On the generalized self-similar singularities for
the Euler and the Navier–Stokes equations

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Abstract

In this paper we study blow-up rates and the blow-up profiles of possible asymptotically self-similar singularities of the Euler and the Navier–Stokes equations, where the sense of convergence and self-similarity are considered in various generalized senses. We improve substantially, in particular, the previous nonexistence results of self-similar/asymptotically self-similar singularities. Generalization of the self-similar transforms is also considered, and by appropriate choice of the parameterized transform we obtain new a priori estimates for the Euler and the Navier–Stokes equations depending on a free parameter. © 2010 Elsevier Inc. All rights reserved.

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1. Self-similar singularities

We are concerned on the following Euler equations for the homogeneous incompressible fluid flows in $\mathbb{R}^3$:

\[
\begin{cases}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p, \\
\text{div} v = 0, \\
v(x, 0) = v_0(x),
\end{cases} 
\quad (x, t) \in \mathbb{R}^3 \times (0, \infty),
\]

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where \( v = (v_1, v_2, v_3) \), \( v_j = v_j(x, t) \), \( j = 1, 2, 3 \), is the velocity of the flow, \( p = p(x, t) \) is the scalar pressure, and \( v_0 \) is the given initial velocity, satisfying \( \text{div} v_0 = 0 \). The system (E) is first modeled by Euler in [13]. The local well-posedness of the Euler equations in \( H^m(\mathbb{R}^3) \), \( m > 5/2 \), is established by Kato in [17], which says that given \( v_0 \in H^m(\mathbb{R}^3) \), there exists \( T \in (0, \infty) \) such that there exists a unique solution to (E), \( v \in C([0, T); H^m(\mathbb{R}^3)) \). The finite time blow-up problem of the local classical solution is known as one of the most important and difficult problems in partial differential equations (see e.g. [20,6,8,7,5] for graduate level texts and survey articles on the current status of the problem). We say a local in time classical solution \( v \in C([0, T); H^m(\mathbb{R}^3)) \) blows up at \( T \) if \( \limsup_{t \to T} \| v(t) \|_{H^m(\mathbb{R}^3)} = \infty \) for all \( m > 5/2 \). The celebrated Beale–Kato–Majda criterion [1] states that the blow-up happens at \( T \) if and only if

\[
\int_0^T \| \omega(t) \|_{L^\infty} \, dt = \infty.
\]

Although the original result of [1] is the blow-up criterion in the \( H^m(\mathbb{R}^3) \) norm with \( m \geq 3 \), it is easy to extend it to the case of \( m > 5/2 \). There are studies of geometric nature for the blow-up criterion [9,7,12]. As another direction of studies of the blow-up problem mathematicians also consider various scenarios of singularities and study carefully their possibility of realization (see e.g. [10,11,2,3] for some of those studies). One of the purposes in this paper, especially in this section, is to study more deeply the notions related to the scenarios of the self-similar singularities in the Euler equations, the preliminary studies of which are done in [2,3]. We recall that system (E) has scaling property that if \((v, p)\) is a solution of the system (E), then for any \( \lambda > 0 \) and \( \alpha \in \mathbb{R} \) the functions

\[
v^{\lambda, \alpha}(x, t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t), \quad p^{\lambda, \alpha}(x, t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1} t)
\]

are also solutions of (E) with the initial data \( v^{\lambda, \alpha}_0(x) = \lambda^\alpha v_0(\lambda x) \). In view of the scaling properties in (1.1), a natural self-similar blowing-up solution \( v(x, t) \) of (E) should be of the form,

\[
v(x, t) = \frac{1}{(T - t)^{\alpha+1}} \nabla \left( \frac{x}{(T - t)^{\frac{1}{\alpha+1}}} \right),
\]

\[
p(x, t) = \frac{\alpha + 1}{(T - t)^{\frac{2\alpha}{\alpha+1}}} \nabla \left( \frac{x}{(T - t)^{\frac{1}{\alpha+1}}} \right)
\]

for \( \alpha \neq -1 \) and \( t \) sufficiently close to \( T \). Substituting (1.2)–(1.3) into (E), we obtain the following stationary system:

\[
\begin{cases}
\alpha \nabla + (y \cdot \nabla) \nabla + (\alpha + 1)(\nabla \cdot \nabla) \nabla = -\nabla P, \\
\text{div} \nabla = 0,
\end{cases}
\]

the Navier–Stokes equations version of which has been studied extensively after Leray’s pioneering paper [19,23,24,22,3,16]. Existence of solution of the system (1.4) is equivalent to the existence of solutions to the Euler equations of the form (1.2)–(1.3), which blows up in a self-similar fashion. Given \((\alpha, p) \in (-1, \infty) \times (0, \infty) \), we say the blow-up is \( \alpha \)-asymptotically
self-similar in the sense of $L^p$ if there exists $\vec{V} = \vec{V}_\alpha \in \dot{W}^{1,p}(\mathbb{R}^3)$ such that the following convergence holds true:

$$\lim_{t \to T} (T - t) \left\| \nabla v(\cdot, t) - \frac{1}{T-t} \nabla \left( \frac{\cdot}{(T-t)^{\frac{1}{\alpha+1}}} \right) \right\|_{L^\infty} = 0$$

if $p = \infty$, while

$$\lim_{t \to T} (T - t)^{1 - \frac{3}{(\alpha+1)p}} \left\| \omega(\cdot, t) - \frac{1}{T-t} \vec{\Omega} \left( \frac{\cdot}{(T-t)^{\frac{1}{\alpha+1}}} \right) \right\|_{L^p} = 0$$

if $0 < p < \infty$, where and hereafter we denote

$$\Omega = \text{curl} V \quad \text{and} \quad \vec{\Omega} = \text{curl} \vec{V}.$$ 

The above limit function $\vec{V} \in L^p(\mathbb{R}^3)$ with $\vec{\Omega} \neq 0$ is called the blow-up profile. We observe that the self-similar blow-up given by (1.2)–(1.3) is trivial case of $\alpha$-asymptotic self-similar blow-up with the blow-up profile given by the representing function $\vec{V}$. We say a blow-up at $T$ is of type I, if

$$\limsup_{t \to T} (T - t) \left\| \nabla v(t) \right\|_{L^\infty} < \infty.$$ 

If the blow-up is not of type I, we say it is of type II. For the use of terminology, type I and type II blow-ups, we followed the literatures on the studies of the blow-up problem in the semilinear heat equations (see e.g. [21,15,14], and the references therein). The use of $\left\| \nabla v(t) \right\|_{L^\infty}$ rather than $\left\| v(t) \right\|_{L^\infty}$ in our definition of types I and II is motivated by Beale–Kato–Majda’s blow-up criterion.

