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Effective non-vanishing of global sections of multiple adjoint bundles for polarized 3-folds ☆

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ABSTRACT

Let X be a smooth complex projective variety of dimension 3 and let L be an ample line bundle on X. In this paper, we provide a lower bound for $h^0(m(K_X + L))$ under the assumption that $\kappa(K_X+L) \geq 0$. In particular, we get the following: (1) if $0 \leq \kappa(K_X+L) \leq 2$, then $h^0(K_X + L) > 0$ holds. (2) If $\kappa(K_X + L) = 3$, then $h^0(2(K_X + L)) \geq 3$ holds. Moreover we get a classification of (X, L) with $\kappa(K_X + L) = 3$ and $h^0(2(K_X + L)) = 3$ or 4.

1. Introduction

Let X be a smooth projective variety of dimension n and let L be an ample (resp. a nef and big) line bundle on X. Then the pair (X, L) is called a polarized (resp. quasi-polarized) manifold. When we study this (X, L), we find that adjoint bundles $K_X + tL$ play important roles (for example, see [4, Chapters 7, 9, and 11]), where K_X is the canonical line bundle of X. In particular, it is important to know the value of $h^0(K_X + tL)$.

In relation to the positivity of $h^0(K_X + tL)$, there are the following conjectures.

Conjecture 1. (i) ***(Beltrametti–Sommese [4, Conjecture 7.2.7])*** *Let (X, L) be a polarized manifold of dimension n. Assume that $K_X + (n-1)L$ is nef. Then $h^0(K_X + (n-1)L) > 0$.*

(ii) *(**Ionescu** [25, Open problems, P.321]) Let (X, L) be a quasi-polarized manifold of dimension n. Assume that $K_X + L$ is nef. Then $h^0(K_X + L) > 0$.*

(iii) *(**Ambro** [1, Section 4], **Kawamata** [23, Conjecture 2.1]) Let X be a complex normal variety, B an effective $\mathbb{R}$-divisor on X such that the pair (X, B) is KLT, and D a Cartier divisor on X. Assume that D is nef, and that $D - (K_X + B)$ is nef and big. Then $h^0(D) > 0$.*

These conjectures have been studied by several authors (see [14,23,8,17,18,6,7]). In particular it is known that Conjecture 1 (i) is true if $\dim X \leq 3$, and Conjecture 1 (ii) and (iii) are true if $\dim X \leq 2$.

Here we note that if $K_X + L$ is nef, then by [29] there exists a positive integer m such that $h^0(m(K_X + L)) > 0$, that is, $\kappa(K_X+L) \geq 0$. So, as a generalization of Conjecture 1 (ii), it is natural and interesting to study the following problem, which was proposed in [18, Problem 3.2]:

Problem 1. For any fixed positive integer n, determine the smallest positive integer p, which depends only on n, such that the following $(*)$ is satisfied:

$(*)$ $h^0(p(K_X + L)) > 0$ for any polarized manifold (X, L) of dimension n with $\kappa(K_X + L) \geq 0$.

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The aim of this paper is to study Problem 1. It is known that $p = 1$ if X is a curve or surface (see [18, Theorem 2.8]). In the case of $n = 3$, as we said in [20, Corollary 5.2], we can prove $p \leq 2$. Specifically, in [20, Theorem 5.4 (2)], we proved that $h^0(2(K_X + L)) \geq 3$ holds if $n = 3$ and $\kappa(K_X + L) = 3$. Moreover in [20, Theorem 5.4 (1)], we announced that we would prove that $h^0(K_X + L) > 0$ if $n = 3$ and $0 \leq \kappa(K_X + L) \leq 2$. So in this paper, we will show that $h^0(K_X + L) > 0$ if $n = 3$ and $0 \leq \kappa(K_X + L) \leq 2$. Furthermore we will also study a lower bound for $h^0(m(K_X + L))$ under the assumption that $n = 3$ and $\kappa(K_X + L) \geq 0$.

The paper is organized as follows. In Sections 2 and 3, we will state some definitions and results which will be used later. In particular, in Section 3, we review the sectional geometric genus. In Section 4, we will treat special cases. If $\dim X = 3$ and $\kappa(K_X + L) = 1$ (resp. 2), then there exist a polarized manifold (M, A), a normal projective variety Y of dimension 1 (resp. 2), a fiber space $f : M \to Y$ and an ample line bundle H on Y such that $h^0(m(K_X + L)) = h^0(m(K_M + A))$ for any positive integer m and $K_M + A = f^*(H)$. (This (M, A) is called a reduction of (X, L). See Definition 2.1.) Hence it is important to consider the following case: Let (X, L) be a polarized manifold of dimension $n \geq 3$ and let Y be a normal projective variety of dimension 1 or 2. Assume that there exists a fiber space $f : X \to Y$ such that $K_X + L = f^*(H)$ for some ample line bundle H on Y. In Section 4, we consider (X, L) like this and we will give a lower bound for $h^0(m(K_X + L))$. In particular, we see that $h^0(K_X + L) > 0$ in this case. In Section 5, we will study the case of $\dim X = 3$. In particular, we will give a lower bound for $h^0(m(K_X + L))$ under the following assumptions:

(a) $0 \leq \kappa(K_X + L) \leq 2$ and $m \geq 1$.
(b) $\kappa(K_X + L) = 3$ and $m \geq 2$.

In particular we get $h^0(K_X + L) > 0$ if $0 \leq \kappa(K_X + L) \leq 2$, and $h^0(2(K_X + L)) \geq 3$ if $\kappa(K_X + L) = 3$ (see also [20, Theorem 5.4 (2)]). Moreover we will also classify (X, L) with $\kappa(K_X + L) = 3$ and $h^0(2(K_X + L)) = 3$ or 4 (see Theorems 5.3 and 5.4).

In this paper, we shall study mainly a smooth projective variety X over the field of complex numbers $\mathbb{C}$. We will employ the customary notation in algebraic geometry.

2. Preliminaries

Here we list up several results which will be used later.

Definition 2.1. (i) Let X (resp. Y) be an n-dimensional projective manifold, and L (resp. A) an ample line bundle on X (resp. Y). Then (X, L) is called a *simple blowing up of* (Y, A) if there exists a birational morphism $\pi : X \to Y$ such that π is a blowing up at a point of Y and $L = \pi^*(A) - E$, where E is the π-exceptional effective reduced divisor.

(ii) Let X (resp. M) be an n-dimensional projective manifold, and L (resp. A) an ample line bundle on X (resp. M). Then we say that (M, A) is a *reduction of* (X, L) if there exists a birational morphism $\mu : X \to M$ such that μ is a composition of simple blowing ups and (M, A) is not obtained by a simple blowing up of any polarized manifold. The map $\mu : X \to M$ is called the *reduction map*.

Remark 2.1. Let (X, L) be a polarized manifold and let (M, A) be a reduction of (X, L). Let $\mu : X \to M$ be the reduction map.

(i) If (X, L) is not obtained by a simple blowing up of another polarized manifold, then (X, L) is a reduction of itself.
(ii) Reduction exists provided that $K_X + (n - 1)L$ is nef and big (see [4, Definition 7.3.3], [11, Chapter II, (11.11)]).

Definition 2.2. A quasi-polarized surface (S, L) is said to be *L-minimal* if $LE > 0$ for every (-1)-curve E on S.

Lemma 2.1. *Let X be a complete normal variety of dimension n, and let D_1 and D_2 be effective Cartier divisors on X. Then $h^0(D_1 + D_2) \geq h^0(D_1) + h^0(D_2) - 1$.*

Proof. See [15, Lemma 1.10] or [24, 15.6.2 Lemma]. □

Proposition 2.1. *Let X be a projective variety of dimension n and let D_i be $\mathbb{Q}$-Cartier divisors on X for $0 \leq i \leq k$. Assume that $n \geq 2$ and that D_i is nef for every integer i with $1 \leq i \leq k$. If $n_1 + \cdots + n_k = n - 1$ and $n_1 \geq 1$, then we have*

$$(D_0 D_1^{n_1} \cdots D_k^{n_k})^2 \geq (D_0^2 D_1^{n_1 - 1} \cdots D_k^{n_k})(D_1^{n_1 + 1} \cdots D_k^{n_k}).$$

Proof. See [4, Proposition 2.5.1]. □

Proposition 2.2. *Let X be a normal projective surface and let $\pi : S \to X$ be a resolution of singularities of X. Then $\chi(\mathcal{O}_S) + h^0(R^1\pi_*(\mathcal{O}_S)) = \chi(\mathcal{O}_X)$. In particular $\chi(\mathcal{O}_S) \leq \chi(\mathcal{O}_X)$ holds.*

Proof. By using Leray's spectral sequence for $\pi^*(\mathcal{O}_X)$, we have

$$\chi(\pi^*\mathcal{O}_X) = \sum_{q \geq 0} (-1)^q \chi(R^q\pi_*(\pi^*\mathcal{O}_X)).$$

Since $R^q\pi_*(\pi^*\mathcal{O}_X) \cong R^q\pi_*(\mathcal{O}_S)$ and $R^q\pi_*(\mathcal{O}_S) = 0$ for every integer q with $q \geq 2$, we have

$$\chi(\pi^*\mathcal{O}_X) = \chi(\pi_*(\mathcal{O}_S)) - \chi(R^1\pi_*(\mathcal{O}_S)).$$

Here we also note that $\pi_*(\mathcal{O}_S) = \mathcal{O}_X$ because π is birational and X is normal (see [22, Corollary 11.4 in Chapter III]). Moreover $\chi(R^1\pi_*(\mathcal{O}_S)) = h^0(R^1\pi_*(\mathcal{O}_S))$ because $\dim \mathrm{Supp}(R^1\pi_*(\mathcal{O}_S)) \le 0$. Therefore since $\mathcal{O}_S = \pi^*(\mathcal{O}_X)$, we get the assertion. □

Lemma 2.2. *Let X be a smooth projective variety of dimension n and let Y be a normal projective variety of dimension m with $n > m \ge 1$. Assume that $q(X) = q(Y)$ and there exists a fiber space $f : X \to Y$, that is, f is a surjective morphism with connected fibers. Then for any resolution of singularities of Y, $\pi : Z \to Y$, we have $q(Z) = q(Y)$. In particular, if $q(Y) \ge 1$, then the Albanese map of Y can be defined.*

Proof. By assumption, there exist smooth projective varieties X_1 and Y_1, birational morphisms $\mu_1 : X_1 \to X$ and $\nu_1 : Y_1 \to Y$, and a fiber space $f_1 : X_1 \to Y_1$ such that $f \circ \mu_1 = \nu_1 \circ f_1$. Here we note that $q(X) = q(X_1)$ and $q(X_1) \ge q(Y_1)$. Moreover we have $q(Y_1) \ge q(Y)$ holds. Hence we get $q(Y_1) \ge q(Y) = q(X) = q(X_1) \ge q(Y_1)$ and we have $q(Y_1) = q(Y)$. On the other hand let Z be any resolution of singularities of Y. Then $q(Z) = q(Y_1)$ because Z is birationally equivalent to Y_1. In particular, by [30, (0.3.3) Lemma] or [4, Lemma 2.4.1 and Remark 2.4.2], the Albanese map of Y can be defined. Hence we get the assertion of Lemma 2.2. □

3. Review on the sectional geometric genus

In this section, we review the definition and some properties of the sectional geometric genus of polarized manifolds, which will be used later.

