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Meromorphic functions that share a set with their derivatives

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Abstract

There exists a set S with three elements such that if a meromorphic function f, having at most finitely many simple poles, shares the set S CM with its derivative f', then $f' \equiv f$.

Keywords: Meromorphic functions; Shared values

1. Introduction

For f a nonconstant entire function in the plane domain D and S a set of complex numbers, let

$$E_D(S, f) = \bigcup_{a\in S}\{z \in D:\ f(z) - a = 0\},$$

where zero of multiplicity m is counted m times in the set $E_D(S, f)$. When $D = \mathbb{C}$, we simply write $E(S, f)$.

In [2], Fang and Zalcman proved

Theorem A. *There exists a finite set S containing 3 elements such that if f is a nonconstant entire function and $E(S, f) = E(S, f')$, then $f \equiv f'$.*

It is natural to ask whether Theorem A remains valid for meromorphic functions. In this paper, we prove the following generalization of Theorem A.

Theorem 1. *There exists a set S with three elements such that if f is a meromorphic function f with at most finitely many simple poles and $E(S, f) = E(S, f')$, then $f' \equiv f$.*

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Theorem 1 follows from the following more precise result.

Theorem 2. *Let f be a nonconstant meromorphic function with at most finitely many simple poles; and let $S = \{0, a, b\}$, where a and b are distinct nonzero complex numbers. If f and its derivative f' satisfy $E(S, f) = E(S, f')$, then either*

(i) $f(z) = Ce^{z}$; *or*
(ii) $f(z) = Ce^{-z} + \frac{2}{3}(a + b)$ *and either* $a + b = 0$ *or* $2a^2 - 5ab + 2b^2 = 0$; *or*
(iii) $f(z) = Ce^{\frac{-1\pm i\sqrt{3}}{2}z} + \frac{3\pm i\sqrt{3}}{6}(a + b)$ *and* $a^2 - ab + b^2 = 0$,

where C is a nonzero constant.

Throughout this paper, we use the standard notions and notation of Nevanlinna theory [3,6]. In particular, the spherical derivative of a meromorphic function f is given by

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2},$$

and the order of f is defined by

$$\rho = \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r}.$$

2. Auxiliary results

Lemma 1. *Let f be a meromorphic function on $\mathbb{C}$. If f has bounded spherical derivative on $\mathbb{C}$, f is of order at most* 2. *If, in addition, f is entire, then the order of f is at most* 1.

Remark. The first part of the lemma follows from the formula for the Ahlfors–Shimizu characteristic

$$T_0(r, f) = \int_0^r \frac{1}{t}\left(\frac{1}{\pi}\iint_{|z|\leqslant t} \left[f^{\#}(z)\right]^2 dx\, dy\right) dt$$

and the fact that $T(r, f)$ and $T_0(r, f)$ differ by a bounded quantity (independent of r). The result for entire functions is more subtle; it is a special case of Theorem 3 in [1].

It is not difficult to extend Lemma 1 as follows.

Lemma 2. *Let f be a meromorphic function on $\mathbb{C}$ with finitely many poles. If f has bounded spherical derivative on $\mathbb{C}$, f is of order at most* 1.

Recently, using Zalcman's Lemma [5] (cf. [7]), Liu and Pang obtained the following normality criterion [4].

Lemma 3. *(See [4].) Let $\mathcal{F}$ be a family of functions meromorphic on the unit disk Δ. If there exists a set S with three elements such that $E_\Delta(S, f) = E_\Delta(f', S)$ for every $f \in \mathcal{F}$, then $\mathcal{F}$ is normal on Δ.*

As an almost immediate consequence, we have

Lemma 4. *Let f be a function meromorphic on $\mathbb{C}$. If there exists a set S with three elements such that $E(S, f) = E(S, f')$, then $f^{\#}(z)$ is bounded on $\mathbb{C}$.*

Proof. Set $\mathcal{F} = \{f_w\colon w \in \mathbb{C}\}$, where $f_w(z) = f(z + w)$. By Lemma 3, $\mathcal{F}$ is normal on Δ; so by Marty's Theorem, $f^{\#}(w) = f_w^{\#}(0) \leqslant M$ for some $M > 0$ and all $w \in \mathbb{C}$. □

Lemma 5. *(See [3, p. 56].) Let f be a meromorphic function of finite order on the plane $\mathbb{C}$. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r)$$

for each positive integer k.

