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Parametrizing the abstract Ellentuck theorem

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Abstract

We give a parametrization with perfect subsets of 2^{∞} of the abstract Ellentuck theorem (see [T.J. Carlson, S.G. Simpson, Topological Ramsey Theory, in: J. Neŝetr̂il, V. Rödl (Eds.), Mathematics of Ramsey Theory, Algorithms and Combinatorics, vol. 5, Springer, Berlin, 1990, pp. 172–183], [S. Todorcevic, Introduction to Ramsey spaces, to appear] or [S. Todorcevic, Lecture notes from a course given at the Fields Institute in Toronto, Canada, Autumn 2002]). The main tool for achieving this goal is a sort of parametrization of an abstract version of the Nash–Williams theorem. As corollaries, we obtain some known classical results like the parametrized version of the Galvin–Prikry theorem due to Miller and Todorcevic [A.W. Miller, Infinite combinatorics and definability, Ann. Pure Appl. Logic 41 (1989) 179–203], and the parametrized version of Ellentuck's theorem due to Pawlikowski [Parametrized Ellentuck theorem, Topology Appl. 37 (1990) 65–73]. Also, we obtain parametized vesions of nonclassical results such as Milliken's theorem [K.R. Milliken, Ramsey's theorem with sums or unions, J. Combin. Theory (A) 18 (1975) 276–290], and we prove that the family of perfectly Ramsey subsets of $2^{\infty} \times \mathrm{FIN}_k^{[\infty]}$ is closed under the Souslin operation.

Keywords: Ramsey space; Abstract Ellentuck theorem; Perfect set

1. Introduction

In [4], Ellentuck considers the space $\mathbb{N}^{[\infty]}$, of all the infinite sets of natural numbers, with the *exponential* (or *Ellentuck's*) *topology* to obtain a topological proof of Silver's theorem [15]. The basic open sets of this topology are the neighborhoods $[a, \alpha] = \{\beta \in \mathbb{N}^{[\infty]} : a \subset \beta \subseteq \alpha\}$, where a (resp. α) is a finite (resp. infinite) subset of $\mathbb{N}$. We recall that a set $\mathscr{X} \subseteq \mathbb{N}^{[\infty]}$ is *Ramsey* (*completely Ramsey* in [4]) if for every nonempty Ellentuck neighborhood $[a, \alpha]$ there exists an infinite $\beta \in [a, \alpha]$ such that either $[a, \beta] \subseteq \mathscr{X}$ or $[a, \beta] \cap \mathscr{X} = \emptyset$. In [4], Ellentuck proved that a set is Ramsey if and only if it has the Baire property relative to Ellentuck's topology, proving in this way that the family of Ramsey sets is closed under the Souslin operation and hence obtaining a simpler topological proof of Silver's theorem [15] that classical analytic subsets of $\mathbb{N}^{[\infty]}$ are Ramsey, and generalizing the Galvin–Prikry theorem [6] that classical Borel subsets of $\mathbb{N}^{[\infty]}$ are Ramsey.

Theorems analogous to that of Ellentuck have also been proven on other spaces and different contexts, provided an analogous "exponential" topology is given in each case (see for instance [3,5,11]). These are the so-called *Ellentuck-type theorems*. Spaces where an Ellectuck-type theorem can be proven are called (topological) *Ramsey spaces*. The main tool for proving such theorems (including Ellentuck's) is devised for each particular case as a sort of "combinatorial forcing",

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inspired by the works of Galvin–Prikry [6] and Nash–Williams [13]. Recently, based on a previous work of Carlson and Simpson (see [1]), Todorcevic has isolated abstract combinatorial features which are conditions of sufficiency for topological spaces provided with a suitable "exponential" topology, to guarantee that Ellentuck-type theorems can be obtained in such spaces. These combinatorial features enable to devise a sort of "abstract Galvin–Prikry machinery" which constitutes the main means for obtaining an abstract Ellectuck theorem. It turns out that spaces satisfying these conditions are Ramsey spaces. We present a summary of Todorcevic's recent presentation of the abstract topological Ramsey theory in Section 2.

In this work we will show that the conditions proposed in [16], and summarized in the next section, are also sufficient to obtain a parametrized version of the abstract Ellentuck theorem. In this way, we obtain as corollaries some known classical results like the parametrized version of the Galvin–Prikry theorem due to Miller and Todorcevic [10], and the parametrized version of the Ellentuck theorem due to Pawlikowski [14] which makes use of the notion of abstract Baire property of Morgan [12]. These results refer to the *perfectly Ramsey property* for subsets of $2^\infty \times \mathbb{N}^{[\infty]}$, whose consistency relative to the consistency of the existence of an inaccessible cardinal was proved by Di Prisco in [2]. Here 2^∞ is the space of all the infinite sequences of 0's and 1's, with the product topology regarding $2 = \{0, 1\}$ as a discrete space. Also, we obtain parametized versions of nonclassical results such as Milliken's theorem [11], and we prove that the family of perfectly Ramsey subsets of $2^\infty \times \mathrm{FIN}_k^{[\infty]}$ is closed under the Souslin operation, and hence that analytic subsets of $2^\infty \times \mathrm{FIN}_k^{[\infty]}$ are perfectly Ramsey. The space $\mathrm{FIN}_k^{[\infty]}$, of infinite block basic sequences of functions $p : \mathbb{N} \to \{0, 1, \ldots, k\}$ of finite domain and with $p(n) = k$ for some n, will be described in Section 4.

2. Abstract topological Ramsey theory. Summary of main facts

All the definitions and results throughout this section were taken from [17] and are expected to appear in [16]. A previous presentation of the following notions can also be found in [1].

We will consider triplets of the form $(\mathscr{R}, \leqslant, (p_n)_{n\in\mathbb{N}})$, where $\mathscr{R}$ is a set, $\leqslant$ is a quasi order on $\mathscr{R}$, $\mathbb{N}$ is the set of natural numbers, and for every $n \in \mathbb{N}$, $p_n : \mathscr{R} \to \mathscr{P}_n$ is a function with range $\mathscr{P}_n$. For each $A \in \mathscr{R}$, we say that $p_n(A)$ is *the* nth *approximation of* A. To capture the combinatorial structure we need in order to prove an Ellentuck-type theorem, we will impose some assumptions on $(\mathscr{R}, \leqslant, (p_n)_n)$. The first three of them are the following:

(A1) For any $A \in \mathscr{R}$, $p_0(A) = \emptyset$.
(A2) For any $A, B \in \mathscr{R}$, if $A \neq B$ then $(\exists n)\ p_n(A) \neq p_n(B)$.
(A3) If $p_n(A) = p_m(B)$ then $n = m$ and $(\forall i < n)\ p_i(A) = p_i(B)$.