**Theorem 1.1.** Let $m > 5/2$, and $v \in C([0, T); H^m(\mathbb{R}^3))$ be a solution to (E) with $v_0 \in H^m(\mathbb{R}^3)$, $\text{div} v_0 = 0$. We set

$$\limsup_{t \to T} (T - t) \left\| \nabla v(t) \right\|_{L^\infty} := M(T). \tag{1.5}$$

Then, either $M(T) \geq 1$, or the solution does not blow up at time $T$, which implies that $M(T) = 0$. Hence, one can only have type I blow-up at time $T$ if $M(T) \geq 1$.

An immediate implication of the above theorem on the self-similar blow-up is the following.

**Corollary 1.1.** There exists no self-similar blow-up for the solution of the 3D Euler equations with the blow-up profile $V$ satisfying $\left\| \nabla V \right\|_{L^\infty} < 1$.

**Proof.** We just observe that if $v(x, t) = \frac{1}{(T-t)^{\frac{1}{\alpha+1}}} V(\frac{x}{(T-t)^{\frac{1}{\alpha+1}}})$, then

$$(T - t) \left\| \nabla v(t) \right\|_{L^\infty} = \left\| \nabla V \right\|_{L^\infty}, \quad \forall t \in (0, T). \qed$$
Proof of Theorem 1.1. It suffices to show that $M(T) < 1$ implies non-blow-up at $T$, which, in turn, leads to $M(T) = 0$, since $\|\nabla v(t)\|_{L^\infty} \in C([0, T])$ in this case. We suppose $M(T) < 1$. Then, there exists $t_0 \in (0, T)$ such that

$$\sup_{t_0 < t < T} (T - t) \|\nabla v(t)\|_{L^\infty} := M_0 < 1.$$  

Taking curl of the evolution part of (E), we have the vorticity equation,

$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla) \omega = (\omega \cdot \nabla)v.$$  

This, taking dot product with $\xi = \omega/|\omega|$, leads to

$$\frac{\partial |\omega|}{\partial t} + (v \cdot \nabla)|\omega| = (\xi \cdot \nabla)v \cdot \xi |\omega|.$$  

Integrating this over $[t_0, t]$ along the particle trajectories $\{X(a, t)\}$ defined by $v(x, t)$, we have

$$|\omega(X(a, t), t)| = |\omega(X(a, t_0), t_0)| \exp \left[ \int_{t_0}^{t} (\xi \cdot \nabla)v \cdot \xi (X(a, s), s) \, ds \right],$$  

from which we estimate

$$\|\omega(t)\|_{L^\infty} \leq \|\omega(t_0)\|_{L^\infty} \exp \left[ \int_{t_0}^{t} \|\nabla v(\tau)\|_{L^\infty} \, d\tau \right]$$

$$< \|\omega(t_0)\|_{L^\infty} \exp \left[ M_0 \int_{t_0}^{t} (T - \tau)^{-1} \, d\tau \right]$$

$$= \|\omega(t_0)\|_{L^\infty} \left( \frac{T - t_0}{T - t} \right)^{M_0}.$$  

(1.7)

Since $M_0 < 1$, we have $\int_{t_0}^{T} \|\omega(t)\|_{L^\infty} \, dt < \infty$, and thanks to the Beale–Kato–Majda criterion there exists no blow-up at $T$, and we can continue our classical solution beyond $T$.  

The following is our main theorem in this section.

**Theorem 1.2.** Let a classical solution $v \in C([0, T); H^m(\mathbb{R}^3))$ of the 3D Euler equations with initial data $v_0 \in H^m(\mathbb{R}^3) \cap \dot{W}^{1, p}(\mathbb{R}^3)$, $\text{div} \, v_0 = 0$, $\omega_0 \neq 0$ blows up with type I. Let $M = M(T)$ be as in Theorem 1.1. Suppose $(\alpha, p) \in (-1, \infty) \times (0, \infty]$ satisfies

$$M < \left| 1 - \frac{3}{(\alpha + 1)p} \right|.$$  

(1.8)
Then, there exists no $\alpha$-asymptotically self-similar blow-up at $t = T$ in the sense of $L^p$ if $\omega_0 \in L^p(\mathbb{R}^3)$. Hence, for any type I blow-up and for any $\alpha \in (-1, \infty)$ there exists $p_1 \in (0, \infty)$ such that it is not $\alpha$-asymptotically self-similar in the sense of $L^{p_1}$.

**Remark 1.1.** We note that the case $p = \infty$ of the above theorem follows from Theorem 1.1, which states that there is no singularity at all at $t = T$ in this case. The above theorem can be regarded an improvement of the main theorem in [3], in the sense that we can consider the $L^p$ convergence only to exclude nontrivial blow-up profile $V$, where $p$ depends on $M$. Moreover, we do not need to use the Besov space $\dot{B}^{0}_{\infty, 1}$ in the statement of the theorem, and the continuation principle of local solution in the Besov space in the proof.

**Proof of Theorem 1.2.** We assume asymptotically self-similar blow-up happens at $T$. Let us introduce similarity variables defined by

$$y = x \left(\frac{T}{T-t}\right)^{\frac{1}{\alpha+1}}, \quad s = \frac{1}{\alpha+1} \log \left(\frac{T}{T-t}\right),$$

and transformation of the unknowns $(v, p) \rightarrow (V, P)$ according to

$$v(x, t) = \frac{1}{(T-t)^{\frac{\alpha}{\alpha+1}}} V(y, s), \quad p(x, t) = \frac{1}{(T-t)^{\frac{2\alpha}{\alpha+1}}} P(y, s). \quad (1.9)$$

Substituting $(v, p)$ into the (E) we obtain the equivalent evolution equation for $(V, P)$,

$$(E_1) \begin{cases} 
V_s + \alpha V + (y \cdot \nabla) V + (\alpha + 1)(V \cdot \nabla) V = -\nabla P, \\
\text{div } V = 0, \\
V(y, 0) = V_0(y) = T^{\frac{\alpha}{\alpha+1}} v_0 \left( T^{\frac{1}{\alpha+1}} y \right).
\end{cases}$$

Then the assumption of asymptotically self-similar singularity at $T$ implies that there exists $V = \overline{V}_\alpha \in \dot{W}^{1, p}(\mathbb{R}^3)$ such that

$$\lim_{s \to \infty} \left\| \Omega(\cdot, s) - \overline{\Omega} \right\|_{L^p} = 0. \quad (1.10)$$

Now the hypothesis (1.8) implies that there exists $t_0 \in (0, T)$ such that

$$\sup_{t_0 < t < T} (T-t) \left\| \nabla v(t) \right\|_{L^\infty} := M_0 < \left| 1 - \frac{3}{(\alpha + 1)p} \right|. \quad (1.11)$$

