Meromorphic functions that share a set with their derivatives

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Abstract
There exists a set $S$ with three elements such that if a meromorphic function $f$, having at most finitely many simple poles, shares the set $S$ CM with its derivative $f'$, then $f' \equiv f$.

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1. Introduction

For $f$ a nonconstant entire function in the plane domain $D$ and $S$ a set of complex numbers, let

$$E_D(S, f) = \bigcup_{a \in S} \{ z \in D : f(z) - a = 0 \},$$

where zero of multiplicity $m$ is counted $m$ times in the set $E_D(S, f)$. When $D = \mathbb{C}$, we simply write $E(S, f)$.

In [2], Fang and Zalcman proved

**Theorem A.** There exists a finite set $S$ containing 3 elements such that if $f$ is a nonconstant entire function and $E(S, f) = E(S, f')$, then $f \equiv f'$.

It is natural to ask whether Theorem A remains valid for meromorphic functions. In this paper, we prove the following generalization of Theorem A.

**Theorem 1.** There exists a set $S$ with three elements such that if $f$ is a meromorphic function $f$ with at most finitely many simple poles and $E(S, f) = E(S, f')$, then $f' \equiv f$.

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Theorem 1 follows from the following more precise result.

**Theorem 2.** Let \( f \) be a nonconstant meromorphic function with at most finitely many simple poles; and let \( S = \{0, a, b\} \), where \( a \) and \( b \) are distinct nonzero complex numbers. If \( f \) and its derivative \( f' \) satisfy \( E(S, f) = E(S, f') \), then either

(i) \( f(z) = C e^{z} \); or
(ii) \( f(z) = C e^{-z} + \frac{z}{2}(a + b) \) and either \( a + b = 0 \) or \( 2a^2 - 5ab + 2b^2 = 0 \); or
(iii) \( f(z) = C e^{\frac{-z+\sqrt{3}i}{2}} + \frac{3+i\sqrt{3}}{6}(a + b) \) and \( a^2 - ab + b^2 = 0 \).

where \( C \) is a nonzero constant.

Throughout this paper, we use the standard notions and notation of Nevanlinna theory [3,6]. In particular, the spherical derivative of a meromorphic function \( f \) is given by

\[
 f^\#(z) = \frac{|f'(z)|}{1 + |f(z)|^2},
\]
and the order of \( f \) is defined by

\[
 \rho = \limsup_{r \to \infty} \frac{\log^+ T(r, f)}{\log r}.
\]

2. Auxiliary results

**Lemma 1.** Let \( f \) be a meromorphic function on \( \mathbb{C} \). If \( f \) has bounded spherical derivative on \( \mathbb{C} \), \( f \) is of order at most 2. If, in addition, \( f \) is entire, then the order of \( f \) is at most 1.

**Remark.** The first part of the lemma follows from the formula for the Ahlfors–Shimizu characteristic

\[
 T_0(r, f) = \int_0^r \frac{1}{t} \left( \frac{1}{\pi} \int_{|z| \leq t} \left[ f^\#(z) \right]^2 \, dx \, dy \right) \, dt
\]
and the fact that \( T(r, f) \) and \( T_0(r, f) \) differ by a bounded quantity (independent of \( r \)). The result for entire functions is more subtle; it is a special case of Theorem 3 in [1].

It is not difficult to extend Lemma 1 as follows.

**Lemma 2.** Let \( f \) be a meromorphic function on \( \mathbb{C} \) with finitely many poles. If \( f \) has bounded spherical derivative on \( \mathbb{C} \), \( f \) is of order at most 1.

Recently, using Zalcman’s Lemma [5] (cf. [7]), Liu and Pang obtained the following normality criterion [4].

**Lemma 3.** (See [4].) Let \( \mathcal{F} \) be a family of functions meromorphic on the unit disk \( \Delta \). If there exists a set \( S \) with three elements such that \( E_\Delta(S, f) = E_\Delta(f', S) \) for every \( f \in \mathcal{F} \), then \( \mathcal{F} \) is normal on \( \Delta \).

