A power of an entire function sharing one value with its derivative

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ABSTRACT

In this paper, we investigate uniqueness problems of entire functions that share one value with one of their derivatives. Let f be a non-constant entire function, n and k be positive integers. If $f^n$ and $(f^n)'$ share 1 CM and $n \geq k + 1$, then $f^n = (f^n)'$, and f assumes the form $f(z) = ce^{\lambda z}$, where c is a non-zero constant and $\lambda^k = 1$. This result shows that a conjecture given by Brück is true when $F = f^n$, where $n \geq 2$ is an integer.

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1. Introduction

In what follows, a meromorphic (resp. entire) function always means a function which is meromorphic (resp. analytic) in the whole complex plane. We will use the standard notation in Nevanlinna’s value distribution theory of meromorphic functions; see, e.g. [1].

We say that two meromorphic functions f and g share $\alpha \in \mathbb{C}$ IM (ignoring multiplicities) when $f - \alpha$ and $g - \alpha$ have the same zeros. If $f - \alpha$ and $g - \alpha$ have the same zeros with the same multiplicities, then we say that f and g share $\alpha$ CM (counting multiplicities). Let m and p be positive integers. We denote by $N_p(r, 1/(f - \alpha))$ the counting function of the zeros of $f - \alpha$ where m-fold zeros are counted m times if $m \leq p$ and p times if $m > p$.

Recently, a widely studied subtopic of the uniqueness theory has been the consideration of shared value problems relative to a meromorphic function $F$ and its kth derivative $F^{(k)}$. In order to get the uniqueness of sharing one value of $F$ and $F^{(k)}$, some deficient assumption is needed. The reader is invited to see the recent papers [2–7].

The purpose of this paper is to study a power of an entire function sharing one value with its derivative. We will give some results concerning Brück’s Conjecture, which is mentioned later.

Let f be a non-constant entire function and n be a positive integer. If $f^n$ and $(f^n)'$ share 1 CM, then there exists an entire function $\alpha$ such that

$$\frac{(f^n)' - 1}{f^n - 1} = e^{\alpha}.$$  

Rewriting above equation, we have

$$g_1 + g_2 + g_3 = 1,$$  

where $g_1 = (f^n)'$, $g_2 = -e^{\alpha}f^n$, $g_3 = e^{\alpha}$.

There are many results on a combination of three meromorphic functions

$$f_1 + f_2 + f_3 = 1$$  

in uniqueness theory. The following result is a useful one. As for the proof; see, e.g. [8].

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Theorem 1.1. Let \( f_j \) (\( j = 1, 2, 3 \)) be meromorphic functions satisfying (1.2). If \( f_j \) is not a constant, and
\[
\sum_{j=1}^{3} N_2(r, 1/f_j) + \sum_{j=1}^{3} N(r, f_j) < (\lambda + o(1)) T(r), \quad r \in I,
\]
where \( \lambda < 1 \), \( T(r) = \max\{T(r, f_1), T(r, f_2), T(r, f_3)\} \), \( I \) denotes a set of \( r \in (0, \infty) \) with infinite linear measure. Then either \( f_2 = 1 \) or \( f_3 = 1 \).

Applying Theorem 1.1 on (1.1), the present authors [8] got:

Theorem 1.2. Let \( f \) be a non-constant entire function, \( n \geq 7 \) be an integer. If \( f^n \) and \((f^n)\)' share 1 CM and, then \( f^n = (f^n)' \), and \( f \) assumes the form
\[
f(z) = ce^{\lambda z},
\]
where \( c \) is a non-zero constant.

It is natural to ask whether \( n \) can be reduced in Theorem 1.2. In fact, there are much more relations between \( g_i \) in (1.1) than \( f_i \) in (1.2). By studying this, we give a result improving Theorem 1.2 in Section 2. In Section 3, we consider a power of an entire function sharing 1 IM with its derivative. We provide some concluding remarks in Section 4.

2. Sharing 1 CM

In order to get a general result, we consider \( f^n \) sharing 1 CM with its \( k \)th derivative, where \( k \) is a positive integer, and obtain the following result:

Theorem 2.1. Let \( f \) be a non-constant entire function, \( n \) and \( k \) be positive integers. If \( f^n \) and \((f^n)^k \) share 1 CM and \( n \geq k + 1 \), then \( f^n = (f^n)^k \), and \( f \) assumes the form
\[
f(z) = ce^{\lambda z},
\]
where \( c \) is a non-zero constant and \( \lambda^k = 1 \).