Taking $L^p(\mathbb{R}^3)$ norm of (1.6), taking into account the following simple estimates,

$$-\left\| \nabla v(\cdot, t) \right\|_{L^\infty} \leq (\xi \cdot \nabla)v \cdot \xi(x, t) \leq \left\| \nabla v(\cdot, t) \right\|_{L^\infty}, \quad \forall (x, t) \in \mathbb{R}^3 \times [t_0, T),$$

we obtain, for all $p \in (0, \infty)$,
\[ \| \omega(t_0) \|_{L^p} \exp \left[ - \int_{t_0}^{t} \| \nabla v(\cdot, s) \|_{L^\infty} \, ds \right] \leq \| \omega(t) \|_{L^p} \]
\[ \leq \| \omega(0) \|_{L^p} \exp \left[ \int_{t_0}^{t} \| \nabla v(\cdot, s) \|_{L^\infty} \, ds \right], \quad (1.12) \]

where we use the fact that \( a \mapsto X(a, t) \) is a volume preserving map. From the fact
\[ \int_{t_0}^{t} \| \nabla v(\cdot, s) \|_{L^\infty} \, ds \leq M_0 \int_{t_0}^{t} (T - \tau)^{-1} \, d\tau = -M_0 \log \left( \frac{T - t}{T - t_0} \right), \]

and
\[ \frac{\| \omega(t) \|_{L^p}}{\| \omega(0) \|_{L^p}} = \left( \frac{T - t}{T - t_0} \right)^{-3 (\alpha + 1) p / (\alpha (\alpha + 1))} \frac{\| \Omega(s) \|_{L^p}}{\| \Omega(s_0) \|_{L^p}}, \]

where we set
\[ s_0 = \frac{1}{\alpha + 1} \log \left( \frac{T}{T - t_0} \right), \]
we find that (1.12) leads us to
\[ \left( \frac{T - t}{T - t_0} \right)^{M_0 + 1 - \frac{3}{(\alpha + 1) p}} \leq \frac{\| \Omega(s) \|_{L^p}}{\| \Omega(s_0) \|_{L^p}} \leq \left( \frac{T - t}{T - t_0} \right)^{-M_0 + 1 - \frac{3}{(\alpha + 1) p}} \quad (1.13) \]

for all \( p \in (0, \infty] \). Passing \( t \to T \), which is equivalent to \( s \to \infty \) in (1.13), we have from (1.10)
\[ \lim_{s \to \infty} \frac{\| \Omega(s) \|_{L^p}}{\| \Omega(s_0) \|_{L^p}} = \frac{\| \Omega \|_{L^p}}{\| \Omega(s_0) \|_{L^p}} \in (0, \infty). \quad (1.14) \]

By (1.11), \( M_0 + 1 - \frac{3}{(\alpha + 1) p} < 0 \) or \( -M_0 + 1 - \frac{3}{(\alpha + 1) p} > 0 \). In the former case we have
\[ \lim_{t \to T} \left( \frac{T - t}{T - t_0} \right)^{M_0 + 1 - \frac{3}{(\alpha + 1) p}} = \infty, \quad (1.15) \]
while in the latter case
\[ \lim_{t \to T} \left( \frac{T - t}{T - t_0} \right)^{-M_0 + 1 - \frac{3}{(\alpha + 1) p}} = 0. \quad (1.16) \]

Both of (1.15) and (1.16) contradict with (1.14). If the blow-up is of type I, and \( M(T) \to \infty \), then one can always choose \( p_1 \in (0, p_0) \) so small that (1.8) is valid for \( p = p_1 \). With such \( p_1 \) it is not \( \alpha \)-asymptotically self-similar in \( L^{p_1} \). \( \square \)
For the self-similar blowing-up solution of the form (1.2)–(1.3) we observe that in order to be consistent with the energy conservation, \( \|v(t)\|_{L^2} = \|v_0\|_{L^2} \) for all \( t \in [0, T) \), we need to fix \( \alpha = 3/2 \). Since the self-similar blowing-up solution corresponds to a trivial convergence of the asymptotically self-similar blow-up, the following is immediate from Theorem 1.2.

**Corollary 1.2.** Given \( p \in (0, \infty] \), there exists no self-similar blow-up with the blow-up profile \( V \) satisfying \( \Omega \in L^p(\mathbb{R}^3) \) if

\[
\|\nabla V\|_{L^\infty} < \left| 1 - \frac{6}{5p} \right|.
\]

**Remark 1.2.** The above corollary implies that we can exclude self-similar singularity of the Euler equations under the assumption of \( \Omega \in L^p(\mathbb{R}^3) \) if \( p \) satisfies the condition (1.17).

The following is, in turn, immediate from the above corollary, which is essentially Theorem 1.1 in [2]. Note that here we do not need the diffeomorphism condition for the particle trajectory mapping generated by a local classical solution before the blow-up time, which was necessary in [2] in order to guarantee the existence of the back-to-label map.

**Corollary 1.3.** There exists no self-similar blow-up with the blow-up profile \( V \) satisfying \( \Omega \in L^p(\mathbb{R}^3) \) for all \( p \in (0, p_0) \) for some \( p_0 > 0 \).

**Proof.** Suppose, on the contrary, there exists a self-similar blow-up with \( \|\nabla V\|_{L^\infty} < \infty \). Then, there exists \( p_1 > 0 \) such that

\[
\|\nabla V\|_{L^\infty} < \left| 1 - \frac{6}{5p} \right|, \quad \forall p \in (0, p_1).
\]

Due to Corollary 1.2 there should be no \( p \in (0, p_1) \) such that \( \text{curl} \ V = \Omega \in L^p(\mathbb{R}^3) \), which contradicts the hypothesis of the current corollary.

The following theorem is concerned on the possibility of type II asymptotically self-similar singularity of the Euler equations, for which the blow-up rate near the possible blow-up time \( T \) is

\[
\left\| \nabla v(t) \right\|_{L^\infty} \sim \frac{1}{(T-t)^\gamma}, \quad \gamma > 1.
\]

**Theorem 1.3.** Let \( v \in C([0, T); H^m(\mathbb{R}^3)) \), \( m > 5/2 \), be local classical solution of the Euler equations. Suppose there exists \( \gamma > 1 \) and \( R_1 > 0 \) such that the following convergence holds true:

\[
\lim_{t \to T} (T - t)^{\left( \frac{3}{2} - \frac{1}{\gamma} \right)} \left\| v(\cdot, t) - \frac{1}{(T-t)^{\frac{\gamma}{\alpha+\gamma}}} \nabla \left( \frac{\cdot}{(T-t)^{\frac{1}{\alpha+1}}} \right) \right\|_{L^2(B_{R_1})} = 0.
\]
where $B_{R_1} = \{ x \in \mathbb{R}^3 \mid |x| < R_1 \}$. Then, the blow-up profile $\vec{V} \in L^2_{\text{loc}}(\mathbb{R}^3)$ is a weak solution of the following stationary Euler equations,

\begin{equation}
(\vec{V} \cdot \nabla)\vec{V} = -\nabla \bar{P}, \quad \text{div } \vec{V} = 0. \tag{1.20}
\end{equation}

**Remark 1.3.** Eq. (1.20) seems to not have any immediate applicability in the Euler setting, but see its counterpart (1.27) and Theorem 1.4 below, where it is used to rule out type II asymptotically self-similar singularities for the Navier–Stokes equations.

