ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 310 (2007) 413–427
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Large restricted Lie algebras

Yuri Bahturin [a,c,1], Alexander Olshanskii [b,c,*,2]

[a] *Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL, A1C 5S7, Canada*
[b] *Department of Mathematics, 1326 Stevenson Center, Vanderbilt University, Nashville, TN 37240, USA*
[c] *Faculty of Mathematics and Mechanics, Moscow State University, 119899 Moscow, Russia*

Received 25 September 2006
Available online 4 January 2007
Communicated by Evgenii Khukhro

Abstract

We establish some results about large restricted Lie algebras similar to those known in the Group Theory. As an application, we use this group-theoretic approach to produce some examples of restricted as well as ordinary Lie algebras which can serve as counterexamples for various Burnside-type questions.

Keywords: Lie algebra; Generator; Defining relation; Nil-algebra

1. Introduction

In this paper we consider *restricted* Lie algebras over a *perfect* field F of characteristic $p > 0$. For the basic information see [4] and [1]. All subalgebras and ideals considered will also be restricted, that is, closed under the p-operation $x \mapsto x^{[p]}$.

Definition 1. A restricted Lie algebra G is called *large* if there is a subalgebra H in G of finite codimension such that H admits a surjective homomorphism on a nonabelian free restricted Lie algebra.

* Corresponding author.
E-mail address: alexander.olshanskiy@vanderbilt.edu (A. Olshanskii).
1 Partially supported by NSERC grant #227060-04 and URP grant, Memorial University of Newfoundland.
2 Partially supported by NSF grants DMS-0245600 and DMS-0455881 and by RFBR grant 05-01-00895.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.12.018

The main goal of this paper is to prove three theorems about restricted Lie algebras presented in terms of generators and defining relations. The first two deal with the large restricted Lie algebras while the third one is an application of the methods used in the first two theorems to the construction of finitely generated nil restricted Lie algebras of infinite dimension and their generalizations. A restricted Lie algebra G is called *nil* if for any $g \in G$ there is natural n such that $g^{[p^n]} = 0$.

We introduce some more notation. Given a set X of elements of a restricted Lie algebra G we denote by $\mathrm{alg}^p\{X\}$ the restricted subalgebra H of G generated by X (if $X = \{g\}$ then we simply write $\mathrm{alg}^p\{g\}$). Any element of H has the form of $\sum_{i=1}^m [f_i(x_1, \ldots, x_n)]^{[p^{k_i}]}$ where each $f_i(x_1, \ldots, x_n)$ is an ordinary Lie polynomial in $x_1, \ldots, x_n \in X$. Another notation, $\mathrm{id}_G^p\{X\}$, will be used to denote the restricted ideal I of G generated by X. Again, each element of I will look like $\sum_{i=1}^m w_i^{[p^{k_i}]}$ where each w_i is in the ordinary ideal of G generated by X. While obtaining these remarks, it is important to remember the main identity of restricted Lie algebras:

$$\left[g^{[p]}, h\right] = (\mathrm{ad}\, g)^p(h) = \Big[\underbrace{g, \ldots, g}_{p}, h\Big]. \tag{1}$$

We recall that given $g \in G$ and n a nonnegative integer, one defines $g^{[p^n]}$ by induction if one sets $g^{[p^0]} = g$ and $g^{[p^n]} = (g^{[p^{n-1}]})^{[p]}$ if $n \geqslant 1$. We call g *nilpotent* if $g^{[p^n]} = 0$ for some $n \in \mathbb{N}$. In this case, by (1), we also have that the linear transformation $\mathrm{ad}\, g : G \to G$ defined by $(\mathrm{ad}\, g)(x) = [g, x]$ for any $x \in G$ is also nilpotent. A nonzero element $g \in G$ is called *algebraic* if $\dim(\mathrm{alg}^p\{g\}) = n < \infty$. In this case $\mathrm{alg}^p\{g\} = \mathrm{Span}\{g, g^{[p]}, \ldots, g^{[p^{n-1}]}\}$, for some $n \in \mathbb{N}$. If, additionally, g is nilpotent then p^n is called the *nil-index* of g. Thus n is the least natural number such that $g^{[p^n]} = 0$. A restricted Lie algebra G is called *cyclic* if $G = \mathrm{alg}^p\{g\}$ for some $g \in G$.

Our first result is an analogue of a group-theoretic theorem due to B. Baumslag and S. Pride [2].

Theorem 1. *Let G be a restricted Lie algebra over a perfect field F of characteristic $p > 0$ given by a presentation with n generators and m relations, where $m \leqslant n - 2$. Then G is large.*

A technically useful form of Theorem 1, immediate from its proof, is as follows.

Proposition 1. *Let G be a restricted Lie algebra over a perfect field F of characteristic $p > 0$, given by a presentation with n generators and m relations, where $m \leqslant n - 2$. Then for any cyclic restricted Lie algebra A of sufficiently large dimension, G has a restricted ideal M, with $G/M \cong A$, such that M maps homomorphically on a nonabelian free restricted Lie algebra.*

In Section 4 we give two examples of one-relator restricted Lie algebras, which cannot be mapped homomorphically onto a nonabelian free restricted Lie algebra. We need this to show that the conclusion of Theorem 1 cannot be strengthened to "G can be mapped onto a nonabelian free restricted Lie algebra."

The next theorem is an analogue of a group-theoretic theorem due to M. Lackenby [6], with a simplified proof due to A. Olshanskii–D. Osin [7].

Theorem 2. *Let G be a restricted Lie algebra over a perfect field F of characteristic $p > 0$, H an ideal of finite codimension in G admitting a homomorphism on a nonabelian free restricted*

Lie algebra, $g_1, \ldots, g_k$ a set of elements of H. Let I_n be a (restricted) ideal of G generated by $g_1^{[p^n]}, \ldots, g_k^{[p^n]}$. Then G/I_n is large for all but finitely many $n \in \mathbb{N}$.

An important particular case of this theorem, with some more information, reads like this.

Proposition 2. *Let L be a free restricted Lie algebra of rank at least two, N an ideal of finite codimension in L, $g \in L \setminus N$ and $h \in \mathrm{alg}^p\{g\} \cap N$. Then $N/\mathrm{id}_L^p\{h\}$ is a large restricted Lie algebra with a presentation in which the number of generators exceeds the number of defining relations at least by 2. As a consequence, also $L/\mathrm{id}_L^p\{h\}$ is a large restricted Lie algebra.*

The above results allow us to construct some examples in the spirit of the Unrestricted Burnside Problem for groups. In [4, Chapter V, Exercise 17] the author asked for the proof of the finite-dimensionality (probably under certain conditions) of finitely generated nil restricted Lie algebras. Examples of infinite-dimensional finitely generated nil restricted Lie algebras can be derived from E. Golod's original example of finitely generated Engel Lie algebras [3]. In distinction with the situation in the Group Theory, where the example giving negative solutions to the Unrestricted Burnside problem are abundant, in the case of Lie algebras until now we had just one Golod's example and its derivatives.[3]

Before we formulate our results, we recall some terminology.

We call an algebra *residually finite-dimensional* (respectively, *residually finite-dimensional nilpotent*) if for every nonzero element $g \in G$ there is a homomorphism φ of G onto a finite-dimensional (respectively, finite-dimensional nilpotent) algebra such that $\varphi(g) \neq 0$. Equivalently, one can say that G is residually finite-dimensional (nilpotent) if G has a set of ideals $\{I_\alpha\}$ with trivial intersection $\bigcap_\alpha I_\alpha$ and such that each quotient algebra G/I_α is finite-dimensional (nilpotent). The same definition applies to ordinary restricted Lie algebras. A *subfactor* of a restricted Lie algebra G is a restricted Lie algebra H/K where K is an ideal of H and H is a subalgebra of G.