In order to prove Theorem 2.1, we need the following lemma.

Lemma 2.2 ([1, Lemma 4.3]). Let \( f \) be a non-constant meromorphic function and \( P(f) \) be a polynomial in \( f \) with constant coefficients. Let \( b_j \) (\( j = 1, \ldots, q \)) be distinct finite values. If \( q > \deg P \), then
\[
m\left( r, \frac{P(f)f'}{(f - b_1)(f - b_2) \cdots (f - b_q)} \right) = S(r, f).
\]

We begin to prove Theorem 2.1:

Proof. Denote
\[
F = f^n.
\]
Since \( F \) and \((F)^k \) share 1 CM, then there exists an entire function \( \alpha \), such that
\[
F^{(k)} - 1 = e^{\alpha}(F - 1).
\]
Suppose first that \( e^{\alpha} \) is a non-constant entire function. By differentiation, we have
\[
F^{(k+1)} = \alpha' e^{\alpha}(F - 1) + e^{\alpha} F'.
\]
Combining (2.3) with (2.4) yields
\[
F^{(k+1)} - \alpha' F^{(k)} F - F^{(k)} F' = F^{(k+1)} - \alpha' (F^{(k)} + F) - F' + \alpha'.
\]
By induction, we deduce from (2.2) that
\[
F^{(k)} = \sum_{\lambda} c_\lambda f^{(\lambda)} f^{(\lambda)} \cdots f^{(\lambda)}\frac{F}{k},
\]
where \( l_0, \ldots, l_k \) are non-negative integers satisfying \( \sum_{j=0}^{k} l_j = n, n - k \leq l_0 \leq n - 1 \) and \( c_\lambda \) are constants.
Substituting (2.2) and (2.6) into (2.5), we have
\[ f^n \cdot f^{n-k-1} P = Q, \quad (2.7) \]
where \( Q \) is a differential polynomial in \( f \) of the degree \( n \), \( P \) is a differential polynomial in \( f \) of the degree \( k + 1 \) and the coefficients of \( P \) are the polynomials in \( \alpha' \). In particular, every monomial of \( P \) has the form
\[ R(\alpha' f^{l_0} (f')^{l_1} \cdots (f^{(k+1)})^{l_{k+1}}), \]
where \( l_0, \ldots, l_{k+1} \) are non-negative integers satisfying \( \sum_{j=0}^{k+1} l_j = k + 1 \) and \( l_0 \leq k \) (since \( n \geq k + 1 \)). \( R(\alpha') \) is a polynomial in \( \alpha' \) with constant coefficients. From this and logarithmic derivative lemma, we obtain
\[ m\left( r, \frac{p}{f f'} \right) = S(r, f). \quad (2.8) \]

If \( P \neq 0 \), we get from (2.7) and Clunie lemma (for the proof, see, e.g. [9, Chapter 2.4]) that
\[ T(\alpha f^n, f^n f^{n-k-1}) = m(r, f^n f^{n-k-1}) = S(r, f). \]
Combining this with (2.8), we have
\[
m\left( r, \frac{1}{f^{n-k-1}} \right) \leq m\left( r, \frac{f^{n-k-1} P}{f^n f^{n-k-1}} \right) + m\left( r, \frac{1}{f^{n-k-1}} \right) \\
\leq m\left( r, \frac{p}{f f'} \right) + T(\alpha f^n, f^n f^{n-k-1}) + O(1) \\
= S(r, f).
\]
From the above inequality and Lemma 2.2, we have
\[
m\left( r, \frac{1}{f^n} \right) = \frac{1}{n} m\left( r, \frac{1}{f^n} \right) \\
\leq \frac{1}{n} m\left( r, \frac{f^{n-1} f'}{f^n} \right) + \frac{1}{n} m\left( r, \frac{1}{f^{n-1} f'} \right) \\
\leq S(r, f)
\]
and
\[
m\left( r, \frac{1}{f^n - 1} \right) \leq m\left( r, \frac{f^{n-1} f'}{f^n - 1} \right) + m\left( r, \frac{1}{f^n - 1} \right) = S(r, f).
\]
From (2.3) and (2.10), we get
\[ m(r, e^{\alpha}) \leq m\left( r, \frac{F(k)}{F - 1} \right) + m\left( r, \frac{1}{f^n - 1} \right) + O(1) \leq S(r, f), \]
which means that \( T(r, e^{\alpha}) = S(r, f) \).