**Proof of Theorem 1.3.** We introduce a self-similar transform defined by

\begin{align*}
v(x, t) &= \frac{1}{(T - t)^{\frac{\alpha}{\alpha + 1}}} V(y, s), \quad p(x, t) = \frac{1}{(T - t)^{\frac{2\alpha}{\alpha + 1}}} P(y, s) \tag{1.21}
\end{align*}

with

\begin{align*}
y &= \frac{1}{(T - t)^{\frac{\gamma}{\gamma + 1}}} x, \quad s &= \frac{1}{(\gamma - 1)T^{\gamma - 1}} \left[ \frac{T^{\gamma - 1}}{(T - t)^{\gamma - 1}} - 1 \right]. \tag{1.22}
\end{align*}

Substituting $(v, p)$ in (1.21)–(1.22) into the (E), we have

\begin{equation}
(E_2) \quad \begin{cases}
-\frac{\gamma}{s(\gamma - 1) + T^{1-\gamma}} \left[ \frac{\alpha}{\alpha + 1} V + \frac{1}{\alpha + 1} (y \cdot \nabla) V \right] = V_s + (V \cdot \nabla) V + \nabla P,
\text{div } V = 0, \\
V(y, 0) = V_0(y) = T^{\frac{\alpha}{\alpha + 1}} v_0(T^{\frac{\gamma}{\alpha + 1}} y).
\end{cases} \tag{1.23}
\end{equation}

The hypothesis (1.19) is written as

\begin{equation}
\lim_{s \to \infty} \| V(\cdot, s) - \vec{V}(\cdot) \|_{L^2(B_{R(s)})} = 0, \quad R(s) = \left[ (\gamma - 1)s + \frac{1}{T^{\gamma - 1}} \right]^\frac{\gamma}{(\alpha + 1)(\gamma - 1)}, \tag{1.24}
\end{equation}

which implies that

\begin{equation}
\lim_{s \to \infty} \| V(\cdot, s) - \vec{V} \|_{L^2(B_R)} = 0, \quad \forall R > 0, \tag{1.25}
\end{equation}

where $V(y, s)$ is defined by (1.21). Similarly to [16,3], we consider the scalar test function $\xi \in C^1_0(0, 1)$ with $\int_0^1 \xi(s) ds \neq 0$, and the vector test function $\phi = (\phi_1, \phi_2, \phi_3) \in C^1_0(\mathbb{R}^3)$ with $\text{div } \phi = 0$.

We multiply the first equation of $(E_2)$, in the dot product, by $\xi(s - n)\phi(y)$, and integrate it over $\mathbb{R}^3 \times [n, n + 1]$, and then we integrate by parts to obtain
\[ + \frac{\alpha}{\alpha + 1} \int_0^1 \int_{\mathbb{R}^3} g(s + n) \xi(s) V(y, s + n) \cdot \phi(y) \, dy \, ds \]
\[ - \frac{1}{\alpha + 1} \int_0^1 \int_{\mathbb{R}^3} g(s + n) \xi(s) V(y, s + n) \cdot (y \cdot \nabla) \phi(y) \, dy \, ds \]
\[ = \int_0^1 \int_{\mathbb{R}^3} \xi(s) \phi(y) \cdot V(y, s + n) \, dy \, ds \]
\[ + \int_0^1 \int_{\mathbb{R}^3} \xi(s) \left[ V(y, s + n) \cdot (V(y, s + n) \cdot \nabla) \phi(y) \right] \, dy \, ds = 0, \]

where we set
\[ g(s) = \frac{\gamma}{s(\gamma - 1) + T^{1-\gamma}}. \]

Passing to the limit \( n \to \infty \) in this equation, using the facts \( \int_0^1 \xi_s(s) \, ds = 0, \int_0^1 \xi(s) \, ds \neq 0, \)
\( V(\cdot, s + n) \to \overline{V} \) in \( L^2_{\text{loc}}(\mathbb{R}^3) \), and finally \( g(s + n) \to 0 \), we find that \( \overline{V} \in L^2_{\text{loc}}(\mathbb{R}^3) \) satisfies
\[ \int_{\mathbb{R}^3} \overline{V} \cdot (\overline{V} \cdot \nabla) \phi(y) \, dy = 0 \]
for all vector test function \( \phi \in C^1_0(\mathbb{R}^3) \) with \( \text{div} \phi = 0 \). On the other hand, we can pass \( s \to \infty \) directly in the weak formulation of the second equation of (E2) to have
\[ \int_{\mathbb{R}^3} \overline{V} \cdot \nabla \psi(y) \, dy = 0 \]
for all scalar test function \( \psi \in C^1_0(\mathbb{R}^3) \).

In the following theorem we rule out the possibility of the blow-up rate given by (1.18) in the setting of the Navier–Stokes equations.

**Theorem 1.4.** Let \( p \in [3, \infty) \) and \( v \in C([0, T); L^p(\mathbb{R}^3)) \) be a local classical solution of the Navier–Stokes equations constructed by Kato [18]. Suppose there exist \( \gamma > 1 \) and \( \overline{V} \in L^p(\mathbb{R}^3) \) such that the following convergence holds true:
\[ \lim_{t \to T} (T - t) \frac{(p-3)\gamma}{2p} \left\| v(\cdot, t) - (T - t) \frac{(p-3)\gamma}{2p} \overline{V} \frac{\cdot}{(T - t)^{\frac{\gamma}{2}}} \right\|_{L^p} = 0. \quad (1.26) \]
If the blow-up profile \( \overline{V} \) belongs to \( \dot{H}^1(\mathbb{R}^3) \), then \( \overline{V} = 0 \).
Proof. Since the main part of the proof is essentially identical to that of Theorem 1.3, we will be brief. Introducing the self-similar variables of the form (1.21)–(1.23) with $\alpha = \frac{1}{2}$, and substituting $(v, p)$ into the Navier–Stokes equations,

\[
\begin{aligned}
&\frac{\partial v}{\partial t} + (v \cdot \nabla) v = \Delta v - \nabla p, \\
&\text{div } v = 0, \\
&v(x, 0) = v_0(x),
\end{aligned}
\]

we find that $(V, P)$ satisfies

\[
\begin{aligned}
&-\frac{\gamma}{2\nu(\gamma - 1) + 2T^{1-\nu}} \left[ V + (y \cdot \nabla) V \right] = V_s + (V \cdot \nabla) V - \Delta V + \nabla P, \\
&\text{div } V = 0, \\
&V(y, 0) = V_0(y) = T^{\frac{\nu}{2}} v_0(T^{\frac{\nu}{2}} y).
\end{aligned}
\]

The hypothesis (1.26) is now translated as

\[
\lim_{s \to \infty} \| V(\cdot, s) - \overline{V}(\cdot) \|_{L^p} = 0.
\]