Theorem 3. *Let F be a perfect at most countable field F of characteristic $p > 0$. Then for any finitely generated restricted Lie algebra G with an ideal P of finite codimension that can be mapped homomorphically onto a nonabelian free restricted Lie algebra there exists an infinite-dimensional homomorphic image $\widetilde{G}$ in which the image $\widetilde{P}$ of P is a nil restricted Lie algebra, and $\widetilde{G}/\widetilde{P} \cong G/P$. One can choose $\widetilde{G}$ residually finite-dimensional and the direct limit of large restricted Lie algebras.*

Corollary 1. *Let F be a perfect at most countable field of characteristic $p > 0$. Then for any restricted Lie algebra G, with a presentation where the number of generators exceeds the number of relations at least by two, there exists an infinite-dimensional homomorphic image $\widetilde{G}$ which is a nil restricted Lie algebra. One can choose $\widetilde{G}$ residually finite-dimensional nilpotent and the direct limit of large restricted Lie algebras.*

[3] We are thankful to the referee who indicated that an example of another character has been published in a recent paper by V. Petrogradsky [8] in the case of fields of characteristic 2. We also learned that this was later generalized to other fields of positive characteristic by I. Shestakov and E. Zelmanov.

Corollary 2. *Let F be a perfect at most countable field of characteristic $p > 0$. Then there exist infinite-dimensional finitely generated nil restricted Lie algebras over F. One can choose such algebras residually finite-dimensional and direct limits of large restricted Lie algebras.*

Corollary 3 *(E.S. Golod). Let F be an at most countable field of characteristic $p > 0$. Then there exist infinite-dimensional finitely generated Engel Lie algebras. One can choose such algebras residually finite-dimensional nilpotent.*

2. Some properties of large restricted Lie algebras

First, we want to mention a couple of obvious properties of large restricted Lie algebras, following from the additivity of codimension.

Proposition 3. *The following are true.*

(i) *If a restricted Lie algebra G has a homomorphic image which is large then also G is large*;
(ii) *If a subalgebra of finite codimension in a restricted Lie algebra G is large then also G is large.*

The next result requires a little more sophistication.

Proposition 4. *The following are true.*

(i) *If a restricted Lie algebra can be mapped onto a nonabelian free restricted Lie algebra then any subalgebra of finite codimension has the same property. A subalgebra of finite codimension in a large restricted Lie algebra is itself large.*
(ii) *If a subalgebra H of finite codimension in a restricted Lie algebra G can be mapped onto a nonabelian free restricted Lie algebra then an ideal K of finite codimension in G also has this property. One can choose K with $K \subset H$.*

Proof. To prove (i), we notice that if a restricted Lie algebra G can be mapped onto a free restricted Lie algebra L by means of a surjective homomorphism ε and H is a subalgebra of finite codimension in G then $M = \varepsilon(H)$ is a subalgebra of finite codimension in L. If r is the number of generators in L (could be an infinite cardinal) and $d = \dim L/M$ then an analogue of Schreier's formula for groups, due to G.P. Kukin, see for example [1, 2.7.5], says that the number of free generators for M is given by $p^d(r-1)+1$. Obviously, if r is greater than 1, this latter number is greater than 1, proving that M is indeed a nonabelian free restricted Lie algebra.

The second claim in (i) now follows since if a restricted Lie algebra G is large, G_1 a restricted subalgebra of G of finite codimension and a restricted subalgebra H of finite codimension in G can be mapped on a free restricted Lie algebra then $G_1 \cap H$ is a restricted subalgebra of finite codimension in G_1 which by what we have just proved can be mapped onto a free restricted Lie algebra. Thus G_1 is large.

As for claim (ii), it easily follows from claim (i) and the result, apparently due to G.P. Kukin [5], according to which every subalgebra H of finite codimension in a restricted Lie algebra G contains an ideal K of G of finite codimension. In [9] V. Petrogradsky gives a comodule proof of this result whose original proof is combinatorial. Below is a pure module-theoretic proof of this result.

Lemma 1. *Let S be a unital subalgebra of an associative algebra R with 1 over a field F, such that R is generated as a left regular S-module by a finite subset T. Assume that U is a unital left R-module, V an S-submodule of V such that $\dim_F U/V < \infty$. Then there is an R-submodule W such that $W \subset V$ and still $\dim_F U/W < \infty$.*

Proof. For each $r \in R$ we consider a linear mapping $\Phi(r): V \to U/V$ given by $\Phi(r)(v) = rv + V$, for any $r \in R$ and $v \in V$. Obviously, $\Phi(s)(V) = sV = \{V\}$, for any $s \in S$. The set $W = \bigcap_{r\in R} \operatorname{Ker} \Phi(r)$ is easily seen to be an R-submodule of U contained in V. Also, if $\Phi(t)(v) = V$ for all $t \in T$ then $v \in W$. This follows because for any $s \in S$, any $t \in T$ and v as just above we have $\Phi(st)(v) = (st)v + V = s(tv) + V = V$. As a result, W contains the intersections of the kernels of the finite set of linear mappings $\Phi(t)$, $t \in T$, into a finite-dimensional space U/V. Each such kernel is of finite codimension by the Isomorphism Theorem, proving that $\dim V/W < \infty$. □

Now we can continue with the proof of claim (ii). A subspace V of a restricted Lie algebra G is a restricted ideal of G if and only if V satisfies two conditions. First, V must be a submodule under the natural R-module structure of G, R the restricted enveloping algebra of G. Second, V must be closed under the p-operation of G. If G has a restricted subalgebra H of finite codimension then by PBW-theorem [4, Chapter 5] R is a (free) finitely generated left (and right!) module over the associative subalgebra S generated by H. Now Lemma 1 with $U = G$ and $V = H$ applies and provides us with a subspace W of finite codimension in H. Since H is closed under the p-map, the p-closure K of W is a restricted ideal of G contained between W and H. By part (i) of this lemma it follows that K can be mapped onto a free restricted Lie algebra, as required. □

In view of the last result one can define a large restricted Lie algebra as one with an *ideal* of finite codimension which can be homomorphically mapped onto a nonabelian free restricted Lie algebra.

3. Baumslag–Pride's theorem for restricted Lie algebras

Let F be a perfect field of characteristic $p > 0$ and $L = L(X)$ a free restricted Lie algebra over F with a set of free generators $X = \{x_1, \ldots, x_n\}$. Let also $W = \{w_1, \ldots, w_m\}$ be a set of elements in L, I a restricted ideal in L generated by W, and $G = L/I$. We then say that G has a presentation $G = \langle x_1, \ldots, x_n \mid w_1, \ldots, w_m \rangle$ with n generators and m relations $w_1 = 0, \ldots, w_m = 0$. Sometimes the left-hand sides of the relations, that is, the elements of W are called the *relators*. In [2] it was established that a *group* which can be presented by n generators and m defining relations is large provided that $m \leqslant n - 2$. We want to adapt this result to our situation.

Before we formulate our first result, we recall [1] that in a free restricted Lie algebra $L(X)$ any element w can be uniquely written as a linear combination of p-powers of generators (the power component) plus a linear combination of commutators of degree at least two in the generators and their p-powers (the commutator component).