Notation 3.1. *Let X be a projective variety of dimension n and let L be a line bundle on X. Let $\chi(tL)$ be the Euler–Poincaré characteristic of tL, where t is an indeterminate. Then we put*

$$\chi(tL) = \sum_{j=0}^{n} \chi_j(X, L)\binom{t+j-1}{j}.$$

Definition 3.1. Let X be a projective variety of dimension n and let L be a line bundle on X. Then for every integer i with $0 \le i \le n$, the *i-th sectional H-arithmetic genus* $\chi_i^H(X, L)$ and the *i-th sectional geometric genus* $g_i(X, L)$ *of* (X, L) are defined by the following:

$$\chi_i^H(X, L) := \chi_{n-i}(X, L),$$

$$g_i(X, L) := (-1)^i(\chi_i^H(X, L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X).$$

Remark 3.1. (1) Since $\chi_{n-i}(X, L) \in \mathbb{Z}$, we see that $\chi_i^H(X, L)$ and $g_i(X, L)$ are integers by definition.
(2) If $i = 0$, then $\chi_0^H(X, L)$ and $g_0(X, L)$ are equal to the degree of (X, L).
(3) If $i = 1$, then $g_1(X, L)$ is equal to the sectional genus $g(X, L)$ of (X, L).
(4) If $i = n$, then $\chi_n^H(X, L) = \chi(\mathcal{O}_X)$ and $g_n(X, L) = h^n(\mathcal{O}_X)$.

Theorem 3.1. *Let (X, L) be a quasi-polarized manifold with $\dim X = n$. For every integer i with $0 \le i \le n-1$, we have*

$$g_i(X, L) = \sum_{j=0}^{n-i-1}(-1)^j\binom{n-i}{j}h^0(K_X + (n-i-j)L) + \sum_{k=0}^{n-i}(-1)^{n-i-k}h^{n-k}(\mathcal{O}_X).$$

Proof. See [15, Theorem 2.3]. □

The following theorem will be often used later.

Theorem 3.2. *Let (X, L) be a polarized 3-fold. Assume that $\kappa(K_X + L) \ge 0$. Then $g_2(X, L) \ge h^1(\mathcal{O}_X)$.*

Proof. See [16, Theorem 3.3.1 (2)]. □

Notation 3.2. *Let X be a projective variety of dimension n, let i be an integer with $0 \le i \le n-1$, and let $L_1, \ldots, L_{n-i}$ be line bundles on X. Then $\chi(\sum_{j=1}^{n-i} t_jL_j)$ is a polynomial in $t_1, \ldots, t_{n-i}$ of total degree at most n. So we can write $\chi(\sum_{j=1}^{n-i} t_jL_j)$ uniquely as follows.*

$$\chi\left(\sum_{j=1}^{n-i} t_jL_j\right) = \sum_{p=0}^{n}\sum_{\substack{p_1\ge 0,\ldots,p_{n-i}\ge 0\\ p_1+\cdots+p_{n-i}=p}} \chi_{p_1,\ldots,p_{n-i}}(L_1, \ldots, L_{n-i})\binom{t_1+p_1-1}{p_1}\cdots\binom{t_{n-i}+p_{n-i}-1}{p_{n-i}}.$$

Definition 3.2 (*[19, Definition 2.1 and Remark 2.2 (2)]*). Let X be a projective variety of dimension n, let i be an integer with $0 \le i \le n$, and let $L_1, \ldots, L_{n-i}$ be line bundles on X.

(1) The *i-th sectional H-arithmetic genus* $\chi_i^H(X, L_1, \ldots, L_{n-i})$ is defined by the following:

$$\chi_i^H(X, L_1, \ldots, L_{n-i}) = \begin{cases} \chi_{\underbrace{1,\ldots,1}_{n-i}}(L_1, \ldots, L_{n-i}) & \text{if } 0 \le i \le n-1,\\ \chi(\mathcal{O}_X) & \text{if } i = n. \end{cases}$$

(2) The *i-th sectional geometric genus* $g_i(X, L_1, \ldots, L_{n-i})$ is defined by the following:

$$g_i(X, L_1, \ldots, L_{n-i}) = (-1)^i(\chi_i^H(X, L_1, \ldots, L_{n-i}) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X).$$

Remark 3.2. (1) Let X be a projective variety of dimension n and let L be a line bundle on X. Let i be an integer with $0 \leq i \leq n-1$. Then

$$\chi_i^H(X, L, \ldots, L) = \chi_i^H(X, L)$$

and

$$g_i(X, L, \ldots, L) = g_i(X, L).$$

(See [19, Corollary 2.1].)

(2) Let X be a smooth projective variety of dimension n, and let $L_1, \ldots, L_{n-1}$ be line bundles on X. Then

$$g_1(X, L_1, \ldots, L_{n-1}) = 1 + \frac{1}{2}\left(K_X + \sum_{j=1}^{n-1} L_j\right) L_1 \cdots L_{n-1}.$$

(See [19, Corollary 2.7] or [21, Proposition 6.1.1].)

Theorem 3.3. *Let i be an integer with $1 \leq i \leq n$. Let $A, B, L_1, \ldots, L_{n-i-1}$ be line bundles on X. Then*

$$\chi_i^H(X, A+B, L_1, \ldots, L_{n-i-1}) = \chi_i^H(X, A, L_1, \ldots, L_{n-i-1}) + \chi_i^H(X, B, L_1, \ldots, L_{n-i-1}) - \chi_{i-1}^H(X, A, B, L_1, \ldots, L_{n-i-1})$$

$$\begin{aligned} g_i(X, A+B, L_1, \ldots, L_{n-i-1}) &= g_i(X, A, L_1, \ldots, L_{n-i-1}) + g_i(X, B, L_1, \ldots, L_{n-i-1}) \\ &\quad + g_{i-1}(X, A, B, L_1, \ldots, L_{n-i-1}) - h^{i-1}(\mathcal{O}_X). \end{aligned}$$

Proof. See [19, Corollary 2.4]. □

Proposition 3.1. *Let X be a smooth projective variety with $\dim X = n \geq 2$, let $L_1, \ldots, L_m$ be nef and big line bundles on X and let L be a nef line bundle, where $m \geq 1$. Then*

$$\begin{aligned} h^0(K_X + L_1 + \cdots + L_m + L) - h^0(K_X + L_1 + \cdots + L_m) &= \sum_{s=0}^{n-1} \sum_{(k_1, \ldots, k_{n-s-1}) \in A_{n-s-1}^m} g_s(X, L_{k_1}, \ldots, L_{k_{n-s-1}}, L) \\ &\quad - \sum_{s=0}^{n-2} \binom{m-1}{n-s-2} h^s(\mathcal{O}_X). \end{aligned}$$

Here $A_t^p := \{(k_1, \ldots, k_t) \mid k_l \in \{1, \ldots, p\}, k_i < k_j \text{ if } i < j\}$*, and we set*

$$\sum_{(k_1, \ldots, k_{n-s-1}) \in A_{n-s-1}^m} g_s(X, L_{k_1}, \ldots, L_{k_{n-s-1}}, L) = \begin{cases} 0 & \text{if } n-s-1 > m, \\ g_{n-1}(X, L) & \text{if } s = n-1. \end{cases}$$

Proof. See [20, Theorem 5.1]. □

4. Special cases

In this section, we will investigate the dimensions of multiples of adjoint linear systems for special cases. First we prove the following.

Theorem 4.1. *Let (X, L) be a polarized manifold of dimension $n \geq 2$ and let C be a smooth projective curve. Assume that there exists a fiber space $f : X \to C$ such that $K_X + L = f^*(H)$ for some ample line bundle H on C. Then for every positive integer m*

$$h^0(m(K_X + L)) \geq \begin{cases} (m-1)(g(C)-1) + mg(C) & \text{if } g(C) \geq 1, \\ m+1 & \text{if } g(C) = 0. \end{cases}$$

In particular $h^0(K_X + L) > 0$ holds.

Proof. In this case

$$\begin{aligned} h^0(m(K_X + L)) &= h^0(f^*(mH)) \\ &= h^0(mH) \\ &= h^1(mH) + \deg(mH) + (1 - g(C)). \end{aligned}$$

On the other hand, by [15, Lemma 1.13], we have $\deg H \geq 2g(C) - 1$. Hence if $g(C) \geq 1$, then

$$\begin{aligned} h^0(mH) &\geq m(2g(C)-1) + 1 - g(C) \\ &= (2m-1)g(C) - (m-1) \\ &= (m-1)(g(C)-1) + mg(C). \end{aligned}$$

If $g(C) = 0$, then $h^1(mH) = 0$ and $h^0(mH) = \deg(mH) + 1 \geq m + 1$. Therefore

$$h^0(m(K_X+L)) \geq \begin{cases} (m-1)(g(C)-1) + mg(C) & \text{if } g(C) \geq 1, \\ m+1 & \text{if } g(C) = 0. \end{cases}$$

This completes the proof. □

Corollary 4.1. *Let (X, L) be a polarized manifold of dimension $n \geq 2$ and let C be a smooth projective curve. Assume that there exists a fiber space $f : X \to C$ such that $K_X + L = f^*(H)$ for some ample line bundle H on C. Then for every positive integer m*

$$h^0(m(K_X+L)) \geq \begin{cases} m & \text{if } g(C) \geq 1, \\ m+1 & \text{if } g(C) = 0. \end{cases}$$

Theorem 4.2. *Let (X, L) be a polarized manifold of dimension $n \geq 2$ and let C be a smooth projective curve. Assume that there exists a fiber space $f : X \to C$ such that $K_X + L = f^*(H)$ for some ample line bundle H on C.*

(1) *If $g(C) \geq 1$ and $h^0(m(K_X + L)) = m$ for some positive integer m, then $g(C) = 1$ and $\deg H = 1$.*
(2) *If $g(C) = 0$ and $h^0(m(K_X + L)) = m + 1$ for some positive integer m, then $(C, H) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$.*

Proof. (2.1) Assume that $g(C) \geq 1$ and $h^0(m(K_X + L)) = m$. Then by the proof of Theorem 4.1 we have $g(C) = 1$ and $\deg H = 1$.

(2.2) Assume that $g(C) = 0$ and $h^0(m(K_X + L)) = m + 1$. Then the proof of Theorem 4.1 implies that $\deg H = 1$, that is, $H = \mathcal{O}_{\mathbb{P}^1}(1)$. Therefore $(C, H) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. So we get the assertion. □

Next we consider the following case.

Theorem 4.3. *Let (X, L) be a polarized manifold of dimension $n \geq 3$ and let Y be a normal projective surface. Assume that there exists a fiber space $f : X \to Y$ such that $K_X + L = f^*(H)$ for some ample line bundle H on Y. Then for every positive integer m*

$$h^0(m(K_X+L)) \geq \begin{cases} \binom{m+1}{2} - (m-1)\chi(\mathcal{O}_Y) & \text{if } \chi(\mathcal{O}_Y) \leq 0, \\ \binom{m}{2} + \chi(\mathcal{O}_Y) & \text{if } \chi(\mathcal{O}_Y) > 0. \end{cases}$$

In particular $h^0(K_X + L) > 0$ holds.

Proof. In this case $h^0(m(K_X + L)) = h^0(mH)$. Here we note the following.