Lemma 6. *Let f be a nonconstant meromorphic function of finite order, and let A, B be two constants satisfying $A^2 - 4B \neq 0$ and $B \neq 0$. Then*

$$m\left(r, \frac{(f')^3 + A(f')^2 + Bf'}{f^3 + Af^2 + Bf}\right) = O(\log r).$$

Proof. Since $A^2 - 4B \neq 0$ and $B \neq 0$, we have

$$f^3 + Af^2 + Bf = f(f+\alpha)(f+\beta),$$

where α and β are two distinct nonzero constants. Then

$$\begin{aligned}\frac{1}{f^3 + Af^2 + Bf} &= \frac{1}{f(f+\alpha)(f+\beta)}\\ &= \frac{1}{\beta-\alpha}\left(\frac{1}{f(f+\alpha)} - \frac{1}{f(f+\beta)}\right)\\ &= \frac{1}{\alpha\beta}\cdot\frac{1}{f} - \frac{1}{\alpha(\beta-\alpha)}\cdot\frac{1}{f+\alpha} + \frac{1}{\beta(\beta-\alpha)}\cdot\frac{1}{f+\beta}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{(f')^3 + A(f')^2 + Bf'}{f^3 + Af^2 + Bf} &= \frac{(f')^3}{f(f+\alpha)(f+\beta)} + \frac{A}{\beta-\alpha}\left(\frac{(f')^2}{f(f+\alpha)} - \frac{(f')^2}{f(f+\beta)}\right) + \frac{B}{\alpha\beta}\cdot\frac{f'}{f}\\ &\quad - \frac{B}{\alpha(\beta-\alpha)}\cdot\frac{f'}{f+\alpha} + \frac{B}{\beta(\beta-\alpha)}\cdot\frac{f'}{f+\beta}.\end{aligned}$$

Lemma 6 now follows from Lemma 5. □

Lemma 7. *Let f be a nonconstant meromorphic function satisfying the equation*

$$a_0 f^n + a_1 f^{n-1} + \cdots + a_n = 0,$$

where a_j are meromorphic functions with $a_0 \not\equiv 0$. Then

$$m(r, f) \leqslant m\left(r, \frac{1}{a_0}\right) + \sum_{j=1}^{n} m(r, a_j) + O(1).$$

Proof. By the equation,

$$f^n = -\frac{1}{a_0}\left(a_1 f^{n-1} + \cdots + a_n\right).$$

So

$$\begin{aligned}nm(r, f) = m\left(r, f^n\right) &\leqslant m\left(r, \frac{1}{a_0}\right) + m\left(r, a_1 f^{n-1} + \cdots + a_n\right)\\ &\leqslant m\left(r, \frac{1}{a_0}\right) + m\left(r, a_1 f^{n-1} + \cdots + a_{n-1} f\right) + m(r, a_n) + O(1)\\ &\leqslant m\left(r, \frac{1}{a_0}\right) + m(r, f) + m\left(r, a_1 f^{n-2} + \cdots + a_{n-1}\right) + m(r, a_n) + O(1)\end{aligned}$$

$$\begin{aligned}&\leqslant m\left(r,\frac{1}{a_0}\right)+m(r,f)+m\big(r,a_1f^{n-2}+\cdots+a_{n-2}f\big)+m(r,a_{n-1})+m(r,a_n)+O(1)\\&\leqslant\cdots\\&\leqslant m\left(r,\frac{1}{a_0}\right)+(n-1)m(r,f)+m(r,a_1)+\cdots+m(r,a_n)+O(1).\end{aligned}$$