These three assumptions allow us to identify each $A \in \mathscr{R}$ with the sequence $(p_n(A))_n$ of its approximations. In this way, if we consider the space $\mathscr{P} = \cup_n \mathscr{P}_n$ with the discrete topology, we can identify $\mathscr{R}$ with a subspace of the (metric) space $\mathscr{P}^\infty$ (with the product topology) of all the sequences of elements of $\mathscr{P}$. Via this identification, we will regard $\mathscr{R}$ as a subspace of $\mathscr{P}^\infty$, and we will say that $\mathscr{R}$ is *metrically closed* if it is a closed subspace of $\mathscr{P}^\infty$.

Also, for $a \in \mathscr{P}$ we define the *length* of a, $|a|$, as the unique n such that $a = p_n(A)$ for some $A \in \mathscr{R}$.

We also consider on $\mathscr{R}$ the *Ellentuck-type neighborhoods*

$$[a, A] = \{B \in \mathscr{R} : (\exists n)(a = p_n(B)) \quad \text{and} \quad B \leqslant A\},$$

where $a \in \mathscr{P}$ and $A \in \mathscr{R}$. If $[a, A] \neq \emptyset$ we will say that a is *compatible* with A (or A is compatible with a). Let $\mathscr{P}[A] = \{a \in \mathscr{P} : a \text{ is compatible with } A\}$.

We write $[n, A]$ for $[p_n(A), A]$, and $\mathrm{Exp}(\mathscr{R})$ for the family of all the neighborhoods $[n, A]$. This family generates the natural "exponential" topology on $\mathscr{R}$ which is finer than the product topology.

Definition 1. A sequence $([n_k, A_k])_k$ of elements of $\mathrm{Exp}(\mathscr{R})$ is called a fusion sequence of neighborhoods if it is infinite and if

(i) $(n_k)_k \subseteq \mathbb{N}$ is nondecreasing and $\lim_{k\to\infty} n_k = \infty$,
(ii) $A_{k+1} \in [n_k, A_k]$ for all k.

When $\mathscr{R}$ is metrically closed, building fusion sequences constitute a very useful procedure for defining desired elements of $\mathscr{R}$: for every fusion sequence $([n_k, A_k])_k$ we have that $\cap_k [n_k, A_k] \neq \emptyset$ (and a singleton). The *limit* of the fusion sequence is the unique element A_∞ of $\cap_k [n_k, A_k]$. Note that $p_{n_k}(A_\infty) = p_{n_k}(A_k)$, for all k.

We define now, for subsets of $\mathscr{R}$, a notion analogous to the Ramsey property of subsets of $\mathbb{N}^{[\infty]}$:

Definition 2. A set $\mathscr{X} \subseteq \mathscr{R}$ is Ramsey if for every neighborhood $[a, A] \neq \emptyset$ there exists $B \in [a, A]$ such that $[a, B] \subseteq \mathscr{X}$ or $[a, B] \cap \mathscr{X} = \emptyset$. A set $\mathscr{X} \subseteq \mathscr{R}$ is Ramsey null if for every neighborhood $[a, A]$ there exists $B \in [a, A]$ such that $[a, B] \cap \mathscr{X} = \emptyset$.

Definition 3. We say that $(\mathscr{R}, \leqslant, (p_n)_n)$ is a (topological) Ramsey space if subsets of $\mathscr{R}$ with the Baire property are Ramsey and meager subsets of $\mathscr{R}$ are Ramsey null. (Here "having the Baire property" and "being meager" are relative to the topology generated by the family $\mathrm{Exp}(\mathscr{R})$).

In [16] it is shown that (A1)–(A3), together with the following three assumptions are conditions of sufficiency for a triplet $(\mathscr{R}, \leqslant, (p_n)_n)$, with $\mathscr{R}$ metrically closed, to be a Ramsey space (see also [1]):

(A4)(*Finitization*) There is a quasi order $\leqslant_{\text{fin}}$ on $\mathscr{P}$ such that:

(i) $A \leqslant B$ iff $\forall n\ \exists m\ \ p_n(A) \leqslant_{\text{fin}} p_m(B)$.
(ii) $\{b \in \mathscr{P} : b \leqslant_{\text{fin}} a\}$ is finite, for every $a \in \mathscr{P}$.

Given compatible a and A, we define the *depth of a in A*, $\mathrm{depth}_A(a)$, as the minimal n such that $a \leqslant_{\text{fin}} p_n(A)$.

(A5) (*Amalgamation*) Given compatible a and A with $\mathrm{depth}_A(a) = n$, the following holds:
 (i) $\forall B \in [n, A]\ ([a, B] \neq \emptyset)$.
 (ii) $\forall B \in [a, A]\ \exists A' \in [n, A]\ ([a, A'] \subseteq [a, B])$.
(A6) (*Pigeon hole principle*) Given compatible a and A with $\mathrm{depth}_A(a) = n$, for each partition $\phi : \mathscr{P}_{|a|+1} \to \{0, 1\}$ there is $B \in [n, A]$ such that ϕ is constant in $p_{|a|+1}[a, B]$.

2.1. Abstract Ellentuck theorem

Theorem 1 (*Carlson–Simpson [1], Todorcevic [16]*). *Any $(\mathscr{R}, \leqslant, \mathscr{P}_n, (p_n)_n)$ with $\mathscr{R}$ metrically closed and satisfying* (A1)–(A6) *is a Ramsey space*.