Following exactly same argument as in the proof of Theorem 1.3, we can deduce that $\overline{V}$ is a stationary solution of the Navier–Stokes equations, namely there exists $\overline{P}$ such that

\[
(\overline{V} \cdot \nabla) \overline{V} = \Delta \overline{V} - \nabla \overline{P}, \quad \text{div } \overline{V} = 0. \tag{1.27}
\]

In the case $\overline{V} \in \dot{H}^1 \cap L^p(\mathbb{R}^3)$, we easily obtain from (1.27) that $\int_{\mathbb{R}^3} |\nabla \overline{V}|^2 \, dy = 0$, which implies $\overline{V} = 0$. □

2. Generalized self-similar singularities

Let us consider a classical solution to (E) $v \in C([0, T); H^m(\mathbb{R}^3))$, $m > 5/2$, where we assume $T \in (0, \infty]$ is the maximal time of existence of the classical solution. Let $p(x, t)$ be the associated pressure. Let $\mu(\cdot) \in C^1([0, T))$ be a scalar function such that $\mu(t) > 0$ for all $t \in [0, T)$. We transform from $(v, p)$ to $(V, P)$ according to the formula,

\[
v(x, t) = \mu(t) \frac{1}{\alpha + 1} V \left( \mu(t)^{\frac{1}{\alpha + 1}} x, \int_0^t \mu(\sigma) \, d\sigma \right), \tag{2.1}
\]

\[
p(x, t) = \mu(t) \frac{2\alpha}{\alpha + 1} P \left( \mu(t)^{\frac{1}{\alpha + 1}} x, \int_0^t \mu(\sigma) \, d\sigma \right). \tag{2.2}
\]

where $\alpha \in (-1, \infty)$ as previously. This means that the space–time variables are transformed from $(x, t) \in \mathbb{R}^3 \times [0, T)$ into $(y, s) \in \mathbb{R}^3 \times [0, \infty)$ as follows:

\[
y = \mu(t)^{\frac{1}{\alpha + 1}} x, \quad s = \int_0^t \mu(\sigma) \, d\sigma. \tag{2.3}
\]
Substituting (2.1)–(2.3) into the Euler equations, we obtain the equivalent equations satisfied by \((V, P)\)

\[
\begin{align*}
(E_\ast) \quad &\left\{ 
\begin{array}{ll}
-\frac{\mu'(t)}{\mu(t)^2} \left[ \frac{\alpha}{\alpha+1} V + \frac{1}{\alpha+1} (y \cdot \nabla)V \right] = V_s + (V \cdot \nabla)V + \nabla P, \\
\text{div } V = 0, \\
V(y, 0) = V_0(y) = \mu(0) \frac{\alpha}{\alpha+1} v_0(\mu(0) \frac{1}{\alpha+1} y).
\end{array}
\right.
\end{align*}
\]

We note that the special cases

\[\mu(t) = \frac{1}{T - t}, \quad \mu(t) = \frac{1}{(T - t)^\gamma}, \quad \gamma > 1,\]

are considered in the previous section. Let us choose \(\mu(t) = \exp[\pm \gamma \int_0^t \| \nabla v(\tau) \|_{L^\infty} \, d\tau], \gamma \geq 1.\) Then,

\[
\begin{align*}
v(x, t) &= \exp \left[ \pm \frac{\gamma \alpha}{\alpha+1} \int_0^t \| \nabla v(\tau) \|_{L^\infty} \, d\tau \right] V(y, s), \quad (2.4) \\
p(x, t) &= \exp \left[ \pm \frac{2 \gamma \alpha}{\alpha+1} \int_0^t \| \nabla v(\tau) \|_{L^\infty} \, d\tau \right] P(y, s) \tag{2.5}
\end{align*}
\]

with

\[
\begin{align*}
y &= \exp \left[ \pm \frac{\gamma}{\alpha+1} \int_0^t \| \nabla v(\tau) \|_{L^\infty} \, d\tau \right] x,
\end{align*}
\]

\[s = \int_0^t \exp \left[ \pm \frac{\gamma}{\alpha+1} \int_0^\tau \| \nabla v(\sigma) \|_{L^\infty} \, d\sigma \right] \, d\tau \tag{2.6}
\]

respectively for the signs \(\pm.\) Substituting \((v, p)\) in (2.4)–(2.6) into the \((E_\ast),\) we find that \((E_\ast)\) becomes

\[
\begin{align*}
(E_{\pm}) \quad &\left\{ 
\begin{array}{ll}
\mp \gamma \| \nabla V(s) \|_{L^\infty} \left[ \frac{\alpha}{\alpha+1} V + \frac{1}{\alpha+1} (y \cdot \nabla)V \right] = V_s + (V \cdot \nabla)V + \nabla P, \\
\text{div } V = 0, \\
V(y, 0) = V_0(y) = v_0(y)
\end{array}
\right.
\end{align*}
\]

respectively for \(\pm.\) Similar equations to the system \((E_{\pm}),\) without the term involving \((y \cdot \nabla)V,\)

are introduced and studied in [4], where similarity type of transform with respect to only time variables was considered. The argument of the global/local well-posedness of the system \((E_{\pm})\)
respectively from the local well-posedness result of the Euler equations is as follows. We define
\[
S^\pm = \int_0^T \exp \left[ \pm \gamma \int_0^\tau \| \nabla v(\sigma) \|_{L^\infty} d\sigma \right] d\tau.
\]

Then, \( S^\pm \) is the maximal time of existence of classical solution for the system \((E^\pm)\). Indeed, by the BKM criterion we have
\[
S^+ = \int_0^T \exp \left[ \gamma \int_0^\tau \| \nabla v(\sigma) \|_{L^\infty} d\sigma \right] d\tau \geq \frac{1}{\| \omega_0 \|_{L^\infty}} \int_0^T \| \omega(\tau) \|_{L^\infty} d\tau = \infty.
\]

We also note the following integral invariant of the transform,
\[
\int_0^T \| \nabla v(t) \|_{L^\infty} dt = \int_0^{S^\pm} \| \nabla V^\pm(s) \|_{L^\infty} ds.
\]
Below we fix \( \mu(t) := \exp[\int_0^t \| \nabla v(\tau) \|_{L^\infty} d\tau] \).

We assume our local classical solution in \( H^m(\mathbb{R}^3) \) blows up at \( T \), and hence \( \mu(T - 0) = \exp[\int_0^T \| \nabla v(\tau) \|_{L^\infty} d\tau] = \infty \). Given \( (\alpha, p) \in (-1, \infty) \times (0, \infty] \), as previously, we say the blow-up is \( \alpha \)-asymptotically self-similar in the sense of \( L^p \) if there exists \( \nabla = \nabla_\alpha \in \dot{W}^{1,p}(\mathbb{R}^3) \) such that the following convergence holds true:
\[
\lim_{t \to T} \mu(t)^{-1} \| \nabla v(\cdot, t) - \nabla (\mu(t)^{\frac{1}{\alpha+1}} \cdot) \|_{L^\infty} = 0 \tag{2.7}
\]
for \( p = \infty \), and
\[
\lim_{t \to T} \mu(t)^{-1+\frac{3}{(\alpha+1)p}} \| \omega(\cdot, t) - \mu(t)^{-\frac{3}{(\alpha+1)p}} \nabla \left( \mu(t)^{\frac{1}{\alpha+1}} \cdot \right) \|_{L^p} = 0 \tag{2.8}
\]
for \( p \in (0, \infty) \). The above limiting function \( \nabla \) with \( \nabla \neq 0 \) is called the blow-up profile as previously.