Claim 4.1. *$h^i(mH) = 0$ for $i = 1, 2$.*

Proof. Since $f^*(mH) - K_X = (m-1)K_X + mL = (m-1)(K_X + L) + L$ is ample, we have $R^i f_*(f^*(mH)) = 0$ for every $i > 0$ by [15, Theorem 1.7]. Hence by [22, Exercise 8.1 page 252 in Chapter III] we have $h^i(f^*(mH)) = h^i(f_*f^*(mH)) = h^i(mH)$. Therefore for every $i > 0$

$$\begin{aligned} h^i(mH) &= h^i(f^*(mH)) \\ &= h^i(m(K_X+L)) \\ &= h^i(K_X + (m-1)(K_X+L) + L) \\ &= 0. \end{aligned}$$

This completes the proof of Claim 4.1. □

By Claim 4.1, we have $h^0(m(K_X + L)) = h^0(mH) = \chi(mH)$. Here we use Notation 3.1. Then $\chi_0(Y, H) = \chi(\mathcal{O}_Y)$, $\chi_1(Y, H) = 1 - g(Y, H)$ and $\chi_2(Y, H) = H^2$, where $g(Y, H)$ denotes the sectional genus of (Y, H). Let $\delta : S \to Y$ be a minimal resolution of Y. Then there exist a smooth projective variety X_1, a birational morphism $\mu_1 : X_1 \to X$ and a fiber space $f_1 : X_1 \to S$ such that $f \circ \mu_1 = \delta \circ f_1$.

(I) The case where $\chi(\mathcal{O}_Y) \leq 0$.

Then

$$\begin{aligned} \chi(mH) - m\chi(H) &= \sum_{j=0}^{2} \chi_j(Y, H)\binom{m+j-1}{j} - m\sum_{j=0}^{2}\chi_j(Y, H) \\ &= -(m-1)\chi(\mathcal{O}_Y) + \left(\binom{m+1}{2} - m\right)H^2 \\ &\geq \binom{m+1}{2} - m - (m-1)\chi(\mathcal{O}_Y) \\ &= \binom{m}{2} - (m-1)\chi(\mathcal{O}_Y). \end{aligned} \tag{1}$$

Therefore $\chi(mH) \geq m\chi(H) + \binom{m}{2} - (m-1)\chi(\mathcal{O}_Y) = mh^0(H) + \binom{m}{2} - (m-1)\chi(\mathcal{O}_Y)$.

Next we prove the following claim.

Claim 4.2. $h^0(H) > 0$.

Proof. Since $\chi(\mathcal{O}_Y) \leq 0$ in this case, we see that $h^1(\mathcal{O}_Y) > 0$. Because $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_Y)$ in this case, by Lemma 2.2 we see that Y has the Albanese map. Let $\alpha : Y \to \mathrm{Alb}(Y)$ be the Albanese map of Y and let $h := \alpha \circ f$. Here we note that $\dim h(X) = 1$ or 2.

(a) First we consider the case where $\dim h(X) = 2$. By [22, Corollary 10.7 in Chapter III] any general fiber F_h of h can be written as follows: $F_h = \cup_{i=1}^r F_i$, where F_i is a smooth projective variety of dimension $n-2$. We note that F_i is a fiber of f for every i. Since $(K_X + L)|_{F_i} = f^*(H)|_{F_i} \cong \mathcal{O}_{F_i}$, we have

$$h^0((K_X + L)|_{F_h}) = \sum_{i=1}^r h^0(K_{F_i} + L_{F_i}) = \sum_{i=1}^r h^0(\mathcal{O}_{F_i}) > 0.$$

By [8, Lemma 4.1] we have $h^0(H) = h^0(K_X + L) > 0$.

(b) Next we consider the case where $\dim h(X) = 1$. Then we note that h has connected fibers. Let F_h (resp. F_α) be a general fiber of h (resp. α). Then $f|_{F_h} : F_h \to F_\alpha$ is a fiber space such that $K_{F_h} + L_{F_h} = f^*(H)|_{F_h} = (f|_{F_h})^*(H|_{F_\alpha})$. Here we note that F_h and F_α are smooth projective varieties. Since H is ample, so is H_{F_α} on F_α. Since $\dim F_\alpha = 1$, by Theorem 4.1 we have $h^0(K_{F_h} + L_{F_h}) > 0$. Therefore by [8, Lemma 4.1] we get $h^0(H) = h^0(K_X + L) > 0$. This completes the proof. □

Claim 4.2 implies that by (1)

$$\begin{aligned}
\chi(mH) &\geq mh^0(H) + \binom{m}{2} - (m-1)\chi(\mathcal{O}_Y) \\
&\geq m + \binom{m}{2} - (m-1)\chi(\mathcal{O}_Y) \\
&\geq \binom{m+1}{2} - (m-1)\chi(\mathcal{O}_Y).
\end{aligned}$$

(II) Next we consider the case where $\chi(\mathcal{O}_Y) > 0$. First we prove the following lemma.

Lemma 4.1. $\chi_1(Y, H) + \chi_2(Y, H) \geq 0$.

Proof. First we note that $K_{X_1} + \mu_1^*(L) \geq \mu_1^*(K_X + L) = \mu_1^* f^*(H) = f_1^* \delta^*(H)$. Hence for a general fiber F_1 of f_1, we have $0 < h^0((K_{X_1} + \mu_1^*(L))|_{F_1}) = h^0(K_{F_1} + (\mu_1^*(L))_{F_1})$. Hence we have $(f_1)_*(K_{X_1/S} + \mu_1^*(L)) \neq 0$. By Hironaka's theory there exist a smooth projective variety X_2 and a birational morphism $\mu_2 : X_2 \to X_1$ such that

$$\mu_2^* f_1^*((f_1)_*(K_{X_1/S} + \mu_1^*(L))) \to \mu_2^*(K_{X_1/S} + \mu_1^*(L) - D) - E_2$$

is surjective, where D is an effective divisor on X_1 and E_2 is a μ_2-exceptional effective divisor on X_2. Since $(f_1)_*(K_{X_1/S} + \mu_1^*(L))$ is weakly positive ([13, Theorem A′ in Appendix]), we see that $\mu_2^*(K_{X_1/S} + \mu_1^*(L) - D) - E_2$ is pseudo effective (see the proof of (1) in [13, Remark 1.3.2]). Here we note that for every positive integer p we have

$$0 \leq (\mu_2^*(K_{X_1/S} + \mu_1^*(L) - D) - E_2)\mu_2^* f_1^* \delta^*(H)(\mu_2^* \mu_1^*(pL))^{n-2}$$

because H is ample. On the other hand

$$\begin{aligned}
(\mu_2^*(K_{X_1/S} + \mu_1^*(L) - D) - E_2)\mu_2^* f_1^* \delta^*(H)(\mu_2^* \mu_1^*(pL))^{n-2} &= (K_{X_1/S} + \mu_1^*(L) - D)(f_1^* \delta^*(H))(\mu_1^*(pL))^{n-2} \\
&\leq (K_{X_1/S} + \mu_1^*(L))(f_1^* \delta^*(H))(\mu_1^*(pL))^{n-2}.
\end{aligned}$$

Since $K_{X_1} = \mu_1^* K_X + E_1$, where E_1 is a μ_1-exceptional effective divisor on X_1, we have

$$\begin{aligned}
(K_{X_1/S} + \mu_1^*(L))(f_1^* \delta^*(H))(\mu_1^*(pL))^{n-2} &= (\mu_1^*(K_X + L) - f_1^*(K_S) + E_1)(f_1^* \delta^*(H))(\mu_1^*(pL))^{n-2} \\
&= (f_1^*(\delta^*(H) - K_S) + E_1)(\mu_1^* f^*(H))(\mu_1^*(pL))^{n-2} \\
&= f_1^*(\delta^*(H) - K_S)(\mu_1^* f^*(H))(\mu_1^*(pL))^{n-2} \\
&= f_1^*(\delta^*(H) - K_S)(f_1^* \delta^*(H))(\mu_1^*(pL))^{n-2}.
\end{aligned}$$

Here we take p as $\mathrm{Bs}|\mu_1^*(pL)| = \emptyset$. Then there exist $(n-2)$-general members $H_1, \ldots, H_{n-2}$ in $|\mu_1^*(pL)|$ such that $H_1 \cap \cdots \cap H_{n-2}$ is a smooth projective surface S_1. Then $f_1|_S : S_1 \to S$ is a surjective morphism and we have

$$\begin{aligned}
f_1^*(\delta^*(H) - K_S)(f_1^* \delta^*(H))(\mu_1^*(pL))^{n-2} &= f_1^*(\delta^*(H) - K_S) f_1^*(\delta^*(H)) S_1 \\
&= (\deg f_1|_{S_1})(\delta^*(H) - K_S)\delta^*(H).
\end{aligned}$$

On the other hand, since $\chi_2(Y, H) = \chi_2(S, \delta^*(H))$ and $\chi_1(Y, H) = \chi_1(S, \delta^*(H))$, we have $(\delta^*(H) - K_S)\delta^*(H) = 2(\chi_1(S, \delta^*(H)) + \chi_2(S, \delta^*(H))) = 2(\chi_1(Y, H) + \chi_2(Y, H))$. Hence we get the assertion. □

Therefore we get

$$\begin{aligned} h^0(mH) = \chi(mH) &= \chi_0(Y,H) + \chi_1(Y,H)m + \chi_2(Y,H)\binom{m+1}{2} \\ &= \chi(\mathcal{O}_Y) + m(\chi_1(Y,H) + \chi_2(Y,H)) + \left(\binom{m+1}{2} - m\right)\chi_2(Y,H) \\ &\geq \chi(\mathcal{O}_Y) + \binom{m}{2}. \end{aligned}$$

Therefore

$$h^0(m(K_X + L)) \geq \binom{m}{2} + \chi(\mathcal{O}_Y).$$

This completes the proof. □

Corollary 4.2. *Let (X, L) be a polarized manifold of dimension $n \geq 3$ and let Y be a normal projective surface. Assume that there exists a fiber space $f : X \to Y$ such that $K_X + L = f^*(H)$ for some ample line bundle H on Y. Then for every positive integer m*

$$h^0(m(K_X + L)) \geq \begin{cases} \binom{m+1}{2} & \text{if } \chi(\mathcal{O}_Y) \leq 0, \\ \binom{m}{2} + 1 & \text{if } \chi(\mathcal{O}_Y) > 0. \end{cases}$$

Theorem 4.4. *Let (X, L) be a polarized manifold of dimension $n \geq 3$ and let Y be a normal projective surface. Assume that there exists a fiber space $f : X \to Y$ such that $K_X + L = f^*(H)$ for some ample line bundle H on Y.*

(1) *If $\chi(\mathcal{O}_Y) \leq 0$ and $h^0(m(K_X + L)) = \binom{m+1}{2}$ for some positive integer $m \geq 2$, then Y is smooth and (Y, H) is a scroll over a smooth elliptic curve C such that $H^2 = 1$.*

(2) *If $\chi(\mathcal{O}_Y) > 0$ and $h^0(m(K_X + L)) = \binom{m}{2} + 1$ for some positive integer $m \geq 2$, then one of the following holds. (Here let $\delta : S \to Y$ be the minimal resolution of Y.)*

(2.0) *$\kappa(S) = 2$, Y has at most canonical singularities with $h^1(\mathcal{O}_Y) = 0$ and $\chi(\mathcal{O}_Y) = 1$, and $H = K_Y + T$ with $H^2 = 1$, where T is a non zero torsion divisor.*

(2.1) *$\kappa(S) = 1$ and there exists an elliptic fibration $f : S \to C$ over a smooth curve C such that $g(C) = 1$, $\chi(\mathcal{O}_S) = 1$, $q(S) = 1$ and $\delta^*(H)F = 1$, where F is a general fiber of f. In this case Y has only rational singularities.*

(2.2) *$\kappa(S) = 1$ and there exists an elliptic fibration $f : S \to C$ over a smooth curve C such that $g(C) = 0$, $\chi(\mathcal{O}_S) = 1$, $q(S) = 0$ and one of the following holds. (Here let t be the number of multiple fibers.)*

$p_g(S)$	$\delta^*(H)F$	t	$(m_1, \ldots, m_t)$
0	6	2	(2, 3)
1	4	2	(2, 4)
0	3	2	(3, 3)
0	2	3	(2, 2, 2)

(2.3) *S is a one point blowing up of an Enriques surface S' and $\delta^*(H) = \mu^*(H') - E_\mu$, where $\mu : S \to S'$ is the blowing up at a point P, H' is an ample line bundle on S' and E_μ is the exceptional divisor.*

(2.4) *$\kappa(S) = -\infty$ and $q(S) = 0$. In this case Y has only rational singularities.*

Proof. Let $\delta : S \to Y$ be the minimal resolution of Y.