Rewriting (2.3), yields
\[ e^{\alpha} - 1 = \frac{F(k) - F}{F - 1} = \frac{f^{n-k}(P_k(f) + f^k)}{f^n - 1}, \]
where \( P_k(f) \) is a differential polynomial in \( f \). Noting that \( n \geq k + 1 \), we get
\[ N\left( r, \frac{1}{e^{\alpha} - 1} \right) \leq N\left( r, \frac{1}{e^{\alpha} - 1} \right) \leq T(r, e^{\alpha}) + O(1) = S(r, f). \]
Combining this with (2.9), we obtain
\[ T(r, f) = T\left( r, \frac{1}{f^n} \right) + O(1) = m\left( r, \frac{1}{f^n} \right) + N\left( r, \frac{1}{f^n} \right) + O(1) = S(r, f), \]
which is a contradiction. Hence \( P = 0 \). Then \( Q = 0 \) from (2.7), where \( Q = F^{(k+1)} - \alpha' (F^{(k)} - F') - F' + \alpha' \). We get from (2.5) that \( F^{(k+1)} F - \alpha' F^{(k)} F - F^{(k)} F' = 0 \). If \( F \) is a polynomial, then \( F - 1 \) and \( F(k) - 1 \) cannot have the same zeros with the same multiplicities. Thus \( FF^{(k)} \neq 0 \). Therefore
\[ \frac{F^{(k+1)}}{F^{(k)}} = \alpha' + \frac{F'}{F}. \]
By integration, we have $F^{(k)} = dFe^z$, where $d$ is a non-zero constant. Substituting this and (2.2) into (2.3), we have

$$(d - 1)f^n = \frac{1 - e^z}{e^z}.$$ 

Obviously, $d \neq 1$ and all zeros of $1 - e^z$ have the multiplicities at least $n$. Noting that $n \geq 2$, we get from the second fundamental theorem that

$$T(r, e^z) \leq N(r, e^z) + \frac{1}{n} N'(r, e^z) + S(r, e^z)$$

which is a contradiction since we suppose first that $e^z$ is a non-constant entire function.

Suppose then that $e^z$ is a non-zero constant. Say $A$. From (2.3), we have

$$F^{(k)} - AF = 1 - A. \tag{2.11}$$

If $A \neq 1$, we claim that 0 is a Picard exceptional value of $f$. Otherwise, suppose that $z_0$ is a zero of $f$ of the multiplicity $p$. Noting that $n \geq k + 1$, $z_0$ is zero of $F^{(k)}$ of the multiplicity $np - k$. Then we get $A = 1$ from (2.11), which is a contradiction. We may assume that $f = e^\beta$, where $\beta$ is a non-constant entire function. Substituting this into (2.11), we obtain

$$(P(\beta') - A)e^{\beta} = 1 - A,$$

where $P(\beta')$ is a differential polynomial in $\beta'$. Obviously, $P(\beta') \neq 1$. Then $nT(r, e^\beta) = T(r, e^{\beta'}) = T(r, (1 - A)/(P(\beta') - A)) = T(r, P(\beta')) + O(1) = S(r, e^\beta)$ from the above equation, which contradicts with the fact that $\beta$ is a non-constant entire function.

Hence $A = 1$. Therefore, $F = F^{(k)}$ from (2.11). By the same arguments as above, we have that 0 is a Picard exceptional value of $f$, then $f$ assumes the form

$$f(z) = ce^{\frac{z}{\lambda}}$$

where $c$ is a non-zero constant and $\lambda^k = 1$. \qed

For the special case $k = 1$, we have the following corollary improving Theorem 1.2:

**Corollary 2.3.** Let $f$ be a non-constant entire function, $n (\geq 2)$ be an integer. If $f^n$ and $(f^n)'$ share $1$ CM, then $f^n = (f^n)'$, and $f$ assumes the form

$$f(z) = ce^{\frac{z}{\lambda^k}}, \tag{2.12}$$

where $c$ is a non-zero constant.