**Proposition 2.1.** Let \( \alpha \neq 3/2 \). Then there exists no \( \alpha \)-asymptotically self-similar blow-up in the sense of \( L^\infty \) with the blow-up profile belonging to \( L^2(\mathbb{R}^3) \).

**Proof.** Let us suppose that there exists \( \nabla \in \dot{W}^{1,\infty}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \) such that (2.7) holds, then we will show that \( \nabla = 0 \). In terms of the self-similar variables (2.7) is translated into
\[
\lim_{s \to \infty} \| \nabla V(\cdot, s) - \nabla \nabla \|_{L^\infty} = 0,
\]
where \( V \) is defined in (2.1). If \( \| \nabla \nabla \|_{L^\infty} = 0 \), then the condition \( \nabla \in L^2(\mathbb{R}^3) \) implies that \( \nabla = 0 \), and there is nothing to prove. Let us suppose \( \| \nabla \nabla \|_{L^\infty} > 0 \). As is done in the proof of Theorem 1.3 the equations satisfied \( \nabla \) are easily shown to be
\[ \begin{cases} -\|\nabla V\|_{L^\infty} \left[ \frac{\alpha}{\alpha + 1} V + \frac{1}{\alpha + 1} (y \cdot \nabla) V \right] = (V \cdot \nabla) V + \nabla \bar{P}, \\
\text{div} V = 0 \end{cases} \] (2.9)

for a scalar function \( \bar{P} \). Taking \( L^2(\mathbb{R}^3) \) inner product of the first equation of (2.9) by \( V \) we obtain

\[ \|\nabla V\|_{L^\infty} \left( \alpha - \frac{3}{2} \right) \|V\|_{L^2} = 0. \]

Since \( \|\nabla V\|_{L^\infty} \neq 0 \) and \( \alpha \neq \frac{3}{2} \), we have \( \|V\|_{L^2} = 0 \), and \( V = 0 \).

**Proposition 2.2.** There exists no \( \alpha \)-asymptotically self-similar blowing-up solution to \((E)\) in the sense of \( L^p \) if \( 0 < p < \frac{3}{2(\alpha + 1)} \).

**Proof.** Suppose there exists \( \alpha \)-asymptotically self-similar blow-up at \( T \) in the sense of \( L^p \). Then, there exists \( \Omega \in L^p(\mathbb{R}^3) \) such that, in terms of the self-similar variables introduced in (2.1)–(2.2), we have

\[ \lim_{s \to \infty} \|\Omega(s)\|_{L^p} = \|\Omega\|_{L^p} < \infty. \] (2.10)

We represent the \( L^p \) norm of \( \|\omega(t)\|_{L^p} \) in terms of similarity variables to obtain

\[ \|\omega(t)\|_{L^p} = \mu(t)^{1 - \frac{3}{(\alpha + 1)p}} \|\Omega(s)\|_{L^p}, \quad \mu(t) = \exp \left[ \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \right]. \] (2.11)

Substituting this into the lower estimate part of (1.12), we have

\[ \mu(t)^{2 + \frac{3}{(\alpha + 1)p}} \leq \frac{\|\Omega(s)\|_{L^p}}{\|\Omega_0\|_{L^p}}. \] (2.12)

If \( -2 + \frac{3}{(\alpha + 1)p} > 0 \), then taking the limit \( t \to T \) in the above inequality we obtain

\[ \infty = \limsup_{t \to T} \mu(t)^{-2 + \frac{3}{(\alpha + 1)p}} \|\Omega_0\|_{L^p} \leq \limsup_{s \to \infty} \|\Omega(s)\|_{L^p} = \|\bar{\Omega}\|_{L^p}, \]

which is a contradiction to (2.10).

**3. New a priori estimates**

One of the important advantages of the formulation of \((E)\) in terms of \((E_{\pm})\) of the previous section is that after representing it by the vorticity formulation, the convection term is dominated by the linear term associated with \( \mp \nabla \|\nabla V(s)\|_{L^\infty} \) (see (3.7) below), which enables us to derive new a priori estimates for \( \|\omega(t)\|_{L^\infty} \) as follows.
Theorem 3.1. Given $m > 5/2$ and $v_0 \in H^m(\mathbb{R}^3)$ with $\text{div} \, v_0 = 0$, let $\omega$ be the vorticity of the solution $v \in C([0, T); H^m(\mathbb{R}^3))$ to the Euler equations (E). Then we have an upper estimate

$$\|\omega(t)\|_{L^\infty} \leq \frac{\|\omega_0\|_{L^\infty} \exp[\gamma \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau]}{1 + (\gamma - 1)\|\omega_0\|_{L^\infty} \int_0^t \exp[\gamma \int_0^\tau \|\nabla v(\sigma)\|_{L^\infty} d\sigma] d\tau},$$

(3.1)

and lower one

$$\|\omega(t)\|_{L^\infty} \geq \frac{\|\omega_0\|_{L^\infty} \exp[-\gamma \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau]}{1 - (\gamma - 1)\|\omega_0\|_{L^\infty} \int_0^t \exp[-\gamma \int_0^\tau \|\nabla v(\sigma)\|_{L^\infty} d\sigma] d\tau},$$

(3.2)

for all $\gamma \geq 1$ and $t \in [0, T)$. Moreover, the denominator of the right-hand side of (3.2) can be estimated from below as

$$1 - (\gamma - 1)\|\omega_0\|_{L^\infty} \int_0^t \exp[-\gamma \int_0^\tau \|\nabla v(\sigma)\|_{L^\infty} d\sigma] d\tau \geq \frac{1}{(1 + \|\omega_0\|_{L^\infty} T)^{\gamma - 1}},$$

(3.3)

which shows that the finite time blow-up does not follow from (3.2).