(I) The case where $\chi(\mathcal{O}_Y) \leq 0$.

Then $h^0(m(K_X + L)) \geq \binom{m+1}{2} - (m-1)\chi(\mathcal{O}_Y)$ by Theorem 4.3.

Assume that $h^0(m(K_X + L)) = \binom{m+1}{2}$. Then, since $m \geq 2$, by the proof of Theorem 4.3, we have $\chi(\mathcal{O}_Y) = 0$, $H^2 = 1$ and $h^0(H) = 1$. Hence by Claim 4.1

$$\begin{aligned} 1 = h^0(H) &= \chi(H) \\ &= \chi(\mathcal{O}_Y) + (1 - g(Y,H)) + H^2 \\ &= 2 - g(Y,H). \end{aligned}$$

Hence $g(Y, H) = 1$. Moreover since $\chi(\mathcal{O}_Y) = 0$, we have $h^1(\mathcal{O}_Y) > 0$. Then $g(S, \delta^*(H)) = g(Y, H) = 1$. In particular $\kappa(S) = -\infty$. Since $\delta^*(H)$ is nef and big, we have $g(S, \delta^*(H)) \geq h^1(\mathcal{O}_S)$ by [12, Theorem 2.1]. Moreover because $h^1(\mathcal{O}_S) \geq h^1(\mathcal{O}_Y)$, we have $1 = g(S, \delta^*(H)) \geq h^1(\mathcal{O}_S) \geq h^1(\mathcal{O}_Y) > 0$. Hence $g(S, \delta^*(H)) = h^1(\mathcal{O}_S)$ and $h^1(\mathcal{O}_S) = h^1(\mathcal{O}_Y) = 1$. Here we note that $\delta^*(H)$ is $\delta^*(H)$-minimal because H is ample and δ is the minimal resolution. Hence by [12, Theorem 3.1], we see that $(S, \delta^*(H))$ is a scroll over a smooth curve. Then we can prove the following.

Claim 4.3. *δ is the identity map.*

Proof. Since $h^1(\mathcal{O}_S) = h^1(\mathcal{O}_Y)$, we see that Y has the Albanese mapping by Lemma 2.2. Then there exists an elliptic curve C and morphisms $\alpha : Y \to C$ and $\alpha' : S \to C$ such that $\alpha' = \alpha \circ \delta$. Here we note that α and α' have connected fibers. Since α' is a $\mathbb{P}^1$-bundle over C, we see that any fiber of α' is irreducible. Assume that δ is not the identity map. Then $\mathrm{Sing}(Y) \neq \emptyset$ and α' has non-irreducible fiber. But this is a contradiction. Therefore δ is the identity map. □

Hence $S \cong Y$, that is, Y is smooth, and (Y, H) is a scroll over a smooth elliptic curve C. In particular, there exists an ample vector bundle $\mathcal{E}$ on C such that $Y = \mathbb{P}_C(\mathcal{E})$ and $H = H(\mathcal{E})$. Then $c_1(\mathcal{E}) = 1$ because $H^2 = 1$. Therefore we see that $\mathcal{E}$ is an indecomposable ample vector bundle on C.

(II) Assume that $\chi(\mathcal{O}_Y) > 0$.

Then we have $h^0(m(K_X + L)) \geq \binom{m}{2} + 1$. We consider (X, L) with $h^0(m(K_X + L)) = \binom{m}{2} + 1$. Then, since $m \geq 2$, by the proof of Theorem 4.3 we obtain $\chi(\mathcal{O}_Y) = \chi_0(Y, H) = 1$, $\chi_1(Y, H) + \chi_2(Y, H) = 0$ and $H^2 = \chi_2(Y, H) = 1$. Hence we have $g(Y, H) = 1 - \chi_1(Y, H) = 2$.

Hence we see that a quasi-polarized surface $(S, \delta^*(H))$ is $\delta^*(H)$-minimal with $g(S, \delta^*(H)) = 2$ (Here we note that quasi-polarized surfaces of this type were studied in [5].) Here we note that $\delta^*(H)^2 = 1$ and $K_S\delta^*(H) = 1$.

Next we study $(S, \delta^*(H))$ with $g(S, \delta^*(H)) = 2$.

(II.a) Assume that $\kappa(S) = 2$. Since $(\delta^*(H))^2 = H^2 = 1$ and $\delta^*(H)K_S = HK_Y = 1$, we see that S is minimal because $(S, \delta^*(H))$ is $\delta^*(H)$-minimal (see Definition 2.2). By the Hodge index theorem we have $\delta^*(H) \equiv K_S$ and $K_S^2 = 1$. Then $h^1(\mathcal{O}_S) = 0$ and $h^1(\mathcal{O}_Y) = 0$. On the other hand $K_S = \delta^*(K_Y) + E_\delta$ holds, where E_δ is a δ-exceptional divisor. Here we note that E_δ is not always effective. Hence $\delta^*(H - K_Y) \equiv E_\delta$. If $E_\delta \neq 0$, then $(E_\delta)^2 < 0$ by Grauert's criterion (e.g. [2, (2.1) Theorem in Chapter III]). But since $\delta^*(H - K_S)E_\delta = 0$, this is impossible. Therefore we have $E_\delta = 0$ and $K_S = \delta^*(K_Y)$. Therefore Y has at most canonical singularities. Namely the singularities of Y are at most rational double points. Therefore Y is Gorenstein and K_Y is a Cartier divisor. Since $\delta^*(H) \equiv \delta^*(K_Y)$, we have $H \equiv K_Y$. If $H = K_Y$, then $h^2(H) = h^2(K_Y) = h^0(\mathcal{O}_Y) = 1$. But this contradicts Claim 4.1. Therefore $H = K_Y + T$, where T is a torsion divisor.

(II.b) Next we consider the case where $\kappa(S) = 1$. Here we use the results of [26]. Let $h : S \to C$ be its elliptic fibration. Then, since $(\delta^*(H))^2 = 1$ and $K_S\delta^*H = 1$, the following cases are possible from [26].

(1) h has no multiple fibers (see [26, Table 3.1]).
  (1.1) $g(C) = 0$, $\chi(\mathcal{O}_S) = 3$, $q(S) = 0$, $p_g(S) = 2$ and $\delta^*(H)F = 1$.
  (1.2) $g(C) = 1$, $\chi(\mathcal{O}_S) = 1$, $q(S) = 1$, $p_g(S) = 1$ and $\delta^*(H)F = 1$. (This is the type (2.1) in Theorem 4.4.)
(2) The case where $(S, \delta^*(H))$ fits into [26, Table 4.1]. (This is the type (2.2) in Theorem 4.4.)
(3) h has only one multiple fiber and its multiplicity is 2. In this case $g(C) = 1$, $\chi(\mathcal{O}_S) = 0$, $q(S) = 1$, $p_g(S) = 0$ and $\delta^*(H)F = 2$ (see the first case of [26, Table 5.1]).
(4) The case where $(S, \delta^*(H))$ fits into [26, Table 5.2].

Lemma 4.2. *The cases* (1.1), (3) *and* (4) *above are impossible.*

Proof. First we consider the case of (1.1). In this case $\chi(\mathcal{O}_S) = 3 > 1 = \chi(\mathcal{O}_Y)$. But this is impossible by Proposition 2.2 because $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X)$.

Next we consider the case (3) above. Since $q(S) = 1$, S has the Albanese fibration $\alpha : S \to B$, where B is an elliptic curve. In this case, since C is also an elliptic curve, by the universality of the Albanese map we see that there exists a morphism $\lambda : B \to C$ such that $h = \lambda \circ \alpha$. Because α and h have connected fibers, we see that λ is an isomorphism. Namely we may assume that $\alpha = h$. Moreover by Lemma 2.2 the Albanese map of Y can be defined, and let $\alpha_Y : Y \to B$ be its morphism. But here h is a quasi-bundle, so α is also a quasi-bundle. (For the definition of quasi-bundle, see [28, Definition 1.1].) Hence δ is an isomorphism because $\alpha = \alpha_Y \circ \delta$. Therefore $Y \cong S$. But then $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_S) = 0$ and this is a contradiction.

Finally we consider case (4). Then by [26, Proposition 5.1], δ^*H is ample. Namely δ is an isomorphism. But then $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_S) = 0$ and this is also impossible.

This completes the proof of Lemma 4.2. □

(II.c) Next we consider the case where $\kappa(S) = 0$. Let $\mu : S \to S'$ be the minimalization of S. If δ is an isomorphism, then $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_Y) = 1$ and S' is an Enriques surface. If δ is not an isomorphism, then since $g(S, \delta^*(H)) = 2$, by [5, Proposition 3.2] we see that S' is either an Enriques surface or a K3-surface. If S' is birationally equivalent to a K3-surface, then $\chi(\mathcal{O}_{S'}) = 2$. But by Proposition 2.2 this is impossible because $\chi(\mathcal{O}_Y) = 1$ in this case. Therefore S' is birationally equivalent to an Enriques surface.

(II.d) Next we consider the case where $\kappa(S) = -\infty$. By Proposition 2.2 we see that $\chi(\mathcal{O}_S) \leq \chi(\mathcal{O}_Y) = 1$. Since $g(S, \delta^*(H)) = 2$, we have $q(S) \leq 2$ by [12, Theorem 2.1]. By Lemma 2.2, we have $q(Y) = q(S)$ and if $q(Y) \geq 1$, then there exist the Albanese map of Y, $\alpha_Y : Y \to \mathrm{Alb}(Y)$, and a morphism $\beta : \mathrm{Alb}(S) \to \mathrm{Alb}(Y)$ such that $\alpha_Y \circ \delta = \beta \circ \alpha_S$ holds, where $\alpha_S : S \to \mathrm{Alb}(S)$ is the Albanese map of S. Then $\alpha_S(S)$ and $\alpha_Y(Y)$ are smooth curves and α_S and α_Y have connected fibers (see [4, Lemma 2.4.5]). Hence $\alpha_S(S) \cong \alpha_Y(Y)$.

(i) If $q(S) = 2$, then $g(S, \delta^*(H)) = q(S)$ implies that $(S, \delta^*(H))$ is a scroll over a smooth curve by [12, Theorem 3.1]. Here we note that δ is an isomorphism because S is a $\mathbb{P}^1$-bundle over $\alpha_S(S)$. But then $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_S) = -1$ and this is impossible.