**Example 2.4.** Let $f$ be a non-constant solution of

$$\frac{f' - 1}{f - 1} = e^z.$$ 

Then $f$ and $f'$ share 1 CM, while $f \neq f'$. This example shows that the assumption $n \geq 2$ in Corollary 2.3 is sharp.

3. Sharing 1 IM

In this section, we consider a power of an entire function sharing 1 IM with its $k$th derivative:

**Theorem 3.1.** Suppose that $f$ is an entire function, $n$ and $k$ are positive integers satisfying $n \geq k + 2$. If $f^n$ and $(f^n)^{(k)}$ share $1$ IM, then $f^n = (f^n)^{(k)}$, and $f$ assumes the form (2.1).

**Proof.** Suppose that $F \neq F^{(k)}$. From the second fundamental theorem, we have

$$T(r, F) \leq \frac{1}{N}(r, 1/F) + \frac{1}{N}(r, 1/(F - 1)) + S(r, F)$$

$$\leq \frac{1}{N}(r, 1/f) + \frac{1}{N}(r, 1/(F^{(k)}/F - 1)) + S(r, F)$$

$$\leq \frac{1}{N}(r, 1/f) + T(r, F^{(k)}/F) + S(r, F)$$

$$= \frac{1}{N}(r, 1/f) + N(r, F^{(k)}/F) + S(r, F)$$
which contradicts with \( n \geq k + 2 \). Thus \( F = F^{(k)} \). Using the same way as in the proof of Theorem 2.1, we get that \( f \) assumes the form (2.1). \( \Box \)

Comparing Theorem 2.1 with Theorem 3.1, we give an open problem as follows:

**Question 1.** What happens if \( n \geq k + 2 \) is replaced by \( n \geq k + 1 \) in Theorem 3.1?

In this paper, we give an answer to Question 1 when \( k = 1 \) by the following result, which also improves Corollary 2.3.

**Theorem 3.2.** Let \( f \) be a non-constant entire function, \( n(\geq 2) \) be an integer. If \( f^n \) and \((f^n)'\) share 1 IM, then \( f^n = (f^n)' \), and \( f \) assumes the form (2.12).

**Proof.** Let \( F \) be given by (2.2). Since \( F \) and \( F' \) share 1 IM, we know that all zeros of \( F - 1 \) are simple zeros. Suppose that \( F \neq F' \). Denote

\[
H := \frac{F'(F' - F)}{F(F - 1)} = \frac{n f^{n-2} f'(n f' - f)}{f^n - 1}.
\]  (3.1)

Then \( H \) is an entire function and

\[
T(r, H) = m(r, H) = m \left( r, \frac{F'}{F - 1} \left( \frac{F'}{F} - 1 \right) \right)
\leq m \left( r, \frac{F'}{F - 1} \right) + m \left( r, \frac{F'}{F} \right) + O(1)
= S(r, f).
\]  (3.2)

Rewriting (3.1) gives

\[
F'^2 - FF = H(F^2 - F).
\]

Differentiating twice, we obtain

\[
2F'F'' - F'^2 - FF'' = H'(F^2 - F) + H(2FF' - F')
\]  (3.3)

and

\[
2F'^2 + 2F' F''' - 3F' F'' - FF''' = H''(F^2 - F) + 2H'(2FF' - F') + H(2F'^2 + 2FF'' - F'').
\]  (3.4)

Let \( z_1 \) be a zero of \( F - 1 \). Then \( F(z_1) = F'(z_1) = 1 \). From (3.3) and (3.4) we have

\[
F''(z_1) = H(z_1) + 1,
\]

\[
F'''(z_1) = 2H'(z_1) - H^2(z_1) + 2H(z_1) + 1.
\]

Set

\[
\phi = \frac{F'' - (H + 1)F'}{F - 1},
\]  (3.5)

\[
\psi = \frac{F''' - (2H' - H^2 + 2H + 1)F'}{F - 1}.
\]  (3.6)