As an almost immediate consequence, we have

**Lemma 4.** Let \( f \) be a function meromorphic on \( \mathbb{C} \). If there exists a set \( S \) with three elements such that \( E(S, f) = E(S, f') \), then \( f^\#(z) \) is bounded on \( \mathbb{C} \).

**Proof.** Set \( \mathcal{F} = \{ f_w : w \in \mathbb{C} \} \), where \( f_w(z) = f(z + w) \). By Lemma 3, \( \mathcal{F} \) is normal on \( \Delta \); so by Marty’s Theorem, \( f^\#(w) = f^\#(0) \) is bounded for some \( M > 0 \) and all \( w \in \mathbb{C} \). \( \square \)
Lemma 5. (See [3, p. 56].) Let $f$ be a meromorphic function of finite order on the plane $\mathbb{C}$. Then

$$m(r, \frac{f^{(k)}}{f}) = O(\log r)$$

for each positive integer $k$.

Lemma 6. Let $f$ be a nonconstant meromorphic function of finite order, and let $A, B$ be two constants satisfying $A^2 - 4B \neq 0$ and $B \neq 0$. Then

$$m(r, \frac{(f')^3 + A(f')^2 + Bf'}{f^3 + Af^2 + Bf}) = O(\log r).$$

Proof. Since $A^2 - 4B \neq 0$ and $B \neq 0$, we have

$$f^3 + Af^2 + Bf = f(f + \alpha)(f + \beta),$$

where $\alpha$ and $\beta$ are two distinct nonzero constants. Then

$$\frac{1}{f^3 + Af^2 + Bf} = \frac{1}{f(f + \alpha)(f + \beta)} = \frac{1}{\beta - \alpha} \left( \frac{1}{f + \alpha} - \frac{1}{f + \beta} \right) = \frac{1}{\alpha\beta} \cdot \frac{1}{f} - \frac{1}{\alpha(\beta - \alpha)} \cdot \frac{1}{f + \alpha} + \frac{1}{\beta(\beta - \alpha)} \cdot \frac{1}{f + \beta}.$$

Thus

$$\frac{(f')^3 + A(f')^2 + Bf'}{f^3 + Af^2 + Bf} = \frac{(f')^3}{f(f + \alpha)(f + \beta)} + \frac{A}{\beta - \alpha} \left( \frac{(f')^2}{f(f + \alpha)} - \frac{(f')^2}{f(f + \beta)} \right) + \frac{B}{\alpha\beta} \cdot \frac{f'}{f} - \frac{B}{\alpha(\beta - \alpha)} \cdot \frac{f'}{f + \alpha} + \frac{B}{\beta(\beta - \alpha)} \cdot \frac{f'}{f + \beta}.$$

Lemma 6 now follows from Lemma 5. □

Lemma 7. Let $f$ be a nonconstant meromorphic function satisfying the equation

$$a_0 f^n + a_1 f^{n-1} + \cdots + a_n = 0,$$

where $a_j$ are meromorphic functions with $a_0 \neq 0$. Then

$$m(r, f) \lesssim m\left(r, \frac{1}{a_0}\right) + \sum_{j=1}^{n} m(r, a_j) + O(1).$$

Proof. By the equation,

$$f^n = -\frac{1}{a_0} \left(a_1 f^{n-1} + \cdots + a_n\right).$$

So

$$nm(r, f) = m(r, f^n) \lesssim m\left(r, \frac{1}{a_0}\right) + m(r, a_1 f^{n-1} + \cdots + a_n) + m(r, a_n) + O(1)$$

$$\lesssim m\left(r, \frac{1}{a_0}\right) + m(r, f) + m\left(r, a_1 f^{n-2} + \cdots + a_{n-1}\right) + m(r, a_n) + O(1)$$
Lemma 8. Let \( f \) be a nonconstant meromorphic function. Set
\[
\gamma = \frac{f'''}{f'}.
\] (1)