(ii) Next we consider the case where $q(S) = 1$. Assume that $K_S + \delta^*(H)$ is not nef. Then there exists an extremal rational curve E on S such that $(K_S + \delta^*(H))E < 0$. If E is a (-1)-curve, then $(K_S + \delta^*(H))E \geq 0$ since $(S, \delta^*(H))$

is $\delta^*(H)$-minimal. Hence S is a $\mathbb{P}^1$-bundle over a smooth elliptic curve C and E is a fiber of this because $q(S) = 1$. Let $f : S \to C$ be its morphism. Moreover we see that $\delta^*(H)F = 1$ for any fiber F of f because $(K_S + \delta^*(H))F < 0$. Then $g(S, \delta^*(H)) = q(S) = 1$. But this contradicts to $g(S, \delta^*(H)) = g(Y, H) = 2$. Hence $K_S + \delta^*(H)$ is nef. So we get $0 \le (K_S + \delta^*(H))^2 = K_S^2 + 2K_S\delta^*(H) + (\delta^*(H))^2 = 3 + K_S^2$, that is, $-3 \le K_S^2$. On the other hand $K_S^2 \le 0$ and $K_S^2 = 0$ if and only if S is minimal. Hence S is at most three points blowing up of a $\mathbb{P}^1$-bundle over C.

(ii.1) Assume that S is a $\mathbb{P}^1$-bundle over C. Then $S \cong Y$ because every exceptional curve of δ is contained in a fiber of α_S. But this is impossible because $\chi(\mathcal{O}_S) = 0 \neq 1 = \chi(\mathcal{O}_Y)$.

(ii.2) Assume that S is one point blowing up of a $\mathbb{P}^1$-bundle over C. Then S has one singular fiber F_1 and $F_1 = C_1 + C_2$, where each C_i is a (-1)-curve and $C_1C_2 = 1$. Since δ is the minimal resolution, we have $S \cong Y$. But this is also impossible by the same reason as in (ii.1).

(ii.3) Assume that S is two point blowing up of a $\mathbb{P}^1$-bundle over C. Then the following two cases possibly occur:

(ii.3.1) α_S has one singular fiber F and $F = C_1 + C_2 + C_3$, where C_1 and C_3 are (-1)-curves and C_2 is a (-2)-curve such that $C_1C_2 = 1$, $C_2C_3 = 1$ and $C_1C_3 = 0$.

(ii.3.2) f has two singular fibers F_1 and F_2 such that $F_1 = C_1 + C_2$, $F_2 = C_3 + C_4$, where each C_i is a (-1)-curve with $C_1C_2 = 1$ and $C_3C_4 = 1$.

By the same argument as (ii.2), (ii.3.2) cannot occur. So we consider the case where (ii.3.1). Then since δ is the minimal resolution, the exceptional curve of δ is C_2. So Y is rational by Artin's criterion [2, (3.2) Theorem in Chapter III]. But this is impossible because $\chi(\mathcal{O}_S) = 0 \neq 1 = \chi(\mathcal{O}_Y)$.

(ii.4) Assume that S is three point blowing up of a $\mathbb{P}^1$-bundle over C. Then the following four cases possibly occur:

(ii.4.1) α_S has one singular fiber F and $F = C_1 + C_2 + C_3 + C_4$, where C_2, C_3 and C_4 are (-1)-curves and C_1 is a (-3)-curve such that $C_1C_i = 1$ for every i with $i = 2, 3, 4$, $C_jC_k = 0$ with $j, k \in \{2, 3, 4\}$ and $j \neq k$.

(ii.4.2) α_S has one singular fiber F and $F = C_1 + C_2 + C_3 + C_4$, where C_1 and C_4 are (-1)-curves, and C_2 and C_3 are (-2)-curves such that $C_iC_{i+1} = 1$ for every i with $i = 1, 2, 3$, $C_jC_k = 0$ with $|j - k| \ge 2$.

(ii.4.3) α_S has two singular fibers F_1 and F_2 such that $F_1 = C_1 + C_2 + C_3$, $F_2 = C_4 + C_5$, where C_i is a (-1)-curve for every $i \neq 2$ and C_2 is a (-2)-curve such that $C_1C_2 = 1$, $C_2C_3 = 1$, $C_1C_3 = 0$ and $C_4C_5 = 1$.

(ii.4.4) f has three singular fibers F_1, F_2 and F_3 such that $F_1 = C_1 + C_2$, $F_2 = C_3 + C_4$ and $F_3 = C_5 + C_6$, where each C_i is a (-1)-curve such that $C_iC_{i+1} = 1$ with $i \in \{1, 3, 5\}$.

By the same argument as above, in these 4 cases we see that δ is an isomorphism or Y has rational singularities. But this is impossible because $\chi(\mathcal{O}_S) = 0 \neq \chi(\mathcal{O}_Y)$.

Therefore the case where $q(S) = 1$ cannot occur. By the above argument, we see that $q(S) = 0$. Then $\chi(\mathcal{O}_S) = 1 = \chi(\mathcal{O}_Y)$ and by Proposition 2.2 we have $h^0(R^1\delta_*(\mathcal{O}_S)) = 0$. So Y has rational singularities. This completes the proof. □

5. Main results

Let (X, L) be a polarized manifold of dimension 3. In this section, we consider $h^0(m(K_X + L))$. First by Theorems 4.1 and 4.3 we have the following.

Theorem 5.1. *Let (X, L) be a polarized manifold of dimension 3.*

(1) *Assume that $\kappa(K_X + L) = 0$. Then $h^0(m(K_X + L)) = 1$ for every positive integer m.*

(2) *Assume that $\kappa(K_X + L) = 1$. Then for every positive integer m the following holds.*

$$h^0(m(K_X + L)) \ge \begin{cases} (m-1)(h^1(\mathcal{O}_X) - 1) + mh^1(\mathcal{O}_X) & \text{if } h^1(\mathcal{O}_X) \ge 1, \\ m + 1 & \text{if } h^1(\mathcal{O}_X) = 0. \end{cases}$$

(3) *Assume that $\kappa(K_X + L) = 2$. Then for every positive integer m the following holds.*

$$h^0(m(K_X + L)) \ge \begin{cases} \binom{m+1}{2} - (m-1)\chi(\mathcal{O}_X) & \text{if } \chi(\mathcal{O}_X) \le 0, \\ \binom{m}{2} + \chi(\mathcal{O}_X) & \text{if } \chi(\mathcal{O}_X) > 0. \end{cases}$$

Proof. Let (M, A) be a reduction of (X, L). Here we note that $h^0(m(K_X + L)) = h^0(m(K_M + A))$ for any positive integer m. If $\kappa(K_X + L) = 0$, then (M, A) is a Mukai manifold, that is, $\mathcal{O}_M(K_M + A) = \mathcal{O}_M$ by [4, Theorem 7.5.3]. This implies that $h^0(m(K_X + L)) = h^0(m(K_M + A)) = 1$.

If $\kappa(K_X + L) = 1$ (resp. 2), then by [4, Theorem 7.5.3] there exist a smooth projective curve C (resp. a normal projective surface Y), and a fiber space $f : M \to C$ (resp. $M \to Y$) such that $K_M + A = f^*(H)$ for some ample line bundle H on C (resp. Y). Moreover we have $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_M) = h^1(\mathcal{O}_C)$ (resp. $h^i(\mathcal{O}_X) = h^i(\mathcal{O}_M) = h^i(\mathcal{O}_Y)$ for $i = 0, 1, 2$ and $h^3(\mathcal{O}_X) = 0$). Hence by Theorems 4.1 and 4.3 we get the assertion. □

Next we consider the case where $\kappa(K_X + L) = 3$. Then the following is obtained.

Theorem 5.2. *Let (X, L) be a polarized manifold of dimension 3. Assume that $\kappa(K_X + L) = 3$. Then we have*

$$h^0(m(K_X + L)) \ge \begin{cases} \frac{1}{8}m^3 + \frac{1}{4}m^2 + 1 & \text{if } m \text{ is even with } m \ge 2, \\ \frac{1}{8}m^3 + \frac{1}{4}m^2 + \frac{1}{8}m + 1 & \text{if } m \text{ is odd with } m \ge 3. \end{cases}$$

Proof. Let (M, A) be a reduction of (X, L). By assumption and [4, Proposition 7.6.9] we see that $K_M + A$ is nef.

(I) The case where m is even with $m \geq 2$.

Then by Proposition 3.1 we have the following.

$$\begin{aligned} h^0(m(K_X + L)) &= h^0(m(K_M + A)) \\ &= h^0\left(\left(\frac{m}{2} + 1\right) K_M + \frac{m}{2}A\right) + g_2\left(M, \left(\frac{m}{2} - 1\right)(K_M + A) + A\right) \\ &\quad - h^1(\mathcal{O}_M) + g_1\left(M, \left(\frac{m}{2} - 1\right)(K_M + A) + A, \frac{m}{2}(K_M + A)\right). \end{aligned}$$

Since $((m/2) - 1)(K_M + A) + A$ is ample and $\kappa(K_M + ((m/2) - 1)(K_M + A) + A) = \kappa(K_M + A) = 3$, we have $g_2(M, ((m/2) - 1)(K_M + A) + A) \geq h^1(\mathcal{O}_M)$ by Theorem 3.2. On the other hand, by Remark 3.2(2) we have

$$\begin{aligned} g_1\left(M, \left(\frac{m}{2} - 1\right)(K_M + A) + A, \frac{m}{2}(K_M + A)\right) &= 1 + \frac{1}{2}\left(K_M + \left(\frac{m}{2} - 1\right)(K_M + A) + A + \frac{m}{2}(K_M + A)\right) \\ &\quad \times \left(\left(\frac{m}{2} - 1\right)(K_M + A) + A\right)\left(\frac{m}{2}(K_M + A)\right) \\ &= 1 + \frac{m^2(m-2)}{8}(K_M + A)^3 + \frac{m^2}{4}(K_M + A)^2 A. \end{aligned}$$

We also note that $(K_M + A)^3 \geq 1$ and $(K_M + A)^2 A \geq 1$.

If $(K_M + A)^2 A = 1$, then by Proposition 2.1 we see that $(K_M + A)A^2 = 1$ and $A^3 = 1$ because $(K_M + A)A^2 > 0$. Hence $g_1(M, A) = 2$. Therefore by [9, (1.10) Theorem and Section 2] we see that $K_M = \mathcal{O}_M$ and $h^0(A) \geq 1$ since $\kappa(K_M + A) = 3$. On the other hand, we have

$$h^0(m(K_M + A)) = h^0(mA) = \chi(mA) = \frac{1}{6}m^3A^3 + \frac{1}{12}mc_2(M)A$$

because $h^i(mA) = h^i(K_M + mA) = 0$ for every $i > 0$. Since $h^0(A) \geq 1$, we get

$$1 \leq h^0(A) = \frac{1}{6}A^3 + \frac{1}{12}c_2(M)A.$$

Hence $(1/12)c_2(M)A \geq 1 - (1/6)A^3 = 5/6$. So we obtain

$$\begin{aligned} h^0(m(K_M + A)) &= \frac{1}{6}m^3A^3 + \frac{1}{12}mc_2(M)A \\ &\geq \frac{1}{6}m^3 + \frac{5}{6}m. \end{aligned}$$

If $(K_M + A)^2 A \geq 2$, then

$$\begin{aligned} h^0(m(K_M + A)) &\geq 1 + \frac{m^2(m-2)}{8} + 2\frac{m^2}{4} \\ &= \frac{1}{8}m^3 + \frac{1}{4}m^2 + 1. \end{aligned}$$

Here we note that $(1/6)m^3 + (5/6)m - ((1/8)m^3 + (1/4)m^2 + 1) = (1/24)(m-2)((m-2)^2 + 8) \geq 0$. So if m is even with $m \geq 2$, then we have $h^0(m(K_M + A)) \geq (1/8)m^3 + (1/4)m^2 + 1$.

(II) The case where m is odd with $m \geq 3$.