Hence

$$m(r,f)\leqslant m\left(r,\frac{1}{a_0}\right)+\sum_{j=1}^{n}m(r,a_j)+O(1).$$

Lemma 7 is proved. □

Lemma 8. *Let f be a nonconstant meromorphic function. Set*

$$\gamma=\frac{f''}{f'}. \tag{1}$$

Then

$$\begin{aligned}&f''=\gamma f',\qquad f'''=\big(\gamma'+\gamma^2\big)f',\qquad f^{(4)}=\big(\gamma''+3\gamma'\gamma+\gamma^3\big)f',\\&\left(\frac{1}{f'}\right)'=-\frac{\gamma}{f'},\qquad\left(\frac{f''}{f'}\right)'=\gamma',\qquad\frac{f''}{(f')^2}=\frac{\gamma}{f'},\\&\left(\frac{f''}{(f')^2}\right)'=\frac{\gamma'-\gamma^2}{f'},\qquad\frac{(f'')^2}{f'}=\gamma^2f',\qquad\left(\frac{(f'')^2}{f'}\right)'=\big(2\gamma\gamma'+\gamma^3\big)f',\\&\frac{f'''}{f'}=\gamma'+\gamma^2,\qquad\left(\frac{f'''}{f'}\right)'=\gamma''+2\gamma\gamma',\qquad\left(\frac{1}{f'}\left(\frac{f''}{f'}\right)'\right)'=\frac{\gamma''-\gamma\gamma'}{f'}.\end{aligned}$$

Proof. Straightforward calculation. □

Lemma 9. *Let f, h be meromorphic functions such that f is nonconstant, and let A, B be two constants. Assume that*

$$f^3+Af^2+Bf=h\big[(f')^3+A(f')^2+Bf'\big]. \tag{2}$$

Then

$$\begin{aligned}6f'={}&h'''\left(f'+A+\frac{B}{f'}\right)+h''\left(6f''+3A\frac{f''}{f'}\right)\\&+h'\left[8f'''+3A\left(\frac{f''}{f'}\right)'+B\left(\frac{f''}{(f')^2}\right)'+\frac{3(f'')^2}{f'}+2A\frac{f'''}{f'}+\frac{B}{f'}\left(\frac{f''}{f'}\right)'\right]\\&+h\left[3\left(\frac{(f'')^2}{f'}\right)'+3f^{(4)}+2A\left(\frac{f'''}{f'}\right)'+B\left(\frac{1}{f'}\left(\frac{f''}{f'}\right)'\right)'\right].\end{aligned} \tag{3}$$

Proof. Differentiating (2), we get

$$\big(3f^2+2Af+B\big)f'=h'\big[(f')^3+A(f')^2+Bf'\big]+h\big[3(f')^2+2Af'+B\big]f'',$$

so that

$$3f^2+2Af+B=h'\big[(f')^2+Af'+B\big]+h\left(3f'f''+2Af''+B\frac{f''}{f'}\right). \tag{4}$$

Differentiating (4), we obtain a new equality. Dividing both sides of the new equality by f' shows that

$$6f + A = h''\left(f' + A + \frac{B}{f'}\right) + h'\left(5f'' + 3A\frac{f''}{f'} + B\frac{f''}{(f')^2}\right) + h\left[3f''' + \frac{3(f'')^2}{f'} + 2A\frac{f'''}{f'} + \frac{B}{f'}\left(\frac{f''}{f'}\right)'\right]. \tag{5}$$

Finally, differentiating (5), we get (3). □

Lemma 10. *Let f, h be meromorphic functions such that f is nonconstant, and let A, B be constants such that* (2) *holds. Then we have*

$$P + \frac{AQ}{f'} + \frac{BR}{(f')^2} = 0, \tag{6}$$

where

$$P = h''' + 6\gamma h'' + \left(8\gamma' + 11\gamma^2\right)h' + \left(3\gamma'' + 15\gamma\gamma' + 6\gamma^3\right)h - 6, \tag{7}$$

$$Q = h''' + 3\gamma h'' + \left(5\gamma' + 2\gamma^2\right)h' + (2\gamma'' + 4\gamma\gamma')h, \tag{8}$$

$$R = h''' + \left(2\gamma' - \gamma^2\right)h' + (\gamma'' - \gamma\gamma')h, \tag{9}$$

and γ is defined in (1).