Then
\[
\begin{align*}
f'' &= \gamma f', & f''' &= (\gamma' + \gamma^2) f', & f^{(4)} &= (\gamma'' + 3\gamma'\gamma + \gamma^3) f', \\
\left(\frac{1}{f'}\right)' &= -\frac{\gamma}{f'}, & \left(\frac{f''}{f'}\right)' &= \gamma', & \frac{f'''}{f'} = \frac{\gamma}{f'}, \\
\left(\frac{f''}{(f')^2}\right)' &= \frac{\gamma' - \gamma^2}{f'}, & (f''')^2 &= \gamma^2 f', & \left(\frac{(f'')^2}{f'}\right)' &= (2\gamma\gamma' + \gamma^3) f', \\
\frac{f''''}{f'} &= \gamma' + \gamma^2, & \left(\frac{f''''}{f'}\right)' &= \gamma'' + 2\gamma\gamma', & \frac{1}{f'} \left(\frac{f''''}{f'}\right)' &= \frac{\gamma'' - \gamma\gamma'}{f'}.
\end{align*}
\]

Proof. Straightforward calculation. □

Lemma 9. Let \( f, h \) be meromorphic functions such that \( f \) is nonconstant, and let \( A, B \) be two constants. Assume that
\[
f^3 + Af^2 + Bf = h[(f')^3 + A(f')^2 + Bf'].
\] (2)

Then
\[
6f' = h'' \left( f' + A + \frac{B}{f'} \right) + h'' \left( 6f'' + 3A \frac{f'''}{f'} \right)
\]
\[
+ h' \left[ 8f''' + 3A \left( \frac{f'''}{f'} \right)' + B \left( \frac{f'''}{(f')^2} \right) + \frac{3(f'')^2}{f'} + 2A \frac{f'''}{f'} + \frac{B}{f'} \left( \frac{f'''}{f'} \right)' \right]
\]
\[
+ h \left[ 3 \left( \frac{(f'')^2}{f'} \right)' + 3 f^{(4)} + 2A \left( \frac{f''''}{f'} \right)' + B \left( \frac{1}{f'} \left( \frac{f''''}{f'} \right)' \right) \right].
\] (3)

Proof. Differentiating (2), we get
\[
(3f'^2 + 2Af + B) f' = h[(f')^3 + A(f')^2 + Bf'] + h[3(f')^2 + 2Af' + B] f''.
\]
so that
\[
3f'^2 + 2Af + B = h[(f')^2 + Af' + B] + h \left( 3 f' f'' + 2Af'' + B \frac{f'''}{f'} \right).
\] (4)

Differentiating (4), we obtain a new equality. Dividing both sides of the new equality by \( f' \) shows that
Lemma 12. Let $f$, $h$ be meromorphic functions such that $f$ is nonconstant, and let $A$, $B$ be constants such that (2) holds. Then we have

\[ 6f + A = h'' \left( f' + A + \frac{B}{f'} \right) + h' \left( 5f'' + 3A \frac{f'''}{f'} + B \frac{f'''}{(f')^2} \right) + h \left[ 3f''' + 3(f'')^2 + 2A \frac{f'''}{f'} + B \left( \frac{f'''}{(f')^2} \right)' \right]. \]

Finally, differentiating (5), we get (3). $\square$

Lemma 10. Let $f$, $h$ be meromorphic functions such that $f$ is nonconstant, and let $A$, $B$ be constants such that (2) holds. Then we have

\[ P + \frac{AQ}{f'} + \frac{BR}{(f')^2} = 0, \]

where

\[ P = h''' + 6 \gamma h'' + (8 \gamma' + 11 \gamma^2)h' + (3 \gamma'' + 15 \gamma \gamma' + 6 \gamma^3)h - 6, \]
\[ Q = h''' + 3 \gamma h'' + (5 \gamma' + 2 \gamma^2)h' + (2 \gamma'' + 4 \gamma \gamma')h, \]
\[ R = h''' + (2 \gamma' - \gamma^2)h' + (\gamma'' - \gamma \gamma')h, \]

and $\gamma$ is defined in (1).

Proof. By Lemma 9, we have (3). Substituting the formulae obtained in Lemma 8 into (3), we obtain (6). $\square$

Lemma 11. Let $f$ be a nonconstant meromorphic function, and let $A$ and $B \neq 0$ be constants. Assume that $f^3 + Af^2 + Bf$ and $(f')^3 + A(f')^2 + Bf'$ have the same zeros with the same multiplicity. Then $f' \neq 0$, so that $1/f'$ is an entire function.