For instance, take $\mathscr{R} = \mathbb{N}^{[\infty]}$, the set of infinite subsets of $\mathbb{N}$, $\leqslant = \subseteq$ and $p_n(A)$ = the first n elements of A, for each $A \in \mathbb{N}^{[\infty]}$. So, the set of approximations is $\mathscr{P} = \mathbb{N}^{[<\infty]}$, the set of finite subsets of $\mathbb{N}$. The family of neighborhoods $[a, A]$, with $a \in \mathbb{N}^{[<\infty]}$ and $A \in \mathbb{N}^{[\infty]}$, is the family of Ellentuck neighborhoods defined in the introduction. Define $\leqslant_{\text{fin}}$ as $a \leqslant_{\text{fin}} b$ iff ($a = b = \emptyset$ or $a \subseteq b$ and $\max(a) = \max(b)$), for $a, b \in \mathbb{N}^{[<\infty]}$. With these definitions, (A1)–(A6) are easily verified. In this case (A6) reduces to a natural variation of the classical pigeon hole principle for finite partitions of an infinite set of natural numbers. Note also that $\mathbb{N}^{[\infty]}$ is easily identified with a closed subspace of $\mathscr{P}^\infty$, namely, the set of all the sequences $(x_n)_n$ of finite sets such that $x_n = x_{n+1} \backslash \{\max(x_{n+1})\}$, for each $n \in \mathbb{N}$. Then $(\mathbb{N}^{[\infty]}, \subseteq, (p_n)_n)$ is a Ramsey space in virtue of the abstract Ellentuck theorem. Hence, Ellentuck's theorem is obtained as corollary:

Corollary 1 (*Ellentuck [4]*). *Given $\mathscr{X} \subseteq \mathbb{N}^{[\infty]}$, the following hold*:

(a) *$\mathscr{X}$ is Ramsey iff $\mathscr{X}$ has the Baire Property, relative to Ellentuck's topology.*
(b) *$\mathscr{X}$ is Ramsey null iff $\mathscr{X}$ is meager, relative to Ellentuck's topology.*

3. Parametrizing with perfect sets

In this section we will present our main result. We recall that 2^∞ denotes the space of infinite sequences of 0's and 1's, with the product topology regarding $2 = \{0, 1\}$ as a discrete space. Also, $2^{<\infty}$ denotes the set of finite sequences of 0's and 1's. Let us consider some features of the perfect subsets of 2^∞, following [14]:

Some notation is needed. For $x = (x_n)_n \in 2^\infty$, $x|_k$ denotes the finite sequence $(x_0, x_1, \ldots, x_{k-1})$. For $u \in 2^{<\infty}$, let $[u] = \{x \in 2^\infty : (\exists k)(u = x|_k)\}$ and let $|u|$ be the length of u. Given a perfect set $Q \subseteq 2^\infty$, let $T_Q = \{x|_k : x \in Q, k \in \mathbb{N}\}$ be its associated perfect tree. Also, for $u, v = (v_0, v_1, \ldots, v_{|v|-1}) \in 2^{<\infty}$ we write $u \sqsubseteq v$ to mean $(\exists k \leqslant |v|)(u = (v_0, v_1, \ldots, v_{k-1}))$. For each $u \in 2^{<\infty}$, let $Q(u) = Q \cap [u(Q)]$, where $u(Q) \in T_Q$ is defined inductively, as follows: $\emptyset(Q) = \emptyset$. Suppose $u(Q)$ defined. Find $\sigma \in T_Q$ such that σ is the $\sqsubseteq$-extension of $u(Q)$ where the first ramification occurs. Then, set $(u * i)(Q) = \sigma * i$, $i = 0, 1$. Here "$*$" denotes "concatenation". Note that for each n, $Q = \bigcup\{Q(u) : u \in 2^n\}$.

Given $n \in \mathbb{N}$, and perfect sets S and Q we say that $S \subseteq_n Q$ if $S(u) \subseteq Q(u)$, for every $u \in 2^n$. The relation "$\subseteq_n$" is a partial order. If for every $u \in 2^n$ we have chosen $S_u \subseteq Q(u)$, then $S = \cup_u S_u$ is perfect and we have $S(u) = S_u$ and $S \subseteq_n Q$. As pointed out in [14], the most important feature of this partial order is the *property of fusion*: if $Q_{n+1} \subseteq_{n+1} Q_n$, $n \in \mathbb{N}$, then $Q = \cap_n Q_n$ is a perfect set and $Q \subseteq_n Q_n$, for each n. A sequence $\{Q_n\}_n$ of perfect subsets of 2^∞ such that $Q_{n+1} \subseteq_{n+1} Q_n$ is called *fusion sequence of perfect sets*.

Our goal in this section is proving a parametrized version of the abstract Ellentuck theorem of the previous section. This constitutes the main result contained in this work and is stated as Theorem 2.

We introduce now the abstract version of the notion of "perfectly Ramsey" (see [14]). We will use the same name: given a triplet $(\mathcal{R}, \leqslant, (p_n)_n)$ as defined at the previous section, we say that a set $\mathcal{X} \subseteq 2^\infty \times \mathcal{R}$ is *perfectly Ramsey* if for every perfect set $Q \subseteq 2^\infty$ and every neighborhood $[a, A] \neq \emptyset$ there exist a perfect set $S \subseteq Q$ and $B \in [a, A]$ such that $S \times [a, B] \subseteq \mathcal{X}$ or $S \times [a, B] \cap \mathcal{X} = \emptyset$. A set $\mathcal{X} \subseteq 2^\infty \times \mathcal{R}$ is *perfectly Ramsey null* if for every perfect set $Q \subseteq 2^\infty$ and every neighborhood $[a, A] \neq \emptyset$ there exist a perfect set $S \subseteq Q$ and $B \in [a, A]$ such that $S \times [a, B] \cap \mathcal{X} = \emptyset$.

Also, we will need to generalize the notion of abstract Baire property (see [12]) to this context:

Let $\mathbb{P}$ be the family of perfect subsets of 2^∞. We will say that a set $\mathcal{X} \subseteq 2^\infty \times \mathcal{R}$ has the $\mathbb{P} \times \mathrm{Exp}(\mathcal{R})$-*Baire property* if for every perfect set $Q \subseteq 2^\infty$ and every neighborhood $[a, A]$ there exist a perfect set $S \subseteq Q$ and a neighborhood $[b, B] \subseteq [a, A]$ such that $S \times [b, B] \subseteq \mathcal{X}$ or $S \times [b, B] \cap \mathcal{X} = \emptyset$. A set $\mathcal{X} \subseteq 2^\infty \times \mathcal{R}$ is $\mathbb{P} \times \mathrm{Exp}(\mathcal{R})$-*meager* if for every perfect set $Q \subseteq 2^\infty$ and every neighborhood $[a, A]$ there exist a perfect set $S \subseteq Q$ and a neighborhood $[b, B] \subseteq [a, A]$ such that $S \times [b, B] \cap \mathcal{X} = \emptyset$.