Remark 3.1. We observe that for $\gamma = 1$, the estimates (3.1)–(3.2) reduce to the well-known ones in (1.12) with $p = \infty$. In this sense the above estimates seem to be a natural extension from the known ones, but their use is not clear at this point. Moreover, combining (3.1)–(3.2) together, we easily derive another new estimate,

$$\sinh[\gamma \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau] \int_0^t \cosh[\gamma \int_0^\tau \|\nabla v(\sigma)\|_{L^\infty} d\sigma] d\tau \geq (\gamma - 1)\|\omega_0\|_{L^\infty}.$$

(3.4)

Proof of Theorem 3.1. Below we denote $V^\pm$ for the solutions of $(E^\pm)$ respectively, and $\Omega^\pm = \text{curl} \, V^\pm$. Note that $V^\pm_0 = v_0 := V_0$ and $\Omega^\pm_0 = \omega_0 := \Omega_0$. We will first derive the following estimates for the system $(E^\pm)$:

$$\|\Omega^+(s)\|_{L^\infty} \leq \frac{\|\Omega_0\|_{L^\infty}}{1 + (\gamma - 1)s\|\Omega_0\|_{L^\infty}},$$

(3.5)

$$\|\Omega^-(s)\|_{L^\infty} \geq \frac{\|\Omega_0\|_{L^\infty}}{1 - (\gamma - 1)s\|\Omega_0\|_{L^\infty}},$$

(3.6)

as long as $V^\pm(s) \in H^m(\mathbb{R}^3)$. Taking curl of the first equation of $(E^\pm)$, we have

$$\mp \gamma \|\nabla V\|_{L^\infty} \left[ \Omega - \frac{1}{\alpha + 1} (\gamma \cdot \nabla) \Omega \right] = \Omega_s + (V \cdot \nabla) \Omega - (\Omega \cdot \nabla) V.$$
Multiplying \( \Xi = \Omega/|\Omega| \) on the both sides of (3.7), we deduce
\[
|\Omega|_s + (V \cdot \nabla)|\Omega| \equiv \frac{\|\nabla V(s)\|_{L^\infty}}{\alpha + 1} (\nabla \cdot \nabla)|\Omega|
\]
\[
\equiv (\Xi \cdot \nabla V \cdot \Xi \equiv \|\nabla V\|_{L^\infty})|\Omega|
\]
\[
\equiv (\gamma - 1)\|\nabla V\|_{L^\infty}|\Omega|
\]
\[
\leq (\gamma - 1)\|\nabla V\|_{L^\infty}|\Omega| \quad \text{for (E_+),}
\]
\[
\geq (\gamma - 1)\|\nabla V\|_{L^\infty}|\Omega| \quad \text{for (E_-),}
\] (3.8)
since \( |\Xi \cdot \nabla V \cdot \Xi| \leq |\nabla V| \leq \|\nabla V\|_{L^\infty} \). Given smooth solution \( V(y, s) \) of (E_\pm), we introduce the particle trajectories \( \{Y_\pm(a, s)\} \) defined by
\[
\frac{\partial Y(a, s)}{\partial s} = V_\pm(Y(a, s), s) \equiv \frac{\|\nabla V(s)\|_{L^\infty}}{\alpha + 1} Y(a, s); \quad Y(a, 0) = a.
\]
Recalling the estimate
\[
\|\nabla V(s)\|_{L^\infty} \geq \|\Omega(s)\|_{L^\infty} \geq |\Omega(y, s)|, \quad \forall y \in \mathbb{R}^3,
\]
we can further estimate from (3.8)
\[
\frac{\partial}{\partial s}\left|\Omega(Y(a, s), s)\right| \begin{cases} 
\leq -(\gamma - 1)|\Omega(Y(a, s), s)|^2 & \text{for (E_+),} \\
\geq (\gamma - 1)|\Omega(Y(a, s), s)|^2 & \text{for (E_-).}
\end{cases}
\] (3.9)
Solving these differential inequalities (3.9) along the particle trajectories, we obtain that
\[
|\Omega(Y(a, s), s)| \begin{cases} 
\leq \frac{|\Omega_0(a)|}{1+(\gamma - 1)|\Omega_0(a)|} & \text{for (E_+),} \\
\geq \frac{|\Omega_0(a)|}{1-(\gamma - 1)|\Omega_0(a)|} & \text{for (E_-).}
\end{cases}
\] (3.10)
Writing the first inequality of (3.10) as
\[
|\Omega^+(Y(a, s), s)| \leq \frac{1}{\|\Omega_0(a)\|_{L^\infty}^\alpha + (\gamma - 1)s} \leq \frac{1}{\|\Omega_0\|_{L^\infty}^\alpha + (\gamma - 1)s},
\]
and then taking supremum over \( a \in \mathbb{R}^3 \), which is equivalent to taking supremum over \( Y(a, s) \in \mathbb{R}^3 \) due to the fact that the mapping \( a \mapsto Y(a, s) \) is a diffeomorphism (although not volume preserving) on \( \mathbb{R}^3 \) as long as \( V \in C([0, S]; H^m(\mathbb{R}^3)) \), we obtain (3.5). In order to derive (3.6) from the second inequality of (3.10), we first write
\[
\|\Omega^-(s)\|_{L^\infty} \geq |\Omega(Y(a, s), s)| \geq \frac{1}{\|\Omega_0^-(a)\|_{L^\infty}^\alpha - (\gamma - 1)s},
\]
and then take supremum over \( a \in \mathbb{R}^3 \). Finally, in order to obtain (3.1)–(3.2), we just change variables from (3.5)–(3.6) back to the original physical ones, using the fact
Ω⁺(y, s) = \exp \left[ -\gamma \int_0^t \| \nabla v(\tau) \|_{L^\infty} \, d\tau \right] \omega(x, t),

s = \int_0^t \exp \left[ \gamma \int_0^\tau \| \nabla v(\sigma) \|_{L^\infty} \, d\sigma \right] \, d\tau

for (3.1), while in order to deduce (3.2) from (3.6) we substitute

Ω⁻(y, s) = \exp \left[ \gamma \int_0^t \| \nabla v(\tau) \|_{L^\infty} \, d\tau \right] \omega(x, t),

s = \int_0^t \exp \left[ -\gamma \int_0^\tau \| \nabla v(\sigma) \|_{L^\infty} \, d\sigma \right] \, d\tau.

Now we can rewrite (3.2) as

\| \omega(t) \|_{L^\infty} \geq -\frac{1}{\gamma - 1} \frac{d}{dt} \log \left\{ 1 - (\gamma - 1)\| \omega_0 \|_{L^\infty} \int_0^t \exp \left[ -\gamma \int_0^\tau \| \nabla v(\sigma) \|_{L^\infty} \, d\sigma \right] \, d\tau \right\}.

Thus,

\int_0^t \| \nabla v(\tau) \|_{L^\infty} \, d\tau \geq \int_0^t \| \omega(\tau) \|_{L^\infty} \, d\tau

\geq -\frac{1}{\gamma - 1} \log \left\{ 1 - (\gamma - 1)\| \omega_0 \|_{L^\infty} \int_0^t \exp \left[ -\gamma \int_0^\tau \| \nabla v(\sigma) \|_{L^\infty} \, d\sigma \right] \, d\tau \right\}.

(3.11)

Set

y(t) := 1 - (\gamma - 1)\| \omega_0 \|_{L^\infty} \int_0^t \exp \left[ -\gamma \int_0^\tau \| \nabla v(\sigma) \|_{L^\infty} \, d\sigma \right] \, d\tau.

We find further integrable structure in (3.11), which is

y'(t) \geq -(\gamma - 1)\| \omega_0 \|_{L^\infty} y(t)^{\frac{\gamma}{\gamma - 1}}.

