MATHEMATICS

DUAL AND TRIPLE SEQUENCE EQUATIONS INVOLVING ORTHOGONAL POLYNOMIALS

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ABSTRACT

We extend certain results of Richard Askey concerning dual sequence equations involving Jacobi and Laguerre polynomials to dual and triple sequence equations involving general orthogonal polynomials. We approach the problem via the polynomial expansions of Fields and Wimp, and Fields and Ismail.

1. INTRODUCTION

In [2] ASKEY posed and solved the following problems:

PROBLEM 1. Find $f(x)$ if

$$a_n = \begin{cases} \int_0^\infty x^c f(x)\, x^\alpha e^{-x} L_n^{(\alpha)}(x)\, dx & n=0, 1, \ldots, N \\ \int_0^\infty f(x)\, x^\beta e^{-x} L_n^{(\beta)}(x)\, dx & n=N+1, N+2, \ldots \end{cases}$$

where the L's are Laguerre polynomials and the a's are given.

PROBLEM 2. Find $f(x)$ if

$$a_n = \begin{cases} \int_{-1}^{1} (1-x)^c f(x)(1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x)\, dx & n=0, 1, \ldots, N \\ \int_{-1}^{1} f(x)(1-x)^\gamma (1+x)^\delta P_n^{(\gamma,\delta)}(x)\, dx & n=N+1, N+2, \ldots \end{cases}$$

where the P's are Jacobi polynomials and the a's are given.

Of course some restrictions are imposed on α, β, γ, δ and c in the above problems.

At the end of [2] ASKEY posed the analogous discrete problem. Section 2 of the present work generalizes Askey's work. We also consider some triple series equations. As illustrations we treat the corresponding problems involving the Bessel, Charlier, and Meixner polynomials in section 3. Our analysis is formal and no attempt is made to supply rigorous proofs.

2. The dual and triple sequence equations

Let $P_n(x, a)$ be a set of polynomials orthogonal with respect to the weight function $w(x, a)$, where a is a parameter or set of parameters in vector form. Assume that the interval of orthogonality of $\{P_n(x, a)\}_0^\infty$ is the same for all permissible values of the parameter a. Typical examples of these polynomials are the polynomials of Jacobi, Laguerre, Bessel, Charlier, and Meixer. We also assume that the system $\{P_n(x, a)\}_0^\infty$ is complete.

Consider the following dual sequence equations

$$a_n = \begin{cases} \int f(x)\, w(x, c)\, P_n(x, a)\, dx & n = 0, 1, \dots, N \\ \int f(x)\, w(x, b)\, P_n(x, b)\, dx & n = N+1, N+2, \dots \end{cases} \tag{2.1}$$

where the integration is over the common interval of orthogonality. Let

$$P_n(x, c) = \sum_{k=0}^{n} \lambda_{n,k}(c, a)\, P_k(x, a). \tag{2.2}$$

Relations of the type (2.2) may be found without using the orthogonality relations for $\{P_n(x, a)\}_0^\infty$. Indeed Fields and Wimp [5] established a large class of such relationships when the P's are hypergeometric functions. Fields and Ismail [4] established them for a much larger class. This approach will become clearer in the examples.

Setting

$$g_n(a) = \int \{P_n(x, a)\}^2 w(x, a)\, dx, \tag{2.3}$$

we get

$$\lambda_{n,k}(c, a) = \int P_n(x, c)\, P_k(x, a)\, w(x, a)\, dx / g_k(a), \qquad k \leqslant n. \tag{2.4}$$

Therefore

$$b_n = \int f(x)\, w(x, c)\, P_n(x, c)\, dx, \qquad n = 0, 1, \dots, N, \tag{2.5}$$

where

$$b_n = \sum_{k=0}^{n} \lambda_{n,k}(c, a)\, a_k, \qquad n = 0, 1, \dots, N. \tag{2.6}$$

Clearly the function $w(x, c)\, P_n(x, c) - \sum_{j=n}^{\infty} \mu_{j,n}(b, c)\, w(x, b)\, P_j(x, b)$ is orthogonal to $P_k(x, b)$ for all k if

$$\mu_{j,n}(b, c) = \int w(x, c)\, P_n(x, c)\, P_j(x, b)\, dx / g_j(b), \qquad j \geqslant n.$$

Thus by (2.4), and completeness of $\{P_n(x, a)\}_0^\infty$ we get

$$w(x, c)\, P_n(x, c) = \sum_{j=n}^{\infty} \frac{g_n(c)}{g_j(b)}\, \lambda_{j,n}(b, c)\, w(x, b)\, P_j(x, b),$$

which implies

$$b_n = \int f(x)\, w(x, c)\, P_n(x, c)\, dx, \qquad n = N+1, N+2, \dots \tag{2.7}$$

with

$$b_n = \sum_{j=n}^{\infty} \frac{g_n(c)}{g_j(b)} \lambda_{j,n}(b, c)\, a_j, \qquad n = N+1, N+2, \ldots. \tag{2.8}$$

Therefore we proved the following theorem.