Now we are ready to state our main result:

Theorem 2. *Given* $(\mathcal{R}, \leqslant, (p_n)_n)$, *with* $\mathcal{R}$ *metrically closed and satisfying* (A1)–(A6), *the following are true*:

(a) $\mathcal{X} \subseteq 2^\infty \times \mathcal{R}$ *is perfectly Ramsey iff* $\mathcal{X}$ *has the* $\mathbb{P} \times \mathrm{Exp}(\mathcal{R})$-*Baire property*.
(b) $\mathcal{X} \subseteq 2^\infty \times \mathcal{R}$ *is perfectly Ramsey null iff* $\mathcal{X}$ *is* $\mathbb{P} \times \mathrm{Exp}(\mathcal{R})$-*meager*.

The main tool for proving Theorem 2 is the following fact which is a sort of parametrization of an abstract version of Nash–Williams' theorem:

Theorem 3. *Given* $(\mathcal{R}, \leqslant, (p_n)_n)$, *with* $\mathcal{R}$ *metrically closed and satisfying* (A1)–(A6), *the following is true*:

For every $\mathcal{F} \subseteq 2^{<\infty} \times \mathcal{P}$, *perfect* $P \subseteq 2^\infty$ *and* $A \in \mathcal{R}$ *there exist a perfect* $S \subseteq P$ *and* $D \leqslant A$ *such that one of the following holds*:

(a) *for every* $x \in S$ *and every* $C \leqslant D$ *there exist integers* k *and* $m > 0$ *such that* $(x|_k, p_m(C)) \in \mathcal{F}$.
(b) $(T_S \times \mathcal{P}[D]) \cap \mathcal{F} = \emptyset$.

Theorem 3 is inspired on a parametrized version of the semiselective Nash–Williams theorem proved by Farah (see [5, Theorems 2.2 and 2.3]). Before proving it, we need to set up our parametrized combinatorial machinery, based on the techniques used in [5,14,17].

Combinatorial forcing 1. Fix $\mathcal{F} \subseteq 2^{<\infty} \times \mathcal{P}$. For a perfect $Q \subseteq 2^\infty$, $A \in \mathcal{R}$ and a pair $(u, a) \in 2^{<\infty} \times \mathcal{P}$, we say that (Q, A) *accepts* (u, a) if for every $x \in Q(u)$ and for every $B \in [a, A]$ there exist integers k and m such that $(x|_k, p_m(B)) \in \mathcal{F}$. We say that (Q, A) *rejects* (u, a) if for every perfect $S \subseteq Q(u)$ and every $B \leqslant A$, compatible with a, (S, B) does not accepts (u, a). Also, we say that (Q, A) *decides* (u, a) if it accepts or rejects it.

Combinatorial Forcing 2. Fix $\mathcal{X} \subseteq 2^\infty \times \mathcal{R}$. For a perfect $Q \subseteq 2^\infty$, $A \in \mathcal{R}$ and a pair $(u, a) \in 2^{<\infty} \times \mathcal{P}$, we say that (Q, A) *accepts* (u, a) if $Q(u) \times [a, A] \subseteq \mathcal{X}$. We say that (Q, A) *rejects* (u, a) if for every perfect $S \subseteq Q(u)$ and every $B \leqslant A$, compatible with a, (S, B) does not accepts (u, a). And as before, we say that (Q, A) *decides* (u, a) if it accepts or rejects it.

Note: Lemmas 1–3 hold *for both* combinatorial forcings defined above.

Lemma 1. *The following are true*:

(a) *If* (Q, A) *accepts* (*rejects*) (u, a) *then* (S, B) *also accepts* (*rejects*) (u, a), *for every perfect* $S \subseteq Q(u)$ *and every* $B \leqslant A$ *compatible with* a.
(b) *If* (Q, A) *accepts* (*rejects*) (u, a) *then* (Q, B) *also accepts* (*rejects*) (u, a), *for every* $B \leqslant A$ *compatible with* a.
(c) *For all* (u, a) *and* (Q, A) *such that* A *is compatible with* a, *there exist a perfect* $S \subseteq Q$ *and* $B \leqslant A$, *compatible with* a, *such that* (S, B) *decides* (u, a).
(d) *If* (Q, A) *accepts* (u, a) *then* (Q, A) *accepts* (u, b) *for every* $b \in p_{|a|+1}[a, A]$.
(e) *If* (Q, A) *rejects* (u, a) *then there exist* $B \in [\mathrm{depth}_A(a), A]$ *such that* (Q, A) *does not accept* (u, b) *for every* $b \in p_{|a|+1}[a, B]$.
(f) (Q, A) *accepts* (*rejects*) (u, a) *iff* (Q, A) *accepts* (*rejects*) (v, a), *for every* $v \in 2^{<\infty}$ *with* $u \sqsubseteq v$.

Proof. (a)–(d) and (f) follow from the definitions. Now to proof (e), take (u, a) with $|a| = m$ and suppose (Q, A) rejects it. Define $\phi : \mathscr{P}_{m+1} \to 2$ such that $\phi(b) = 1$ iff (Q, A) accepts (u, b). Let $n = \mathrm{depth}_A(a)$. By (A6), there exists $B \in [n, A]$ such that ϕ is constant on $p_{m+1}[a, B]$.

If ϕ takes value 1 on $p_{m+1}[a, B]$ then (Q, B) accepts (u, a). So, in virtue of part (b), ϕ must take value 0 on $p_{m+1}[a, B]$ since (Q, A) rejects (u, a). Then B is as required. $\square$

Lemma 2. *For every perfect* $P \subseteq 2^\infty$ *and* $A \in \mathscr{R}$ *there exist a perfect* $Q \subseteq P$ *and* $B \leqslant A$ *such that* (Q, B) *decides* (u, a), *for every* $(u, a) \in 2^{<\infty} \times \mathscr{P}[B]$ *with* $\mathrm{depth}_B(a) \leqslant |u|$.

Proof. Let $\langle\rangle$ be the empty sequence of 0's and 1's, and recall from Section 2 that $\emptyset \in \mathscr{P}$. Using Lemma 1(c), find a perfect $Q_0 \subseteq P$ and $B_0 \leqslant A$ such that (Q_0, B_0) decides $(\langle\rangle, \emptyset)$.