The first result we would like to start with is the following.

Proposition 5. *Let a restricted Lie algebra G be presented in terms of generators and defining relations as above, with $m \leqslant n - 1$. Then another presentation can be chosen so that one of the generators is not involved in the power components of the defining relations.*

Proof. Given a set of elements $a_1, \dots, a_d$ of a restricted Lie algebra G, the following transformation is called an *elementary transformation*: $a_i \to \lambda a_i + f(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_d)$, $a_k \to a_k$, for some $i \neq j$, and any $k \neq i, j$, f a p-polynomial, λ a nonzero element of F.

It is well known [1, Chapter 2] that any set $b_1, \dots, b_d$ that can be obtained by a finite sequence of these transformations from $a_1, \dots, a_d$ generates the same restricted subalgebra of G. In the case where $a_1, \dots, a_d$ is the set of free generators of a free restricted Lie algebra G the set $b_1, \dots, b_d$ is again the set of free generators of G. Each elementary transformation extends to an automorphism of G and it is known that the group of automorphisms of a free restricted Lie algebra is generated by such automorphisms.

We need an easy result about free abelian restricted Lie algebras.

Lemma 2. *Suppose we are given a free abelian restricted Lie algebra A with free generators $y_1, \dots, y_n$ and a restricted subalgebra B generated by a set of elements $v_1, \dots, v_m$. Then there is another free generating set $z_1, \dots, z_n$ for A and a set of elements $w_1, \dots, w_k$ generating B such that $k \leqslant \min(m, n)$, and each w_i is a p-polynomial in z_i, $i = 1, \dots, k$.*

Proof. It is well known from [4] that any abelian restricted Lie algebra A is a left module over a skew polynomial algebra Λ in one variable t in the sense that $\alpha^p t = t\alpha$, for any $\alpha \in F$. The action is given by $t * a = a^{[p]}$. The left and right analogues of the Division Algorithm work in Λ provided that the base field is perfect. If A is a free abelian restricted Lie algebra as above then A is a free left Λ-module with the free generating set $y_1, \dots, y_n$. The Λ-submodules of A are precisely the subalgebras of A.

For the proof of our lemma we need to write the matrix $(f_{ij}(t))$ of coefficients of the expression of v_i in terms of y_j. As in the case of the Fundamental Theorem of Finitely Generated Modules Over Principal Ideal Domains, we apply elementary transformations to the rows and columns of this matrix. The elementary transformations of the rows of this matrix correspond to elementary transformations of $v_1, \dots, v_m$, which replace one generating set of B by another, in particular, replacing v_i by $v_i + g(t) * v_j$, for $i \neq j$, corresponds to adding to the ith row the jth one multiplied on the left by $g(t)$. Elementary transformations of the columns correspond to the replacement of one free generating set of A by another. For example, if we modify the jth column by subtracting from it the ith one, $i \neq j$, then we obtain the matrix of coefficients of the generators of B with respect to the new free generating set where y_i is replaced by $y_i + g(t) * y_j$ while the remaining ones are left intact.

If we apply the natural versions of the left and the right division algorithms in Λ to the above matrix it becomes obvious that using the elementary transformations we can reduce our matrix to the form where the only nonzero elements are the first k diagonal elements $f_{11}(t), \dots, f_{kk}(t)$ where $k \leqslant \min(m, n)$. □

Now we can continue with the proof of Proposition 5. Let us assume that L is freely generated by $x_1, \dots, x_n$ and J the restricted ideal of L generated by $u_1, \dots, u_m$ so that $G = L/J$. Recall that $m \leqslant n - 1$. We can work modulo the commutator subalgebra $[L, L]$ of L. Suppose $A = L/[L, L]$ is the respective free abelian restricted Lie algebra. Let $y_1, \dots, y_n$, $v_1, \dots, v_m$, and B be the images of $x_1, \dots, x_n$, $u_1, \dots, u_m$, and J under the natural homomorphism of L onto A. Then we apply Lemma 2. As a result, we obtain the free generators $z_1, \dots, z_n$ of A and the generators $w_1, \dots, w_k$ of B, each w_i being a p-polynomial of z_i.

If we go back to the generators $x_1, \dots, x_n$ of L and the relators $u_1, \dots, u_m$ of $G = L/J$ and apply the same transformations as we did to their images $y_1, \dots, y_n$ and $v_1, \dots, v_m$ then, according to what was mentioned about the automorphisms of the free restricted Lie algebras before the proof of Lemma 2, we obtain the desired generators and relations for G. □

Now we are ready to complete the proof of Theorem 1.

Proof. It follows from Lemma 2 that we can choose a presentation for $G = L/I$ in such a way that one of the generators, say t, is not involved in the p-power portions of defining relations. We denote this generator by t and other generators by $a_1, \dots, a_{n-1}$. Let $w_1, \dots, w_m$ be the defining relators of G.

Let M_k be a restricted ideal of L generated by $a_1, \dots, a_{n-1}$ and $t^{[p^k]}$, where $k = 1, 2, \dots$. Then it follows from [1, 2.7.4] that M_k is a free restricted Lie algebra with free generators $a_{li} = (\operatorname{ad} t)^l(a_i)$, $0 \leqslant l < p^k$, $1 \leqslant i \leqslant n-1$ and $t^{[p^k]}$.

Now we consider an ideal M of L generated by $a_1, \dots, a_{n-1}$. This ideal is a restricted Lie algebra whose free generators are $a_{li} = (\operatorname{ad} t)^l(a_i)$, $l = 0, 1, \dots,$ $1 \leqslant i \leqslant n-1$. This follows because $M = \bigcap_{k=1}^{\infty} M_k$ and each finite subset of the generating set of M mentioned above is a subset of the free generating set of M_k, for an appropriate k.

By Proposition 5, $I \subset M$. The image of M in G is defined by the relations $w_{lj} = 0$ where each w_{lj} is a p-polynomial $(\operatorname{ad} t)^l(w_j)$, rewritten in terms of the generators a_{lj}, where l is as above and $1 \leqslant j \leqslant m$. Applying the Leibniz rule for the derivations we can easily rewrite each w_j in terms of a finite subset of the latter set of generators a_{li}. Thus me may assume that there is a number q such that only a_{li} with $l < p^q$ are involved in the expression of w_j as the elements of M.

Now let us choose k so that $p^q < p^k$ and an element s, which is a p-polynomial in t with leading term $t^{[p^k]}$. Let $M(s)$ be the ideal of L generated by $a_1, \dots, a_{n-1}$ and s. Again by [1, Section 2.7], $M(s)$ is freely generated by $a_{li} = (\operatorname{ad} t)^l(a_i)$, $0 \leqslant l < p^k$, $1 \leqslant i \leqslant n-1$ and s. The image P of $M(s)$ in G is defined by the relations $w_{lj} = (\operatorname{ad} t)^l(w_j)$, l is as above, $1 \leqslant j \leqslant m$, which have to be rewritten through the new free generating system. (This is a known fact but it follows also from our Lemma 4 below.) If we perform derivation in the relations w_{lj} with the use of the Leibniz rule then we observe that the maximum value of index l in the generators of $M(s)$ that are involved in w_{lj} is less than $p^q + p^k$. The generators a_{li} of M with $p^k \leqslant l < p^k + p^q$ are no longer on the above list of the free generators of $M(s)$. For these we have $a_{p^k+j,i} = (\operatorname{ad} t)^{p^k}(a_{ji}) = [t^{[p^k]}, a_{ji}]$, where $0 \leqslant j < p^q$, and so if $s = t^{[p^k]} + \sum_{m=0}^{k-1} \alpha_m t^{[p^m]}$ for some $\alpha_m \in F$, then each $a_{p^k+j,i}$ should be replaced by $[s, a_{ji}] - \sum_{m=0}^{k-1} \alpha_m a_{j+p^m,i}$. Since $q, m < k$, each generator $a_{j+p^m,i}$ is on the list of the free generators of $M(s)$.

