega(\text{sob}(P))\) is the set \(\text{sob}(P)\) considered as a poset with the specialization order. Noticing that a space and its sobrification have isomorphic lattices of open sets, by the well-known theorem of Lawson in [3], we thus deduce that the sobrification \(\text{sob}(P)\) of \(P\) is a continuous dcpo equipped with the Scott topology in the specialization order. For the continuous dcpo \(\Omega(\text{sob}(P))\), define \(D_y = \{ j(x) : j(x) \ll y, x \in P \} \subseteq j(P), \forall y \in \text{sob}(P)\). By the continuity of \(\Omega(\text{sob}(P))\), the strictness of \(j\) in Lemma 4.3 and the interpolation property of \(\ll\) in \(\Omega(\text{sob}(P))\), it is easy to show that \(D_y\) is directed and \(\sup D_y = y\). That is to say, \(j(P)\) is a basis for \(\Omega(\text{sob}(P))\). Since \(j\) is also an embedding with respect to the specialization orders of \(P\) and \(\text{sob}(P)\), \((P, j)\) is an embedded basis for \(\text{sob}(P)\) by Definition 2.3. Thus by Theorem 3.7, \(P\) is continuous, as desired. \(\square\)

We remark that Theorems 4.7 and 4.8 can also be proved in a way by using the concepts of minimal sets in [8] and of directed completion of posets in [9] as well as the subtle result [11, Theorem 1.7] of Zhao. We leave the details to the reader.

With the above results, we immediately have our second characterization theorem:

**Theorem 4.9 (The Characterization theorem).** A poset \(P\) is continuous iff its Scott topology is completely distributive.

**Theorem 4.10.** Let \(P\) be a poset. Then \(P\) is continuous iff \(RI(P)\) is continuous and with the Scott topology, \((RI(P), j)\) is a sobrification of \((P, \sigma(P))\).

**Proof.** (\(\Rightarrow\)) If \(P\) is continuous, then by Corollary 4.5, \(j^{-1}: \sigma(\text{RI}(P)) \cong \sigma(P)\), i.e., \((RI(P), j)\) is a sobrification of \((P, \sigma(P))\).

(\(\Leftarrow\)) If \(RI(P)\) is continuous and \((RI(P), j)\) is a sobrification of \((P, \sigma(P))\), then \(\sigma(P) \cong \sigma(\text{RI}(P))\) is completely distributive. By Theorem 4.9, \(P\) is continuous. \(\square\)

Passing to topological spaces, we have also the following interesting theorems.

**Theorem 4.11.** A topological \(T_0\) space is a continuous poset equipped with the Scott topology iff its topology is completely distributive and is coarser than or equal to the Scott topology in the specialization order.

**Proof.** (\(\Rightarrow\)) Trivial.

(\(\Leftarrow\)) Let \(X\) be a \(T_0\) space with a completely distributive topology which is coarser than or equal to the Scott topology in the specialization order. Let \(\text{sob}(X)\) be a sobrification of \(X\) with the embedding map \(j:X \rightarrow \text{sob}(X)\). Then similar argument in the proof of Theorem 4.8 shows that \(\Omega(\text{sob}(X))\) is a continuous dcpo. And the sobrification topology coincides with the Scott topology of the specialization order in \(\text{sob}(X)\). Moreover, \(j(X)\) is a basis for \(\Omega(\text{sob}(X))\). Since the Scott topology of \(X\) with the specialization order is finer than or equal to the original topology of \(X\), \(j : (X, \sigma(X)) \rightarrow \Omega(\text{sob}(X))\) is also continuous. Thus, by Definition 2.3, \(j(X)\) is an embedded basis for \(\Omega(\text{sob}(X))\) and \(X\) is a continuous
poset in the specialization order. Since by Lemma 4.2, the sobrification topology coincides with the Scott topology of \(\Omega(\text{sob}(X))\), the original topology of \(X\) coincides with the Scott topology of the specialization order, i.e., \(X\) with the original topology is a continuous poset equipped with the Scott topology in the specialization order, as desired.

Theorem 4.12. A topological \(T_1\) space is a discrete space iff its topology is completely distributive.

Proof. Note that any \(T_1\) topology induces a discrete specialization order and the Scott topology of this order is a discrete topology which is the finest one. Thus the corollary is clear.

Example 4.13. Let \(P\) be any non-continuous poset. Let \(\Lambda(P)\) be the Alexandroff topology of \(P\) consisting of all the upper sets of \(P\). Then \(\Lambda(P)\) is \(T_0\) and closed with arbitrary unions and arbitrary intersections in the completely distributive lattice \(\mathcal{P}(P)\), the power set of \(P\). Hence \(\Lambda(P)\) is also completely distributive. In this case, by Theorem 4.9, \(\sigma(P)\) is not completely distributive and \(\Lambda(P) \neq \sigma(P)\), for \(P\) is not continuous. It is easy to see that the specialization order induced by \(\Lambda(P)\) is exactly the original partial order on \(P\). Thus \((P, \Lambda(P))\) cannot be a continuous poset equipped with the Scott topology. This example shows that the phrase “coarser than or equal to” in Theorem 4.11 cannot be omitted.

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