Proof. Since $f^3 + Af^2 + Bf$ and $(f')^3 + A(f')^2 + Bf'$ have the same zeros with the same multiplicity, we see that

\[ h = \frac{f^3 + Af^2 + Bf}{(f')^3 + A(f')^2 + Bf'} \]

is an entire function, and $h(z) = 0$ if and only if $f(z) = \infty$. By (10), we have (2) and hence (4). It follows that $f' \neq 0$. Indeed, at any zero of $f'$, the left side of (4) is holomorphic while the right side fails to be holomorphic since $h(z) = 0$ if and only if $f(z) = \infty$. The lemma is proved. $\square$

Lemma 12. Let $f$ be a nonconstant rational function satisfying $f' \neq 0$. Then $f(z) = az + b$ or $f(z) = \frac{a}{(z+c)^n} + b$, where $a (\neq 0)$, $b, c$ are constants and $n$ is a positive integer.

Proof. If $f$ is a polynomial, then clearly $f(z) = az + b$ for some constants $a \neq 0$ and $b$.

If $f$ is not a polynomial, it has at least one pole in $\mathbb{C}$; moreover, since $f' \neq 0$, all zeros of $f - C$ are simple for any $C \in \mathbb{C}$. Thus

\[ f(z) = b + a \prod_{j=1}^{m} \frac{1}{(z - w_j)^{p_j}}, \]

where $a \neq 0$ and $b$ are constants, $m \geq 0$ and $n$, $p_j > 1$ are integers, and all $z_j$ and $w_j$ are distinct complex numbers. Furthermore, we may assume that $m \neq \sum_{j=1}^{n} p_j$, since if $m = \sum_{j=1}^{n} p_j$, we can consider the function $f(z) - a$ instead of $f(z)$.

Now direct calculation shows that $f'(z) = a P(z)/ \prod_{j=1}^{n} (z - z_j)^{p_j + 1}$, where

\[ P(z) = \prod_{j=1}^{n} (z - z_j) \cdot \sum_{j=1}^{m} (z - w_j) - \prod_{j=1}^{m} (z - w_j) \cdot \sum_{j=1}^{n} p_j \prod_{s \neq j} (z - z_s). \]
We claim that \( n + m - 1 = 0 \). Suppose that \( n + m - 1 \neq 0 \). Then since the coefficient of the leading term \( z^{n+m-1} \) of \( P = m - \sum_{j=1}^{n} p_j \neq 0 \), we see that \( P \) has at least one zero. However, \( f' \neq 0 \), so each zero of \( P \) must be one of the \( z_j \).

Suppose \( P(z_1) = 0 \). Then \( \prod_{j=1}^{n} (z_1 - w_j) \prod_{k \geq 2} (z_1 - z_k) = 0 \). But this is impossible, as all \( z_j \) and \( w_j \) are distinct.

Thus \( n + m - 1 = 0 \). Since \( n \geq 1 \) and \( m \geq 0 \), \( m = 0 \) and \( n = 1 \). The lemma is proved. \( \Box \)

3. Proof of Theorem 2

Since \( f \) and \( f' \) share the set \( S = \{0, a, b\} \) CM, by Lemma 4, \( f \) is of order \( \leq 2 \). We also see that \( f^3 + Af^2 + Bf \) and \( (f')^3 + A(f')^2 + Bf' \) have the same zeros with the same multiplicity, where \( A = -(a + b) \) and \( B = ab \neq 0 \). Note \( A^2 - 4B \neq 0 \). So by Lemma 11, \( 1/f' \) is an entire function, and there exists an entire function \( h \), whose zeros are the poles of \( f \) and have multiplicity 3, such that

\[
f^3 + Af^2 + Bf = h[(f')^3 + A(f')^2 + Bf'].
\]

By Lemma 6,

\[
m\left( r, \frac{1}{h} \right) = O(\log r),
\]

and by Lemma 10,

\[
P + \frac{AQ}{f'} + \frac{BR}{(f')^2} = 0,
\]

where \( P, Q, R \) are defined in (7)–(9). We claim that \( P, Q \) and \( R \) are entire functions. Note that the possible poles of \( P, Q \) and \( R \) must be poles of \( f \) since \( f' \neq 0 \). So we only need to show that \( P, Q \) and \( R \) are holomorphic at every pole of \( f \).