Now let us impose additional relations on P by setting $a_{li} = 0$ for all $i = 1, \dots, n-1$, $0 \leqslant l < p^q$. Let Q be the quotient algebra of P obtained in this way. Then removing such superfluous generators we will be left with $(n-1)(p^k - p^q) + 1$ generators s and a_{li}, i as always, and $p^q \leqslant l < p^k$, and still mp^k relations obtained by replacing some generators by 0. Since s was involved in the relations of P only inside the commutators $[s, a_{li}]$, where $0 \leqslant l < p^q$, none of the newly obtained relations of Q involves s.

As a result, Q is the free product of the subalgebra generated by s and the subalgebra K generated by $(n-1)(p^k - p^q)$ generators a_{li}, i as always, $p^q \leqslant l < p^k$, with mp^k relations among them. The difference between the number of generators and relations for K now takes the form $(n-m-1)p^k - (n-1)p^q$. It is now obvious that if we choose k sufficiently large this latter number can be made positive and then by Lemma 2 K can be mapped onto a free restricted algebra of rank 1. The free product Q can then be mapped onto a free restricted algebra of rank 2.

Therefore P can be mapped onto a free restricted Lie algebra of rank 2. It remains to notice that the codimension of P in G equals the codimension k of $M(s)$ in L and so is a finite number. $\square$

4. Examples of one-relator restricted Lie algebras

It is essential in Theorem 1 that a restricted subalgebra that maps onto a nonabelian free restricted Lie algebra need not be the whole algebra. Here we give two different constructions of restricted Lie algebras, defined by n generators and $r = 1$ relation, which cannot be mapped homomorphically onto a nonabelian free restricted Lie algebra, although the difference $n - r$ can be made arbitrarily high.

The first one works in the case of finite fields, while the second can be applied in the case of arbitrary fields of characteristic $p > 0$, and is an analogue of a group-theoretical example by J. Stallings [10].

Let F be a finite field of characteristic $p > 0$ and G a nonabelian simple finite-dimensional restricted Lie algebra with two generators. Let L_n be a free restricted Lie algebra of any rank $n \geqslant 2$ and N the intersection of the finite number of the kernels of all homomorphisms of L_n onto G. Choose a minimal subset $\{\varphi_1, \ldots, \varphi_m\}$ in the set of all the above homomorphisms such that $\operatorname{Ker}\varphi_1 \cap \cdots \cap \operatorname{Ker}\varphi_m = N$. Set

$$K_i = \operatorname{Ker}\varphi_i, \qquad N_i = \bigcap_{j \neq i} K_j, \quad \text{and} \quad \overline{N}_i = N_i/N \quad \text{for any } i = 1, \ldots, m.$$

Let P be the direct product of m copies of G, and θ a homomorphism of L_n into P given by $\theta(x) = (\varphi_1(x), \ldots, \varphi_m(x))$. Then θ induces an injective homomorphism $\bar{\theta} : L_n/N \to P$ given by $\bar{\theta}(x) = (\varphi_1(x + N), \ldots, \varphi_m(x + N))$. Obviously, the sum $Q = \sum_{i=1}^{m} \overline{N}_i$ is direct and under $\bar{\theta}$ each $\overline{N}_i$ maps isomorphically onto the ith component of P. Thus $\bar{\theta}$ is surjective and $L_n/N = Q = \bigoplus_{i=1}^{m} \overline{N}_i$. Now we choose any element $v \in L_n/N$ all of whose components in the direct sum are nonzero. Let w be an arbitrary preimage of v in L_n, that is, $v = w + N$.

Proposition 6. *A restricted Lie algebra $M = L_n/\operatorname{id}^p_{L_n}\{w\}$ is an example of an n-generator, 1-relator algebra that cannot be mapped homomorphically onto L_2.*

Proof. Let us show that the image of w under every homomorphism φ of L_n onto G is different from zero. Indeed, if $\varphi(w) = 0$ and $K = \operatorname{Ker}\varphi$ then K/N is a nonzero ideal in the direct sum $L_n/N = \bigoplus_{i=1}^{m} \overline{N}_i$. Each of the summands is a nonabelian simple algebra, isomorphic to G. A standard argument (see, e.g., [4, Chapter III, Section 5]) shows that if an element of K/N has a nontrivial projection on certain $\overline{N}_i$ then $\overline{N}_i \subset K/N$. But $v = w + N$ has nontrivial projections on all summands. Hence $K/N = L_n$ and $\varphi = 0$, a contradiction.

Suppose now, to the contrary of the claim of Proposition 6, that $\alpha : M \to L_2$ is a homomorphism of M onto L_2. Since G is a 2-generator restricted Lie algebra, there is an epimorphism $\psi : M \to G$. If $\varepsilon : L_n \to M$ is the natural homomorphism then $\psi\varepsilon : L_n \to G$ is a homomorphism of L_n onto G such that $\psi\varepsilon(\operatorname{id}^p_{L_n}\{w\}) = \{0\}$, hence $\psi\varepsilon(w) = 0$. This is a contradiction with the above mentioned property of w. $\square$

Remark 1. It is quite obvious that the same argument applies also in the case of ordinary Lie algebras and associative algebras and allows us to produce an n-generator 1-relator algebra over a finite field or over $\mathbb{Z}$, which cannot be mapped homomorphically onto a free 2-generator algebra.

Moreover, since the group A_5 is a noncommutative 2-generator simple group, a similar argument gives an example of an n-generator 1-relator group, which cannot be mapped homomorphically onto a nonabelian free group.

In the case of groups, examples of n-generator 1-relator groups which cannot be mapped homomorphically onto a nonabelian free group, with a longer argument, are due to Lyndon–Stallings [10]. We use a simplified version of this argument to produce similar examples of restricted Lie algebras over any fields (finite or infinite) of characteristic $p > 0$.

Before we go over to the second construction, we need to mention a simple fact, about the vector space bases of free nilpotent restricted Lie algebras. If $\{x_1, \ldots, x_n\}$ is a free generating set of a free restricted Lie algebra L then according to [1, Section 2.7.2], a basis of L can be given in the form of the set W of all elements $w^{[p^k]}$ where w runs through the set of usual Hall's monomials (which form a vector space basis of the ordinary free Lie algebra with the same free generating set), and $k = 0, 1, 2, \ldots$. A subset T_c consisting of all $w^{[p^k]}$ where the degree of w is at least $c + 1$ can be easily shown to span an ideal J_c such that L/J_c is nilpotent of class c. It immediately follows that in the free nilpotent restricted Lie algebras of class c all monomials $w^{[p^k]}$ with w of degree at most c and $k = 0, 1, 2, \ldots$ are linearly independent.

