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NOTE

LINEAR ORDERS AND SEMIORDERS CLOSE TO AN INTERVAL ORDER

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The purpose of this paper is to study the properties of the linear orders and semiorders at minimum symmetric difference distance from a given interval order on a finite set.

1. Introduction

Let X be a non-empty finite set with elements $x, y, \ldots$. Let P be an asymmetric ($xPy \to \text{not } yPx$) binary relation on X. The symmetric complement of P on X is defined by

$$xIy \leftrightarrow \text{not } xPy \text{ and not } yPx.$$

If P is transitive (xPy and $yPz \to xPz$), then it is a *partial order*, and if both P and I are transitive, then P is called a *weak order*. A complete (xPy or yPx, for all $x, y \in X$) partial order is a *linear order*.

If the partial order satisfies for all $x, y, z, w \in X$,

$$xPy \text{ and } zPw \to xPw \text{ or } zPy,$$

it is an *interval order* (Fishburn [1]).

This is one of the two conditions which define a *semiorder*.

The second condition is (Scott and Suppes [4]):

$$xPy \text{ and } yPz \to xPw \text{ or } wPz.$$

The relation E, defined by xEy iff $xIz \Leftrightarrow yIz$, for all $z \in X$, is an *equivalence* (reflexive symmetric and transitive relation). Remark: in what follows, we shall always suppose that, for all $x, y \in X$, $x \neq y \to \text{not } xEy$.

The symmetric difference distance between two relations P and P' is equal to

$$d(P, P') = \#\{(x, y) : xPy \text{ and not } xP'y\} + \#\{(x, y) : xP'y \text{ and not } xPy\}$$

where $\#A$ is the cardinality of set A.

P' is *close* to a given relation P iff P' minimizes the distance $d(P, P')$.

The *opinion tableau* related to P is a matrix M with elements m_{xy}, where

$$\begin{cases} m_{xx} = 0, & \forall x \in X, \\ m_{xy} = 1, & \text{iff } x P y, \\ m_{xy} = 0, & \text{iff not } x P y. \end{cases}$$

M is *step-typed* iff $m_{xy} = 1$, $l_z \le l_x$, $c_y \le c_w \to m_{zw} = 1$, where l_x and c_x represent the indexes for the line and the column of x in M.

For given partial order P on X, and any element $x \in X$, let us consider

$$L(x) = \{z : x P z\}, \qquad M(x) = \{z : z P x\}.$$

If P denotes preference, $L(x)$ and $M(x)$ are respectively the subsets of X whose alternatives are less preferred than, more preferred than x.

Let us consider the following linear orders T_L and T_C related to a given interval order P:

$$x T_L y \quad \text{iff} \quad \# L(x) > \# L(y) \quad \text{or} \quad \begin{cases} \# L(x) = \# L(y), \\ \# M(x) < \# M(y). \end{cases}$$

$$x T_C y \quad \text{iff} \quad \# M(x) < \# M(y) \quad \text{or} \quad \begin{cases} \# M(x) = \# M(y), \\ \# L(x) > \# L(y). \end{cases}$$

Note. The case $\#L(x) = \#L(y)$ and $\#M(x) = \#M(y)$, corresponding to $x E y$, is excluded by the remark made in the introduction.

It is clear that T_L and T_C are linear orders extending P, so that they are close to P.

If P is a semiorder, then $T_L = T_C = T_S$ (Sharp [5], Monjardet [3]).

2. On semiorders close to a given interval order

Let P be a given interval order with two underlying linear orders T_L and T_C. As mentioned in the introduction, we suppose that the equivalence relation E, associated to P, is empty (in the other case, we consider X/E). Let P_S be a semiorder and E_S the associated equivalence relation. Let T_S' be the underlying linear order in the set X/E_S. Define, $\forall A, B \in X/E_S$:

$$x T_S y \quad \text{iff} \quad x \in A,\ y \in B \text{ and } A T_S' B.$$

So, $\forall x, y \in X$:

$$x T_S y \quad \text{or} \quad x E_S y \quad \text{or} \quad y T_S x.$$

The matrix M related to P is step-typed if its lines and columns are respectively rearranged according to T_L and T_C:

$$l_x < l_y \quad \text{iff} \quad x\,T_L\,y,$$
$$c_x < c_y \quad \text{iff} \quad x\,T_C\,y.$$

The matrix M_S related to P_S is step-typed if its lines and columns satisfy

$$x\,T_S\,y \to l_x < l_y \text{ and } c_x < c_y.$$

The following theorem shows that the underlying linear order related to a semiorder close to a given interval order is, in some sense, compatible with the intersection of the underlying linear orders related to this interval order.

Theorem. *If P_S is close to P, then for all $x, y \in X$.*

$$x(T_L \cap T_C)\,y \to x\,T_S\,y \text{ or } x\,E_S\,y.$$

Proof. Let us suppose that $y\,T_S\,x$. Starting from P_S and its opinion tableau M_S, we consider the relation P' defined by the matrix M' which is identical to M_S but where $l'_x = l_y$, $l'_y = l_x$, $c'_x = c_y$ and $c'_y = c_x$. So:

$$m^s_{xz} = 1 \text{ and } y\,T_S\,x \to m^s_{yz} = 1 \to m'_{xz} = 1,$$
$$m^s_{yz} = 0 \text{ and } y\,T_S\,x \to m^s_{xz} = 0 \to m'_{yz} = 0,$$
$$m^s_{xz} = m_{xz} = 0,\ m'_{xz} = 1 \text{ and } x(T_L \cap T_C)\,y \to m_{yz} = m'_{yz} = 0,\ m^s_{yz} = 1,$$
$$m^s_{yz} = m_{yz} = 1,\ m'_{yz} = 0 \text{ and } x(T_L \cap T_C)\,y \to m_{xz} = m'_{xz} = 1,\ m^s_{xz} = 0,$$

and the same results hold when x and y are second indexes. The reader will easily verify that these properties imply

$$d(P', P) < d(P_S, P),$$

which is contradictory. $\square$

To close this second part, we propose the following

Conjecture. If P_S is close to P, then $T_S = T_{LC}$, where T_{LC} is a linear order which verifies

$$x(T_L \cap T_C)\,y \to x\,T_{LC}\,y \text{ or } x\,E_S\,y$$

and which minimizes

$$\sum_{(x,y):\,x\,T_L\,y,\ y\,T_{LC}\,x} \#\{z : x\,P\,z,\ \text{not } y\,P\,z\} + \sum_{(x,y):\,x\,T_C\,y,\ y\,T_{LC}\,x} \#\{z : z\,P\,y,\ \text{not } z\,P\,x\}.$$

Remark. As

$$x\,P\,y \to x(T_L \cap T_C)\,y \to x\,T_{LC}\,y,$$

T_{LC} is a linear order close to P.

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