Here we use the following equality which is obtained from Proposition 3.1.

$$\begin{aligned} h^0(m(K_X + L)) &= h^0(m(K_M + A)) \\ &= h^0\left(\left(\frac{m+1}{2} + 1\right) K_M + \frac{m+1}{2}A\right) + g_2\left(M, \left(\frac{m-1}{2} - 1\right)(K_M + A) + A\right) - h^1(\mathcal{O}_M) \\ &\quad + g_1\left(M, \left(\frac{m-1}{2} - 1\right)(K_M + A) + A, \frac{m+1}{2}(K_M + A)\right). \end{aligned}$$

Since $(-1 + (m-1)/2)(K_M + A) + A$ is ample and $\kappa(K_M + (-1 + (m-1)/2)(K_M + A) + A) = \kappa(((m-1)/2)(K_M + A)) = \kappa(K_M + A) = 3$, we have $g_2(M, (-1 + (m-1)/2)(K_M + A) + A) \geq h^1(\mathcal{O}_M)$ by Theorem 3.2. On the other hand,

$$\begin{aligned} &g_1\left(M, \left(\frac{m-1}{2} - 1\right)(K_M + A) + A, \frac{m+1}{2}(K_M + A)\right) \\ &= 1 + \frac{1}{2}\left(K_M + \left(\frac{m-1}{2} - 1\right)(K_M + A) + A + \frac{m+1}{2}(K_M + A)\right) \end{aligned}$$

$$\times\left(\left(\frac{m-1}{2}-1\right)(K_M+A)+A\right)\left(\frac{m+1}{2}(K_M+A)\right)$$
$$=1+\frac{m(m+1)(m-3)}{8}(K_M+A)^3+\frac{m(m+1)}{4}(K_M+A)^2A.$$

If $(K_M+A)^2A=1$, then by the same argument as above we see that

$$h^0(m(K_M+A))\geq\frac{1}{6}m^3+\frac{5}{6}m.$$

If $(K_M+A)^2A\geq 2$, then we have

$$\begin{aligned}h^0(m(K_M+A)) &\geq 1+\frac{m(m+1)(m-3)}{8}+\frac{m(m+1)}{2}\\ &=\frac{1}{8}m^3+\frac{1}{4}m^2+\frac{1}{8}m+1.\end{aligned}$$

Here we note that $(1/6)m^3+(5/6)m-((1/8)m^3+(1/4)m^2+(1/8)m+1)=(1/24)(m-3)((m-(3/2))^2+23/4)\geq 0$. So if m is odd with $m\geq 3$, then we have $h^0(m(K_M+A))\geq(1/8)m^3+(1/4)m^2+(1/8)m+1$. This completes the proof of Theorem 5.2. □

Remark 5.1. By Theorem 5.2 we see that if $\kappa(K_X+L)=3$, then for every integer m with $m\geq 2$, we have

$$h^0(m(K_X+L))\geq\frac{1}{8}m^3+\frac{1}{4}m^2+1.$$

If $\kappa(K_X+L)=3$ and $m=2$, then by Theorem 5.2 or [20, Theorem 5.4 (2)] we have $h^0(2(K_X+L))\geq 3$. So it is interesting to study (X,L) with $\kappa(K_X+L)=3$ and small $h^0(2(K_X+L))$. The following results (Theorems 5.3 and 5.4) give a classification of these (X,L).

First we note the following which will be used later.

Proposition 5.1. *Let (X,L) be a polarized manifold of dimension 3. Then the following equalities hold.*

$$h^0(2K_X+2L)-h^0(2K_X+L)=g_2(X,L)-h^1(\mathcal{O}_X)+g_1(X,K_X+L,L), \tag{2}$$
$$h^0(2K_X+2L)-h^0(K_X+L)=g_2(X,K_X+L)-h^1(\mathcal{O}_X)+g_1(X,K_X+L,L). \tag{3}$$

Proof. These equalities are obtained from Proposition 3.1. □

Notation 5.1. *Let (X,L) be a polarized manifold of dimension 3 and let (M,A) be a reduction of (X,L). Set $d_1:=g_2(M,A)-h^1(\mathcal{O}_M)$ and $d_2:=g_2(M,K_M+A)-h^1(\mathcal{O}_M)$. Then we see that*

$$\begin{aligned}d_2-d_1 &=\frac{1}{12}(K_M+A)(6K_M+6A)K_M+\frac{1}{12}c_2(M)K_M\\ &=\frac{1}{12}(K_M+A)(6K_M+6A)K_M-2\chi(\mathcal{O}_M).\end{aligned}$$

Therefore

$$d_2-d_1+2\chi(\mathcal{O}_M)=\frac{1}{2}(K_M+A)^2K_M. \tag{4}$$

Theorem 5.3. *Let (X,L) be a polarized manifold of dimension 3. Assume that $\kappa(K_X+L)=3$. Then $h^0(2(K_X+L))=3$ if and only if (X,L) satisfies $L^3=1$, $\mathcal{O}_X(K_X)=\mathcal{O}_X$, $h^1(\mathcal{O}_X)=0$ and $h^0(L)=1$.*

Proof. (α) Assume that $h^0(2(K_X+L))=3$.

Let (M,A) be a reduction of (X,L). Then by assumption we see that K_M+A is nef and big. First we prove the following claim.

Claim 5.1. $h^0(K_M+A)\leq 2$.

Proof. Assume that $h^0(K_M+A)\geq 3$. Then by Lemma 2.1 we have $h^0(2(K_M+A))\geq 2h^0(K_M+A)-1\geq 5$. This is a contradiction. □

By Proposition 5.1 (2) and Theorem 3.2, we see that

$$\begin{aligned}3=h^0(2K_M+2A) &\geq g_2(M,A)-h^1(\mathcal{O}_M)+g_1(M,K_M+A,A)\\ &\geq g_1(M,K_M+A,A)\\ &=1+(K_M+A)^2A.\end{aligned}$$

Hence we have $(K_M + A)^2 A \le 2$. On the other hand, since $1 \le (K_M + A)^2 A$ we get

$$1 \le (K_M + A)^2 A \le 2. \tag{5}$$

Namely the following holds.

$$2 \le g_1(M, K_M + A, A) \le 3. \tag{6}$$

Since $g_1(M, K_M + A, A) \le 3$, by Proposition 5.1 (3) we get $h^0(2(K_M + A)) - h^0(K_M + A) \le g_2(M, K_M + A) - h^1(\mathcal{O}_M) + 3$. By Claim 5.1 and $h^0(2(K_M + A)) = 3$, we see that

$$\begin{aligned} 3 - 2 &\le h^0(2(K_M + A)) - h^0(K_M + A) \\ &\le g_2(M, K_M + A) - h^1(\mathcal{O}_M) + 3. \end{aligned}$$

Namely,

$$g_2(M, K_M + A) - h^1(\mathcal{O}_M) \ge -2. \tag{7}$$

From Proposition 5.1 (3), (6) and the assumption that $h^0(2(K_M + A)) = 3$, we have

$$\begin{aligned} 3 &\ge h^0(2(K_M + A)) - h^0(K_M + A) \\ &= g_2(M, K_M + A) - h^1(\mathcal{O}_M) + g_1(M, K_M + A, A) \\ &\ge g_2(M, K_M + A) - h^1(\mathcal{O}_M) + 2. \end{aligned}$$

Hence we have

$$g_2(M, K_M + A) - h^1(\mathcal{O}_M) \le 1. \tag{8}$$

By (6) and Proposition 5.1 (2), we have

$$\begin{aligned} 3 &\ge h^0(2(K_M + A)) - h^0(2K_M + A) \\ &= g_2(M, A) - h^1(\mathcal{O}_M) + g_1(M, K_M + A, A) \\ &\ge g_2(M, A) - h^1(\mathcal{O}_M) + 2. \end{aligned}$$

Hence $1 \ge g_2(M, A) - h^1(\mathcal{O}_M)$. From this and Theorem 3.2 we have

$$d_1 = 0, 1. \tag{9}$$

We also note that

$$-2 \le d_2 \le 1 \tag{10}$$

by (7) and (8).

(I) If $(K_M + A)^2 A = 1$, then $(K_M + A)A^2 = 1$ and $A^3 = 1$ by Proposition 2.1. Therefore we get $g_1(M, A) = 2$. Since $\kappa(K_M + A) = 3$, by [9, (1.10) Theorem and Section 2] we see that $K_M = \mathcal{O}_M$, $h^1(\mathcal{O}_M) = 0$ and $h^0(A) = 1$. By the Riemann–Roch theorem we get $\chi(tA) = (1/6)A^3 t^3 + (1/12)c_2(M)At$. Since $h^0(2K_M + 2A) = \chi(2K_M + 2A) = \chi(2A)$, we get $h^0(2K_M + 2A) = (4/3)A^3 + (1/6)c_2(M)A$. Therefore $3 = h^0(2K_M + 2A) = (4/3)A^3 + (1/6)c_2(M)A = (4/3) + (1/6)c_2(M)A$. Namely $c_2(M)A = 10$. Here we note that $(M, A) \cong (X, L)$ because $A^3 = 1$.

(II) Next we assume that

$$(K_M + A)^2 A = 2. \tag{11}$$

We will prove that this case cannot occur. Since $(K_M + A)^2 A = 2$, by Proposition 2.1 we have

$$1 \le (K_M + A)^3 \le 4. \tag{12}$$

By using (4) and (9)–(12), we can determine the value of $\chi(\mathcal{O}_M)$. For example, assume that $d_1 = 0$ and $d_2 = -2$. Then $d_2 - d_1 = -2$ and $(K_M + A)^2 K_M = 4\chi(\mathcal{O}_M) - 4$ by (4). Since $(K_M + A)^2 A = 2$, we have $(K_M + A)^3 = 4\chi(\mathcal{O}_M) - 2$. By considering (12) we have $\chi(\mathcal{O}_M) = 1$. By the same argument as this, we can get the following list:

d_1	d_2	$d_2 - d_1$	$(K_M + A)^2 K_M$	$(K_M + A)^3$	$\chi(\mathcal{O}_M)$
0	−2	−2	$4\chi(\mathcal{O}_M) - 4$	$4\chi(\mathcal{O}_M) - 2$	1
0	−1	−1	$4\chi(\mathcal{O}_M) - 2$	$4\chi(\mathcal{O}_M)$	1
0	0	0	$4\chi(\mathcal{O}_M)$	$4\chi(\mathcal{O}_M) + 2$	0
0	1	1	$4\chi(\mathcal{O}_M) + 2$	$4\chi(\mathcal{O}_M) + 4$	0
1	−2	−3	$4\chi(\mathcal{O}_M) - 6$	$4\chi(\mathcal{O}_M) - 4$	2
1	−1	−2	$4\chi(\mathcal{O}_M) - 4$	$4\chi(\mathcal{O}_M) - 2$	1
1	0	−1	$4\chi(\mathcal{O}_M) - 2$	$4\chi(\mathcal{O}_M)$	1
1	1	0	$4\chi(\mathcal{O}_M)$	$4\chi(\mathcal{O}_M) + 2$	0

By this list, we see that $(K_M + A)^3 = 2$ or 4.

Assume that $(K_M+A)^3=4$. Then by Proposition 2.1 we have

$$\begin{aligned}4 &= ((K_M+A)^2A)^2\\ &\geq ((K_M+A)^3)((K_M+A)A^2)\\ &\geq 4(K_M+A)A^2.\end{aligned}$$

Since K_M+A is nef and big, we see that $(K_M+A)A^2\geq 1$. Therefore $(K_M+A)A^2=1$. But by Proposition 2.1, we have $1=((K_M+A)A^2)^2\geq ((K_M+A)^2A)A^3=2A^3\geq 2$, and this is impossible.