Suppose we have defined Q_n and B_n such that (Q_n, B_n) decides every $(u, a) \in 2^n \times \mathscr{P}[B_n]$ with $\mathrm{depth}_{B_n}(a) = n$. Let $u_0, u_1, \ldots, u_{2^{n+1}-1}$ be a list of the elements of 2^{n+1}; and let $b_0, b_1, \ldots, b_r$ be a list of the $b \in \mathscr{P}[B_n]$ such that $\mathrm{depth}_{B_n}(b) = n + 1$.

Using Lemma 1(c), find a perfect $Q_n^{0,0} \subseteq Q_n(u_0)$ and $B_n^{0,0} \leqslant B_n$ compatible with b_0 such that $(Q_n^{0,0}, B_n^{0,0})$ decides (u_0, b_0). We can suppose $B_n^{0,0} \in [b_0, B_n]$. Hence, we can assume $B_n^{0,0} \in [n+1, B_n]$ in virtue of (A5)(ii) and Lemma 1(b).

In a similar way, for every $(i, j) \in \{0, 1, \ldots, 2^{n+1} - 1\} \times \{0, 1, \ldots, r\}$, we can find $Q_n^{i,j}$ and $B_n^{i,j}$ with: $Q_n^{i,j+1} \subseteq Q_n^{i,j}(u_i)$, $B_n^{i,j+1} \in [b_{j+1}, B_n^{i,j}]$, $Q_n^{i+1,0} \subseteq Q_n(u_{i+1})$, $B_n^{i+1,0} \in [b_0, B_n^{i,r}]$; and such that $(Q_n^{i,j}, B_n^{i,j})$ decides (u_i, b_j). (Notice that this construction is possible in virtue of (A5)(i). Again, we can assume $B_n^{i,j} \in [n+1, B_n]$ in virtue of (A5)(ii) and Lemma 1(b)).

Let $Q_{n+1} = \cup_{i=0}^{2^{n+1}-1} Q_n^{i,r}$ and $B_{n+1} = B_n^{2^{n+1}-1,r}$. Then, (Q_{n+1}, B_{n+1}) decides (u, b), for every $(u, b) \in 2^{n+1} \times \mathscr{P}[B_{n+1}]$ with $\mathrm{depth}_{B_{n+1}}(b) = n + 1$: for such (u, b), there exist $(i, j) \in \{0, 1, \ldots, 2^{n+1} - 1\} \times \{0, 1, \ldots, r\}$ such that $u = u_i$ and $b = b_j$. Then $(Q_n^{i,j}, B_n^{i,j})$ decides (u, b). Notice that

$$Q_{n+1}(u_i) = Q_n^{i,r} \subseteq Q_n^{i,r-1}(u_i) \subseteq \cdots \subseteq Q_n^{i,j}(u_i)$$

and

$$B_{n+1} = B_n^{2^{n+1}-1,r} \leqslant B_n^{i,j}.$$

Hence (Q_{n+1}, B_{n+1}) decides (u_i, b_j). Besides, $Q_{n+1} \subseteq_{n+1} Q_n$ and $B_{n+1} \in [n+1, B_n]$.

Now let $Q = \cap_n Q_n$ and take $B \in \cap_n [n+1, B_n]$. A similar argument shows that (Q, B) decides (u, b), for every $(u, b) \in 2^{<\infty} \times \mathscr{P}[B]$ with $\mathrm{depth}_B(b) = |u|$. Then in virtue of Lemma 1(f), Q and B are as required. $\square$

Lemma 3. *Let Q and B be as in Lemma 2. Suppose (Q, B) rejects $(\langle\rangle, \emptyset)$. Then there exists $D \leqslant B$ such that (Q, D) rejects (u, b), for every $(u, b) \in 2^{<\infty} \times \mathscr{P}[D]$ with* $\mathrm{depth}_D(b) \leqslant |u|$.

Proof. Let us build a fusion sequence $([n, D_n])_n$. Let $D_0 = B$. Then by hypothesis (Q, D_0) rejects $(\langle\rangle, \emptyset)$. Suppose D_n is given such that (Q, D_n) rejects (u, b), for every $(u, b) \in 2^n \times \mathscr{P}[D_n]$ with $\mathrm{depth}_{D_n}(b) = n$.

Now, let $u_0, u_1, \ldots, u_{2^{n+1}-1}$ be a list of the elements of 2^{n+1}; and let $b_0, b_1, \ldots, b_r$ be a list of the $b \in \mathscr{P}[D_n]$ such that $\mathrm{depth}_{D_n}(b) = n$. By Lemma 1(f), (Q, D_n) rejects (u_i, b_j) for every $(i, j) \in \{0, 1, \ldots, 2^{n+1} - 1\} \times \{0, 1, \ldots, r\}$. Now, by Lemma 1(e) there exists $D_n^{0,0} \in [n, D_n]$ such that $(Q, D_n^{0,0})$ rejects (u_0, b) for every $b \in p_{|b_0|+1}[b_0, D_n^{0,0}]$.

In the same way, for every $(i, j) \in \{0, 1, \ldots, 2^{n+1} - 1\} \times \{0, 1, \ldots, r\}$, we can find a $D_n^{i,j}$ with: $D_n^{i,j+1} \in [n, D_n^{i,j}]$, $D_n^{i+1,0} \in [n, D_n^{i,r}]$; and such that $(Q, D_n^{i,j})$ rejects (u_i, b) for every $b \in p_{|b_j|+1}[b_j, D_n^{i,j}]$. Let $D_{n+1} = D_n^{2^{n+1}-1,r}$. Notice that if $(u, b) \in 2^{n+1} \times \mathscr{P}[D_{n+1}]$ and $\mathrm{depth}_{D_{n+1}}(b) = n + 1$ then $u = u_i$ and $b \in p_{|b_j|+1}[b_j, D_n^{i,j}]$ for some $(i, j) \in \{0, 1, \ldots, 2^{n+1} - 1\} \times \{0, 1, \ldots, r\}$. Hence (Q, D_{n+1}) rejects (u, b). Besides, $D_{n+1} \in [n, D_n]$.