Proof. By Lemma 9, we have (3). Substituting the formulae obtained in Lemma 8 into (3), we obtain (6). □

Lemma 11. *Let f be a nonconstant meromorphic function, and let A and $B \neq 0$ be constants. Assume that $f^3 + Af^2 + Bf$ and $(f')^3 + A(f')^2 + Bf'$ have the same zeros with the same multiplicity. Then $f' \neq 0$, so that $1/f'$ is an entire function.*

Proof. Since $f^3 + Af^2 + Bf$ and $(f')^3 + A(f')^2 + Bf'$ have the same zeros with the same multiplicity, we see that

$$h = \frac{f^3 + Af^2 + Bf}{(f')^3 + A(f')^2 + Bf'} \tag{10}$$

is an entire function, and $h(z) = 0$ if and only if $f(z) = \infty$. By (10), we have (2) and hence (4). It follows that $f' \neq 0$. Indeed, at any zero of f', the left side of (4) is holomorphic while the right side fails to be holomorphic since $h(z) = 0$ if and only if $f(z) = \infty$. The lemma is proved. □

Lemma 12. *Let f be a nonconstant rational function satisfying $f' \neq 0$. Then $f(z) = az + b$ or $f(z) = \frac{a}{(z+c)^n} + b$, where a ($\neq 0$), b, c are constants and n is a positive integer.*

Proof. If f is a polynomial, then clearly $f(z) = az + b$ for some constants $a \neq 0$ and b.

If f is not a polynomial, it has at least one pole in $\mathbb{C}$; moreover, since $f' \neq 0$, all zeros of $f - C$ are simple for any $C \in \mathbb{C}$. Thus

$$f(z) = b + a\frac{\prod_{j=1}^{m}(z - w_j)}{\prod_{j=1}^{n}(z - z_j)^{p_j}},$$

where $a \neq 0$ and b are constants, $m \geqslant 0$ and n, $p_j \geqslant 1$ are integers, and all z_j and w_j are distinct complex numbers. Furthermore, we may assume that $m \neq \sum_{j=1}^{n} p_j$, since if $m = \sum_{j=1}^{n} p_j$, we can consider the function $f(z) - a$ instead of $f(z)$.

Now direct calculation shows that $f'(z) = aP(z)/\prod_{j=1}^{n}(z - z_j)^{p_j+1}$, where

$$P(z) = \prod_{j=1}^{n}(z - z_j) \cdot \sum_{j=1}^{m}\prod_{l \neq j}(z - w_l) - \prod_{j=1}^{m}(z - w_j) \cdot \sum_{j=1}^{n} p_j \prod_{s \neq j}(z - z_s).$$

We claim that $n+m-1=0$. Suppose that $n+m-1\neq 0$. Then since the coefficient of the leading term z^{n+m-1} of P is $m-\sum_{j=1}^{n} p_j \neq 0$, we see that P has at least one zero. However, $f'\neq 0$, so each zero of P must be one of the z_j. Suppose $P(z_1)=0$. Then $\prod_{j=1}^{m}(z_1-w_j)\prod_{s\geqslant 2}(z_1-z_s)=0$. But this is impossible, as all z_j and w_j are distinct.

Thus $n+m-1=0$. Since $n\geqslant 1$ and $m\geqslant 0$, $m=0$ and $n=1$. The lemma is proved. □

3. Proof of Theorem 2

Since f and f' share the set $S=\{0,a,b\}$ CM, by Lemma 4, f is of order $\leqslant 2$. We also see that f^3+Af^2+Bf and $(f')^3+A(f')^2+Bf'$ have the same zeros with the same multiplicity, where $A=-(a+b)$ and $B=ab\neq 0$. Note $A^2-4B\neq 0$. So by Lemma 11, $1/f'$ is an entire function, and there exists an entire function h, whose zeros are the poles of f and have multiplicity 3, such that

$$f^3+Af^2+Bf=h\big[(f')^3+A(f')^2+Bf'\big]. \tag{11}$$

By Lemma 6,

$$m\left(r,\frac{1}{h}\right)=O(\log r), \tag{12}$$

and by Lemma 10,

$$P+\frac{AQ}{f'}+\frac{BR}{(f')^2}=0, \tag{13}$$

where P, Q, R are defined in (7)–(9). We claim that P, Q and R are entire functions. Note that the possible poles of P, Q and R must be poles of f since $f'\neq 0$. So we only need to show that P, Q and R are holomorphic at every pole of f.