THEOREM 1. The solution of the system (2.1) is

$$f(x) = \sum_{n=0}^{\infty} b_n P_n(x, c)/g_n(c),$$

where the b's, g's and λ's are defined by (2.6) and (2.8), (2.3), and (2.2).

We now consider the triple sequence equations

$$a_n = \begin{cases} \int f(x)\, w(x, c)\, P_n(x, a)\, dx, & n = 0, 1, \ldots, N \\ \int f(x)\, w(x, c)\, P_n(x, d)\, dx, & n = N+1, N+2, \ldots, M \\ \int f(x)\, w(x, b)\, P_n(x, b)\, dx, & n = M+1, M+2, \ldots. \end{cases} \tag{2.9}$$

For $N < n \leqslant M$ set

$$P_n(x, c) = \sum_{k=0}^{N} \xi_{n,k}(a, d, c)\, P_k(x, a) + \sum_{k=N+1}^{n} \eta_{n,k}(c, d)\, P_k(x, d). \tag{2.10}$$

Clearly

$$\eta_{n,k}(c, d) = \int P_n(x, c) P_k(x, d) \frac{w(x, d)}{g_k(d)}\, dx,$$

that is,

$$\eta_{n,k}(c, d) = \lambda_{n,k}(c, d), \qquad k = N+1, \ldots, n. \tag{2.11}$$

From (2.10) we get

$$\xi_{n,k}(a, d, c) = \frac{1}{g_k(a)} \{\int P_n(x, c)\, w(x, a)\, P_k(x, a)\, dx$$
$$- \sum_{j=N+1}^{n} \lambda_{n,j}(c, d) \int P_j(x, d)\, P_k(x, a)\, w(x, a)\, dx\}.$$

Therefore

$$\begin{cases} \xi_{n,k}(a, d, c) = \lambda_{n,k}(c, a) - \sum_{j=N+1}^{n} \lambda_{n,j}(c, d)\, \lambda_{j,k}(a, d), \\ \qquad\qquad k = 0, 1, \ldots, N, \quad n > N. \end{cases} \tag{2.12}$$

Thus we proved

THEOREM 2. The formal solution of (2.9) is

$$f(x) = \sum_{n=0}^{\infty} \frac{b_n}{g_n(c)} P_n(x, c),$$

where b_n is given by (2.6) for $n = 0, 1, \ldots, N$, by (2.8) for $n = M+1, M+2, \ldots$ and for $n = N+1, N+2, \ldots, M$ by

$$b_n = \sum_{k=0}^{N} \{\lambda_{n,k}(c, a) - \sum_{j=N+1}^{n} \lambda_{n,j}(c, d)\, \lambda_{j,k}(a, d)\}\, a_k + \sum_{k=N+1}^{n} \lambda_{n,k}(c, d)\, a_k.$$

3. Examples

In this section we calculate the λ's and g's for the polynomials under consideration. The Charlier Polynomials $c_n(x, a)$ may be defined by the generating relation [3, p. 266]

$$\sum_{n=0}^{\infty} c_n(x, a)\frac{(at)^n}{n!} = e^{-at}(1-t)^x,$$

or explicitly as [3, p. 226]

$$c_n(x, a) = {}_2F_0\left(-n, -x; -; -\frac{1}{a}\right).$$

Formula (1.6) of [5] implies

$$c_n(x, c) = \left(\frac{a-c}{-c}\right)^n \sum_{k=0}^{n} \binom{n}{k}\left(\frac{-a}{a-c}\right)^k c_k(x, a),$$

that is, the λ's of (2.2) are given by

$$\lambda_{n,k}(c, a) = \left(\frac{a-c}{-c}\right)^n \binom{n}{k}\left(\frac{-a}{a-c}\right)^k. \tag{3.1}$$

The g's of (2.3) are given by [3, p. 226]

$$g_n(a) = a^{-n}n!, \quad a > 0. \tag{3.2}$$

The Charlier polynomials are orthogonal with respect to the discrete distribution $e^{-a}a^x/x!$, $a>0$, $x=0, 1, 2, \ldots$. Substituting for $\lambda_{n,k}$ and g_n from (3.1) and (3.2) in theorem 1 or 2 we obtain the solution of the dual or triple sequence equations.