Then \( \phi \) and \( \psi \) are entire functions since all zeros of \( F - 1 \) are simple. Hence, we get from (3.2) that

\[
T(r, \phi) = m(r, \phi)
\leq m \left( r, \frac{F''}{F - 1} \right) + m \left( r, \frac{F'}{F - 1} \right) + m(r, H) + O(1)
= S(r, f).
\]

Similarly,

\[
T(r, \psi) = S(r, f).
\]

From (3.5), we obtain

\[
F'' = (H + 1)F' + \phi(F - 1).
\]  (3.7)
Differentiating the above equation gives
\[ F''' = H'F' + (H + 1)F' + \phi'(F - 1) + \phi F'. \]  
(3.8)

Combining (3.7), (3.8) and (3.6) yields
\[ F'(2H^2 - H' + \phi) = (F - 1)\left(\psi - \phi' - (1 - H)\phi\right). \]

Namely,
\[ n\phi^-f'(2H^2 - H' + \phi) = (f^n - 1)\left(\psi - \phi' - (1 - H)\phi\right). \]

If \(2H^2 - H' + \phi \neq 0\), the last two equations imply
\[ N\left(r, \frac{1}{2H^2 - H' + \phi}\right) \leq N\left(r, \frac{1}{H} \right) = S(r, f), \]
\[ N\left(r, \frac{1}{H} \right) \leq N\left(r, \frac{1}{\psi - \phi' - (1 - H)\phi}\right) = S(r, f). \]

By the second fundamental theorem, we obtain
\[ T(r, f^n) \leq \bar{N}\left(r, \frac{1}{f^n - 1}\right) + \bar{N}\left(r, \frac{1}{H}\right) + S(r, f) = S(r, f), \]
which is a contradiction. Therefore
\[ 2H^2 - H' + \phi = 0. \]  
(3.9)

Let \(z_0\) be a zero of \(f\). Then \(F(z_0) = F'(z_0) = 0\) since \(n \geq 2\). Substituting this into (3.4) and (3.5), we get
\[ F''(z_0)(2F''(z_0) + H(z_0)) = 0, \]
(3.10)
\[ F''(z_0) = -\phi(z_0). \]  
(3.11)

We claim that
\[ 2F''(z_0) = -H(z_0). \]  
(3.12)

In fact, if \(n \geq 3\), then \(F''(z_0) = 0\). Furthermore, we get from (3.1) that \(H(z_0) = 0\). Hence (3.12) holds. If \(n = 2\), then \(F''(z_0) = 2f''(z_0) + 2f(z_0)f''(z_0) = 2f''^2(z_0)\). If \(F''(z_0) = 0\), then \(f'(z_0) = 0\). From (3.1), we get \(H(z_0) = 0\), and so (3.12) holds. If \(F''(z_0) \neq 0\), (3.12) comes immediately from (3.10).

Substituting (3.11) and (3.12) into (3.9), we obtain
\[ 2H^2(z_0) + \frac{1}{2}H(z_0) - H'(z_0) = 0. \]  
(3.13)

If \(2H^2 + \frac{1}{2}H - H' \neq 0\), we get from (3.2) and (3.13) that
\[ \bar{N}\left(r, \frac{1}{H}\right) \leq N\left(r, \frac{1}{2H^2 + \frac{1}{2}H - H'}\right) = S(r, f). \]

Noting that
\[ N\left(r, \frac{1}{F - 1}\right) \leq N\left(r, \frac{1}{F'} - 1\right) \leq T\left(r, \frac{F'}{F}\right) + O(1) \]
\[ = N\left(r, \frac{F'}{F}\right) + m\left(r, \frac{F'}{F}\right) + O(1) \]
\[ \leq \bar{N}\left(r, \frac{1}{F}\right) + S(r, f) \]
\[ = S(r, f). \]

By the second fundamental theorem, we have a contradiction. Hence
\[ 2H^2 + \frac{1}{2}H - H' = 0. \]  
(3.14)