Now take $D \in \cap_n [n, D_n]$. Then D is as required. $\square$

Proof of Theorem 3. Given $\mathscr{F} \subseteq 2^{<\infty} \times \mathscr{P}$, perfect $P \subseteq 2^\infty$ and $A \in \mathscr{R}$, consider the combinatorial forcing 1. Let $Q \subseteq P$ and $B \leqslant A$ be as in Lemma 2. If (Q, B) accepts $(\langle\rangle, \emptyset)$ then part (a) of Theorem 3 holds by the definition of "accepts". So suppose (Q, B) does not accept (and hence, rejects) $(\langle\rangle, \emptyset)$. By Lemma 3, find $D \leqslant B$ such that (Q, D) rejects (u, b), for every $(u, b) \in 2^{<\infty} \times \mathscr{P}[D]$ with $\mathrm{depth}_D(b) \leqslant |u|$. Suppose towards a contradiction that there exist (t, b) in $(T_Q \times \mathscr{P}[D]) \cap \mathscr{F}$. Find $u_t \in 2^{<\infty}$ such that $Q(u_t) \subseteq Q \cap [t]$. Then (Q, D) accepts (u_t, b): for $x \in Q(u_t)$ and $C \in [b, D]$, let $k = |t|$ and m be such that $p_m(C) = b$. Then $(x|_k, p_m(C)) = (t, b) \in \mathscr{F}$.

But then, by Lemma 1(f), (Q, D) accepts (v, b), for every $v \in 2^{<\infty}$ such that $u_t \sqsubseteq v$ and $|v| \geqslant \mathrm{depth}_D(b)$. This is a contradiction with the choice of D. Therefore, for $S = Q$ and D part (b) of Theorem 3 holds. $\square$

Now we are ready to prove our main result:

Proof of Theorem 2. (a) The implication from left to right is obvious. So suppose $\mathscr{X} \subseteq 2^\infty \times \mathscr{R}$ has the $\mathbb{P} \times \mathrm{Exp}(\mathscr{R})$-Baire property, and let $P \times [a, A]$ be given. In order to make the proof notationally simpler, we will assume $a = \emptyset$ without a loss of generality.

Claim. *Given $\hat{\mathscr{X}} \subseteq 2^\infty \times \mathscr{R}$, perfect $\hat{P} \subseteq 2^\infty$ and $\hat{A} \in \mathscr{R}$, there exist a perfect $Q \subseteq \hat{P}$ and $B \leqslant \hat{A}$ such that for each $(u, b) \in 2^{<\infty} \times \mathscr{P}[B]$ with $|u| \geqslant \mathrm{depth}_B(b)$ one of the following holds*:

(i) $Q(u) \times [b, B] \subseteq \hat{\mathscr{X}}$.
(ii) $R \times [b, C] \not\subseteq \hat{\mathscr{X}}$, *for every perfect $R \subseteq Q(u)$ and every $C \leqslant B$ compatible with b.*

Proof. Consider the Combinatorial Forcing 2 and apply Lemma 2.

Apply the claim to $\mathscr{X}$, P and A to find a perfect $Q_1 \subseteq P$ and $B_1 \leqslant A$ such that for each $(u, b) \in 2^{<\infty} \times \mathscr{P}[B_1]$ with $|u| \geqslant \mathrm{depth}_{B_1}(b)$ one of the following holds:

(1) $Q_1(u) \times [b, B_1] \subseteq \mathscr{X}$ or
(2) $R \times [b, C] \not\subseteq \mathscr{X}$, for every perfect $R \subseteq Q_1(u)$ and every $C \leqslant B_1$ compatible with b.

For each $t \in T_{Q_1}$, choose $u_1^t \in 2^{<\infty}$ such that $u_1^t(Q_1) \sqsubseteq t$.

Let

$$\mathscr{F}_1 = \{(t, b) \in T_{Q_1} \times \mathscr{P}[B_1] : Q_1(u_1^t) \times [b, B_1] \subseteq \mathscr{X}\}.$$

Now, pick $S_1 \subseteq Q_1$ and $D_1 \leqslant B_1$ satisfying Theorem 3. If (a) of Theorem 3 holds then $S_1 \times [0, D_1] \subseteq \mathscr{X}$ and we are done. So suppose (b) holds.

Apply the claim to $\mathscr{X}^{\mathrm{c}}$, S_1 and D_1 to find a perfect $Q_2 \subseteq S_1$ and $B_2 \leqslant D_1$ such that for each $(u, b) \in 2^{<\infty} \times \mathscr{P}[B_2]$ with $|u| \geqslant \mathrm{depth}_{D_2}(b)$ one of the following holds:

(3) $Q_2(u) \times [b, B_2] \subseteq \mathscr{X}^{\mathrm{c}}$ or
(4) $R \times [b, C] \nsubseteq \mathscr{X}^{\mathrm{c}}$, for every perfect $R \subseteq Q_2(u)$ and every $C \leqslant B_2$ compatible with b.

As before, for each $t \in T_{Q_2}$, choose $u_2^t \in 2^{<\infty}$ such that $u_2^t(Q_2) \sqsubseteq t$.
Let

$$\mathscr{F}_2 = \{(t, b) \in T_{Q_2} \times \mathscr{P}[B_2] : Q_2(u_2^t) \times [b, B_2] \subseteq \mathscr{X}^{\mathrm{c}}\}.$$

Again, pick $S_2 \subseteq Q_2$ and $D_2 \leqslant B_2$ satisfying Theorem 3. If (a) of Theorem 3 holds then $S_2 \times [0, D_2] \cap \mathscr{X} = \emptyset$ and we are done. So suppose (b) holds again. Let us see that this contradicts the fact that $\mathscr{X}$ has the $\mathbb{P} \times \mathrm{Exp}(\mathscr{R})$-Baire property.

Notice that for every $(t, b) \in T_{S_2} \times \mathscr{P}[D_2]$ the following holds:

(i) $Q_1(u_1^t) \times [b, B_1] \nsubseteq \mathscr{X}$, and
(ii) $Q_2(u_2^t) \times [b, B_2] \nsubseteq \mathscr{X}^{\mathrm{c}}$.