Let \( z_0 \) be a pole of \( f \). Then elementary computation shows

\[
P = O(z - z_0),
\]

\[
Q = -\frac{(n - 1)(2n - 1)}{n^3} + O(z - z_0),
\]

\[
R = \frac{2(n - 2)(n + 2)}{n^3} + O(z - z_0),
\]

as \( z \to z_0 \). Thus \( P, Q \) and \( R \) are entire functions.

Next we consider two cases.

Case 1. We have \( R \equiv 0 \). Then by (13), we have

\[
P \frac{1}{h} + \frac{AQ}{h} f' + \frac{BR}{h} \left( \frac{1}{f'} \right)^2 = 0.
\]

Thus, by Lemma 7,

\[
m\left( r, \frac{1}{f'} \right) \leq m\left( r, \frac{h}{R} \right) + m\left( r, \frac{Q}{h} \right) + m\left( r, \frac{P}{h} \right) + O(1).
\]

By Lemma 5, we have \( m(r, Q/h) = O(\log r) \) and \( m(r, R/h) = O(\log r) \); and by Lemma 5 and (12), \( m(r, P/h) = O(\log r) \). Thus

\[
m\left( r, \frac{1}{f'} \right) \leq m\left( r, \frac{h}{R} \right) + O(\log r).
\]

It follows that

\[
N(r, f') \leq T\left( r, \frac{1}{f'} \right) + O(1) = m\left( r, \frac{1}{f'} \right) + O(1) \leq m\left( r, \frac{h}{R} \right) + O(\log r) \leq T\left( r, \frac{h}{R} \right) + O(\log r) \leq T\left( r, \frac{R}{h} \right) + O(\log r).
\]
Let \( N_p(r, f) \) for each \( p \in \mathbb{N} \) be the counting function of the poles of \( f \) with multiplicity exact \( p \), each pole counted only once. Then by (15), we have

\[
N \left( r, \frac{R}{h} \right) \leq 3N_1(r, f) + 2N_2(r, f) + 3 \sum_{p \geq 3} N_p(r, f).
\]  

(17)

We also have

\[
N(r, f') = \sum_{p \geq 1} (p + 1)N_p(r, f).
\]  

(18)

By hypothesis, \( N_1(r, f) = O(\log r) \). Then by (16)–(18)

\[
N_2(r, f) + \sum_{p \geq 3} (p - 2)N_p(r, f) \leq O(\log r).
\]  

(19)

It follows that \( f \) has finitely many poles. Thus by Lemmas 4 and 2, the order of \( f \) satisfies \( \rho(f) \leq 1 \), and we can write

\[
f'(z) = \frac{e^{cz}}{M(z)},
\]  

(20)

where \( c \) is a constant and \( M(z) (\neq 0) \) is a polynomial.

Since all zeros of \( h \) are poles of \( f \), \( h \) has finitely many zeros. Thus by (12), \( h \) is a polynomial.

If \( c = 0 \), then \( f \) is a rational function. By Lemma 12, this case cannot occur.

Thus \( c \neq 0 \). We claim that

\[
f(z) = R_1(z)e^{cz} + R_2(z)e^{-cz} + R_3(z),
\]  

(21)

where \( R_j(z) \) are rational functions. Indeed, by (20), \( f' = e^{cz}/M \). Thus \( 1/f' = Me^{-cz} \), \( f'' = M_1e^{cz} \), \( f''' = M_2e^{cz} \), \( f''/f' = c - M'/M \), \( f''/f'' = MM_2 \) and \( (f''/f')' = -(M'/M)' \), where \( M_1 \) and \( M_2 \) are rational functions. Since \( h \) is a polynomial, (21) follows from (5), proving the claim.