Since \(F \neq F'\), we have \(H \neq 0\). Then
\[ 2H = H' - \frac{1}{2}. \]
Noting that $H$ is an entire function, we have
\[ T(r, H) = m(r, H) \leq m\left(r, \frac{H'}{H}\right) + O(1) = S(r, H), \]
which means that $H$ is a constant. From (3.14), we know that $H = -\frac{1}{4}$. From (3.1), we obtain
\[ (2F' - F)^2 = F. \]
Set
\[ \gamma = 2F' - F \quad \text{or} \quad -\gamma = 2F' - F. \]
Then $F = \gamma^2$ and $F' = 2\gamma \gamma'$. Thus $4\gamma' = \gamma + 1$ or $4\gamma' = \gamma - 1$. If $4\gamma' = \gamma + 1$, by integration,
\[ \gamma = Ae^{\frac{1}{2}z} - 1, \]
where $A$ is a non-zero constant. Let $z^* = 4\pi i - 4 \log A$. Then $\gamma(z^*) = -2$ and $\gamma'(z^*) = -\frac{1}{4}$. Thus $F'(z^*) = 1$ and $F(z^*) = 4$, which contradicts with $F$ and $F'$ sharing $1$ IM. If $4\gamma' = \gamma - 1$, by integration, $\gamma = Be^{\frac{1}{2}z} + 1$, where $B$ is a non-zero constant. Let $z^* = -4 \log A$. We obtain a contradiction by the same reasoning. Therefore, $F = F'$, and there exists a non-zero constant $c$ such that $f = ce^{\frac{1}{2}z}$. This completes the proof of Theorem 3.2. □

4. Concluding remarks

Now, we introduce the definition of weighted sharing: let $l$ be a non-negative integer or infinite. For any $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_l(a, f)$ the set of all $a$-points of $f$ where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq l$ and $l + 1$ times if $m > l$. If $E_l(a, f) = E_l(a, g)$, we say that $f$ and $g$ share the value $a$ with weight $l$ (see [5]).

We say that $f$ and $g$ share $(a, l)$ if $f$ and $g$ share the value $a$ with weight $l$. It is easy to see that $f$ and $g$ share $(a, l)$ implies $f$ and $g$ share $(a, p)$ for $0 \leq p \leq l$. Also we note that $f$ and $g$ share a value $a$ IM or CM if and only if $f$ and $g$ share $(a, 0)$ or $(a, \infty)$ respectively.

We recall the following result which is a corollary of Theorem 1.1 in [10]:

**Theorem 4.1.** Let $f$ be a non-constant entire function and $k$ be a positive integer. Suppose that $F$ and $F^{(k)}$ share (1, 2). If
\[ \delta_2(0, F) + \delta_{2+k}(0, F) > 1, \]
where $\delta_p(0, F) = 1 - \limsup_{r \to \infty} \frac{N_p(r, 1/F)}{T(r, F)}$, then $F = F^{(k)}$.

If $F = f^n$, where $f$ is a non-constant entire function and $n$ is a positive integer. Then
\[ \delta_p(0, F) = 1 - \limsup_{r \to \infty} \frac{N_p(r, 1/F)}{T(r, F)} \geq 1 - \frac{p}{n}. \]

Noting this, from Theorem 4.1 we have the following corollary:

**Corollary 4.2.** Let $f$ be a non-constant entire function and $n$, $k$ be two positive integers. Denote $F = f^n$. Suppose that $F$ and $F^{(k)}$ share 1 CM. If $n \geq k + 5$, then $F = F^{(k)}$.

Obviously, Theorem 2.1 improves Corollary 4.2.

Without deficient assumption, Brück [11] proposed the following conjecture:

**Brück Conjecture 1.** Let $F$ be a non-constant entire function. Suppose that
\[ \rho_2(F) := \limsup_{r \to \infty} \frac{\log^+ \log^+ T(r, F)}{\log r} \]
is not a positive integer or infinite. If $F$ and $F'$ share a finite value $b$ CM, then
\[ \frac{F' - b}{F - b} = c \]
for some non-zero constant $c$.

The conjecture has been verified in special cases only: (1) $\rho_2(F) < \frac{1}{2}$, see [12]; (2) $b = 0$, see [11]; (3) $N(r, 1/F') = S(r, F)$, see [11].

In this paper, Corollary 2.3 tells us Conjecture 1 is true when $F = f^n$, where $n \geq 2$ is an integer.
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