Lemma 3. *Let F be a field of characteristic $p > 0$, P_n a free nilpotent of class 2 restricted Lie algebra with $n \geqslant 2$ free generators $y_1, y_2, \ldots, y_n$. Let*

$$w = \sum_{1 \leqslant i < j \leqslant n} [y_i, y_j]^{[p^{k_{ij}}]}$$

where all k_{ij} are pairwise different natural numbers. Then under every homomorphism of P_n onto P_2 the element w maps onto a nonzero element.

Proof. Suppose that z_1, z_2 are free generators of F_2. Let φ be a homomorphism of P_n onto P_2. Let also $\bar{\varphi}$ be an induced homomorphism of P_n onto $\overline{P_2} = P_2/([P_2, P_2] + P^{[p]})$, which is a free abelian restricted Lie algebra with zero p-map and free generators $\bar{z}_1, \bar{z}_2$. By composing with an appropriate automorphism of $\overline{P_2}$, we may assume that $\bar{\varphi}(y_1) = \bar{z}_1$ and $\bar{\varphi}(y_2) = \bar{z}_2$, and $\bar{\varphi}(y_i) = \alpha_{i1}\bar{z}_1 + \alpha_{i2}\bar{z}_2$, for some scalars α_{i1}, α_{i2}, and $i = 3, \ldots, n$. Therefore, $\varphi(y_1) = z_1 + u_1$, $\varphi(y_2) = z_2 + u_2$ and $\varphi(y_i) = \alpha_{i1}z_1 + \alpha_{i2}z_2 + u_i$ where all $u_1, \ldots, u_n$ are linear combinations of $[z_1, z_2]$ and p-powers of z_1 and z_2. This follows because in the case of free nilpotent restricted Lie algebras, any automorphism (essentially, a nonsingular linear transformation) of $\overline{P_2}$ extends to an automorphism of P_2. Every substitution of a u_i into a commutator, thanks to the nilpotency class 2 and the main identity (1) of restricted Lie algebras gives us zero. And so the image w, possibly twisted by certain automorphism of P_2 takes the form of

$$\sum_{1 \leqslant i < j \leqslant n} \gamma_{ij}[z_1, z_2]^{[p^{k_{ij}}]}$$

where γ_{ij} are some scalars, and $\gamma_{12} = 1$. All $[z_1, z_2]^{[p^{k_{ij}}]}$ are pairwise distinct elements of a standard basis of P_2, hence the image of w is nonzero, as claimed. □

Now the following is true.

Proposition 7. *Let L_n be a free restricted Lie algebra, with free generators $x_1, \dots, x_n$. Choose an element*

$$v = \sum_{1 \leqslant i < j \leqslant n} [x_i, x_j]^{[p^{k_{ij}}]} \in L_n$$

where all k_{ij} are pairwise different natural numbers. Then the n-generator, 1-relator restricted Lie algebra $M = L_n / \mathrm{id}^p_{L_n}\{v\}$ cannot be mapped homomorphically onto a nonabelian free restricted Lie algebra.

Proof. If M could be mapped homomorphically onto a nonabelian free restricted Lie algebra, then we would also have a homomorphism $\psi : M \to P_2$ where P_2 as in the previous Lemma 3. Under ψ we would have that all the commutators of degree three or larger are mapped into zero. Thus we have a homomorphism of L_n onto P_2 which annihilates v and all commutators of degree three or larger. This, in its turn, induces a homomorphism of P_n onto P_2 mapping w, the image of v under the natural homomorphism of L_n onto P_n, into zero. By Lemma 3, this is not possible. □

5. Lackenby–Olshanskii–Osin theorem for restricted Lie algebras

Our aim in this section is the proof of Theorem 2, which is an analogue of some group-theoretical results in [6] and [7] in the case of restricted Lie algebras.

Before we prove this theorem we need few lemmas.

Lemma 4. *Let G be a Lie algebra, N an ideal of G, g an element of N. Let $C \subset C_G(g)$ where $C_G(g)$ is the centralizer of g in G. Suppose that T is any totally ordered subset of G whose union with $C + N$ spans G as a vector space. Denote by Z the set*

$$\{(\mathrm{ad}\, t_1) \cdots (\mathrm{ad}\, t_k)(g) \mid t_1 \leqslant \cdots \leqslant t_k \in T\}.$$

Then the ideal of G generated by g coincides with the ideal of N generated by Z. In the case where G is a restricted Lie algebra over a field of characteristic $p > 0$, and all the ideals are restricted, we can replace Z by a subset Z_p consisting of all monomial in which the degree of any t_i is at most $p - 1$.

Proof. Since N is an ideal of G, the above elements are in N. Now by the definition of the universal enveloping algebra $U(G)$, any ideal of G is a left module for the adjoint representation of $U(G)$. Thus the ideal of G generated by g is a submodule of the left $U(G)$-module G generated by g. Using PBW-theorem [4, Chapter 5], if we choose a totally ordered basis of G, in which the elements n_α of N precede some elements t_β of T and these precede some elements c_γ of C, then any element of $U(G)$ is a linear combination of the ordered monomials of the form

$$n_{\alpha_1} \cdots n_{\alpha_k} t_{\beta_1} \cdots t_{\beta_l} c_{\gamma_1} \cdots c_{\gamma_m}.$$

The action of $U(G)$ on G is the unique extension of the adjoint representation. If we apply the above monomial to g and recall that C is the centralizer of g we will see that in N the ideal in question is generated by the elements $(\mathrm{ad}\, t_{\beta_1}) \cdots (\mathrm{ad}\, t_{\beta_l})(g)$ with $t_{\beta_1} \leqslant \cdots \leqslant t_{\beta_l}$, as claimed.

In the case where G is a restricted Lie algebra over a field of characteristic $p > 0$ the universal enveloping algebra should be replaced by the restricted enveloping algebra $u_p(G)$. Then the restricted ideals of G are left $u_p(G)$-submodules closed under the p-operation. As mentioned in the Introduction, when we generate an ideal we can first apply the action of $u_p(G)$ and then take all possible p-powers. Thus the argument as just above applies also in this case. By PBW-theorem for the restricted enveloping algebras [4, Chapter 5] any element x_α, t_β, and c_γ enters the monomials of the basis to the degree at most $p-1$, as claimed. Thus the proof is complete. □

Our next lemma is as follows.

Lemma 5. *For any finite collection of nonzero elements $g_1, \dots, g_k$ of a free restricted Lie algebra L and any number $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ with the following property. For every $q \geqslant m$ there is a restricted ideal N of finite codimension in L such that for all $1 \leqslant i \leqslant k$ we have* $\mathrm{Span}\{g_i, g_i^{[p]}, \dots, g_i^{[p^{n-1}]}\} \cap N = \{0\}$ *but* $g_i^{[p^q]} \in N$.

Proof. It is sufficient to prove this lemma in the case where L is finitely generated. If $x_1, \dots, x_r$ is the set of free generators of L then there is a natural filtration on L in which an element $w^{[p^n]}$ of the canonical basis has filtration $p^n d$ if w is a commutator in $x_1, \dots, x_r$ of degree d. An arbitrary $g \in L$ has filtration f if f is the least filtration of the basic elements in its expression through the basis. The set of elements of filtration at least f is a restricted ideal of L which we denote by I_f. Obviously each such ideal is of finite codimension in L. Now suppose m_0 is the maximum filtration of the elements $g_1, \dots, g_k$. Choose $m = m_0 p^n + 1$. For each $q \geqslant m$, we set $N = I_m$. If d_i is the filtration of g_i then the filtration of $g_i^{[p^s]}$ equals $d_i p^s$. By our choice of m, for each i, $1 \leqslant i \leqslant k$, the elements $g_i, g_i^{[p]}, \dots, g_i^{[p^{n-1}]}$ are linearly independent modulo N. But if $q \geqslant m$ then for any element $a \in L$ we always have $a^{[p^q]} \in I_m$. □

One more result we need for the proof of Theorem 2 is the following.