By (20) and (21), we have

\[
\left( R_1' + cR_1 - \frac{1}{M} \right)e^{2cz} + R_3'e^{cz} + R_2' - cR_2 = 0.
\]  

(22)

It follows that \( R_3' = 0 \), \( R_2' - cR_2 = 0 \) and

\[
R_1' + cR_1 - \frac{1}{M} = 0.
\]  

(23)

Thus \( R_2 = 0 \) and \( R_2 \) is a constant, say \( R_3 = d \). So

\[
f(z) = R_1(z)e^{cz} + d.
\]  

(24)

Substituting (20) and (24) into (11), we get

\[
\left( (R_1)^3 - \frac{h}{M^3} \right) + \left( [3d + A](R_1)^2 - \frac{Ah}{M^2} \right)e^{2cz} + \left( [3d^2 + 2dA + B]R_1 - \frac{Bh}{M} \right)e^{cz} + d^3 + Ad^2 + Bd = 0.
\]  

(25)

It follows that

\[
(R_1)^3 - \frac{h}{M^3} = 0,
\]  

(26)

\[
(3d + A)(R_1)^2 - \frac{Ah}{M^2} = 0,
\]  

(27)

\[
(3d^2 + 2dA + B)R_1 - \frac{Bh}{M} = 0,
\]  

(28)

\[
d^3 + Ad^2 + Bd = 0.
\]  

(29)

A tedious calculation, which we defer to Appendix A, then shows that \( f \) assumes one of the following forms:
(i) \( f(z) = Ce^z; \)
(ii) \( f(z) = Ce^{-z} - \frac{2}{3}A \) and either \( A = 0 \) or \( B = \frac{2}{3}A^2; \)
(iii) \( f(z) = Ce^{\frac{-1+i\sqrt{3}}{2}z} - \frac{3+i\sqrt{3}}{6}A \) and \( B = \frac{1}{3}A^2; \)

where \( C \) is a nonzero constant. Since \( A = -(a + b) \) and \( B = ab \), this completes the proof of Theorem 2 in Case 1.

Case 2. We have \( R \equiv 0 \). Then by (15), all poles of \( f \) are double. Thus

\[
\begin{equation}
\gamma = \frac{f''}{f'} = \alpha' - \beta.
\end{equation}
\]

By (31), we also have

\[
\begin{equation}
h'(z)h(z) = e^{\alpha(z)},
\end{equation}
\]

where \( \alpha \) is a polynomial of degree \( \leq 2 \). Set

\[
\beta = \frac{h'}{h}.
\]

Then by (30) and (31),

\[
\gamma = \frac{f''}{f'} = \alpha' - \beta.
\]

Substituting (32) and (33) into (9) and setting \( R \equiv 0 \), we obtain

\[
\begin{equation}
(2\beta^2 + \beta'')\alpha' + \left[ 3\alpha'' - (\alpha')^2 \right] \beta + \alpha'' - \alpha' \alpha'' = 0.
\end{equation}
\]

Let \( z_0 \) be a pole of \( f \). Then some computation shows that near \( z_0 \),

\[
2\beta^2 + \beta' = \frac{15}{(z - z_0)^2}\left[ 1 + O(z - z_0) \right].
\]

From (34) and (35), it follows that \( \alpha'(z_0) = 0 \). Thus if \( \alpha' \neq 0 \), then since \( \alpha \) is a polynomial of degree at most 2, \( f \) has at most one pole. Thus by Lemma 4, \( f \) is of order at most 1 and hence \( h \) has at most a single zero. Thus \( N(r, 1/h) = O(\log r) \); so by (12) and Nevanlinna’s First Fundamental Theorem,

\[
T(r, h) = T \left( r, \frac{1}{h} \right) + O(1) = O(\log r).
\]

Since \( h \) is entire, it must be a polynomial. An argument similar to that in Case 1 now shows that \( f \) must have one of the forms listed at the end of Case 1.