Let z_0 be a pole of f. Then elementary computation shows

$$P=O(z-z_0),$$
$$Q=-\frac{(n-1)(2n-1)}{n^3}+O(z-z_0), \tag{14}$$
$$R=\frac{2(n-2)(n+2)}{n^3}+O(z-z_0), \tag{15}$$

as $z\to z_0$. Thus P, Q and R are entire functions.

Next we consider two cases.

Case 1. We have $R\not\equiv 0$. Then by (13), we have

$$\frac{P}{h}+\frac{AQ}{h}\cdot\frac{1}{f'}+\frac{BR}{h}\left(\frac{1}{f'}\right)^2=0.$$

Thus, by Lemma 7,

$$m\left(r,\frac{1}{f'}\right)\leqslant m\left(r,\frac{h}{R}\right)+m\left(r,\frac{Q}{h}\right)+m\left(r,\frac{P}{h}\right)+O(1).$$

By Lemma 5, we have $m(r,Q/h)=O(\log r)$ and $m(r,R/h)=O(\log r)$; and by Lemma 5 and (12), $m(r,P/h)=O(\log r)$. Thus

$$m\left(r,\frac{1}{f'}\right)\leqslant m\left(r,\frac{h}{R}\right)+O(\log r).$$

It follows that

$$\begin{aligned}N(r,f')&\leqslant T\left(r,\frac{1}{f'}\right)+O(1)=m\left(r,\frac{1}{f'}\right)+O(1)\leqslant m\left(r,\frac{h}{R}\right)+O(\log r)\\&\leqslant T\left(r,\frac{h}{R}\right)+O(\log r)\leqslant T\left(r,\frac{R}{h}\right)+O(\log r)\leqslant N\left(r,\frac{R}{h}\right)+O(\log r).\end{aligned} \tag{16}$$

Let $N_p(r, f)$ for each $p \in \mathbb{N}$ be the counting function of the poles of f with multiplicity exact p, each pole counted only once. Then by (15), we have

$$N\left(r, \frac{R}{h}\right) \leqslant 3N_1(r, f) + 2N_2(r, f) + 3\sum_{p\geqslant 3} N_p(r, f). \tag{17}$$

We also have

$$N(r, f') = \sum_{p\geqslant 1} (p+1)N_p(r, f). \tag{18}$$

By hypothesis, $N_1(r, f) = O(\log r)$. Then by (16)–(18)

$$N_2(r, f) + \sum_{p\geqslant 3} (p-2)N_p(r, f) \leqslant O(\log r). \tag{19}$$

It follows that f has finitely many poles. Thus by Lemmas 4 and 2, the order of f satisfies $\rho(f) \leqslant 1$, and we can write

$$f'(z) = \frac{e^{cz}}{M(z)}, \tag{20}$$

where c is a constant and $M(z)$ ($\not\equiv 0$) is a polynomial.

Since all zeros of h are poles of f, h has finitely many zeros. Thus by (12), h is a polynomial.

If $c = 0$, then f is a rational function. By Lemma 12, this case cannot occur.

Thus $c \neq 0$. We claim that

$$f(z) = R_1(z)e^{cz} + R_2(z)e^{-cz} + R_3(z), \tag{21}$$

where $R_j(z)$ are rational functions. Indeed, by (20), $f' = e^{cz}/M$. Thus $1/f' = Me^{-cz}$, $f'' = M_1e^{cz}$, $f''' = M_2e^{cz}$, $f''/f' = c - M'/M$, $f'''/f' = MM_2$ and $(f''/f')' = -(M'/M)'$, where M_1 and M_2 are rational functions. Since h is a polynomial, (21) follows from (5), proving the claim.