Solving this differential inequality, we obtain (3.3). □

Similar method can also be applied to derive new a priori estimates for the 3D Navier–Stokes equations.
Theorem 3.2. Given \( v_0 \in H^1(\mathbb{R}^3) \) with \( \text{div} \, v_0 = 0 \), let \( \omega \) be the vorticity of the classical solution \( v \in C([0, T); H^1(\mathbb{R}^3)) \cap C((0, T); C^\infty(\mathbb{R}^3)) \) to the Navier–Stokes equations (NS). Then, there exists an absolute constant \( C_0 > 1 \) such that for all \( \gamma \geq C_0 \) the following enstrophy estimate holds true:

\[
\| \omega(t) \|_{L^2} \leq \frac{\| \omega_0 \|_{L^2} \exp[\frac{\gamma}{4} \int_0^t \| \omega(\tau) \|_{L^2}^4 d\tau]}{\{1 + (\gamma - C_0) \| \omega_0 \|_{L^2}^4 \int_0^t \exp[\gamma \int_0^{\tau} \| \omega(\sigma) \|_{L^2}^4 d\sigma] d\tau\}^{\frac{1}{4}}}. \tag{3.12}
\]

The denominator of (3.12) is estimated from below by

\[
1 + (\gamma - C_0) \| \omega_0 \|_{L^2}^4 \int_0^t \exp[\gamma \int_0^\tau \| \omega(\sigma) \|_{L^2}^4 d\sigma] d\tau \leq \frac{1}{(1 - C_0 \| \omega_0 \|_{L^2}^4 t)^{\frac{\gamma - C_0}{C_0}}} \quad (3.13)
\]

for all \( \gamma \geq C_0 \).

Proof. Let \((v, p)\) be a classical solution of the Navier–Stokes equations, and \( \omega \) be its vorticity. We transform from \((v, p)\) to \((V, P)\) according to the formula, given by (2.1)–(2.3), where

\[
\mu(t) = \exp[\gamma \int_0^t \| \omega(\tau) \|_{L^2}^4 d\tau].
\]

Substituting (2.1)–(2.3) with such \( \mu(t) \) into (NS), we obtain the equivalent equations satisfied by \((V, P)\)

\[
\begin{align*}
\text{(NS)} \quad & -\gamma \| \Omega(s) \|_{L^2}^4 \frac{1}{2}[V + (y \cdot \nabla)V] = V_s + (V \cdot \nabla)V - \Delta V - \nabla P, \\
\text{div} V &= 0, \\
V(y, 0) &= V_0(y) = v_0(y).
\end{align*}
\]

Operating curl on the evolution equations of \((\text{NS})\), we obtain

\[
-\gamma \| \Omega(s) \|_{L^2}^4 \frac{1}{2}[2\Omega + (y \cdot \nabla)\Omega] = \Omega_s + (V \cdot \nabla)\Omega - (\Omega \cdot \nabla)V - \Delta \Omega. \tag{3.14}
\]

Taking \( L^2(\mathbb{R}^3) \) inner product of (3.14) by \( \Omega \), and integrating by part, we estimate

\[
\frac{1}{2} \frac{d}{ds} \| \Omega \|_{L^2}^2 + \| \nabla \Omega \|_{L^2}^2 + \frac{\gamma}{4} \| \Omega \|_{L^6}^6 = \int_{\mathbb{R}^3} (\Omega \cdot \nabla)V \cdot \Omega \, dy \\
\leq \| \Omega \|_{L^3} \| \nabla V \|_{L^2} \| \Omega \|_{L^6} \leq C \| \Omega \|_{L^2}^3 \| \nabla \Omega \|_{L^2}^3 \\
\leq \| \nabla \Omega \|_{L^2}^2 + \frac{C_0}{4} \| \Omega \|_{L^2}^6 \tag{3.15}
\]
for an absolute constant $C_0 > 1$, where we used the fact $\| \Omega \|_{L^2} = \| \nabla V \|_{L^2}$, the Sobolev imbedding, $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, the Gagliardo–Nirenberg inequality in $\mathbb{R}^3$, 

$$
\| f \|_{L^3} \leq C \| f \|_{L^2}^{\frac{1}{2}} \| \nabla f \|_{L^2}^{\frac{1}{2}};
$$

and Young’s inequality of the form $ab \leq a^p/p + b^q/q$, $1/p + 1/q = 1$. Absorbing the term $\| \nabla \Omega \|_{L^2}^2$ to the left-hand side, we have from (3.15)

$$
\frac{d}{ds} \| \Omega \|_{L^2}^2 \leq -\frac{\gamma - C_0}{2} \| \Omega \|_{L^2}^6.
$$

(3.16)

Solving the differential inequality (3.16), we have

$$
\| \Omega(s) \|_{L^2} \leq \frac{\| \Omega_0 \|_{L^2}}{[1 + (\gamma - C_0)s \| \Omega_0 \|_{L^2}^4]^{\frac{1}{2}}},
$$

(3.17)

Transforming back to the original variables and functions, using the relations

$$
s = \int_0^t \exp \left[ \int_0^\tau \| \omega(\sigma) \|_{L^2}^4 \ d\sigma \right] d\tau,
$$

$$
\| \omega(t) \|_{L^2} = \| \Omega(s) \|_{L^2} \exp \left[ \frac{\gamma}{4} \int_0^t \| \omega(\sigma) \|_{L^2}^4 \ d\sigma \right],
$$

we obtain (3.12). Next, we observe (3.12) can be written as

$$
\| \omega(t) \|_{L^2}^4 \leq \frac{1}{(\gamma - C_0)} \frac{d}{dt} \log \left\{ 1 + (\gamma - C_0) \| \omega_0 \|_{L^2}^4 \int_0^t \exp \left[ \gamma \int_0^\tau \| \omega(\sigma) \|_{L^2}^4 \ d\sigma \right] d\tau \right\},
$$

which, after integration over $[0, t]$, leads to

$$
\int_0^t \| \omega(\tau) \|_{L^2}^4 \ d\tau \leq \frac{1}{(\gamma - C_0)} \log \left\{ 1 + (\gamma - C_0) \| \omega_0 \|_{L^2}^4 \int_0^t \exp \left[ \gamma \int_0^\tau \| \omega(\sigma) \|_{L^2}^4 \ d\sigma \right] d\tau \right\}
$$

(3.18)

for all $\gamma > C_0$. Setting

$$
y(t) := 1 + (\gamma - C_0) \| \omega_0 \|_{L^2}^4 \int_0^t \exp \left[ \gamma \int_0^\tau \| \omega(\sigma) \|_{L^2}^4 \ d\sigma \right] d\tau,
$$
we find that (3.18) can be written in the form of a differential inequality,

\[ y'(t) \leq (\gamma - C_0) \left\| \omega_0 \right\|_{L^2}^4 \frac{\gamma}{y(t)^{\gamma-C_0}}, \]

which can be integrated to provide us with (3.13). \( \square \)

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**References**