So we consider the case that \( \alpha' \equiv 0 \), i.e., \( \alpha \) constant. Set \( e^{\alpha} = c \). Then by (30),

\[
\begin{equation}
f(z) = \frac{c}{h(z)}.
\end{equation}
\]

So \( \gamma = -\beta \). In this case, (13) becomes

\[
A(\beta \beta' + \beta'')h^2 - 2c(2\beta \beta' - \beta'')h + 6c = 0.
\]

Differentiating (37), we obtain

\[
A[(\beta')^2 + 3\beta \beta'' + 2\beta'^2 + \beta'']h - 2c[2(\beta')^2 + \beta \beta'' + 2\beta'^2 + \beta''] = 0.
\]

However, near a zero \( z_0 \) of \( h \), we have

\[
(\beta')^2 + 3\beta \beta'' + 2\beta'^2 + \beta'' = \frac{27}{(z - z_0)^2}\left[ 1 + O(z - z_0) \right],
\]

\[
2(\beta')^2 + \beta \beta'' + 2\beta'^2 + \beta' = \frac{9}{(z - z_0)^4}\left[ 1 + O(z - z_0) \right].
\]

It follows from (38)-(40) that \( c = 0 \). But then \( f' = 0 \), so \( f \) is constant, which contradicts the assumptions of Theorem 2. Thus \( h \) does not vanish. By (12), \( T(r, h) = T(r, 1/h) + O(1) = m(r, 1/h) + O(1) = O(\log r) \), so that \( h \) is a polynomial, and hence constant. Thus \( f \) is a linear function. Again, this contradicts the assumptions of Theorem 2.

This completes the proof.
Appendix A

Here we give the details of the derivation of (i)–(iii) from (23) and (26)–(29). By (26) and (27),

\[ AR_1 M = 3d + A. \]  

(\*)

If \( A = 0 \), then by (\*), \( d = 0 \). Thus by (28), \( h = MR_1 \); and hence \( M^2 (R_1)^2 = 1 \) by (26). So \( \frac{1}{M} = \pm R_1 \). Thus by (23), we get \( R'_1 + (c \pm 1) R_1 = 0 \). Since \( R_1 \) is a rational function, \( R_1 \) is a constant and \( c = \pm 1 \). Set \( R_1 = C \). If \( c = 1 \), we have \( f(z) = Ce^z \); and if \( c = -1 \), \( f(z) = Ce^{-z} \).

If \( A \neq 0 \), then by (\*), \( 3d + A \neq 0 \) and \( \frac{1}{M} = \frac{A}{3d + A} R_1 \). Thus by (23),

\[ R'_1 + \left( c - \frac{A}{3d + A} \right) R_1 = 0. \]

Since \( R_1 \) is a rational function, \( R_1 \) is a constant and \( c = A/(3d + A) \). Set \( R_1 = C \). Then \( 1/M = cC \) and \( h = 1/c \) by (26).

From (27), we obtain

\[ 3d + \left( 1 - \frac{1}{c} \right) A = 0 \]  

(\**)

and from (28),

\[ 3d^2 + 2Ad + \left( 1 - \frac{1}{c^2} \right) B = 0. \]  

(\***)

**Case 1:** \( c = 1 \). Then by (\**), \( d = 0 \). So \( f(z) = Ce^z \).

**Case 2:** \( c = -1 \). Then by (\**), \( d = -\frac{2}{3} A \). So \( f(z) = Ce^{-z} - \frac{2}{3} A \). By (29), we have

\[ \frac{8}{27} A^3 + \frac{4}{9} A^3 - \frac{2}{3} AB = 0. \]

It follows that either \( A = 0 \) or \( B = \frac{2}{3} A^2 \).

**Case 3:** \( c \neq \pm 1 \). Then by (\**), \( A = \frac{3cd}{1-c} \); and then by (\***),

\[ B = \frac{c^2}{1-c^2} (3d^2 + 2Ad) = \frac{3c^2d^2}{(1-c^2)^2}. \]

Since \( B \neq 0 \), we have \( d \neq 0 \). Thus by (29), we get

\[ d^3 + \frac{3cd^3}{1-c} + \frac{3c^2d^3}{(1-c^2)^2} = 0. \]

It follows that \( 1 + c + c^2 = 0 \). Thus \( c = \frac{-1 \pm i\sqrt{3}}{2} \), and hence \( d = \frac{1-c}{3c} A = -\frac{3 \pm i\sqrt{3}}{6} A \) and \( B = \frac{1}{3} A^2 \). So

\[ f(z) = Ce^{-\frac{1\pm i\sqrt{3}}{2}z} - \frac{3 \pm i\sqrt{3}}{6} A. \]

References