By (20) and (21), we have

$$\left(R_1' + cR_1 - \frac{1}{M}\right)e^{2cz} + R_3'e^{cz} + R_2' - cR_2 = 0. \tag{22}$$

It follows that $R_3' = 0$, $R_2' - cR_2 = 0$ and

$$R_1' + cR_1 - \frac{1}{M} = 0. \tag{23}$$

Thus $R_2 = 0$ and R_3 is a constant, say $R_3 = d$. So

$$f(z) = R_1(z)e^{cz} + d. \tag{24}$$

Substituting (20) and (24) into (11), we get

$$\left((R_1)^3 - \frac{h}{M^3}\right) + \left([3d + A](R_1)^2 - \frac{Ah}{M^2}\right)e^{2cz} + \left(\left[3d^2 + 2dA + B\right]R_1 - \frac{Bh}{M}\right)e^{cz} + d^3 + Ad^2 + Bd = 0. \tag{25}$$

It follows that

$$(R_1)^3 - \frac{h}{M^3} = 0, \tag{26}$$

$$(3d + A)(R_1)^2 - \frac{Ah}{M^2} = 0, \tag{27}$$

$$\left(3d^2 + 2dA + B\right)R_1 - \frac{hB}{M} = 0, \tag{28}$$

$$d^3 + Ad^2 + Bd = 0. \tag{29}$$

A tedious calculation, which we defer to Appendix A, then shows that f assumes one of the following forms:

(i) $f(z) = Ce^z$;
(ii) $f(z) = Ce^{-z} - \frac{2}{3}A$ and either $A = 0$ or $B = \frac{2}{9}A^2$;
(iii) $f(z) = Ce^{\frac{-1\pm i\sqrt{3}}{2}z} - \frac{3\pm i\sqrt{3}}{6}A$ and $B = \frac{1}{3}A^2$,

where C is a nonzero constant. Since $A = -(a+b)$ and $B = ab$, this completes the proof of Theorem 2 in Case 1.

Case 2. We have $R \equiv 0$. Then by (15), all poles of f are double. Thus

$$f'(z)h(z) = e^{\alpha(z)}, \tag{30}$$

where α is a polynomial of degree $\leqslant 2$. Set

$$\beta = \frac{h'}{h}. \tag{31}$$

Then by (30) and (31),

$$\gamma = \frac{f''}{f'} = \alpha' - \beta. \tag{32}$$

By (31), we also have

$$h' = \beta h, \qquad h'' = \left(\beta' + \beta^2\right)h, \qquad h''' = \left(\beta'' + 3\beta\beta' + \beta^3\right)h. \tag{33}$$

Substituting (32) and (33) into (9) and setting $R \equiv 0$, we obtain

$$\left(2\beta^2 + \beta'\right)\alpha' + \left[3\alpha'' - (\alpha')^2\right]\beta + \alpha''' - \alpha'\alpha'' = 0. \tag{34}$$

Let z_0 be a pole of f. Then some computation shows that near z_0,

$$2\beta^2 + \beta' = \frac{15}{(z - z_0)^2}\left[1 + O(z - z_0)\right]. \tag{35}$$

From (34) and (35), it follows that $\alpha'(z_0) = 0$. Thus if $\alpha' \not\equiv 0$, then since α is a polynomial of degree at most 2, f has at most one pole. Thus by Lemma 4, f is of order at most 1 and hence h has at most a single zero. Thus $N(r, 1/h) = O(\log r)$; so by (12) and Nevanlinna's First Fundamental Theorem,

$$T(r, h) = T\left(r, \frac{1}{h}\right) + O(1) = O(\log r).$$

Since h is entire, it must be a polynomial. An argument similar to that in Case 1 now shows that f must have one of the forms listed at the end of Case 1.