Assume that $(K_M+A)^3=2$. Then by Proposition 2.1 we have

$$\begin{aligned}4 &= ((K_M+A)^2A)^2\\ &\geq ((K_M+A)^3)((K_M+A)A^2)\\ &= 2(K_M+A)A^2.\end{aligned}$$

Hence we have $(K_M+A)A^2\leq 2$. By Proposition 2.1 we see that $((K_M+A)A^2)^2\geq ((K_M+A)^2A)A^3=2A^3\geq 2$. Therefore $(K_M+A)A^2=2$ and $A^3\leq 2$ because $(K_M+A)A^2\geq 1$. But since $(K_M+A)A^2=2$, we have $A^3=2$ because $(K_M+2A)A^2$ is even. Therefore $((K_M+A)A^2)^2=4=((K_M+A)^2A)A^3$ holds and $K_M+A\equiv A$ by [4, Corollary 2.5.4] since A is ample. Namely $K_M\equiv 0$. Now since $g_1(M,A,K_M+A)=1+(K_M+A)^2A=3$, we see that $h^0(2K_M+A)=-d_1$ by Proposition 5.1 (2). Since $d_1=0$ or 1 by (9), we have $d_1=0$. On the other hand, $h^i(K_M+K_M+A)=0$ for every integer i with $i>0$ because K_M+A is nef and big. So by the Riemann–Roch theorem we have $h^0(2K_M+A)=\chi(2K_M+A)=\chi(A)=(1/6)A^3+(1/12)c_2(M)A$. Since $A^3=2$, we have $c_2(M)A=-4$ if $d_1=0$. Here we calculate $h^0(2(K_M+A))$. Since K_M+2A is ample, then $h^i(2K_M+2A)=0$ for $i>0$. Therefore

$$\begin{aligned}h^0(2(K_M+A)) &= \chi(2(K_M+A))\\ &= \chi(2A)\\ &= \frac{4}{3}A^3+\frac{1}{6}c_2(M)A\\ &= 2.\end{aligned}$$

But this is impossible because we assume that $h^0(2(K_M+A))=3$.

(β) Assume that (X,L) satisfies $L^3=1$, $\mathcal{O}_X(K_X)=\mathcal{O}_X$, $h^1(\mathcal{O}_X)=0$ and $h^0(L)=1$. Then $h^0(2K_X+L)=h^0(K_X+L)=h^0(L)=1$ and $h^2(\mathcal{O}_X)=h^1(K_X)=h^1(\mathcal{O}_X)=0$. Hence $g_2(X,L)=h^0(K_X+L)-h^0(K_X)+h^2(\mathcal{O}_X)=0$. Moreover $g_1(X,K_X+L,L)=1+L^3=2$. Therefore by Proposition 5.1 (2) we have

$$\begin{aligned}h^0(2(K_X+L)) &= h^0(2K_X+L)+g_2(X,L)-h^1(\mathcal{O}_X)+g_1(X,K_X+L,L)\\ &= 3.\end{aligned}$$

This completes the proof. □

Remark 5.2. (i) By Theorem 5.3, we see that if $\kappa(K_X+L)=3$ and $h^0(2(K_X+L))=3$, then $h^0(K_X+L)=1$.

(ii) There exists an example of (X,L) which satisfies $\kappa(K_X+L)=3$ and $h^0(2K_X+2L)=3$. See [18, Example 3.1 (4)].

Next we consider the case where (X,L) satisfies $\kappa(K_X+L)=3$ and $h^0(2K_X+2L)=4$.

Theorem 5.4. *Let (X,L) be a polarized manifold of dimension 3 and let (M,A) be a reduction of (X,L). Assume that $\kappa(K_X+L)=3$. Then $h^0(2(K_X+L))=4$ if and only if (M,A) is one of the following.*

(1) $K_M\equiv 0$, $A^3=2$, $\chi(\mathcal{O}_M)=0$ *and* $h^0(A)=1$.
(2) $(K_M+A)^2A=3$, $(K_M+A)^3=1$, $g_2(M,A)=h^1(\mathcal{O}_M)=1$, $h^2(\mathcal{O}_M)=0$, $h^3(\mathcal{O}_M)=0$ *and (M,K_M+A) is birationally equivalent to a scroll over an elliptic curve.*

Proof. (α) Assume that $h^0(2(K_X+L))=4$.

First we prove the following claim.

Claim 5.2. *One of the following holds:*

(i) $g(M,A)=2$.
(ii) (M,A) *satisfies* (1) *in Theorem* 5.4.
(iii) (M,A) *satisfies* (2) *in Theorem* 5.4.

Proof. If $h^0(K_M+A)\geq 3$, then by Lemma 2.1 we see that $h^0(2K_M+2A)\geq 2h^0(K_M+A)-1\geq 5$ and this is impossible. Hence

$$h^0(K_M+A)\leq 2. \tag{13}$$

We note that

$$1\leq (K_M+A)^2A. \tag{14}$$

Since $g_2(M, A) \geq h^1(\mathcal{O}_M)$ by Theorem 3.2 and $g_1(M, K_M + A, A) = 1 + (K_M + A)^2 A$, we have

$$\begin{aligned} h^0(2K_M + 2A) - h^0(2K_M + A) &\geq g_1(M, K_M + A, A) \\ &= 1 + (K_M + A)^2 A \end{aligned} \tag{15}$$

and

$$(K_M + A)^2 A \leq 3 \tag{16}$$

by Proposition 5.1 (2) since $h^0(2K_M + 2A) = 4$.

Here we divide the argument into three cases.

(i) Assume that $(K_M + A)^2 A = 1$. Then $(K_M + A)A^2 = 1$ and $A^3 = 1$ by Proposition 2.1. So we get $g(M, A) = 2$ and this is the type (i) in Claim 5.2.

(ii) Assume that $(K_M + A)^2 A = 2$. Then $g_1(M, K_M + A, A) = 3$. By Proposition 2.1, we have $1 \leq (K_M + A)^3 \leq 4$. Hence by Proposition 5.1 (2) and Theorem 3.2 we have

$$d_1 = 0, 1. \tag{17}$$

By (13), Proposition 5.1 (3) and the assumption $h^0(2K_M + 2A) = 4$ we have

$$\begin{aligned} 2 &\leq h^0(2(K_M + A)) - h^0(K_M + A) \\ &= d_2 + g_1(M, K_M + A, A) \\ &= d_2 + 3. \end{aligned}$$

Namely we have

$$-1 \leq d_2. \tag{18}$$

By Proposition 5.1 (3) and the assumption $h^0(2K_M + 2A) = 4$ we have

$$\begin{aligned} 4 &\geq h^0(2(K_M + A)) - h^0(K_M + A) \\ &= d_2 + 3. \end{aligned}$$

Namely we have

$$1 \geq d_2. \tag{19}$$

So we get the following table by the same argument as in the proof of Theorem 5.3.

	d_1	d_2	$d_2 - d_1$	$(K_M + A)^2 K_M$	$(K_M + A)^3$	$\chi(\mathcal{O}_M)$
(2.1)	0	−1	−1	$4\chi(\mathcal{O}_M) - 2$	$4\chi(\mathcal{O}_M)$	1
(2.2)	0	0	0	$4\chi(\mathcal{O}_M)$	$4\chi(\mathcal{O}_M) + 2$	0
(2.3)	0	1	1	$4\chi(\mathcal{O}_M) + 2$	$4\chi(\mathcal{O}_M) + 4$	0
(2.4)	1	−1	−2	$4\chi(\mathcal{O}_M) - 4$	$4\chi(\mathcal{O}_M) - 2$	1
(2.5)	1	0	−1	$4\chi(\mathcal{O}_M) - 2$	$4\chi(\mathcal{O}_M)$	1
(2.6)	1	1	0	$4\chi(\mathcal{O}_M)$	$4\chi(\mathcal{O}_M) + 2$	0

(ii.1) First we consider the case (2.4). Then $(K_M + A)^3 = 2$. By Proposition 2.1 we have

$$\begin{aligned} 4 &= ((K_M + A)^2 A)^2 \\ &\geq ((K_M + A)^3)((K_M + A)A^2) \\ &= 2(K_M + A)A^2. \end{aligned}$$

Hence $(K_M + A)A^2 \leq 2$.

(ii.1.1) If $(K_M + A)A^2 = 2$, then we also see that

$$\begin{aligned} 4 &\geq ((K_M + A)A^2)^2 \\ &\geq (A^3)((K_M + A)^2 A) \\ &= 2A^3. \end{aligned}$$

Therefore $A^3 \leq 2$. But since $(K_M + 2A)A^2$ is even and $A^3 > 0$, we have $A^3 = 2$. Hence $(A^3)((K_M + A)^2 A) = ((K_M + A)A^2)^2$. By [4, Corollary 2.5.4] we have $K_M + A \equiv A$, that is, $K_M \equiv 0$. In particular, $g_2(M, A) = g_2(M, K_M + A)$. But since $d_1 \neq d_2$ in the case (2.4), this is impossible.

(ii.1.2) If $(K_M + A)A^2 = 1$, then $A^3 = 1$ by Proposition 2.1. Hence we see that $g(M, A) = 2$ and this is the type (i) in Claim 5.2.

(ii.2) Next we consider the cases (2.1), (2.3) and (2.5). Then $(K_M + A)^3 = 4$. Since $(K_M + A)^2 A = 2$, by Proposition 2.1, we have $(K_M + A)A^2 = 1$ and by Proposition 2.1 we have $1 = ((K_M + A)A^2)^2 \geq ((K_M + A)^2 A)(A^3) \geq 2A^3$. Since $A^3 > 0$, this is impossible.

(ii.3) Next we consider the cases (2.2) and (2.6). Then $(K_M + A)^3 = 2$. By Proposition 2.1, we have $(K_M + A)A^2 \le 2$ since $(K_M + A)^2 A = 2$.

(ii.3.1) If $(K_M + A)A^2 = 2$, then by the same argument as (ii.1.1) above, we have $K_M \equiv 0$. In this case

$$h^0(2K_M + A) = \chi(2K_M + A) = \chi(A) = \frac{1}{6}A^3 + \frac{1}{12}c_2(M)A$$

and

$$h^0(2K_M + 2A) = \chi(2K_M + 2A) = \chi(2A) = \frac{4}{3}A^3 + \frac{1}{6}c_2(M)A.$$

Since $(K_M + A)^3 = 2$ and $K_M \equiv 0$, we have $A^3 = 2$ and $g_1(M, K_M + A, A) = g(M, A) = 3$. By Proposition 5.1 (2) we have

$$\begin{aligned} h^0(2K_M + A) &= h^0(2K_M + 2A) - d_1 - g_1(M, K_M + A, A) \\ &= 1 - d_1. \end{aligned}$$

Hence we get $d_1 = 0, 1$ because $d_1 \ge 0$ by Theorem 3.2.

(ii.3.1.1) If $d_1 = 1$, then $h^0(2K_M + A) = 0$ and $(1/6)A^3 + (1/12)c_2(M)A = 0$. Therefore $h^0(2K_M + 2A) = A^3 = 2$ and this is impossible.

(ii.3.1.2) If $d_1 = 0$, then $h^0(2K_M + A) = 1$ and $(1/6)A^3 + (1/12)c_2(M)A = 1$. Hence $h^0(2K_M + 2A) = A^3 + 2 = 4$. We note that $h^i(A) = 0$ for every positive integer i because $K_M \equiv 0$. Hence $1 = h^0(2K_M + A) = \chi(2K_M + A) = \chi(A) = h^0(A)$. So this is the type (ii) in Claim 5.2.