So, suppose there is a nonempty $R \times [b, C] \subseteq S_2 \times [\emptyset, D_2] \cap \mathscr{X}$, where R is perfect, and pick $t \in T_R$ with $|u_1^t| \geqslant \mathrm{depth}_{B_1}(b)$. Notice that $R \cap [t] \subseteq Q_1(u_1^t)$. On the one hand, we have that $R \cap [t] \times [b, C] \subseteq R \times [b, C] \subseteq \mathscr{X}$. But in virtue of (i), $Q_1(u_1^t) \times [b, B_1] \nsubseteq \mathscr{X}$ and hence by (2) above we have that $R \cap [t] \times [b, C] \nsubseteq \mathscr{X}$. If we suppose that there is a nonempty $R \times [b, C] \subseteq S_2 \times [\emptyset, D_2] \cap \mathscr{X}^{\mathrm{c}}$ we reach to a similar contradiction in virtue of (ii) and (4) above. So there is neither $R \times [b, C] \subseteq S_2 \times [\emptyset, D_2] \cap \mathscr{X}$ nor $R \times [b, C] \subseteq S_2 \times [\emptyset, D_2] \cap \mathscr{X}^{\mathrm{c}}$. But this is impossible because $\mathscr{X}$ has the $\mathbb{P} \times \mathrm{Exp}(\mathscr{R})$-Baire Property.

(b) Again, the implication from left to right is obvious. Conversely, the result follows easily from (a) and the fact that $\mathscr{X}$ is $\mathbb{P} \times \mathrm{Exp}(\mathscr{R})$-meager.

This completes the proof of Theorem 2. $\square$

4. Some particular cases

Several interesting consequences can be derived from the facts obtained in the previous section. Some of them are known classical results and the others are parametrized versions of known Ellentuck-type theorems in nonclassical spaces.

Let k be a positive integer. For $p : \mathbb{N} \to \{0, 1, \ldots, k\}$, let $\mathrm{supp}(p)$ denote the set $\{n : p(n) \neq 0\}$ and let $\mathrm{rang}(p)$ denote the range of p.

Let us consider the set

$$\mathrm{FIN}_k := \{p : \mathbb{N} \to \{0, 1, \ldots, k\} : \mathrm{supp}(p) \text{ is finite and } k \in \mathrm{rang}(p)\}.$$

A *block basic sequence* is any finite or infinite sequence $X = (x_n)_{n \in I \subseteq \mathbb{N}}$ of elements of FIN_k such that

$$\max(\mathrm{supp}(x_n)) < \min(\mathrm{supp}(x_m)) \quad \text{whenever } n < m.$$

We shall use $a, b, c, \ldots$ for finite block basic sequences, and $A, B, C, \ldots$ for infinite block basic sequences. In this latter case we will assume that the set of indexes is $I = \mathbb{N}$.

Define $T : \mathrm{FIN}_k \to \mathrm{FIN}_{k-1}$ by

$$T(p)(n) = \max\{p(n) - 1, 0\}.$$

In [16] T is called the *tetris operation*. For every $j \in \mathbb{N}$, $T^{(j)}$ is the jth iteration of T, where $T^{(0)}(p) = p$ and $T^{(j+1)}(p) = T(T^{(j)}(p))$.

For a given block basic sequence $X = (x_n)_{n \in I \subseteq \mathbb{N}}$, the *subspace* of FIN_k *generated* by X, denoted by $[X]$, is the set of elements of FIN_k of the form:

$$T^{(j_0)}(x_{n_0}) + T^{(j_1)}(x_{n_1}) + \cdots + T^{(j_r)}(x_{n_r}),$$

where $n_0 < n_1 < \cdots < n_r$ is a finite sequence of elements of I and $j_0 < j_1 < \cdots < j_r$ is a sequence of elements of $\{0, 1, \ldots, k\}$ such that $j_i = 0$ for some $i \in \{0, 1, \ldots, r\}$.

The following result shows us an important feature of FIN_k, which provides us of a pigeon hole principle within this context:

Theorem 4 (Gowers [7]). *For every integer $r > 0$ and every partition $\phi : \mathrm{FIN}_k \to \{0, 1, \dots, r-1\}$ there exists an infinite block basic sequence A such that ϕ is constant in $[A]$.*

In the case $k = 1$, FIN_k is the set FIN of nonempty finite subsets of $\mathbb{N}$, and Theorem 4 is Hindman's theorem [8]. Let $\mathrm{FIN}_k^{[\infty]}$ be the set of infinite block basic sequences and define,

$$A \leqslant B \text{ iff } A \subseteq [B]$$

for $A, B \in \mathrm{FIN}_k^{[\infty]}$. Also, for every $A \in \mathrm{FIN}_k^{[\infty]}$, let the nth approximation of A be

$$p_n(A) = \text{the first } n \text{ elements of } A.$$

Then the set $\mathscr{P}$ of approximations is $\mathrm{FIN}_k^{[<\infty]}$, the set of finite block basic sequences. Now, for $a, b \in \mathrm{FIN}_k^{[<\infty]}$ define $a \leqslant_{\mathrm{fin}} b$ if and only if

$$a = b = \emptyset \text{ or } a \subset [b] \text{ and } \max\left(\operatorname{supp}\bigcup a\right) = \max\left(\operatorname{supp}\bigcup b\right).$$

With this terminology and the obvious definition of the neighborhoods $[a, A]$ (and the family $\mathrm{Exp}(\mathrm{FIN}_k^{[\infty]})$), the triplet $(\mathrm{FIN}_k^{[\infty]}, \leqslant, (p_n)_{n\in\mathbb{N}})$ satisfies (A1)–(A6). Also, $\mathrm{FIN}_k^{[\infty]}$ is readily identified to a closed subset of $\mathscr{P}^\infty$ and hence it is a Ramsey space, in virtue of the Abstract Ellentuck Theorem (Theorem 1). Here (A6) reduces to a natural variation of Gowers' theorem (Theorem 4). This yields the following corollary of Theorem 2:

Corollary 2. *Let $\mathscr{X} \subseteq 2^\infty \times \mathrm{FIN}_k^{[\infty]}$ be given. $\mathscr{X}$ is perfectly Ramsey iff $\mathscr{X}$ has the $\mathbb{P} \times \mathrm{Exp}(\mathrm{FIN}_k^{[\infty]})$-Baire Property. $\mathscr{X}$ is perfectly Ramsey null iff $\mathscr{X}$ is $\mathbb{P} \times \mathrm{Exp}(\mathrm{FIN}_k^{[\infty]})$-meager.*

In the same way, in virtue of Corollary 1, we obtain the following:

Corollary 3. *Let $\mathscr{X} \subseteq 2^\infty \times \mathbb{N}^{[\infty]}$ be given. $\mathscr{X}$ is perfectly Ramsey iff $\mathscr{X}$ has the $\mathbb{P} \times \mathrm{Exp}(\mathbb{N}^{[\infty]})$-Baire Property. $\mathscr{X}$ is perfectly Ramsey null iff $\mathscr{X}$ is $\mathbb{P} \times \mathrm{Exp}(\mathbb{N}^{[\infty]})$-meager.*

Corollary 3 is the parametrization of Ellentuck's theorem [4] obtained by Pawlikowski in [14]. And Corollary 2 gives a parametrized version of Milliken's theorem [11], when $k = 1$, and a parametrized version of the corresponding Ellentuck-type theorem in the Ramsey space $(\mathrm{FIN}_k^{[\infty]}, \leqslant, (p_n)_{n\in\mathbb{N}})$ defined in this section, when $k > 1$.