So we consider the case that $\alpha' \equiv 0$, i.e., α constant. Set $e^\alpha = c$. Then by (30),

$$f(z) = \frac{c}{h(z)}. \tag{36}$$

So $\gamma = -\beta$. In this case, (13) becomes

$$A(\beta\beta' + \beta'')h^2 - 2c(2\beta\beta' - \beta'')h + 6c = 0. \tag{37}$$

Differentiating (37), we obtain

$$A\left[(\beta')^2 + 3\beta\beta'' + 2\beta^2\beta' + \beta'''\right]h - 2c\left[2(\beta')^2 + \beta\beta'' + 2\beta^2\beta' - \beta'''\right] = 0. \tag{38}$$

However, near a zero z_0 of h, we have

$$(\beta')^2 + 3\beta\beta'' + 2\beta^2\beta' + \beta''' = \frac{27}{(z - z_0)^4}\left[1 + O(z - z_0)\right], \tag{39}$$

$$2(\beta')^2 + \beta\beta'' + 2\beta^2\beta' - \beta''' = -\frac{9}{(z - z_0)^4}\left[1 + O(z - z_0)\right]. \tag{40}$$

It follows from (38)–(40) that $c = 0$. But then $f' = 0$, so f is constant, which contradicts the assumptions of Theorem 2. Thus h does not vanish. By (12), $T(r, h) = T(r, 1/h) + O(1) = m(r, 1/h) + O(1) = O(\log r)$, so that h is a polynomial, and hence constant. Thus f is a linear function. Again, this contradicts the assumptions of Theorem 2.

This completes the proof.

Appendix A

Here we give the details of the derivation of (i)–(iii) from (23) and (26)–(29). By (26) and (27),

$$AR_1M = 3d + A. \tag{$*$}$$

If $A = 0$, then by ($*$), $d = 0$. Thus by (28), $h = MR_1$; and hence $M^2(R_1)^2 = 1$ by (26). So $\frac{1}{M} = \pm R_1$. Thus by (23), we get $R_1' + (c \pm 1)R_1 = 0$. Since R_1 is a rational function, R_1 is a constant and $c = \pm 1$. Set $R_1 = C$. If $c = 1$, we have $f(z) = Ce^z$; and if $c = -1$, $f(z) = Ce^{-z}$.

If $A \neq 0$, then by ($*$), $3d + A \neq 0$ and $\frac{1}{M} = \frac{A}{3d+A}R_1$. Thus by (23),

$$R_1' + \left(c - \frac{A}{3d + A}\right)R_1 = 0.$$

Since R_1 is a rational function, R_1 is a constant and $c = A/(3d + A)$. Set $R_1 = C$. Then $\frac{1}{M} = cC$ and $h = 1/c^3$ by (26).

From (27), we obtain

$$3d + \left(1 - \frac{1}{c}\right)A = 0 \tag{$**$}$$

and from (28),

$$3d^2 + 2Ad + \left(1 - \frac{1}{c^2}\right)B = 0. \tag{$***$}$$

Case 1: $c = 1$. Then by ($**$), $d = 0$. So $f(z) = Ce^z$.

Case 2: $c = -1$. Then by ($**$), $d = -\frac{2}{3}A$. So $f(z) = Ce^{-z} - \frac{2}{3}A$. By (29), we have

$$-\frac{8}{27}A^3 + \frac{4}{9}A^3 - \frac{2}{3}AB = 0.$$

It follows that either $A = 0$ or $B = \frac{2}{9}A^2$.

Case 3: $c \neq \pm 1$. Then by ($**$), $A = \frac{3cd}{1-c}$; and then by ($***$),

$$B = \frac{c^2}{1 - c^2}\left(3d^2 + 2Ad\right) = \frac{3c^2d^2}{(1 - c)^2}.$$

Since $B \neq 0$, we have $d \neq 0$. Thus by (29), we get

$$d^3 + \frac{3cd^3}{1 - c} + \frac{3c^2d^3}{(1 - c)^2} = 0.$$

It follows that $1 + c + c^2 = 0$. Thus $c = \frac{-1 \pm i\sqrt{3}}{2}$, and hence $d = \frac{1-c}{3c}A = -\frac{3 \pm i\sqrt{3}}{6}A$ and $B = \frac{1}{3}A^2$. So

$$f(z) = Ce^{\frac{-1 \pm i\sqrt{3}}{2}z} - \frac{3 \pm i\sqrt{3}}{6}A.$$

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