The Meixner polynomials $m_n(x, \alpha, c)$ may be defined by [3, p. 225]

$$m_n(x; \alpha, c) = (\alpha)_n\, {}_2F_1(-n, -x; \beta; 1-c^{-1}), \quad \alpha > 0, \ 0 < c < 1. \tag{3.3}$$

The special case $p=2$, $r=s=q=1$ of (1.6) in [5] with $a_1=-n$, $a_2=d_1=\beta$, $b_1=\alpha$, $c_1=-x$, $z=(1-c^{-1}/1-s^{-1})$, $w=1-s^{-1}$ yields

$$\left\{\begin{aligned} m_n(x; \alpha, c) = (\alpha)_n \sum_{k=0}^{n} \frac{\binom{n}{k}}{(\alpha)_k}\left\{\frac{(c-1)s}{(s-1)c}\right\}^k \\ {}_2F_1\left(\begin{matrix} k-n, k+\beta \\ k+\alpha \end{matrix} \,\middle|\, \frac{(c-1)s}{(s-1)c}\right) m_k(x; \beta, s). \end{aligned}\right. \tag{3.4}$$

We have not been able to simplify the coefficients

$${}_2F_1\left(\begin{matrix} k-n, k+\beta \\ k+\alpha \end{matrix} \,\middle|\, \frac{(c-1)s}{(s-1)c}\right)$$

any further. However they obviously simplify when $\alpha=\beta$ or $c=s$.

Writing the parameters in vector form we get

$$(3.5)\quad \begin{cases} \lambda_{n,k}((\alpha, c); (\beta, s)) = \dfrac{(\alpha)_n}{(\alpha)_k}\dbinom{n}{k}\left(\dfrac{s(c-1)}{c(s-1)}\right)^k \\ \qquad\qquad {}_2F_1\left(\begin{matrix} k+n,\ k+\beta \\ k+\alpha \end{matrix}\,\middle|\, \dfrac{(c-1)s}{(s-1)c}\right), \quad s\leqslant c. \end{cases}$$

For $\alpha>0$, $0<c<1$, the Meixner polynomials are orthogonal with respect to the discrete weight function $c^x((\alpha)_x/x!)$ with

$$(3.6)\qquad g_n(\alpha, c) = n!\,(\alpha)_n c^{-n}(1-c)^{-\alpha},\ \beta>0,\ 0<c<1.$$

The solution of the system under consideration follows by theorem 1 or 2 and, (3.5) and (3.6).

We now come to the Bessel polynomials $Y_n^{(\alpha)}(x)$. We follow AL-SALAM's [1] notation. The Bessel polynomials may be defined as

$$(3.7)\qquad Y_n^{(\alpha)}(x) = {}_2F_0\left(-n,\, n+\alpha+1\,;\, -\,;\, -\frac{x}{2}\right).$$

They are orthogonal with respect to the weight function

$$w(x, \alpha) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha)}{\Gamma(\alpha+n-1)}\left(-\frac{2}{x}\right)^n,$$

on the unit circle $|x|=1$. The g's are given by

$$g_n(\alpha) = \frac{(-1)^{n+1}n!(\alpha+1)}{(2n+\alpha+1)(\alpha+1)_n}.$$

Using (1.4) of [5] with $p=2$, $q=r=1$, $s=0$, $c_1=b_1=\alpha$, $a_1=-n$, $a_2=n+\alpha+1$, $z=1$ we get by (3.7)

$$(3.8)\qquad Y_n^{(\alpha)}(x) = \sum_{k=0}^{\infty}\frac{\dbinom{n}{k}(\alpha+n+1)_k}{(\gamma+n+1)_k}\,\frac{(4k+\gamma-n-\alpha)_{n-k}}{(2k+\gamma+2)_{n-k}}\,Y_k^{(\gamma)}(x),$$

and hence

$$\lambda_{n,k}(\alpha, \gamma) = \binom{n}{k}\frac{(\alpha+n+1)_k}{(\gamma+n+1)_k}\,\frac{(1+\gamma+k-n-\alpha)_{n-k}}{(2k+\gamma+2)_{n-k}}.$$

Relation (3.8) is also in [1].

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