(ii.3.2) If $(K_M + A)A^2 = 1$, then

$$\begin{aligned} 1 &= ((K_M + A)A^2)^2 \\ &\ge ((K_M + A)^2 A)(A^3) \\ &= 2A^3 \end{aligned}$$

and this is impossible.

(iii) Assume that $(K_M + A)^2 A = 3$. Then $(K_M + A)^3 \le 9$ by Proposition 2.1 and $g_1(M, K_M + A, A) = 1 + (K_M + A)^2 A = 4$. Since $h^0(2K_M + 2A) = 4$ in this case, we have $d_1 = 0$ by Proposition 5.1 (2) and Theorem 3.2. Moreover we see that $-2 \le d_2 \le 0$ by (13) and Proposition 5.1 (3). Since $d_2 - d_1 + 2\chi(\mathcal{O}_M) = (1/2)(K_M + A)^2 K_M$ (see (4)), we have

	d_1	d_2	$d_2 - d_1$	$(K_M + A)^2 K_M$	$(K_M + A)^3$
(3.1)	0	-2	-2	$4\chi(\mathcal{O}_M) - 4$	$4\chi(\mathcal{O}_M) - 1$
(3.2)	0	-1	-1	$4\chi(\mathcal{O}_M) - 2$	$4\chi(\mathcal{O}_M) + 1$
(3.3)	0	0	0	$4\chi(\mathcal{O}_M)$	$4\chi(\mathcal{O}_M) + 3$

First we consider the case (3.1). Since $1 \le (K_M + A)^3 \le 9$, we have $(\chi(\mathcal{O}_M), (K_M + A)^3) = (1, 3)$ or $(2, 7)$.

Next we consider the case (3.2). Then we have $(\chi(\mathcal{O}_M), (K_M + A)^3) = (0, 1), (1, 5)$ or $(2, 9)$.

Finally we consider the case (3.3). In this case, we get $(\chi(\mathcal{O}_M), (K_M + A)^3) = (0, 3)$ or $(1, 7)$.

(iii.1) Here we note that if $(K_M + A)^3 \ge 5$, then by Proposition 2.1

$$\begin{aligned} 9 &= ((K_M + A)^2 A)^2 \\ &\ge ((K_M + A)^3)((K_M + A)A^2) \\ &\ge 5(K_M + A)A^2. \end{aligned}$$

and we have $(K_M + A)A^2 = 1$ and $A^3 = 1$ by Proposition 2.1. Hence $g(M, A) = 2$ and this is the type (i) in Claim 5.2.

(iii.2) Next we consider the case where $(K_M + A)^3 = 3$. By Proposition 2.1, we see that $(K_M + A)A^2 \le 3$.

If $(K_M + A)A^2 \le 2$, then $A^3 = 1$ because $(K_M + A)^2 A = 3$. But since $(K_M + 2A)A^2$ is even, we see that $(K_M + A)A^2 = 1$ and $A^3 = 1$. Namely we have $g(M, A) = 2$ and this is the type (i) in Claim 5.2.

So we may assume that $(K_M + A)A^2 = 3$. Then $((K_M + A)A^2)((K_M + A)^3) = ((K_M + A)^2 A)^2 = 9$. Here we will prove the following lemma.

Lemma 5.1. *Let X be a smooth projective variety of dimension 3. Let D_1, D_2 and D_3 be divisors on X. Assume the following:*

(1) $D_1^2 D_3 > 0$.
(2) D_3 *is semiample and big.*
(3) $(D_1^2 D_3)(D_2^2 D_3) = (D_1 D_2 D_3)^2$.
(4) $D_1^2 D_3 = D_2^2 D_3$.

Then $(D_1 - D_2)D_3 D = 0$ holds for any divisor D on X.

Proof. By the assumption (2), there exists a smooth surface $S \in |mD_3|$ for some $m > 0$. Then by the assumption (3) we have $(D_1|_S)^2(D_2|_S)^2 = ((D_1|_S)(D_2|_S))^2$. So by the assumptions (1) and (4) we have $D_1|_S \equiv D_2|_S$. In particular $(D_1|_S)(D|_S) = (D_2|_S)(D|_S)$ for any divisor D on X. Therefore $D_1D(mD_3) = D_2D(mD_3)$. Hence we get the assertion. □

Since K_M+A is semiample and big, we see that $(K_M+A)^2D = A(K_M+A)D$ for any divisor D on M by Lemma 5.1. Therefore $K_MD(K_M+A) = 0$ for any divisor D on X.

Next we calculate $h^0(2K_M+2A)$ and $h^0(K_M+A)$. Then by the Hirzebruch–Riemann–Roch theorem and the Kodaira vanishing theorem we have

$$h^0(2K_M+2A) = 4 + (1/6)c_2(M)A - 3\chi(\mathcal{O}_M),$$

and

$$h^0(K_M+A) = (1/2) + (1/12)c_2(M)A - \chi(\mathcal{O}_M).$$

Since we are considering the case where $(K_M+A)^3 = 3$, we have $\chi(\mathcal{O}_M) = 0$ or 1.

If $\chi(\mathcal{O}_M) = 0$, then $4 = h^0(2K_M+2A) = 4 + (1/6)c_2(M)A$. Hence $c_2(M)A = 0$. But then $h^0(K_M+A) = 1/2$ and this is impossible.

If $\chi(\mathcal{O}_M) = 1$, then (M, A) satisfies the case (3.1) and $4 = h^0(2K_M+2A) = 4+(1/6)c_2(M)A-3\chi(\mathcal{O}_M) = 1+(1/6)c_2(M)A$. Hence $c_2(M)A = 18$ and $h^0(K_M+A) = 1$. On the other hand, by Theorem 3.1 we have $1 = h^0(K_M+A) = g_2(M, A)-h^2(\mathcal{O}_M)+h^3(\mathcal{O}_M)$. Hence $g_2(M, A) = 1 + h^2(\mathcal{O}_M) - h^3(\mathcal{O}_M)$ and $d_1 = \chi(\mathcal{O}_M) = 1$. But $d_1 = 0$ in this case (3.1). Hence this is also impossible.

(iii.3) Next we consider the case where $(K_M+A)^3 = 1$. Then (M, A) satisfies the case (3.2). In particular $g_2(M, A) = h^1(\mathcal{O}_M)$. We also get $(K_M+A)^2K_M = -2$ from the assumption that $(K_M+A)^2A = 3$ or $\chi(\mathcal{O}_M) = 0$. In particular $\kappa(M) = -\infty$ and $h^3(\mathcal{O}_M) = 0$. Here we note $g_1(M, K_M+A) = 1 + (1/2)(3K_M+2A)(K_M+A)^2 = 1$. We also note that $h^1(\mathcal{O}_M) > 0$ because $\kappa(M) = -\infty$ and $\chi(\mathcal{O}_M) = 0$. Hence by [10, (4.9) Corollary] we have $h^1(\mathcal{O}_M) = 1$ and (M, K_M+A) is birationally equivalent to (V, H) which is a scroll over an elliptic curve because K_M+A is nef and big. This is the type (iii) in Claim 5.2.

These complete the proof of Claim 5.2. □

Here we consider the case where $g(M, A) = 2$. In this case by the classification of (M, A) with $g(M, A) = 2$ ([9, (1.10) Theorem and Section 2]) we see that (M, A) is one of the following type: $\mathcal{O}(K_M) = \mathcal{O}_M$, $h^1(\mathcal{O}_M) = 0$, $h^0(A) > 0$ and $A^3 = 1$.

Then $h^0(A) = (1/6)A^3 + (1/12)c_2(M)A$ and $h^0(2A) = (4/3)A^3 + (1/6)c_2(M)A$. Since $4 = h^0(2K_M+2A) = h^0(2A)$, we have $4 = (4/3)A^3+(1/6)c_2(M)A = (4/3)+(1/6)c_2(M)A$. Hence $c_2(M)A = 16$. But then $h^0(A) = 3/2$ and this is impossible.

Therefore (M, A) is one of the types (1) and (2) in Theorem 5.4.

(β) Assume that (M, A) satisfies one of the types (1) and (2) in Theorem 5.4.

(β.1) Assume that (M, A) satisfies the type (1) in Theorem 5.4. Here we note that $h^i(A) = 0$ for every positive integer i. Then

$$\begin{aligned} h^0(A) &= \chi(A) \\ &= \frac{1}{6}A^3 + \frac{1}{12}c_2(M)A. \end{aligned}$$

Hence we have $c_2(M)A = 8$. Therefore

$$\begin{aligned} h^0(2K_M+2A) &= \chi(2K_M+2A) \\ &= \chi(2A) \\ &= \frac{4}{3}A^3 + \frac{1}{6}c_2(M)A \\ &= 4. \end{aligned}$$

(β.2) Assume that (M, A) satisfies the type (2) in Theorem 5.4.

First we note the following.

Claim 5.3. $h^0(2K_M+A) = 0$.

Proof. Since (M, K_M+A) is birationally equivalent to a scroll (V, H) over a smooth elliptic curve B, there exist a smooth projective 3-fold T and birational morphisms $\mu : T \to M$ and $\nu : T \to V$ such that $\mu^*(K_M+A) = \nu^*(H)$. Here we note that V is smooth. Then $h^0(2K_M+A) = h^0(\mu^*(2K_M+A)) = h^0(K_T+\mu^*(K_M+A)) = h^0(K_T+\nu^*(H)) = h^0(\nu^*(K_V+H)) = h^0(K_V+H) = 0$. This completes the proof. □

We also see that $g_1(M, K_M+A, A) = 1 + (K_M+A)^2A = 4$. Hence from Proposition 5.1 (2) we get

$$\begin{aligned} h^0(2(K_M+A)) &= h^0(2K_M+A) + g_2(M, A) - h^1(\mathcal{O}_M) + g_1(M, K_M+A, A) \\ &= 4. \end{aligned}$$

Therefore we get the assertion of Theorem 5.4. □

Remark 5.3. By Theorem 5.4, we see that if $\kappa(K_X+L) = 3$ and $h^0(2(K_X+L)) = 4$, then $h^0(K_X+L) = 1$.

Example 5.1. Here we give an example of this case.

(1) An example of the type (1) in Theorem 5.4. In [3, Theorem 1.1], Beauville gave an example of a polarized Calabi-Yau threefold (X, L) such that $h^0(L) = 1$ and $L^3 = 2$. This is an example. For details, see [3, Theorem 1.1].
(2) An example of the type (2) in Theorem 5.4. Let C be an elliptic curve and let $\mathcal{E}$ be an ample vector bundle of rank 3 on C with $c_1(\mathcal{E}) = 1$. Then $\mathcal{E}$ is indecomposable. We note that such a vector bundle exists. Let $M = \mathbb{P}_C(\mathcal{E})$ and $A = 4H(\mathcal{E}) - f^*(c_1(\mathcal{E}))$, where $f : M \to C$ is the natural map. Then by [27, Theorem 3.1] we see that A is ample, and we also see that $(M, K_M + A)$ is a scroll over a smooth elliptic curve. We can also check that $h^0(K_M + A) = h^0(H(\mathcal{E})) = 1$, $h^2(\mathcal{O}_M) = 0$, $h^1(\mathcal{O}_M) = 1$, $g_2(M, A) = 1$, $g_1(M, K_M + A, A) = 4$ and $h^0(2K_M + A) = 0$. Therefore by Proposition 5.1(2) we have $h^0(2K_M + 2A) = h^0(2K_M + A) + g_2(M, A) - h^1(\mathcal{O}_M) + g_1(M, K_M + A, A) = 4$.

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