Proposition 8. *Let L be a free restricted Lie algebra of rank $r \geqslant 2$, $g_1, \dots, g_k$ arbitrary elements of L. Let J_q be a restricted ideal of L generated by the elements $g_1^{[p^q]}, \dots, g_k^{[p^q]}$, where q is a natural number. Then $\overline{L} = L/J_q$ is large for all but finitely many $q \in \mathbb{N}$.*

Proof. Without loss of generality we may assume that all $g_1, \dots, g_k$ are nonzero. By Lemma 5, there exists $m \in \mathbb{N}$ such that for any $q \geqslant m$ there is a restricted ideal N of finite codimension such that for all $i = 1, \dots, k$ we have the elements $g_i, g_i^{[p]}, \dots, g_i^{[p^k]}$ are linearly independent modulo N, but $g_i^{[p^q]} \in N$. In particular, the codimension of $\mathrm{alg}^p\{g_i\} + N$ is bounded from above by $j - k - 1$ where $\dim L/N = j$. Now we want to show that the image $\overline{N}$ of N in $\overline{L}$ is a large algebra. Let T_i be a minimal set of elements of L such that the union of T_i and $\mathrm{alg}^p\{g_i\} + N$ spans L as a vector space. Then according to Lemma 4, and considering that $\mathrm{alg}^p\{g_i\} \subset C_L(g_i)$, we have that $\overline{N}$ is isomorphic to the quotient algebra of N by the restricted ideal generated by the elements of the set Z where $Z = \bigcup_{i=1}^k Z_i$ and

$$Z_i = \left\{ (\mathrm{ad}\, t_1)^{l_1} \cdots (\mathrm{ad}\, t_{s_i})^{l_{s_i}} \left(g_i^{[p^q]} \right) \,\middle|\, \{t_1, \dots, t_{s_i}\} = T_i,\ 0 \leqslant l_1, \dots, l_{s_i} < p \right\}.$$

Now since $\dim L/N = j$, according to the analogue of Schreier's Formula [1, Theorem 2.7.5] for the number of generators of a subgroup of a free group, the number of generators of N is

$p^j(r-1)+1 \geqslant p^j+1$. The codimension r_i of $\mathrm{alg}^p\{g_i\}+N$ is at most $j-k-1$ and so the number of elements in each Z_i is at most p^{j-k-1}. In this case the total number of defining relations for $\overline{N}$ is at most $kp^{j-k-1} < p^{j-1} \leqslant p^j - 1$. The difference between the number of generators and relations will be at least 2 and so Theorem 1 applies proving that $\overline{N}$ is a large restricted Lie algebra. Since $\overline{N}$ is of finite codimension in $\overline{L}$ this latter is a large restricted Lie algebra, as required. □

Now we can comment on Proposition 2.

Proof. If N is an ideal of L of codimension j, $g \notin N$ and $h \in \mathrm{alg}^p\{g\} \cap N$ then the same proof as just above, with $k=1$, shows that the number of generators of N is still $p^j(r-1)+1 \geqslant p^j+1$ and $\mathrm{id}_L^p\{g^{[p^n]}\}$ is generated as an ideal of N by p^{j-1} elements. The difference is greater than $p^j+1-p^{j-1} \geqslant 2$. As before, we use Theorem 1 to derive that $N/\mathrm{id}_L^p\{h\}$ is large. This proves Proposition 2, which is important in the proof of Theorem 3. □

The following lemma shows that the situation in restricted Lie algebras can be very different from that in groups. In the case of groups, if we are given an element g of a normal subgroup H of index d in a group G, then the normal subgroup of G generated by g is a normal subgroup of H generated by the conjugates $x_1 g x_1^{-1}, \dots, x_d g x_d^{-1} \in H$ for some $x_1, \dots, x_d \in G$. In the case of restricted Lie algebras, we have instead the following.

Lemma 6. *Let G be a restricted Lie algebra, H a restricted ideal of G such that* $\dim G/H = d$, *$g \in H$, $n \in \mathbb{N}$. Let $I = \mathrm{id}_G^p\{g^{[p^n]}\}$ be the restricted ideal of G generated by $g^{[p^n]}$, where $d \leqslant n$, $J = \mathrm{id}_H^p\{g^{[p^{n-d}]}\}$ the restricted ideal of H generated by $g^{[p^{n-d}]}$. Then $I \subset J$.*

Proof. By Lemma 4, I as an ideal of H is generated by the elements of the set defined as follows

$$Z = \left\{ (\mathrm{ad}\, t_1)^{l_1} \cdots (\mathrm{ad}\, t_d)^{l_d} \left(g^{[p^n]}\right) \;\middle|\; \{t_1, \dots, t_d\} = T,\ 0 \leqslant l_1, \dots, l_d < p \right\}, \tag{2}$$

where T spans G with H. Applying induction by d with obvious basis for $d=0$, it is enough to show that $(\mathrm{ad}\, t)^i(g^{[p^n]}) \in \mathrm{id}_H^p\{g^{[p^{n-1}]}\}$, for $i = 0, 1, \dots, p-1$. If $i > 0$ then using the main identity of restricted Lie algebras (1) and the Leibniz rule, one can write a commutator formula as follows. All commutators are left-normed, that is $[u, v, w] = [u, [v, w]]$, for any $u, v, w \in G$.

$$\begin{aligned}
(\mathrm{ad}\, t)^i\left(g^{[p^n]}\right) &= \Big[\underbrace{t, \dots, t}_{i}, g^{[p^n]}\Big] = -\Big[\underbrace{t, \dots, t}_{i-1}, \left[g^{[p^n]}, t\right]\Big] \\
&= -\Big[\underbrace{t, \dots, t}_{i-1}, \Big[\underbrace{g^{[p^{n-1}]}, \dots, g^{[p^{n-1}]}}_{p}, t\Big]\Big] \\
&= \Big[\underbrace{t, \dots, t}_{i-1}, \Big[\underbrace{g^{[p^{n-1}]}, \dots, g^{[p^{n-1}]}}_{p-1}, \left[t, g^{[p^{n-1}]}\right]\Big]\Big].
\end{aligned}$$

Thus, applying the Leibniz rule, we can write

$$(\mathrm{ad}\, t)^i\left(g^{[p^n]}\right) = \sum_{k_1, \dots k_p} \left[(\mathrm{ad}\, t)^{k_1}\left(g^{[p^{n-1}]}\right), \left[\dots, \left[(\mathrm{ad}\, t)^{k_{p-1}}\left(g^{[p^{n-1}]}\right), (\mathrm{ad}\, t)^{k_p}\left(g^{[p^{n-1}]}\right)\right]\right]\right].$$

It is required, in the latter sum, that $k_1 + \cdots + k_p = i$ and $k_p > 0$. Now for each $j = 1, \ldots, p$ one has $(\operatorname{ad} t)^{k_j}(g^{[p^{n-1}]}) \in H$. Also, because $i < p$, some $k_j = 0$. Thus, indeed, if $i > 0$, the expression in question is in the ideal generated by $g^{[p^{n-1}]}$. If $i = 0$ then $(\operatorname{ad} t)^0(g^{[p^n]}) = (g^{[p^{n-1}]})^{[p]}$ is in the restricted ideal generated by $g^{[p^{n-1}]}$. □

Now we can complete the proof of Theorem 2.