5. Closedness under the Souslin operation

In this section we go back to our parametrization of (abstract) Ramsey spaces and study the family of perfectly Ramsey sets in relation to the Souslin operation. First, we proof the following fact which turns out to be crucial for this study.

Lemma 4. *Given $(\mathscr{R}, \leqslant, (p_n)_{n\in\mathbb{N}})$ satisfying (A1)–(A6) and with $\mathscr{R}$ metrically closed, the perfectly Ramsey null subsets of $2^\infty \times \mathscr{R}$ form a σ-ideal.*

Proof. Let $(\mathscr{X}_n)_n$ be a sequence of perfectly Ramsey null subsets of $2^\infty \times \mathscr{R}$ and fix $P \times [a, A]$. We can assume $a = \emptyset$. Also notice that the finite union of perfectly Ramsey null sets yields a perfectly Ramsey null set; so we will assume $(\forall n)\ \mathscr{X}_n \subseteq \mathscr{X}_{n+1}$. Proceeding as in the proof of Lemma 2 we build fusion sequences Q_n, $[n+1, B_n]$ as follows: take $Q_0 \subseteq P$, $B_0 \leqslant A$ such that $Q_0 \times [0, B_0] \cap \mathscr{X}_0 = \emptyset$. Suppose Q_n, $[n+1, B_n]$ have been defined such that

$$Q_n \times [b, B_n] \cap \mathscr{X}_n = \emptyset$$

for every $b \in \mathscr{P}[B_n]$ with $\mathrm{depth}_{B_n}(b) = n$. Since $\mathscr{X}_{n+1}$ is perfectly Ramsey null, applying this fact successively we find $Q_{n+1} \subseteq_{n+1} Q_n$, and $B_{n+1} \in [n+1, B_n]$ such that

$$Q_{n+1} \times [b, B_{n+1}] \cap \mathscr{X}_{n+1} = \emptyset$$

for every $b \in \mathscr{P}[B_{n+1}]$ with $\mathrm{depth}_{B_{n+1}}(b) = n+1$. Let

$$Q = \bigcap_n Q_n \quad \text{and} \quad B = \bigcap_n [n+1, B_n].$$

Then $Q \times [0, B] \cap \cup_n \mathscr{X}_n = \emptyset$: take $(x, C) \in Q \times [0, B]$ and fix arbitrary n. To show that $(x, C) \notin \mathscr{X}_n$ let k be large enough so that $\mathrm{depth}_B(p_k(C)) = m \geqslant n$. Then by construction $Q \times [p_k(C), B] \cap \mathscr{X}_m = \emptyset$ and hence, since $\mathscr{X}_n \subseteq \mathscr{X}_m$, we have $(x, C) \notin \mathscr{X}_n$. This completes the proof. □

Now, we borrow some terminology from [14]:

Let $\mathscr{A}$ be a family of subsets of a set Z. We say that $X, Y \subseteq Z$ are *compatible* (with respect to $\mathscr{A}$) if there exists $W \in \mathscr{A}$ such that $W \subseteq X \cap Y$. Also, we say that $\mathscr{A}$ is *M-like* if for any $\mathscr{B} \subseteq \mathscr{A}$ such that $|\mathscr{B}| < |\mathscr{A}|$, every member of $\mathscr{A}$ which is not compatible with any member of $\mathscr{B}$ is compatible with $Z \setminus \cup \mathscr{B}$.

Notice that the family $\mathbb{P}$ of perfect subsets of 2^∞ is *M*-like, as well as the family $\mathrm{Exp}(\mathscr{R})$ (this is true of any topological basis). Therefore, according to Lemma 2.7 in [14], if we require that $|\mathrm{Exp}(\mathscr{R})| = |\mathbb{P}|$ $(=2^{\aleph_0})$, then the family $\mathbb{P} \times \mathrm{Exp}(\mathscr{R}) = \{P \times [n, A] : P \in \mathbb{P} \text{ and } A \in \mathscr{R}\}$ is also *M*-like. This lead us to the following:

Corollary 4. *Let* $(\mathscr{R}, \leqslant, (p_n)_{n\in\mathbb{N}})$ *satisfying* (A1)–(A6), *with* $\mathscr{R}$ *metrically closed be such that* $|\mathscr{R}| = 2^{\aleph_0}$. *Then, the family of perfectly Ramsey subsets of* $2^\infty \times \mathscr{R}$ *is closed under the Souslin operation.*

Proof. In virtue of Theorem 2, the family of perfectly Ramsey subsets of $2^\infty \times \mathscr{R}$ coincides with the family of subsets of $2^\infty \times \mathscr{R}$ which have the $\mathbb{P} \times \mathrm{Exp}(\mathscr{R})$-Baire property. And as we pointed out in the previous paragraph, $\mathbb{P} \times \mathrm{Exp}(\mathscr{R})$ is *M*-like. So the proof follows from Lemma 4 on top of this section and Lemmas 2.5 and 2.6 of [14] (which refer to a well-known result of [9]).

Corollary 5 (*Pawlikowski [14]*). (*a*) *The family of perfectly Ramsey subsets of* $2^\infty \times \mathbb{N}^{[\infty]}$ *is closed under the Souslin operation*. (b) *The family of perfectly Ramsey subsets of* $2^\infty \times \mathrm{FIN}_k^{[\infty]}$ *is closed under the Souslin operation*.

Corollary 6 (*Miller–Todorcevic [10]*). (*a*) *Analytic subsets of* $2^\infty \times \mathbb{N}^{[\infty]}$ *are perfectly Ramsey*. (b) *Analytic subsets of* $2^\infty \times \mathrm{FIN}_k^{[\infty]}$ *are perfectly Ramsey*.

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