Proof. As just proved, I_n is contained in the ideal J_m generated in H as a restricted ideal by the elements $g_1^{[p^m]}, \ldots, g_k^{[p^m]}$, where $m = n - d$ and $d = \dim G/H$. If we prove that the homomorphic image H/J_m of H/I_n is large, then also H/I_n is large. Since H/I_n is an ideal of finite codimension in G/I_n, we will be able to conclude that G/I_n is a large restricted Lie algebra.

Now let ε be a homomorphism of H onto a nonabelian free restricted Lie algebra L, assumed in the statement of our theorem. Set $K_m = \varepsilon(J_m)$. Then H/J_m admits a surjective homomorphism onto L/K_m. Now K_m is generated in L, as a restricted ideal, by the elements $\varepsilon(g_1)^{[p^m]}, \ldots, \varepsilon(g_k)^{[p^m]}$. By Lemma 8, for any set of elements $\varepsilon(g_1), \ldots, \varepsilon(g_k) \in L$, there is a number M such that if $m > M$ and K_m is the restricted ideal generated by $\varepsilon(g_1)^{[p^m]}, \ldots, \varepsilon(g_k)^{[p^m]}$ then L/K_m is large. Hence H/J_m is large by Proposition 3, claim (i), as desired. Thus the proof is complete. □

6. Constructing nil restricted Lie algebras

In what follows we will use the *derived p-series* $\{\delta_i(G) \mid i = 0, 1, \ldots\}$ of derivation stable ideals of a restricted Lie algebra G defined as follows. We set $\delta_0(G) = G$ and

$$\delta_i(G) = \big[\delta_{i-1}(G), \delta_{i-1}(G)\big] + \big(\delta_{i-1}(G)\big)^{[p]} \quad \text{for } i \geqslant 1.$$

Obviously, $G/\delta_1(G)$ is finite-dimensional as a finitely generated abelian restricted Lie algebra with all elements of nil-index p. By [1, 2.7.5] then $\delta_1(G)$ is a finitely generated restricted Lie algebra. Continuing in the same way, we obtain that each algebra $\delta_{i-1}(G)/\delta_i(G)$ is finite-dimensional and each $\delta_i(G)$ is finitely generated, for $i = 1, 2, \ldots$. Thus each algebra $G/\delta_i(G)$ is finite-dimensional and applying (1) and Engel's theorem [1, 1.7.3] we easily derive that each $G/\delta_i(G)$ is nilpotent as a Lie algebra.

The following lemma is immediate using an argument similar to the one used in Lemma 5.

Lemma 7. *For any finite-dimensional subspace V of a free restricted Lie algebra L there exists $d \in \mathbb{N}$ such that $\delta_d(L) \cap V = \{0\}$.*

The next proposition is a version of Theorem 2.

Proposition 9. *Let G be a finitely generated restricted Lie algebra. Suppose that P is an ideal of finite codimension in G that can be mapped homomorphically onto a nonabelian free restricted Lie algebra. Then for any element $g \in P$ there is $m \in \mathbb{N}$ such that if $g^{[p^n]} \in \delta_m(P)$ then $\delta_m(P)/\operatorname{id}_G^p\{g^{[p^n]}\}$ can be mapped homomorphically onto a nonabelian free restricted Lie algebra.*

Proof. Let $a = \dim G/P$. We start with proving a weaker claim.

Lemma 8. *Given an arbitrary $g \in P$ there exists m such that if $g^{[p^{n+a}]} \in \delta_m(P)$ then $\delta_m(P)/\operatorname{id}_G^p\{g^{[p^n]}\}$ can be mapped homomorphically onto a nonabelian free restricted Lie algebra.*

Let ε be a surjective homomorphism $\varepsilon: P \to L$, L a free restricted Lie algebra and $h = \varepsilon(g)$. If $h = 0$ then we set $m = 1$. Suppose $g^{[p^n]} \in \delta_1(P)$ and $I = \operatorname{id}_G^p\{g^{[p^{n+a}]}\}$. Then by Lemma 6, I is contained in $Q = \operatorname{id}_P^p\{g^{[p^n]}\}$. Thus $\delta_1(P)/I$ is naturally mapped onto $\delta_1(P)/Q$. Since under ε the element g is mapped into 0, ε induces a homomorphism $\bar{\varepsilon}$ of P/Q onto L. Applying Proposition 4, part (i), we find that $\delta_1(P)/Q$ maps homomorphically onto a nonabelian free restricted Lie algebra. So in the case where $h = 0$, the proof is complete.

Thus we may assume that $h \neq 0$. By Lemma 7, there is $d \in \mathbb{N}$ such that $h \notin \delta_d(L)$. We set $M = \delta_d(L)$. Then $h^{[p^d]} \in M$. If we denote by J the ideal of L generated by $h^{[p^d]}$ then by Proposition 2, $\overline{M} = M/J$ is a large Lie algebra with a presentation in which the number of generators exceeds the number of relations at least by 2. In this case, according to Proposition 1 after Theorem 1, there is a restricted ideal $\overline{N}$ of $\overline{M}$ such that $\overline{M}/\overline{N}$ is a nil cyclic restricted Lie algebra of nil-index c, for a natural number c, and there is a surjective homomorphism $\eta: \overline{N} \to L_1$ where L_1 is a nonabelian free restricted Lie algebra. It follows from the definition of the derived p-series that $\delta_c(\overline{M}) \subset \overline{N}$. Since $\delta_c(\overline{M})$ is an ideal of finite codimension in $\overline{N}$, we may apply Proposition 4, Part (i), to derive that $\delta_c(\overline{M})$ can be mapped homomorphically on a free restricted Lie algebra.

Let us set $m = d + c$. Assume $g^{[p^n]} \in \delta_m(P)$. Using Lemma 6, we conclude that $I = \operatorname{id}_G^p\{g^{[p^{n+a}]}\}$ as a restricted ideal of P is contained in the ideal $Q = \operatorname{id}_P^p\{g^{[p^n]}\}$. Thus $\delta_m(P)/I$ maps homomorphically onto $\delta_m(P)/Q$. If we apply a homomorphism induced by $\varepsilon: P \to L$ then $\delta_m(P)/J$ will be mapped onto $\delta_{d+c}(L)/R = \delta_c(M)/R$ where $R = \operatorname{id}_L^p\{h^{[p^n]}\}$. Since $R \subset J$, there is a natural homomorphism of $\delta_c(M)/R$ onto $\delta_c(\overline{M})$ which is mapped onto a free restricted Lie algebra, as it was shown above. Thus we have established Lemma 8.

Now we can complete the proof of Proposition 9. Indeed, once m, as in Lemma 8, has been found, choose n_0 minimal such that $g^{[p^{n_0}]} \in \delta_m(P)$. Then choose a *maximal* number $m' \geqslant m$ such that $g^{[p^{n_0+a}]} \in \delta_{m'}(P)$. Note that $g^{[p^{n_0+a-1}]} \notin \delta_{m'}(P)$. This m' is the number m sought for g in Proposition 9. For, if $g^{[p^{n'}]} \in \delta_{m'}(P)$ then $g^{[p^{n'-a}]} \in \delta_m(P)$ and so $\delta_m(P)/\operatorname{id}_G^p\{g^{[p^{n'}]}\}$ can be mapped homomorphically on a nonabelian free restricted Lie algebra. Since $\delta_{m'}(P)/\operatorname{id}_G^p\{g^{[p^{n'}]}\}$ is a subalgebra of finite codimension in $\delta_m(P)/\operatorname{id}_G^p\{g^{[p^{n'}]}\}$, by Proposition 3(ii), it can be mapped homomorphically on a free restricted Lie algebra. □

Now we can give a construction of infinite-dimensional finitely generated nil restricted Lie algebras and their generalizations claimed in Theorem 3 and Corollaries 1, 2 and 3. This construction is an analogue of a group-theoretic construction due to Olshanskii–Osin [7].

Proof. Let G be a finitely generated restricted Lie algebra and P an ideal that can be mapped homomorphically onto a free restricted Lie algebra. Let $\{f_1, f_2, \ldots\}$ be the list of all elements of P. We set $G_0 = G$, $P_0 = P$ and suppose we already constructed restricted Lie algebras $G_i \supset P_i$ which are the homomorphic images of G and P (and of G_{i-1}, P_{i-1} for $i > 0$), with the same kernel, so that $\delta_{r_i}(P_i)$ maps homomorphically on a nonabelian free restricted Lie algebra for some $r_i \in \mathbb{N}$ and in which all the images of $f_1, \ldots, f_i$ are nilpotent. Let g_{i+1} denote the image of f_{i+1} in G_i. Then we choose $r_{i+1} > r_i$ and $n = n(i) \in \mathbb{N}$, according to Proposition 9, so that $g_{i+1}^{[p^n]} \in \delta_{r_{i+1}}(P_i)$ and also $\delta_{r_{i+1}}(P_i)/\operatorname{id}_{G_i}^p\{g_{i+1}^{[p^n]}\}$ maps homomorphically onto a nonabelian free restricted Lie algebra. We set $G_{i+1} = G_i / \mathrm{id}^p_{G_i}\{g_{i+1}^{[p^n]}\}$ and $P_{i+1} = P_i / \mathrm{id}^p_{G_i}\{g_{i+1}^{[p^n]}\}$. By Proposition 4, Part (i), $\delta_{r_{i+1}}(P_{i+1})$ maps homomorphically on a nonabelian free restricted Lie algebra. Since $\delta_{r_i}(P_i)$ maps homomorphically on a nonabelian free restricted Lie algebra and $r_{i+1} > r_i$, it easily follows that $\delta_{r_{i+1}}(P_i)$ is a proper subalgebra of $\delta_{r_i}(P_i)$. Therefore,

$$\dim P_{i+1}/\delta_{r_{i+1}}(P_{i+1}) = \dim P_i/\delta_{r_{i+1}}(P_i) > \dim P_i/\delta_{r_i}(P_i). \tag{3}$$

Let K_i denote the common kernel of the natural homomorphisms $G \to G_i$ and $P \to P_i$. Clearly, $\overline{P} = P/\bigcup_{i=0}^{\infty} K_i$ is a nil restricted ideal in the restricted Lie algebra $\overline{G} = G/\bigcup_{i=0}^{\infty} K_i$. Further we set $\widetilde{P} = \overline{P}/\bigcap_{r=0}^{\infty} \delta_{r_i}(\overline{P})$, $\widetilde{G} = \overline{G}/\bigcap_{r=0}^{\infty} \delta_{r_i}(\overline{P})$. Then $\widetilde{P}$ is nil and residually finite-dimensional nilpotent, $\widetilde{G}$ is residually finite-dimensional, and $\widetilde{G}/\widetilde{P} \cong G/P$. To show that $\widetilde{G}$ is infinite-dimensional, we observe that $\mathrm{Ker}(P \to P_i) \subset \delta_{r_i}(P_i)$, for every i. Hence $P/\delta_{r_i}(P_i) \cong P_i/\delta_{r_i}(P_i)$. Now by (3), $\dim P_i/\delta_{r_i}(P_i) \to \infty$ as $i \to \infty$. Therefore $\widetilde{P}$ is infinite-dimensional since it maps homomorphically onto $P_i/\delta_{r_i}(P_i)$ for every i. □

Now we derive the Corollaries 1, 2 and 3.

In the case of Corollary 1, we know from Proposition 1 that any restricted Lie algebra G presented as described has an ideal P such that G/P is cyclic nil. If we apply to G and P the construction of the previous theorem, we obtain a finitely generated $\widetilde{G}$ with a nil-ideal $\widetilde{P}$ such that $\widetilde{G}/\widetilde{P} \cong G/P$ is nil. Obviously, then $\widetilde{G}$ is itself nil. Also, by Engel's theorem, any finite-dimensional nil Lie algebra is nilpotent. Whence our claim about $\widetilde{G}$ being residually finite-dimensional nilpotent.

Corollary 2 is a direct consequence of Corollary 1.

As for Corollary 3, we start with a finitely generated infinite-dimensional nil restricted Lie algebra G, over an algebraic closure $\overline{F}$ of F. Then we can consider an ordinary Lie algebra $\widehat{G}$ generated by a finite generating set X of G over F. As mentioned earlier, any element in G is a linear combination with coefficients in $\overline{F}$ of p-powers of the Lie monomials in X, that is, the p-powers of the elements in $\widehat{G}$. Were $\widehat{G}$ finite-dimensional then using another basic identity of restricted Lie algebras, $(x + y)^{[p]} = x^{[p]} + y^{[p]} + w(x, y)$ for any x, y in L and $w(x, y)$ in the ordinary Lie subring generated by x, y, we would easily obtain G finite-dimensional (over $\overline{F}$). Notice, that by (1) each $\mathrm{ad}\, g$ is nilpotent. So $\widehat{G}$ is an example of an infinite-dimensional finitely generated Engel Lie algebra.

References

[1] Y. Bahturin, Identical Relations in Lie Algebras, VNU Science Press, Utrecht, 1987.

[2] B. Baumslag, S.J. Pride, Groups with two more generators than relators, J. London Math. Soc. (2) 17 (1978) 425–426.

[3] E.S. Golod, On nil-algebras and residually finite p-groups, Izv. Akad. Nauk SSSR Ser. Mat. 28 (1964) 273–276 (in Russian).

[4] N. Jacobson, Lie Algebras, John Wiley and Sons, New York, 1962.

[5] G.P. Kukin, The equality problem and free products of Lie algebras and of associative algebras, Siberian Math. J. 24 (1983) 221–231, translation from Sibirsk. Mat. Zh. 24 (2) (1983) 85–96.

[6] M. Lackenby, Adding high powered relations to large groups, preprint, arXiv: math.GR/0512356, 2005.

[7] A.Yu. Olshanskii, D.V. Osin, Large groups and their periodic quotients, preprint, arXiv: math.GR/0601589, 2006.

[8] V.M. Petrogradsky, Examples of self-iterating Lie algebras, J. Algebra 302 (2006) 881–886.

[9] V.M. Petrogradsky, Growth of subalgebras for restricted Lie algebras and transitive actions, Internat. J. Algebra Comput. 15 (2005) 1151–1168.

[10] J.R. Stallings, Quotients of the power of the augmentation ideal in a group ring, in: Knots, Groups and 3-Manifolds, in: Ann. of Math. Stud., vol. 84, Princeton University Press, Princeton, 1975.